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Additive Problems with Primes from a Thin Bohr Set

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Abstract

For an irrational $\alpha \in \mathbb{R}$ and any $\beta \in \mathbb{R}$, we consider additive problems with the set of primes satisfying $\|\alpha p + \beta\| \leq 1/p^{\tau}$ for some fixed $\tau > 0$. In particular, we show that there exist infinitely many non-trivial three-term arithmetic progressions in the set of primes satisfying $\|\alpha p + \beta\| \leq 1/p^{\tau}$ for $\tau \in (0, 1/8)$. We also consider a binary Goldbach-type problem.

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1. *Introduction*

Let $\|\cdot\|$ denote the distance from a real number to the nearest integer(s).

Let α be a fixed irrational number, $\beta \in \mathbb{R}$, and $\tau > 0$ be fixed. In this paper, we study additive problems for primes p satisfying

$$\|\alpha p + \beta\| \leq p^{-\tau}. \tag{1·1}$$

A classical problem in number theory is to measure how closely rationals can approximate an irrational number. A further refinement is to restrict the denominators to special sets, prime numbers being a particularly natural choice. In fact, for almost every irrational α, there are infinitely many primes p for which (1·1) holds with $\tau = 1$ (see, e.g., [**5**]).

When α is fixed, much less is known. For instance, when $\beta = 0$, Matomäki [**14**], building on results of Harman [**9**] and of Heath-Brown and Jia [**10**], showed that for every fixed $\tau \in (0, 1/3)$, there are infinitely many primes p that satisfy (1·1). It is widely conjectured that the same holds for all $\tau \in (0, 1)$. In fact, one expects the number of primes $p \leq N$ satisfying (1·1) to be $\asymp N^{1-\tau}/\log N$ for any fixed $\tau \in (0, 1)$, along an infinite sequence of values of N. Conversely, Harman [**8**] proved that for $\tau = 1$ there are uncountably many irrationals α for which (1·1) holds for at most finitely many primes p.

It is natural to ask whether the primes p satisfying (1·1) exhibit any additional arithmetic structure. For example, Shi [**16**], improving the work of Matomäki [**15**], showed that there are infinitely many Chen primes p satisfying (1·1) for $\tau \in (0, 3/200)$. For other patterns in primes satisfying (1·1), see also [**7, 17, 18**]. These results suggest that the distribution of such primes retains significant structure despite the Diophantine constraint.

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Studying three-term arithmetic progressions within subsets of the primes has long driven advances in additive number theory, giving rise to techniques now central to the field. Van der Corput [3] first applied the circle method to show that the primes contain infinitely many nontrivial three-term progressions. Green [6] later introduced the transference principle, demonstrating that any dense subset of the primes likewise contains infinitely many such progressions. More recently, building on Kelley–Meka's breakthrough [11], Bloom and Sisask [2] proved that if

$$\mathcal{A} \subseteq [1, N] \cap \mathbb{Z}$$

contains no nontrivial three-term arithmetic progressions, then

$$|\mathcal{A}| \leq \exp\left(-c(\log N)^{1/9}\right)N.$$

By the prime number theorem, both Van der Corput's and Green's theorems follow immediately.

Nevertheless, by suitably adapting either the circle method or the transference principle, one can still force three-term progressions in extremely sparse yet highly structured subsets of the primes. For instance, Balog–Friedlander applied the circle method to study additive problems in Piatetski–Shapiro primes and in sparse sets of primes satisfying the inequality $\|p^{\gamma}\| \leq p^{-2(1-\gamma)/3}$ for some fixed γ in $(22/25, 1]$. More recently, Du-Pan-Sun [4, theorem 1·3], using Green's Fourier-analytic transference principle, established that an even thinner set of Piatetski–Shapiro primes contains infinitely many nontrivial three-term arithmetic progressions.

Analogously, one may ask about three-term progressions among the primes p satisfying (1·1), which are known to be sparse. In this work, we adapt the classical circle method to this context and establish the following theorem.

THEOREM 1·1. *Let $\alpha \in \mathbb{R}$ be irrational, $\beta \in \mathbb{R}$, and $\tau \in (0, 1/8)$. Then there exist infinitely many non-trivial triples of primes (p_1, p_2, p_3) satisfying*

$$\left\|\alpha p_j + \beta\right\| \leq \frac{1}{p_j^{\tau}} \quad \text{for } j = 1, 2, 3,$$

and

$$p_1 + p_3 = 2p_2.$$

One might ask whether Green's Fourier-analytic transference principle can be employed to improve the range of τ in the theorem above. However, the construction of a majorant satisfying a suitable restriction estimate appears to be obstructed by the Diophantine constraint, as its indicator function exhibits many large Fourier coefficients. An analogous difficulty arises in the study of additive problems over primes with restricted digits as noted in [13, section 1·1].

We also consider the binary Goldbach problem: expressing even positive integers as the sum of two primes. The triangle inequality prevents such a decomposition for most even integers when both primes simultaneously satisfy (1·1). Therefore, we impose the Diophantine condition on only one of them. Our result is the following:

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THEOREM 1·2. *Let $\alpha \in \mathbb{R}$ be irrational, $\beta \in \mathbb{R}$, and $\tau \in (0, 1/8)$. There exists a sequence $\{N_k\}_{k\geq 1}$ of positive integers going to infinity such that the following holds. For all but $o(N_k)$ even integers $n \leq N_k$, there exist primes p and q such that*

$$p + q = n, \quad \|\alpha p + \beta\| \leq p^{-\tau}.$$

1.1. Organisation of the paper

We begin in Section 2 by introducing the basic notation used throughout the paper. In Section 3, we give an informal outline of the method to orient the reader.

In Section 4, we establish a key frequency repulsion estimate. Since the condition (1·1) does not admit an obvious pseudorandomisation, we are forced to contend with multiple frequencies simultaneously. To manage this, we exploit Diophantine approximation to isolate one frequency at a time. The arguments are elementary in nature, relying on the pigeonhole principle.

We introduce a smooth weight function in Section 5 and examine its properties. This function serves to smooth the abrupt cutoff imposed by the condition (1·1), making the analysis easier.

In Section 6, we present a slight modification of the traditional circle method designed to handle multiple frequencies. In Section 7, we establish major and minor arc estimates that go into our variant of the circle method.

In Sections 8 and 9, we finish the proofs of Theorems 1·1 and 1·2 respectively.

2. *Notation*

Throughout, we write $e(\theta) = e^{2\pi i\theta}$, $\|\alpha\|$ for the distance from α to the nearest integer(s), and $\Lambda(n)$ for the von Mangoldt function. We think of $\mathbb{T}$ as the metric space $(\mathbb{R}/\mathbb{Z}, \|\cdot\|)$. We use the notation $f \ll g$ (or $f = O(g)$) to mean there is a constant C such that $f(x) \leq C\, g(x)$ for all sufficiently large x, and $f \ll_A g$ (or $f = O_A(g)$) when that constant may depend on the parameter A. We say that $f = o(g)$ if $f(n)/g(n)$ goes to 0 as n tends to infinity. We use $f \asymp g$ when $g \ll f$ and $f \ll g$ hold simultaneously. For any set or property $\mathcal{S}$, $\mathbf{1}_{\mathcal{S}}$ denotes its indicator function. If $h \in L^1(\mathbb{T})$, its Fourier coefficients are

$$\widehat{h}(m) := \int_{\mathbb{T}} h(\theta)\, e(-m\theta)\, d\theta, \qquad m \in \mathbb{Z}.$$

Finally, if $f, g \in L^1(\mathbb{T})$, their Fourier convolution is defined by

$$(f * g)(\varphi) = \int_{\mathbb{T}} f(\theta)\, g(\varphi - \theta)\, d\theta.$$

One immediately checks the standard identity

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}. \tag{2·1}$$

3. *Outline of the method*

We sketch the proof of Theorem 1·1 in this section. Let $w_\delta(\|\theta\|)$ be a smoothed indicator for the condition $\|\theta\| \leq \delta$. To find three-term arithmetic progressions in the set of primes p satisfying $\|\alpha p + \beta\| \leq \delta$, by an application of the circle method, it suffices to show that for

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an infinite sequence of values of N going to infinity, we have that

$$\int_{\mathbb{T}} S_w(\theta)^2 S_w(-2\theta)d\theta \gg \delta^2 N^2,$$

where

$$S_w(\theta) := \sum_{n\leq N} w(\|\alpha n+\beta\|)\Lambda(n)e(n\theta).$$

By Fourier expanding (see Lemma 5·1 below), we have that

$$w_\delta(\|\theta\|) \approx \sum_{|\ell|\leq 1/\delta} \widehat{w_\delta}(\ell)e(\ell\theta).$$

Thus,

$$S_w(\theta) \approx \sum_{|\ell|\leq 1/\delta} \widehat{w_\delta}(\ell)e(\ell\beta)S(\theta+\ell\alpha),$$

where

$$S(\theta) := \sum_{n\leq N} \Lambda(n)e(n\theta).$$

In the circle method, one splits $\mathbb{T}$ into arcs where the exponential sums could be large and the complement. Since $S(\theta)$ can only be large when θ is close to rationals with small denominators, thus $S_w(\theta)$ can be large only if at least one of the phases $\theta+\ell\alpha$, with $|\ell|\leq 1/\delta$, lies near a rational with small denominator. Motivated by this, we define the arcs

$$\mathcal{M}_{a,q,\ell} := \left\{\theta\in\mathbb{T} : \left\|\theta+\ell\alpha-\frac{a}{q}\right\| \leq \frac{P}{qN}\right\}.$$

Let

$$\mathcal{M} := \bigcup_{q<P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \bigcup_{|\ell|\leq 1/\delta} \mathcal{M}_{a,q,\ell}.$$

We would expect $S_w(\theta)$ to be only large when $\theta\in\mathcal{M}$. A priori, the arcs $\mathcal{M}_{a,q,\ell}$ may not be disjoint from each other but let's for now assume that they are disjoint. If $\theta\in\mathcal{M}_{a,q,m}$, then

$$S_w(\theta) \approx \widehat{w_\delta}(m)e(m\beta)S(\theta+m\alpha) + O\left(\delta \sum_{\substack{|\ell|\leq 1/\delta\\ \ell\neq m}} |S(\theta+\ell\alpha)|\right).$$

As $\theta+\ell\alpha$ is not close to rationals with small denominators for $\ell\neq m$ by the disjointness of the arcs $\mathcal{M}_{a,q,\ell}$, we can use the minor arcs estimate for $S(\theta+\ell\alpha)$ when $\ell\neq m$.

To use the minor arc estimate efficiently, we introduce a number of scales $N^{2/5}\geq P_1 > P_2 > \ldots > P_J = P$. Let $\mathcal{E}_0 := \{|\ell|\leq 1/\delta\}$, and

$$\mathcal{E}_j := \left\{|\ell|\leq 1/\delta : \left\|\theta+\ell\alpha-\frac{a}{q}\right\| \leq \frac{P_j}{qN} \text{ for some } q<P_j \text{ and } a\leq q \text{ with } (a,q)=1\right\}.$$

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Let r and s be coprime integers such that

$$\left|\alpha - \frac{r}{s}\right| \leq \frac{1}{s^2}.$$

A combinatorial lemma (see Lemma 4·2 below) allows us to bound the cardinalities of the sets $\mathcal{E}_j$. Provided $N \geq 16sP_j^2$, we have that

$$\left|\mathcal{E}_j\right| \leq \frac{16P_j^2}{\delta s} + 1.$$

For $\ell \notin \mathcal{E}_j$, we have that

$$|S(\theta + \ell\alpha)| \ll (\log N)^4 \left(\frac{N}{\sqrt{P_j}} + N^{4/5}\right) \ll \frac{N(\log N)^4}{\sqrt{P_j}}.$$

Thus, we have that

$$S_w(\theta) \approx \widehat{w_\delta}(m)e(m\beta)S(\theta + m\alpha) + O\left(\delta N(\log N)^4 \sum_{1\leq j\leq J} \frac{\left|\mathcal{E}_{j-1} \setminus \mathcal{E}_j\right|}{\sqrt{P_j}}\right).$$

Substituting the bound for $\left|\mathcal{E}_j\right|$, we have that

$$S_w(\theta) \approx \widehat{w_\delta}(m)e(m\beta)S(\theta + m\alpha) + O\left(N(\log N)^4 \left(\frac{1}{\sqrt{P_1}} + \sum_{2\leq j\leq J} \frac{P_{j-1}^2}{s\sqrt{P_j}}\right)\right).$$

We need the term in $O(\cdot)$ above to be $o(\delta N)$, which roughly corresponds to

$$\sqrt{\frac{N}{s}} > P_1 > 1/\delta^2, \quad \text{and} \quad \delta^2 s^2 P_j > P_{j-1}^4.$$

Additionally, recall that we need to ensure disjointness of the arcs $\mathcal{M}_{a,q,\ell}$. For this, we need $\mathcal{E}_J < 2$ and hence

$$\delta s > 16P_j^2.$$

If we set $P_j = N^{p_j}$ and $\delta = N^{-\mu}$, then the above conditions are satisfied when

$$p_j \approx \frac{1}{4} - \frac{(2\cdot 4^j - 5)(1 - 8\mu)}{12} \quad \text{for } 1 \leq j \leq J \quad \text{and} \quad s \approx N^{1-4\mu}.$$

We choose denominator s along a sequence of convergents to α, and take N in terms of s so that $s \approx N^{1-4\mu}$. Additionally, to have the monotonicity condition for the scales $\{P_j\}_j$, we also impose the condition $\mu < 1/8$. A similar analysis can be used to show that $S_w(\theta)$ is small on $\mathbb{T} \setminus \mathcal{M}$. The theorem then follows by using these estimates for $S_w(\theta)$ (see Section 8 below for the details).

The estimates for $S_w(\theta)$ also can be used in various other additive problems concerning primes from thin Bohr sets to which the usual Fourier analytic circle method applies.

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4. *Exponential sum estimates*

In this section, we provide estimates to bound the exponential sums in our analysis. We first recall the well-known minor arc estimate for exponential sums over primes. Let

$$S(\theta; N) = S(\theta) := \sum_{n \le N} \Lambda(n) e(n\theta). \tag{4·1}$$

LEMMA 4·1. *Let $\beta \in \mathbb{R}$ be a real number for which there exist integers a and q, with $(a, q) = 1$, satisfying*

$$\left|\beta - \frac{a}{q}\right| \le \frac{1}{q^2}.$$

Then, for every $N \ge q$, we have

$$|S(\beta)| \ll (\log N)^4 \left(\frac{N}{q^{1/2}} + N^{4/5} + (Nq)^{1/2}\right).$$

Proof. The result follows from [19, theorem 3·1] as the contribution from prime powers is negligible.

In the next lemma, we show that among the phases $\{\theta + m\alpha\}_m$, only a limited number can be simultaneously approximated by rationals with small denominators.

LEMMA 4·2. *Let $\alpha, \theta \in \mathbb{T}$. Suppose r and s are coprime integers satisfying*

$$\left|\alpha - \frac{r}{s}\right| \le \frac{1}{s^2}. \tag{4·2}$$

Let P, M and N be positive real numbers such that $N \ge 16sP^2$. Let $\mathcal{E}(P; \theta) \subseteq [-M, M] \cap \mathbb{Z}$ denote the set of integers m such that there exist coprime integers a_m and q_m with $q_m < P$ such that

$$\left|\theta + m\alpha - \frac{a_m}{q_m}\right| \le \frac{P}{q_m N}.$$

Then

$$|\mathcal{E}(P; \theta)| \le \frac{16MP^2}{s} + 1.$$

Proof. Suppose for the sake of contradiction that

$$|\mathcal{E}(P; \theta)| > \frac{16MP^2}{s} + 1.$$

By the pigeon hole principle, we can find two integers $m_1, m_2 \in \mathcal{E}(P; \theta)$ such that

$$|m_1 - m_2| \le \frac{s}{8P^2}. \tag{4·3}$$

By the definition of $\mathcal{E}(P; \theta)$, for each $j \in \{1, 2\}$, there exist coprime integers a_j and q_j with $q_j < P$ such that

$$\left|\theta + m_j\alpha - \frac{a_j}{q_j}\right| \le \frac{P}{Nq_j}.$$

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Applying the triangle inequality yields

$$\left|(m_1-m_2)\alpha-\Big(\frac{a_1}{q_1}-\frac{a_2}{q_2}\Big)\right|\le\frac{P}{N}\Big(\frac{1}{q_1}+\frac{1}{q_2}\Big).$$

A further application of the triangle inequality combined with (4·2) and (4·3) gives

$$\left|(m_1-m_2)\frac{r}{s}-\Big(\frac{a_1}{q_1}-\frac{a_2}{q_2}\Big)\right|\le\frac{P}{N}\Big(\frac{1}{q_1}+\frac{1}{q_2}\Big)+\frac{1}{8sP^2}.$$

By (4·3), the left-hand side is nonzero. Hence,

$$\frac{1}{sq_1q_2}\le\frac{P}{N}\Big(\frac{1}{q_1}+\frac{1}{q_2}\Big)+\frac{1}{8sP^2}.$$

Multiplying both sides by sq_1q_2 and using $q_1+q_2\le 2P$ and $q_1q_2\le P^2$, we deduce that

$$1\le\frac{2sP^2}{N}+\frac{1}{8}.$$

As $N\ge 16sP^2$, the right-hand side is strictly less than 1, a contradiction.

Next, we show that, aside from a few exceptions, the exponential sums with phases $\{\theta+m\alpha\}_m$ are well controlled.

LEMMA 4·3. *Let α,θ,r,s,P,M,N and $\mathcal{E}(P;\theta)$ be as in Lemma 4·2. Let a and q be coprime integers with $2q<P$. For all integers $m\in[-M,M]\setminus\mathcal{E}(P;\theta)$, one has*

$$\left|\sum_{n\le N}e\Big(n\Big(\theta+m\alpha-\frac{a}{q}\Big)\Big)\right|\ll\frac{Nq}{P}.$$

Proof. Fix $m\in[-M,M]\setminus\mathcal{E}(P;\theta)$. By the Dirichlet approximation theorem and the definition of $\mathcal{E}(P;\theta)$, there exists coprime integers a_m and q_m with $q_m\in[P,N/P]$ such that

$$\left|\theta+m\alpha-\frac{a_m}{q_m}\right|\le\frac{P}{q_mN}\le\frac{1}{q_m^2}.$$

Then, by the triangle inequality,

$$\left\|\theta+m\alpha-\frac{a}{q}\right\|\ge\left\|\frac{a_m}{q_m}-\frac{a}{q}\right\|-\frac{1}{q_m^2}.$$

Since $2q<P\le q_m$, we have $q_m\ne q$ and thus

$$\left\|\frac{a_m}{q_m}-\frac{a}{q}\right\|\ge\frac{1}{qq_m}.$$

It follows that

$$\left\|\theta+m\alpha-\frac{a}{q}\right\|\ge\frac{1}{qq_m}-\frac{1}{q_m^2}\gg\frac{P}{qN}.$$

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Finally, using the standard bound for exponential sums,

$$\left|\sum_{n\le N} e\left(n\left(\theta+m\alpha-\frac{a}{q}\right)\right)\right| \ll \min\left\{N, \frac{1}{\left\|\theta+m\alpha-\frac{a}{q}\right\|}\right\},$$

the result follows.

5. *Weight function*

To introduce a smooth cut-off in the Diophantine condition (1·1), we insert a bump function. This modification will simplify the application of Fourier analysis on the torus $\mathbb{T}$.

Fix a parameter $\delta \in (0, 1/4)$, and let $w_\delta : \mathbb{T} \to [0, 1]$ be a $C^\infty(\mathbb{T})$ bump function supported on those x with $\|x\| \le 2\delta$, and satisfying

$$w_\delta(x) = 1 \quad \text{for } \|x\| \le \delta.$$

The following lemma provides a convenient truncated Fourier expansion of the function w_δ.

LEMMA 5·1. *For $\delta \in (0, 1/4)$, let w_δ be as defined above. For $L, A \ge 1$, the function w_δ admits the Fourier expansion*

$$w_\delta(x) = \sum_{|\ell|\le L} \widehat{w_\delta}(\ell)\, e(\ell x) + O_A\left(\frac{1}{(1+L\delta)^A}\right).$$

Furthermore, the Fourier coefficients satisfy the bound

$$\widehat{w_\delta}(\ell) \ll \delta. \tag{5·1}$$

Proof. Repeated applications of integration by parts yield

$$\widehat{w_\delta}(\ell) = \int_{\mathbb{T}} w_\delta(x) e(-\ell x)\, dx \ll_A \frac{\delta}{(1+|\ell|\,\delta)^{2A}}. \tag{5·2}$$

Using the estimate

$$\sum_{|\ell|>L} \frac{1}{(1+|\ell|\,\delta)^{2A}} \ll_A \frac{1}{(1+L\delta)^A}$$

the first claim follows.

For the second claim, we observe that

$$|\widehat{w_\delta}(\ell)| \le \int_{\mathbb{T}} w_\delta(x)\, dx \le \int_{-2\delta}^{2\delta} 1\, dx \ll \delta.$$

Additionally, using (5·2),

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} |\widehat{w_\delta}(\ell)| &= \sum_{|\ell|\le 4/\delta} |\widehat{w_\delta}(\ell)| + \sum_{|\ell|>4/\delta} |\widehat{w_\delta}(\ell)| \\ &\ll 1 + \int_{1/\delta}^{\infty} \frac{\delta}{(u\delta)^2} du \ll 1.\end{aligned}$$

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We will also need the following convolution estimate for the Fourier coefficients of w_δ, which will be used to obtain a lower bound on the main term when counting three-term arithmetic progressions.

LEMMA 5·2. *For $\delta \in (0, 1/4)$, let w_δ be as defined above. Then for any L, $A \geq 1$, one has the lower bound*

$$\sum_{|\ell| \leq L} \widehat{w_\delta}(\ell)^2 \widehat{w_\delta}(-2\ell) \gg \delta^2 + O_A\left(\frac{1}{(1+L\delta)^A}\right).$$

Proof. Set

$$v_\delta(\|x\|) := \frac{w_\delta(-\|x\|/2)}{2}.$$

Then,

$$\widehat{v_\delta}(m) = \frac{1}{2}\int_{-4\delta}^{4\delta} w_\delta\left(\frac{\theta}{2}\right) e(-m\theta)\, d\theta = \int_{-2\delta}^{2\delta} w_\delta(\varphi) e(-2m\varphi)\, d\varphi = \widehat{w_\delta}(-2m).$$

Applying the convolution identity (2·1) yields

$$\begin{aligned}(w_\delta * w_\delta * v_\delta)(0) &= \sum_{\ell} \widehat{(w_\delta * w_\delta * v_\delta)}(\ell) e(0 \cdot \ell) \\ &= \sum_{|\ell| \leq L} \widehat{w_\delta}(\ell)^2 \widehat{w_\delta}(-2\ell) + \sum_{|\ell| > L} \widehat{w_\delta}(\ell)^2 \widehat{w_\delta}(-2\ell).\end{aligned}$$

The tail sum is controlled by the decay estimate (5·2),

$$\sum_{|\ell| > L} \widehat{w_\delta}(\ell)^2 \widehat{w_\delta}(-2\ell) \ll_A \frac{1}{(1+L\delta)^A}.$$

On the other hand, since $w_\delta \geq \mathbf{1}_{[-\delta,\delta]}$, using direct computation we have

$$(w_\delta * w_\delta * v_\delta)(0) \geq \mathbf{1}_{[-\delta,\delta]} * \mathbf{1}_{[-\delta,\delta]} * \frac{\mathbf{1}_{[-2\delta,2\delta]}}{2} \geq \delta \mathbf{1}_{[-\delta,\delta]} * \frac{\mathbf{1}_{[-2\delta,2\delta]}}{2} \gg \delta^2.$$

Combining these two estimates completes the proof.

6. *Preliminaries for the circle method*

We set up the following notation to be used throughout the paper. Let α be irrational, $\beta \in \mathbb{R}$, and fix exponents

$$0 < \tau < \mu < \frac{1}{8}.$$

Let $J \geq 1$ be the unique integer satisfying

$$4^J > \frac{1-5\mu}{1-8\mu} \geq 4^{J-1}, \tag{6·1}$$

and choose $\eta \in (0, 1/20)$ fixed such that

$$\eta < \frac{4^J(1-8\mu) - (1-5\mu)}{16(4^J-1)} \leq \frac{1-8\mu}{4}. \tag{6·2}$$

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Our choice of J ensures that the expression in the middle above is positive. The reason for this peculiar choice of parameters η and J will be made clear below when we introduce a hierarchy of scales to aid our analysis.

Next, let r and s be coprime integers with s large enough and such that (4·2) holds. Also let N be such that

$$s = \lfloor N^{1-4\mu-2\eta} \rfloor.$$

It is useful to note that for s large enough (or, interchangeably N large enough) we have the bound $s \geq \sqrt{N}$ following from the second upper bound on η in (6·2). Set

$$\delta := N^{-\tau}, \quad \text{and} \quad M := N^{\mu}.$$

For δ as defined above, set $w := w_\delta$.

In our analysis we will frequently invoke Parseval's identity to control second moments, and for that purpose we require an upper bound on the number of $n \leq N$ satisfying $\|n\alpha\| \leq \delta$. Using the exponential-sum estimates established earlier, we now derive such a bound for the smoothed cutoff.

LEMMA 6·1. *With notation as above, one has the bound*

$$\sum_{n \leq N} w(\|n\alpha + \beta\|) \ll \delta N.$$

Proof. By Lemma 5·1 with cutoff $L = M$, for any $A \geq 1$ we have

$$\sum_{n \leq N} w(\|n\alpha + \beta\|) = \sum_{|\ell| \leq M} \widehat{w}(\ell) e(\ell\beta) \sum_{n \leq N} e(n\ell\alpha) + O_A\left(\frac{1}{N^{A(\mu-\tau)}}\right).$$

As $\mu > \tau$, choosing A large enough, we have

$$\sum_{n \leq N} w(\|n\alpha + \beta\|) = \sum_{|\ell| \leq M} \widehat{w}(\ell) e(\ell\beta) \sum_{n \leq N} e(n\ell\alpha) + O(N^{-50}).$$

Set

$$P := \min\left\{\sqrt{\frac{s}{64M}}, \sqrt{\frac{N}{16s}}\right\},$$

With α, r, s, P, M as above, let $\mathcal{E}(P;0)$ be as defined in Lemma 4·2. Using Lemma 4·2 observe that $|\mathcal{E}(P;0)| \leq 1$. It is clear that $0 \in \mathcal{E}(P;0)$, and therefore

$$\mathcal{E}(P;0) = \{0\}.$$

Therefore, for each ℓ satisfying $1 \leq |\ell| \leq M$, applying Lemma 4·3 with $a = q = 1$, we obtain

$$\sum_{n \leq N} e(n\ell\alpha) \ll \frac{N}{P}.$$

Hence, using (5·1),

$$\sum_{n \leq N} w(\|n\alpha + \beta\|) \ll \delta N + O\left(\delta N \frac{M}{P}\right).$$

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On the other hand, using $\eta \in (0, 1/20)$ and $\mu < 1/8$, we have

$$P \gg \min\left\{N^{\frac{1}{2}-\frac{5\mu+2\eta}{2}}, N^{2\mu+\eta}\right\} \gg \min\left\{N^{\frac{9}{20}-\frac{5\mu}{2}}, N^{2\mu}\right\} \gg N^{\mu} = M.$$

Thus the result follows.

To organise our circle method analysis, we introduce a hierarchy of scales

$$P_1 > P_2 > \ldots > P_J,$$

where for each $1 \le j \le J$

$$P_j = \frac{N^{p_j}}{256}, \quad p_j = \frac{1}{4} - \frac{(2\cdot 4^j - 5)}{3}\left(\frac{1-8\mu}{4} - \eta\right). \tag{6·3}$$

Note that, by the second upper bound on η in (6·2), the exponents p_j are strictly decreasing:

$$p_j - p_{j+1} = 2\cdot 4^j\left(\frac{1-8\mu}{4} - \eta\right) > 0.$$

Moreover, using the upper bound on 4^{J-1} from (6·1), we obtain

$$\begin{aligned} p_J - \eta &= \frac{1}{4} - \frac{2\cdot 4^J - 5}{3}\left(\frac{1-8\mu}{4} - \eta\right) - \eta \\ &= \frac{1}{4} - \frac{2\cdot 4^J - 5}{3}\cdot\frac{1-8\mu}{4} + \eta\left(\frac{2\cdot 4^J - 5}{3} - 1\right) \\ &\ge \frac{1}{4} - \frac{8\cdot 4^{J-1} - 5}{3}\cdot\frac{1-8\mu}{4} \\ &\ge \frac{1}{4} - \frac{8(1-5\mu) - 5(1-8\mu)}{12} = 0 \end{aligned} \tag{6·4}$$

Thus, the scales P_j are indeed monotonically decreasing, and each $P_j > 1$ for sufficiently large N (in fact, $P_j \gg N^{\eta}$).

Our strategy will be to partition the set of phases $\{\theta + m\alpha\}_{|m|\le M}$ according to whether each phase admits a good rational approximation with denominator at most P_j. For each such partition, we then apply the appropriate minor arc bound given by Lemma 4·1 to estimate its contribution accordingly.

To control the size of each partition, we apply Lemma 4·2, which requires that $N \ge 16sP_j^2$ for all $j \le J$. A direct computation shows that $p_1 = 2\mu + \eta$, so for sufficiently large N, remembering that $p_J \ge \eta$ from (6·4), we have

$$\frac{N^{\eta}}{256} \le P_j \le \frac{1}{16}\sqrt{\frac{N}{s}} \le N^{\frac{1}{4}} \quad \text{for} \quad 1 \le j \le J. \tag{6·5}$$

To ensure the contribution from each partition is acceptable, we will require the recurrence relation

$$\frac{MP_j^2}{s} = o\left(\sqrt{P_{j+1}}\right)$$

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to hold for all $j < J$, along with the condition that $M = o\left(\sqrt{P_1}\right)$. A direct check shows that

$$p_{j+1} - 4p_j = 10\mu + 5\eta - 2, \quad p_1 = 2\mu + \eta$$

and hence, for large enough N, the relations

$$N^{\eta}\frac{M^2P_j^4}{s^2} \le P_{j+1} \quad (1 \le j < J), \qquad \frac{M^2N^{\eta}}{256} \le P_1 \tag{6·6}$$

hold, implying the above mentioned recurrence relations.

To ensure, among other things, that the modified major arcs in our setting below are disjoint, we will require that at most one element of the set $\{\theta + m\alpha\}_{|m|\le M}$ is well approximated by a rational with denominator at most P_J. By Lemma 4·2, this is guaranteed provided that

$$P_J \le \frac{1}{8}\sqrt{\frac{s}{M}}.$$

To verify this, observe that by the upper bound on η in (6·2), we have

$$\begin{aligned}\frac{1-4\mu-2\eta-\mu}{2} - p_J &= \frac{1-5\mu}{2} - \eta - \frac{1}{4} + \frac{2\cdot 4^J - 5}{3}\left(\frac{1-8\mu}{4} - \eta\right)\\ &= \frac{4^J(1-8\mu)-(1-5\mu)}{6} - \frac{2(4^J-1)}{3}\eta > 0.\end{aligned}$$

Hence, for sufficiently large N,

$$P_J \le \frac{1}{16}\sqrt{\frac{s}{2M}}. \tag{6·7}$$

We now split the unit torus $\mathbb{T}$ into *major arcs* and *minor arcs* as follows. For $q < P_J$, and a coprime to q, and $|m| \le M$, define the *major arc*

$$\mathcal{M}_{a,q,m} := \left\{\theta \in \mathbb{T} : \left\|\theta + m\alpha - \frac{a}{q}\right\| \le \frac{P_J}{qN}\right\},$$

and set the *minor arcs* to be

$$\mathfrak{m} := \mathbb{T} \setminus \bigcup_{q<P_J} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \bigcup_{|m|\le M} \mathcal{M}_{a,q,m}.$$

We now show that the major arcs $\left\{\mathcal{M}_{a,q,m}\right\}_{a,q,m}$ are pairwise disjoint.

LEMMA 6·2. *Let* $q_1, q_2 < P_J$, *and let* $1 \le a_1 \le q_1$, $1 \le a_2 \le q_2$ *satisfy* $(a_1, q_1) = (a_2, q_2) = 1$. *Moreover, let* $m_1, m_2 \in \mathbb{Z}$ *satisfy* $|m_1|, |m_2| \le M$. *Then*

$$\mathcal{M}_{a_1,q_1,m_1} \cap \mathcal{M}_{a_2,q_2,m_2} \ne \emptyset$$

if and only if

$$a_1 = a_2, \quad q_1 = q_2, \quad \text{and} \quad m_1 = m_2. \tag{6·8}$$

Proof. Assume $\theta \in \mathcal{M}_{a_1,q_1,m_1} \cap \mathcal{M}_{a_2,q_2,m_2}$ and (6·8) does not hold. If $m_1 \ne m_2$, then the result immediately follows from the upper bounds on P_J in (6·5) and (6·7), and Lemma 4·2.

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Hence assume $m_1 = m_2 = m$ and that $a_1/q_1 \neq a_2/q_2$ are distinct reduced fractions. By definition of the major arcs, we have

$$\left\| \theta + m\alpha - \frac{a_1}{q_1} \right\| \leq \frac{P_J}{q_1 N} \quad \text{and} \quad \left\| \theta + m\alpha - \frac{a_2}{q_2} \right\| \leq \frac{P_J}{q_2 N}.$$

By the triangle inequality,

$$\frac{1}{q_1 q_2} \leq \left\| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right\| \leq \left\| \theta + m\alpha - \frac{a_1}{q_1} \right\| + \left\| \theta + m\alpha - \frac{a_2}{q_2} \right\| \leq \frac{P_J}{q_1 N} + \frac{P_J}{q_2 N} \leq \frac{2P_J}{N}.$$

Since $q_1, q_2 \leq P_J$, this gives

$$\frac{1}{P_J^2} \leq \frac{2P_J}{N},$$

contradicting (6·5) and hence completing the proof.

By Lemma 6·2, the integral over $\mathbb{T}$ decomposes as

$$\int_{\mathbb{T}} = \sum_{|m| \leq M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} + \int_{\mathfrak{m}}. \tag{6·9}$$

7. *Major and minor arc estimates*

To implement the circle method in our setting, we require major and minor arc estimates for the following exponential sum

$$S_w(\theta) = S_w(\theta; N) := \sum_{n \leq N} \Lambda(n) w\left(\|\alpha n + \beta\|\right) e(n\theta). \tag{7·1}$$

Recall the definition of $S(\theta)$ from (4·1). The following lemma provides a major arc estimate for the exponential sum in (7·1).

LEMMA 7·1. *Let $q < P_J$ and let $1 \leq a \leq q$ be such that $(a, q) = 1$. Suppose $m \in [-M, M] \cap \mathbb{Z}$, and let $\theta \in \mathcal{M}_{a,q,m}$. Then*

$$S_w(\theta) = \widehat{w}(m) e(m\beta) S(\theta + m\alpha) + O(\delta N^{1-\frac{\eta}{3}})$$

and

$$S_w(-2\theta) = \widehat{w}(-2m) e(-2m\beta) S(-2(\theta + m\alpha)) + O(\delta N^{1-\frac{\eta}{3}}).$$

Proof. Apply Lemma 5·1 with $L = 2M$ and $x = n\alpha$ to obtain

$$S_w(\theta) = \sum_{|\ell| \leq 2M} \widehat{w}(\ell) e(\ell\beta) S(\theta + \ell\alpha) + O(N^{-50}).$$

Let $\mathcal{E}_0 := [-2M, 2M] \cap \mathbb{Z}$. For $1 \leq j \leq J$, define $\mathcal{E}_j \subseteq \mathcal{E}_{j-1}$ as the set of integers $\ell \in \mathcal{E}_{j-1}$ such that there exists coprime integers $a,\ q$ with $q < P_j$ satisfying

$$\left| \theta + \ell\alpha - \frac{a}{q} \right| \leq \frac{P_j}{qN}.$$

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By Dirichlet's approximation theorem, for each $\ell \in \mathcal{E}_{j-1} \setminus \mathcal{E}_j$, there exist coprime integers a, q with $q \in [P_j, N/P_j]$ such that

$$\left|\theta + \ell\alpha - \frac{a}{q}\right| \leq \frac{P_j}{qN} \leq \frac{1}{q^2}.$$

Applying Lemma 4·1, for every $\ell \in \mathcal{E}_{j-1} \setminus \mathcal{E}_j$ we have

$$|S(\theta + \ell\alpha)| \ll N(\log N)^4 \left(\frac{1}{\sqrt{P_j}} + \frac{1}{N^{1/5}}\right) \ll \frac{N(\log N)^4}{\sqrt{P_j}}.$$

From Lemma 4·2 and the upper bounds on P_J in (6·5) and (6·7), we have $|\mathcal{E}_J| \leq 1$. Since $\theta \in \mathcal{M}_{a,q,m}$, it follows that if $\mathcal{E}_J$ is non-empty, then $m \in \mathcal{E}_J$.

Using the bound in (5·1), we conclude

$$S_w(\theta) = \widehat{w}(m)e(m\beta)S(\theta + m\alpha) + O\left(\delta N(\log N)^4 \sum_{0\leq j<J} \frac{|\mathcal{E}_j|}{\sqrt{P_{j+1}}}\right) + O(N^{-50}).$$

Finally, invoking Lemma 4·2 to control $|\mathcal{E}_j|$, and using (6·6), we obtain

$$\begin{aligned} S_w(\theta) &= \widehat{w}(m)e(m\beta)S(\theta + m\alpha) + O(\delta N(\log N)^4 J N^{-\frac{\eta}{2}}) \\ &= \widehat{w}(m)e(m\beta)S(\theta + m\alpha) + O(\delta N^{1-\frac{\eta}{3}}) \end{aligned}$$

as required.

For $\theta \in \mathcal{M}_{a,q,m}$, observe that if q is even, the $-2\theta \in \mathcal{M}_{q-a,\frac{q}{2},-2m}$. Hence, the estimate for $S_w(-2\theta)$ follows from estimate for $S_w(\theta)$.

Now suppose q is odd. For this case, define, for any $d < 2P_J$, $1 \leq c \leq d$ with $(c, d) = 1$, and $|\ell| \leq M$,

$$\mathcal{M}'_{c,d,\ell} := \left\{\theta \in \mathbb{T} : \left\|\theta + \ell\alpha - \frac{c}{d}\right\| \leq \frac{2P_J}{dN}\right\}.$$

Then, for $\theta \in \mathcal{M}_{a,q,m}$, there exists an integer $1 \leq b \leq q$ with $(b, q) = 1$, such that $-2\theta \in \mathcal{M}'_{b,q,-2m}$. Now, by replacing the parameters $P_1, \ldots, P_J$ with $2P_1, \ldots, 2P_J$, the desired estimate for $S_w(-2\theta)$ follows by analogous argument as that used for $S_w(\theta)$.

The next lemma provides a point-wise bound for the exponential sum $S_w(\theta)$ on the minor arcs.

LEMMA 7·2. *We have the following uniform bound*:

$$\sup_{\theta\in\mathfrak{m}} |S_w(\theta)| \ll \delta N^{1-\frac{\eta}{3}}.$$

Proof. The argument mirrors the proof of Lemma 7·1, except that for $\theta \in \mathfrak{m}$, $\mathcal{E}_J$ must be empty. Hence,

$$S_w(\theta) = O\left(\delta N(\log N)^4 \sum_{0\leq j<J} \frac{|\mathcal{E}_j|}{\sqrt{P_{j+1}}}\right) + O(N^{-50}) = O(\delta N^{1-\frac{\eta}{3}}).$$

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8. *Proof of Theorem 1·1*

With the major and minor arc estimates established, we now proceed to complete the proof of Theorem 1·1. By the orthogonality relation

$$\int_{\mathbb{T}} e((n-m)\theta)\,d\theta = \mathbf{1}_{n=m}, \tag{8·1}$$

where $m,\ n \in \mathbb{Z}$, it suffices to establish the lower bound

$$\int_{\mathbb{T}} S_w(\theta)^2\, S_w(-2\theta)\,d\theta \gg \delta^2 N^2. \tag{8·2}$$

Indeed, one may increase τ while having it being still strictly below $1/8$ to absorb the loss of 2 in the width of the Bohr set.

Applying (6·9), we have the decomposition

$$\int_{\mathbb{T}} S_w(\theta)^2\, S_w(-2\theta)\,d\theta = R_{\mathcal{M}} + R_{\mathfrak{m}}, \tag{8·3}$$

where

$$R_{\mathcal{M}} := \sum_{q<P_J}\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\ \sum_{|m|\le M} \int_{\mathcal{M}_{a,q,m}} S_w(\theta)^2\, S_w(-2\theta)\,d\theta \tag{8·4}$$

and

$$R_{\mathfrak{m}} := \int_{\mathfrak{m}} S_w(\theta)^2\, S_w(-2\theta)\,d\theta. \tag{8·5}$$

We will show that the major-arc contribution $R_{\mathcal{M}}$ dominates and that the minor-arc contribution is negligible, thereby proving the desired bound.

LEMMA 8·1. *We have*

$$R_{\mathfrak{m}} \ll \delta^2 N^{2-\frac{\eta}{4}}.$$

Proof. From (8·5),

$$R_{\mathfrak{m}} \ll \sup_{\theta\in\mathfrak{m}} |S_w(\theta)| \int_{\mathbb{T}} |S_w(\theta)S_w(-2\theta)|\ d\theta.$$

An application of the Cauchy–Schwarz inequality gives

$$R_{\mathfrak{m}} \ll \sup_{\theta\in\mathfrak{m}} |S_w(\theta)| \left(\int_{\mathbb{T}} |S_w(\theta)|^2\ d\theta\right)^{1/2} \left(\int_{\mathbb{T}} |S_w(-2\theta)|^2\ d\theta\right)^{1/2}.$$

Invoking Parseval's identity together with Lemma 6·1 shows that

$$\int_{\mathbb{T}} |S_w(\theta)|^2\ d\theta, \quad \int_{\mathbb{T}} |S_w(-2\theta)|^2\ d\theta \ll \delta N(\log N)^2.$$

Hence

$$R_{\mathfrak{m}} \ll \sup_{\theta\in\mathfrak{m}} |S_w(\theta)|\, \delta N(\log N)^2,$$

and the claimed bound follows at once from Lemma 7·2.

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LEMMA 8·2. *We have*

$$R_{\mathcal{M}} \gg \delta^2 N^2.$$

Proof. Invoking Lemma 7·1, we see that for every integer $q < P_J$, each a with $1 \le a \le q$ and $(a, q) = 1$, and every $m \in [-M, M] \cap \mathbb{Z}$,

$$\int_{\mathcal{M}_{a,q,m}} S_w(\theta)^2 S_w(-2\theta)\, d\theta = \widehat{w}(m) e(m\beta) \int_{\mathcal{M}_{a,q,m}} S(\theta + m\alpha) S_w(\theta) S_w(-2\theta)\, d\theta \tag{8·6}$$
$$+ O\left(\delta N^{1-\frac{\eta}{3}} \int_{\mathcal{M}_{a,q,m}} |S_w(\theta) S_w(-2\theta)|\, d\theta\right).$$

Since the arcs $\{\mathcal{M}_{a,q,m}\}$ are disjoint (Lemma 6·2), summing over m, q and a yields

$$\sum_{|m| \le M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} |S_w(\theta) S_w(-2\theta)|\, d\theta \ll \int_{\mathbb{T}} |S_w(\theta) S_w(-2\theta)|\, d\theta.$$

By the Cauchy-Schwarz inequality, Parseval's identity and Lemma 6·1,

$$\sum_{|m| \le M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} |S_w(\theta) S_w(-2\theta)|\, d\theta \ll \delta N (\log N)^2. \tag{8·7}$$

Next, a second application of Lemma 7·1 gives

$$\int_{\mathcal{M}_{a,q,m}} S(\theta + m\alpha) S_w(\theta) S_w(-2\theta)\, d\theta = \widehat{w}(m) e(m\beta) \int_{\mathcal{M}_{a,q,m}} S(\theta + m\alpha)^2 S_w(-2\theta)\, d\theta$$
$$+ O\left(\delta N^{1-\frac{\eta}{3}} \int_{\mathcal{M}_{a,q,m}} |S(\theta + m\alpha) S_w(-2\theta)|\, d\theta\right). \tag{8·8}$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{|m| \le M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} |S(\theta + m\alpha) S_w(-2\theta)|\, d\theta$$
$$\ll \left(\sum_{|m| \le M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} |S(\theta + m\alpha)|^2\, d\theta\right)^{1/2}$$
$$\times \left(\sum_{|m| \le M} \sum_{q < P_J} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{a,q,m}} |S_w(-2\theta)|^2\, d\theta\right)^{1/2}.$$

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Using Lemma 6·2, Parseval's identity, and Lemma 6·1, we have

$$\sum_{|m|\le M}\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,m}}|S_w(-2\theta)|^2\ d\theta \ll \delta N(\log N)^2 \quad \text{and}$$

$$\begin{aligned}\sum_{|m|\le M}\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,m}}|S(\theta+m\alpha)|^2\,d\theta &= M\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,0}}|S(\theta)|^2\,d\theta\\ &\le M\int_{\mathbb{T}}|S(\theta)|^2\,d\theta\\ &\ll MN(\log N)^2.\end{aligned}$$

Thus,

$$\sum_{|m|\le M}\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,m}}|S(\theta+m\alpha)S_w(-2\theta)|\ d\theta \ll (\delta M)^{1/2}N(\log N)^2. \tag{8·9}$$

A final application of Lemma 7·1 then gives, for each major arc $\mathcal{M}_{a,q,m}$,

$$\begin{aligned}\int_{\mathcal{M}_{a,q,m}}S(\theta+m\alpha)^2S_w(-2\theta)\,d\theta &= \widehat{w}(-2m)e(-2m\beta)\int_{\mathcal{M}_{a,q,m}}S(\theta+m\alpha)^2S(-2(\theta+m\alpha))\,d\theta\\ &\quad+O\left(\delta N^{1-\frac{\eta}{3}}\int_{\mathcal{M}_{a,q,m}}|S(\theta+m\alpha)|^2\ d\theta\right).\end{aligned} \tag{8·10}$$

By disjointness of the major arcs (Lemma 6·2), together with Parseval's identity, one obtains

$$\sum_{|m|\le M}\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,m}}|S(\theta+m\alpha)|^2\ d\theta \ll MN(\log N)^2. \tag{8·11}$$

Combining (8·4), (8·6), (8·7), (8·8), (8·9), (8·10), (8·11), and choosing μ sufficiently close to τ, we arrive at

$$R_{\mathcal{M}} = \sum_{|m|\le M}\widehat{w}(m)^2\widehat{w}(-2m)\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,0}}S(\theta)^2S(-2\theta)\,d\theta + O(\delta^2N^{2-\frac{\eta}{4}}). \tag{8·12}$$

Set

$$\mathcal{M}^{(0)} := \bigcup_{q<P_J}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathcal{M}_{a,q,0}.$$

Then

$$\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,0}}S(\theta)^2S(-2\theta)\,d\theta = \int_{\mathbb{T}}S(\theta)^2S(-2\theta)\,d\theta - \int_{\mathbb{T}\setminus\mathcal{M}^{(0)}}S(\theta)^2S(-2\theta)\,d\theta.$$

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By Lemma 4·1 and the lower bound on P_J from (6·5),

$$\sup_{\theta\in\mathbb{T}\setminus\mathcal{M}^{(0)}}|S(\theta)|\ll\frac{N(\log N)^4}{\sqrt{P_J}}\ll N^{1-\frac{\eta}{3}}. \tag{8·13}$$

Another application of the Cauchy-Schwarz inequality and Parseval's identity then gives

$$\int_{\mathbb{T}\setminus\mathcal{M}^{(0)}}\left|S(\theta)^2S(-2\theta)\,d\theta\right|\ll\sup_{\theta\in\mathbb{T}\setminus\mathcal{M}^{(0)}}|S(\theta)|\int_{\mathbb{T}}|S(\theta)S(-2\theta)|\,d\theta\ll N^{2-\frac{\eta}{4}}.$$

It is well known (cf. [**12**, theorem 24·1]) that

$$\int_{\mathbb{T}}S(\theta)^2S(-2\theta)\,d\theta=\sum_{\substack{n_1,n_2,n_3\le N\\ n_1+n_3=2n_2}}\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\gg N^2.$$

Hence

$$\sum_{q<P_J}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathcal{M}_{a,q,0}}S(\theta)^2S(-2\theta)\,d\theta\gg N^2.$$

Substituting this into (8·12) and applying Lemma 5·2 completes the proof.

Substituting the bounds from Lemma 8·1 and Lemma 8·2 into (8·3), we obtain the desired lower bound in (8·2), thereby completing the proof of Theorem 1·1.

9. *Proof of Theorem 1·2*

We turn to the proof of Theorem 1·2. To obtain stronger cancellation in $S(\theta)$, we impose an additional constraint on μ. Define a sequence $\{\mu_t^*\}_{t\ge1}$ increasing to $1/8$ by solving

$$4^t=\frac{1-5\mu_t^*}{1-8\mu_t^*}.$$

That is,

$$\mu_t^*=\frac{4^t-1}{8\cdot4^t-5}.$$

Recall the definition of P_J from (6·3). We would like to give a better lower bound on P_J in order to facilitate cancellations in $S(\theta)$. Since $\mu_t^*\nearrow1/8$ as $t\to\infty$, we may choose m so that $\tau<\mu_m^*<1/8$. Then pick $\mu\in(\tau,\mu_m^*)$ sufficiently close to μ_m^* so that $J=m$ and

$$\frac{2(1-5\mu)}{3}-\frac{4^J(1-8\mu)}{6}=\frac{2(1-5\mu)}{3}-\frac{1-8\mu}{1-8\mu_m^*}\,\frac{1-5\mu_m^*}{6}>\frac{1-5\mu}{3}>\mu.$$

It follows that

$$P_J\gg N^{\mu+\eta}. \tag{9·1}$$

Using the orthogonality relation (8·1), it suffices to show

$$\int_{\mathbb{T}}S(\theta)\,S_w(\theta)\,e(-n\theta)\,d\theta\gg\delta\,\mathfrak{S}(n)\,n$$

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for almost all even n, where

$$\mathfrak{S}(n) := 2\prod_{\substack{p\mid n\\ p>2}}\left(\frac{p-1}{p-2}\right)\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Since

$$\int_{\mathbb{T}} S(\theta)^2 e(-n\theta)\,d\theta \gg \sum_{\substack{m_1,m_2\\ m_1+m_2=n}} \Lambda(m_1)\Lambda(m_2) \gg \mathfrak{S}(n)n$$

for almost all even n (cf. [**19**, section 3·2]), Theorem 1·2 will follow from the next lemma.

LEMMA 9·1. *We have*

$$\sum_{n\le N}\left|\int_{\mathbb{T}} S(\theta)(S_w(\theta)-\widehat{w}(0)S(\theta))e(-n\theta)\,d\theta\right|^2 \ll \delta^2 N^{3-\frac{\eta}{4}}.$$

Proof. By Bessel's inequality,

$$\sum_{n\le N}\left|\int_{\mathbb{T}} S(\theta)(S_w(\theta)-\widehat{w}(0)S(\theta))e(-n\theta)\,d\theta\right|^2 \le \int_{\mathbb{T}} |S(\theta)(S_w(\theta)-\widehat{w}(0)S(\theta))|^2\,d\theta.$$

Let

$$\mathcal{M}^{(0)} := \bigcup_{q<P_J}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathcal{M}_{a,q,0},$$

and split the integral as

$$\int_{\mathbb{T}} = \int_{\mathcal{M}^{(0)}} + \int_{\mathbb{T}\setminus\mathcal{M}^{(0)}}.$$

On $\mathcal{M}^{(0)}$, Lemma 7·1 and Parseval's identity gives

$$\begin{aligned}\int_{\mathcal{M}^{(0)}} |S(\theta)(S_w(\theta)-\widehat{w}(0)S(\theta))|^2\,d\theta &\le \sup_{\theta\in\mathcal{M}^{(0)}} |S_w(\theta)-\widehat{w}(0)S(\theta)|^2 \int_{\mathbb{T}} |S(\theta)|^2\,d\theta\\ &\ll \delta^2 N^{3-\frac{\eta}{4}}.\end{aligned}$$

Moreover, on $\mathbb{T}\setminus\mathcal{M}^{(0)}$, Parseval's identity, Lemma 6·1 and (5·1) gives

$$\begin{aligned}\int_{\mathbb{T}\setminus\mathcal{M}^{(0)}} |S(\theta)(S_w(\theta)-\widehat{w}(0)S(\theta))|^2\,d\theta &\le \sup_{\theta\in\mathbb{T}\setminus\mathcal{M}^{(0)}} |S(\theta)|^2 \int_{\mathbb{T}} |S_w(\theta)-\widehat{w}(0)S(\theta)|^2\,d\theta\\ &\ll \sup_{\theta\in\mathbb{T}\setminus\mathcal{M}^{(0)}} |S(\theta)|^2\,\delta N(\log N)^2.\end{aligned}$$

Using (9·1) and Lemma 4·1, we see that

$$\sup_{\theta\in\mathbb{T}\setminus\mathcal{M}^{(0)}} |S(\theta)| \ll N^{1-\frac{(\mu+\eta)}{2}}.$$

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Therefore

$$\int_{\mathbb{T}\setminus\mathcal{M}^{(0)}} |S(\theta)(S_w(\theta) - \widehat{w}(0)S(\theta))|^2 \, d\theta \ll \delta N^{3-\mu-\frac{\eta}{3}} \ll \delta^2 N^{3-\frac{\eta}{4}}.$$

This completes the proof.

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