TENSOR OPERATORS UNDER SEMI-SIMPLE GROUPS

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Communicated by C. A. Coulson

Received 13 January 1961

ABSTRACT. Tensor operators under any group are defined and the theory is developed for semi-simple continuous groups. Coupled tensor operators are introduced and the matrix elements of tensor operators are expressed in terms of the coupling coefficients. The structure of generalized Casimir operators is investigated.

1. *Introduction.* The concept of an irreducible tensor operator under the 3-dimensional rotation group, R_3, was introduced by Racah (9). The present paper deals with the generalization to any semi-simple continuous group, although the definition of a tensor operator in § 2 applies to any group. Such a generalization was recognized by Racah (11, 13) in his work on the energy matrix of atomic systems and by Koster (7), who derived the generalization of the Wigner–Eckart theorem on matrix elements, but no systematic general theory has been given previously.

The theory of semi-simple groups and their representations has been presented by Racah (13) from the fundamental work of Cartan (1), Weyl (17, 18, 19), van der Waerden (16), and Casimir (3). The basic results are stated in § 3. The general theory is more complicated than that obtaining for the simple group R_3 (5, 14, 20) in two main respects: the weights of an irreducible representation are not all simple, and in the reduction of the Kronecker product of two irreducible representations, a given irreducible representation may occur more than once.

The fact that the infinitesimal operators of a continuous group form a tensor operator under the group is a basic property (§§ 2, 3).

The matrices which reduce the Kronecker product of two irreducible representations play an important part in the theory of tensor operators under R_3. Their elements are the coupling coefficients for the addition of angular momentum in quantum mechanics (20) and are also used to form coupled tensor operators (5). The generalization of coupling and recoupling coefficients is treated in §§ 4, 6 and some important coupling coefficients are obtained. In § 5 the general form of the Wigner–Eckart theorem is derived. In § 7 the structure of Racah's generalized Casimir operators (13) is investigated in terms of coupled tensor operators. All operators commuting with the infinitesimal operators are obtained in principle.

2. *Tensor operators.* The definition of an irreducible tensor operator under R_3 was given by Racah (9) in the context of quantum mechanics. Let (k) be an irreducible representation of R_3 by linear transformations in a $(2k+1)$-dimensional space with basis vectors $|k\mu\rangle$ $(\mu = k, k-1, \ldots, -k)$. Let the set of all vectors $|k\mu\rangle$ span an infinite-dimensional Hilbert space in which the linear operator R corresponding to an element ρ of R_3 is represented by the matrix $\|\langle k\nu|R|k\mu\rangle\|$ (diagonal in k). Then the matrix corresponding to ρ in the representation (k) of R_3 has the elements $\langle k\nu| R |k\mu\rangle$.

https://doi.org/10.1017/S0305004100035507 Published online by Cambridge University Press

Now let R be the rotation operator appropriate to a quantum-mechanical system which possesses a set of $2k+1$ operators $T(k\mu)\,(\mu = k, k-1, \ldots, -k)$ transforming under R_3 according to the representation (k). Then

$$R\,|k\mu\rangle = \langle k\nu|\,R\,|k\mu\rangle |k\nu\rangle, \tag{1a}$$

$$RT(k\mu)R^{-1} = \langle k\nu|\,R\,|k\mu\rangle T(k\nu) \tag{1b}$$

(using the summation convention) and the operators $T(k\mu)$ are said to be the (spherical) components of an irreducible tensor operator $T(k)$. This definition is equivalent to Racah's definition in terms of commutators (5, 14) (cf. equation 5). The angular momentum operators, which are the infinitesimal operators of R_3, form an irreducible tensor operator $J(1)$.

The generalization to any group is now straightforward. Let $|k\mu\rangle$ be the basis vectors of a representation (k) (not necessarily irreducible) of any group G, the dimension of the representation being $\dim(k)$. Let R denote the linear operator corresponding to an element ρ of G, in the Hilbert space spanned by all vectors $|k\mu\rangle$. A set of $\dim(k)$ linearly independent operators $T(k\mu)$ in this space will be said to be the components of a tensor operator $T(k)$ under G, belonging to the representation (k), if they transform under G according to the representation (k). Equations (1) then give the transformations of $|k\mu\rangle$ and $T(k\mu)$.

The characterization of representations as reducible, irreducible, or equivalent applies also to tensor operators belonging to those representations, for if $|k\mu\rangle, T(k\mu)$ are replaced by the linear combinations $S(\nu\mu)|\nu\rangle, S(\nu\mu)\,T(\nu)$, respectively, equations (1) hold for the new quantities with the matrix elements

$$\langle\sigma|\,R\,|\tau\rangle = S(\sigma\nu)\,\langle k\nu|\,R\,|k\mu\rangle\,S^{-1}(\mu\tau).$$

The infinitesimal operators X_μ of a continuous group G satisfy the commutation relations

$$[X_\mu, X_\nu] \equiv X_\mu X_\nu - X_\nu X_\mu = c^\sigma_{\mu\nu} X_\sigma \tag{2}$$

(8), where the structure constants satisfy

$$c^\sigma_{\mu\nu} = -c^\sigma_{\nu\mu},$$

$$c^\alpha_{\mu\nu} c^\beta_{\alpha\sigma} + c^\alpha_{\sigma\mu} c^\beta_{\alpha\nu} + c^\alpha_{\nu\sigma} c^\beta_{\alpha\mu} = 0. \tag{3}$$

Any element of the infinitesimal group of G may be written

$$R = 1 + \epsilon_\mu X_\mu, \tag{4}$$

where the ϵ_μ are infinitesimal parameters. Using (4) in (1 b) to the first order in the parameters yields the commutators

$$[X_\mu, T(k\nu)] = \langle k\sigma|\,X_\mu\,|k\nu\rangle T(k\sigma). \tag{5}$$

Equations (5) are sufficient to identify a tensor operator.

If a set of linearly independent operators $T\{i\}$ is such that commutation with the infinitesimal operators of G gives linear combinations of operators of the set

$$[X_\mu, T\{i\}] = \mu(ji)\,T\{j\}, \tag{6}$$

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then the matrices $\mu(ij)$ form an infinitesimal representation (K) of G, which may be taken with numerical factors as

$$\langle Ki|\, X_\mu \,|Kj\rangle = [c(j)/c(i)]\,\mu(ij), \tag{7}$$

since the Jacobi identities

$$[X_\mu, [X_\nu, T\{i\}]] - [X_\nu, [X_\mu, T\{i\}]] = [[X_\mu, X_\nu], T\{i\}]$$

yield the matrix elements of (2) with the matrix elements (7). It follows by comparison of (5), (6) that the operators form a tensor operator with components

$$T(Ki) = c(i)\, T\{i\}, \tag{8}$$

where the $c(i)$ are determined up to a constant factor by (7).

3. *Semi-simple groups.* The infinitesimal operators of a semi-simple continuous group G of rank N can be taken as $H_1, H_2, \ldots, H_N, E_\alpha, E_{-\alpha}, \ldots$, where (13, 18)

$$\left.\begin{aligned} [H_r, H_s] &= 0, \\ [H_r, E_\alpha] &= \alpha_r E_\alpha, \\ [E_\alpha, E_\beta] &= c_{\alpha\beta}^{\alpha+\beta} E_{\alpha+\beta} \quad (\alpha+\beta \neq 0), \\ [E_\alpha, E_{-\alpha}] &= \alpha^r H_r. \end{aligned}\right\} \tag{9}$$

Indices are raised and lowered with the symmetric matrix

$$g_{\mu\nu} = c_{\mu\alpha}^{\beta}\, c_{\nu\beta}^{\alpha}. \tag{10}$$

The existence of the inverse matrix $g^{\mu\nu}$ is a necessary and sufficient condition for a semi-simple group (1). The structure constant $c_{\mu\nu}^{\sigma}$ is zero unless $\mu + \nu = \sigma$. The quantities $c_{\mu\nu\sigma}$ are totally antisymmetric (2).

The vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a *root* in the N-dimensional Euclidean space with metric tensor g_{rs}. Taking rectangular axes,

$$g_{rs} = K\delta_{rs}.$$

The last equation of (9) ensures that the value of the only other non-zero components of $g_{\mu\nu}$ is (13)

$$g_{\alpha,-\alpha} = 1.$$

According to Racah (13) it is possible to choose the matrices of the H_r diagonal and the roots real. H_r is then self-adjoint and equations (9) show that the adjoint $E_\alpha^\dagger$ must be proportional to $E_{-\alpha}$. The infinitesimal operators may thus be chosen so that their adjoints are

$$H_r^\dagger = H_r, \quad E_\alpha^\dagger = E_{-\alpha}$$

or, since H_r corresponds to a zero root,

$$X_\mu^\dagger = X_{-\mu}. \tag{11}$$

This implies

$$c_{\alpha\beta}^{\gamma} = -\bar{c}_{-\alpha\,-\beta}^{-\gamma}, \tag{12}$$

the bar denoting the complex conjugate. This choice will now be assumed.

The representation $(k) = (k_1, k_2, \ldots, k_N)$ of G has *weights* $q = (q_1, q_2, \ldots, q_N)$ in the space of the root vectors and k is the *highest* weight of (k), one weight being higher than

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another if the first non-vanishing difference of their components is positive. The highest weight of an irreducible representation is *simple*, i.e. the weight k occurs only once in (k). For a *multiple* weight q one introduces a further set of numbers ξ to distinguish its different occurrences. A weight of (k) is thus labelled by $kq\xi$ or, adapting a notation of Spiers (15) for angular momentum, by $\mathbf{k} \equiv kq\xi$. If q, $-q$ are both weights of (k) they have the same multiplicity; (k) will then be said to be symmetrical. It is convenient to use the notation

$$-\mathbf{k} \equiv k, -q, \xi; \qquad {}'\mathbf{k} \equiv kq'\xi', \quad \delta(\mathbf{k}, \mathbf{k}') \equiv \delta(k, k')\,\delta(q, q')\,\delta(\xi, \xi'),$$

with the convention that a repetition of $\mathbf{k}$ implies summation over q, ξ if $\mathbf{k}$ occurs on one side of an equation only, $\mathbf{k}$ and $-\mathbf{k}$ counting equally as a repetition of $\mathbf{k}$.

The commutators (5) may now be written

$$[X_\mu, T(\mathbf{k})] = \langle {}'\mathbf{k}|\, X_\mu\, |\mathbf{k}\rangle T({}'\mathbf{k}). \tag{13}$$

Since the H_r are diagonal and E_α shifts q to $q+\alpha$ (18), the commutation relations in detail for a semi-simple group are

$$[H_r, T(kq\xi)] = \langle kq\xi|\, H_r |kq\xi\rangle T(kq\xi) = q_r T(kq\xi), \tag{14}$$
$$[E_\alpha, T(kq\xi)] = \sum_\eta \langle kq+\alpha\eta\,|E_\alpha|\, kq\xi\rangle T(kq+\alpha\eta).$$

The adjoint of (14) shows that an adjoint tensor operator, $T^\dagger(k^\dagger)$, can be defined by

$$T^\dagger(-\mathbf{k}^\dagger) = \delta(\mathbf{k})\, T(\mathbf{k})^\dagger, \quad |\delta(\mathbf{k})| = 1, \tag{15}$$

where $\mathbf{k}^\dagger \equiv k^\dagger q\xi$. There is a $1-1$ correspondence between the weights $\mathbf{k}$, $-\mathbf{k}^\dagger$, so $\dim(k^\dagger) = \dim(k)$. The representations $(k^\dagger)$, (k) are identical if (k) is symmetrical. This is not always the case; for example, labelling the representations of the group A_2 by partitions one has $[a\,a]^\dagger = [a\,0]$. The commutators (13) imply the relation

$$\langle -{}'\mathbf{k}^\dagger|\, X_\mu\, |-\mathbf{k}^\dagger\rangle = -[\delta(\mathbf{k})/\delta({}'\mathbf{k})]\langle \mathbf{k}|\, X_\mu\, |{}'\mathbf{k}\rangle. \tag{16}$$

Since the commutation relations (2) of the infinitesimal operators are a particular case of (6), the X_μ form a tensor operator under G belonging to a representation which will be denoted by (e). If G is simple, (e) is irreducible and e is the highest root of G (19). If G is semi-simple but not simple it is the direct product of n simple invariant subgroups (1)

$$G = G_1 \times G_2 \times \ldots \times G_n.$$

The root diagram of G consists of the root diagrams of the n groups G_r in mutually orthogonal subspaces of the root space ((6), p. 174). The rth root diagram corresponds to the irreducible representation (e_r) of G_r and the irreducible representation

$$(E_r) \equiv (0) \times (0) \times \ldots \times (e_r) \times \ldots \times (0)$$

of G, where (0) denotes the 1-dimensional representation $(0, 0, \ldots, 0)$ of a group. Thus the representation (e) of G is reducible into the direct sum of n irreducible representations.

By (7) the matrices of (e) are

$$\langle e\sigma|\, X_\mu\, |e\nu\rangle = [c(\nu)/c(\sigma)]\, c^\sigma_{\mu\nu} \tag{17}$$

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and by (8) the components of the tensor operator $X(e)$ are

$$X(e\mu) = c(\mu)X_\mu. \tag{18}$$

If the matrices (17) are the matrix representations of the X_μ as linear operators in Hilbert space as well as a representation of G, one must have, corresponding to (11),

$$\langle e\sigma|\, X_{-\mu}\,|e\nu\rangle = \overline{\langle e\nu|\, X_\mu\,|e\sigma\rangle}. \tag{19}$$

This requires that $g_{\mu\,-\mu}\,|c(\mu)|^2$ (no sum) is independent of μ. The choice which will be made here is

$$c(\mu) = +\sqrt{g^{\mu-\mu}},$$

which implies a choice of phases for R_3 which differs from the usual one (5, 9, 14, 20). Equations (17), (18) may now be written

$$\langle e\sigma|\, X_\mu\,|e\nu\rangle = \sqrt{(g^{\nu-\nu}g^{\sigma-\sigma})}\, c_{\mu\nu-\sigma}, \tag{20}$$

$$X(e\mu) = \sqrt{(g^{\mu-\mu})}\, X_\mu. \tag{21}$$

If G' is a subgroup of G and primed quantities refer to G', the representation (e) of G is a reducible representation of G' which contains (e')

$$(e) = (e') + \Sigma(f').$$

The infinitesimal operators of G may thus be expressed, by taking linear combinations, as the infinitesimal operators of G', $X(e')$, together with irreducible tensor operators $T(f')$ under G'.

4. *Coupling coefficients.* If the Kronecker product of two irreducible representations (a), (b) of a group G is completely reducible, as it is for semi-simple groups (19), an irreducible representation (c) occurs an integral number of times in the reduction. These representations (c) will be labelled by a symbol ρ.

If $\psi(\mathbf{a})$, $\phi(\mathbf{b})$ are the basis vectors of (a), (b), respectively, the reduction of $(a) \times (b)$ is achieved by

$$\Psi(\rho\mathbf{c}) = C_\rho(\mathbf{abc})\,\psi(\mathbf{a})\,\phi(\mathbf{b}). \tag{22}$$

The bases can be chosen so that this transformation is unitary. The inverse transformation is then

$$\psi(\mathbf{a})\,\phi(\mathbf{b}) = \sum_{c,\,\rho} \bar{C}_\rho(\mathbf{abc})\,\Psi(\rho\mathbf{c}). \tag{23}$$

The transformations (22), (23) imply the following orthogonality relations for the coupling coefficients

$$C_\rho(\mathbf{abc})\,\bar{C}_\sigma(\mathbf{abc'}) = \delta(\mathbf{c},\mathbf{c'})\,\delta(\rho,\sigma), \tag{24a}$$

$$\sum_{c,\,\rho} C_\rho(\mathbf{abc})\,\bar{C}_\rho(\mathbf{'a'bc}) = \delta(\mathbf{a},\mathbf{'a})\,\delta(\mathbf{b},\mathbf{'b}). \tag{24b}$$

More generally, the Kronecker product of two reducible representations can be reduced to a direct sum of reducible representations. Hence the restriction of (a), (b), (c) to irreducible representations can be lifted, the corresponding coupling coefficients again satisfying equations (24).

By operating on (22) with any element of G, the coupling coefficients are found to satisfy the relation

$$C_\rho(\mathbf{ab'c})\,\langle\mathbf{'c}|\,R|\mathbf{c}\rangle = \langle\mathbf{a}|\,R\,|\mathbf{'a}\rangle\,\langle\mathbf{b}|\,R\,|\mathbf{'b}\rangle\,C_\rho(\mathbf{'a'bc}).$$

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By virtue of this relation and (1*b*), tensor operators may be coupled with the same coefficients:

$$[T(a)\,U(b)]\,(\rho\mathbf{c}) = C_\rho(\mathbf{abc})\,T(\mathbf{a})\,U(\mathbf{b}).$$

It is important to show that the coupling of two irreducible tensor operators to give the identity representation (0) is unique for semi-simple groups. Since $[T(k)\,U(j)]\,(0)$ commutes with each X_μ and the components of the two tensor operators are linearly independent,

$$\langle'\mathbf{k}|\,X_\mu\,|\mathbf{k}\rangle\,C(\mathbf{kj}0) = -\,C('\mathbf{k}'\mathbf{j}0)\,\langle\mathbf{j}|\,X_\mu\,|'\mathbf{j}\rangle,$$

using (13). If $\mathbf{k} = kq\xi$, $\mathbf{j} = jm\eta$, taking $X_\mu = H_r$ shows that $C(\mathbf{kj}0)$ vanishes unless $m = -q$. Then by (16),

$$\langle'\mathbf{k}|X_\mu|\,\mathbf{k}\rangle\langle\mathbf{k};\,-\mathbf{j}^\dagger\rangle = \langle'\mathbf{k};\,-'\mathbf{j}^\dagger\rangle\langle-'\mathbf{j}^\dagger|\,X_\mu\,|-\mathbf{j}^\dagger\rangle,$$

where

$$\langle\mathbf{k};\,-\mathbf{j}^\dagger\rangle \equiv C(\mathbf{kj}0)\,\delta(\mathbf{j}).$$

Hence, by Schur's lemma, (k) is equivalent to $(j^\dagger)$, and there is then a $1-1$ correspondence between the weights $\mathbf{k}$, $-\mathbf{j}$. It follows that $k = j^\dagger$ and by Schur's lemma $\langle\mathbf{k};\,-\mathbf{j}^\dagger\rangle$ is a multiple of the unit matrix. Normalizing the coupling coefficients by (24*a*) and using (15) yields

$$C(\mathbf{kj}0) = \delta(\mathbf{j},\,-\mathbf{k}^\dagger)/[\delta(\mathbf{k})\sqrt{\dim(k)}], \tag{25}$$

$$[T(k)\,U^\dagger(k^\dagger)]\,(0) = T(\mathbf{k})\,U(\mathbf{k})^\dagger/\sqrt{\dim(k)}.$$

An important example of coupled tensor operators is Casimir's operator (3)

$$\mathscr{C} \equiv g^{\mu\nu}X_\mu X_\nu \tag{26}$$

$$= X(\mathbf{e})\,X(-\mathbf{e}).$$

If G is simple, (e) is irreducible and (25) gives, with $\delta(\mathbf{e}) = 1$, which is a possible choice according to (16), (20),

$$C(\mathbf{e}'\mathbf{e}0) = \delta('\mathbf{e},\,-\mathbf{e})/\sqrt{\dim(e)}. \tag{27}$$

Hence

$$\mathscr{C} = \sqrt{(\dim(e))}\,[X(e)\,X(e)]\,(0). \tag{28}$$

$\mathscr{C}$ commutes with every X_μ (3), as is obvious from the fact that $\mathscr{C}$ belongs to the representation (0).

If G is only semi-simple, $X(e)$ is the direct sum of n commuting irreducible tensor operators $X(E_r)$ (cf. § 3) and

$$\mathscr{C} = \sum_r X(\mathbf{E}_r)\,X(-\mathbf{E}_r).$$

Each term in the sum over r is a Casimir operator for a G_r, but commuting with every infinitesimal operator of G. Thus the representation (0) occurs n times in the reduction of $(e)\times(e)$ and a set of independent coupling coefficients is given by

$$C_r(\mathbf{e}'\mathbf{e}0) = \delta('\mathbf{e},\,-\mathbf{e})\,\delta(\mathbf{e},\mathbf{E}_r)/\sqrt{\dim(e_r)}, \quad (r = 1, 2, \ldots, n).$$

It is convenient to introduce $C_0(\mathbf{e}'\mathbf{e}0)$ as the coupling coefficient given by (27); then $\mathscr{C}$ is always given by (28) with this coupling.

Since $\mathscr{C}$ commutes with every X_μ its matrix in an irreducible representation (k) is a multiple $\mathscr{C}(k)$ of the unit matrix, by Schur's lemma. Whether (e) is irreducible or not, the matrix of $\mathscr{C}$ in the representation (e) is, by (20), (21),

$$\langle e\mu|\,X(\mathbf{e})\,|e\sigma\rangle\langle e\sigma|\,X(-\mathbf{e})\,|e\nu\rangle = \delta(\mu,\nu).$$

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Hence the eigenvalue $\mathscr{C}(e)$ is always given by

$$\mathscr{C}(e) = 1. \tag{29}$$

It will now be shown that there exist coupling coefficients given by

$$C_1(eee; \mu\nu\sigma) = \langle e\nu | X(e, -\mu) | e\sigma \rangle \tag{30a}$$

$$= \sqrt{(g^{\mu-\mu} g^{\sigma-\sigma} g^{\nu-\nu})}\, c_{-\mu\sigma-\nu}. \tag{30b}$$

The normalization and orthogonality relations (24a) are satisfied since, by (19), (26), (29), (suppressing some e's)

$$C_1(\mu\nu\sigma)\, \bar{C}_1(\mu\nu\tau) = \langle e\tau | \mathscr{C} | e\sigma \rangle = \delta(\sigma, \tau).$$

The coupling of $X(e)$ is given by

$$\begin{aligned}[X(e)\, X(e)]\,(1e\sigma) &= \tfrac{1}{2} C_1(\mu\nu\sigma)\, [X(\mu), X(\nu)] \\ &= \tfrac{1}{2} C_1(\mu\nu\sigma)\, C_1(-\mu\tau\nu)\, X(\tau) \\ &= \tfrac{1}{2} X(e\sigma),\end{aligned}$$

using (24a) and the symmetries of C_1.

5. *Matrix elements.* The matrix elements of the components of a tensor operator $T(a)$ under a group G may be expressed in a form which is a generalization of the Wigner–Eckart theorem for R_3 (4, 20). The proof follows the method of Edmonds (5).

It is possible to couple $T(a)$ with the basis vectors $|\mathbf{b}\rangle$:

$$|T(a)\, b\rho\mathbf{c}\rangle \equiv C_\rho(\mathbf{abc})\, T(\mathbf{a})\, |\mathbf{b}\rangle.$$

The inverse transformation is

$$T(\mathbf{a})\, |\mathbf{b}\rangle = \sum_{c,\rho} \bar{C}_\rho(\mathbf{abc}) |T(a)\, b\rho\mathbf{c}\rangle.$$

The matrix elements of $T(\mathbf{a})$ are thus

$$\langle \mathbf{c}| T(\mathbf{a})\, |\mathbf{b}\rangle = \sum_\rho \bar{C}_\rho(\mathbf{abc})\, \langle \mathbf{c}| T(a)\, b\rho\, |\mathbf{c}\rangle,$$

but a transformation coefficient of the form $\langle x\mathbf{c}|y\mathbf{c}\rangle$ is the same for each component of $\mathbf{c}$, as may be seen by operating on the transformation equations with any element of G. Hence one may introduce reduced (double-barred) matrix elements and write

$$\langle \mathbf{c}| T(\mathbf{a})\, |\mathbf{b}\rangle = \sum_\rho \bar{C}_\rho(\mathbf{abc})\, \langle c \| \rho T(a) \| b \rangle. \tag{31}$$

This result was stated by Racah (11) and has been proved independently by Koster (7).

Comparing (30a) with (31) and using (12) shows that

$$\langle e \| \rho X(e) \| e \rangle = \delta(\rho, 1).$$

6. *Recoupling coefficients.* The generalization of Racah's recoupling coefficients (10) is easily obtained. Three irreducible representations, (a), (b), (c), may be coupled in two different ways

$$\Psi(ab(\rho f)\, c\sigma\mathbf{d}) = C_\rho(\mathbf{abf})\, C_\sigma(\mathbf{fcd})\, \psi(\mathbf{a})\, \phi(\mathbf{b})\, \chi(\mathbf{c}),$$

$$\begin{aligned}\Phi(a, bc(\lambda g)\, \mu\mathbf{d}) &= C_\lambda(\mathbf{bcg})\, C_\mu(\mathbf{agd})\, \psi(\mathbf{a})\, \phi(\mathbf{b})\, \chi(\mathbf{c}) \\ &= \sum_{f\rho\sigma} C(abcd; fg; \rho\sigma\lambda\mu)\, \Psi(ab(\rho f)\, c\sigma\mathbf{d}),\end{aligned}$$

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where the recoupling coefficient C is the same for each component of $\mathbf{d}$, as in § 5. These equations yield the relation

$$C_\lambda(\mathbf{bcg})\,C_\mu(\mathbf{agd}) = \sum_{f\rho\sigma} C(abcd; fg; \rho\sigma\lambda\mu)\,C_\rho(\mathbf{abf})\,C_\sigma(\mathbf{fcd}). \tag{32}$$

Relations involving sums of three and four coupling coefficients follow easily from (32) by use of the orthogonality relations (24).

Putting $d = 0$ in (32) and using (25) yields

$$C_\lambda(\mathbf{b}-\mathbf{c}-\mathbf{a}) = \frac{\delta(\mathbf{a}^\dagger)}{\delta(\mathbf{c}^\dagger)}\left[\frac{\dim(a)}{\dim(c)}\right]^{\frac{1}{2}} \sum_\rho C(a^\dagger bc0; c^\dagger a; \rho 0\,\lambda 0)\,C_\rho(\mathbf{a}^\dagger\mathbf{b}\mathbf{c}^\dagger).$$

Equation (32) also holds if the representations involved are not irreducible.

7. *Generalized Casimir operators.* Racah (12) has shown that there exist N polynomials $F(X)$ in the X_μ, which commute with each X_μ and whose eigenvalues completely specify the irreducible representations $(k_1\,k_2 \ldots k_N)$ of a semi-simple group of rank N. By the work of § 2, a necessary and sufficient condition for $F(X)$ to commute with each X_μ is that it should be a tensor operator belonging to the representation (0). It follows that the operators $F(X)$ are given by the independent couplings of the infinitesimal operators to (0).

The simplest of the operators $F(X)$ is Casimir's operator (26). The generalized Casimir operators of Racah (13) are

$$\mathscr{C}_p \equiv g^{\mu\nu\ldots\tau} X_\mu X_\nu \ldots X_\tau,$$
$$g_{\mu\nu\ldots\tau} = c^\beta_{\mu\alpha} c^\gamma_{\nu\beta} \ldots c^\alpha_{\tau\theta},$$

where $g_{\mu\nu\ldots\tau}$ has p indices. The non-zero coefficients have $\mu+\nu+\ldots+\tau = 0$. The coefficients are left unchanged by a cyclic permutation of indices and the relation

$$c^\beta_{\mu\alpha} = -g^{\beta-\beta} c_{\mu-\beta\alpha}$$

leads to

$$g_{\mu\nu\ldots\tau} = (-1)^p g_{\tau\ldots\nu\mu}. \tag{33}$$

By (26), $\mathscr{C}_2 \equiv \mathscr{C}$. The coefficients of $\mathscr{C}_3$ are totally antisymmetric by (33) and hence the relation

$$c_{\mu\nu\sigma} = g_{\mu\sigma\nu} - g_{\mu\nu\sigma}$$

(2), which follows from (3), (10), reduces to

$$g_{\mu\nu\sigma} = -\tfrac{1}{2} c_{\mu\nu\sigma}.$$

Hence, by (30b), (21),

$$\mathscr{C}_3 = \tfrac{1}{2} C_1(\mu\nu-\sigma)\,X(\mu)\,X(\nu)\,X(\sigma)$$
$$= \tfrac{1}{4}\mathscr{C}_2.$$

Combining (30b) with (21) yields the general formula

$$\mathscr{C}_p = C_1(\mu\beta\alpha)\,C_1(\nu\gamma\beta)\ldots C_1(\tau\alpha\theta)\,X(\mu)\,X(\nu)\ldots X(\tau).$$

On recoupling with (32), $\mathscr{C}_p$ may be written as a linear combination of operators

$$[X(e)\,X(e)\,|\,(\rho k)\,X(e)\ldots X(e)|\,(\rho' e)\,X(e)]\,(\sigma 0).$$

The term for which each $(\rho k) = (1e)$ and $\sigma = 0$ reduces to a multiple of $\mathscr{C}_2$.

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By expressing $\mathscr{C}_3$ in different ways it can be shown that, in an obvious notation,

$$C(e^4; e^2; 1^4) = \tfrac{1}{2}.$$

Thanks are due to Mr S. Sankaran for criticism of the presentation.

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https://doi.org/10.1017/S0305004100035507 Published online by Cambridge University Press