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DEAD-CORE PROBLEM FOR A CLASS OF INFINITY LAPLACIAN

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Abstract

For $h > 1$, we consider the reaction-diffusion equation:

$$\Delta_\infty^h u(x) = f(x, u(x), Du(x)), \quad x \in \Omega,$$

where Δ_∞^h denotes the h-degree infinity Laplacian, $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies $0 \le f(x, \delta t, p) \le \Lambda(x)\delta^\gamma f(x, t, p)$, a positive function $\Lambda(x) \in C(\overline{\Omega})$, $\gamma \in [0, h)$, $t > 0$, and $\delta > 0$ is small enough. Such an equation may cause a dead-core region, that is, an unknown region where the nonnegative solution vanishes completely. We establish a flattening estimate for the viscosity solution and obtain sharp $C^{(h+1)/(h-\gamma)}$-regularity along the free boundary $\partial\{u > 0\} \cap \Omega$. Using the sharp regularity, we prove Liouville-type theorems for the global solution and give the porosity of the free boundary. In the end, for the limit case $\gamma = h$, we show that if the viscosity solution vanishes at a point, then the dead-core region must vanish.

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1. Introduction

In this paper, we study the Dirichlet problem of the reaction-diffusion equation involving h-degree infinity Laplacian

$$\begin{cases} \Delta_\infty^h u(x) = f(x, u(x), Du(x)) & \text{in } \Omega, \\ u(x) = \phi(x) & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\phi(x) \in C(\partial\Omega)$ with $\phi(x) \ge 0$, $f(x, t, p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, $f(x, t, p) = 0$ in $\Omega \times (-\infty, 0] \times \mathbb{R}^n$, $f(x, t, p) \ge 0$ is non-decreasing with respect to t and satisfies

($\mathbf{f}_b$) $\sup_{\Omega \times I \times \mathbb{R}^n} f(x, t, p)$ is bounded in every compact interval $I \subseteq \mathbb{R}$;

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($\mathbf{f}_{\Lambda}$) $0 \le f(x, \delta t, p) \le \Lambda(x)\delta^{\gamma} f(x, t, p)$, where $\Lambda(x) \in C(\overline{\Omega})$ is positive, $\gamma \in [0, h)$, $t > 0$, and $\delta > 0$ is small enough.

The h-degree infinity Laplacian Δ_{∞}^{h} is given by

$$\Delta_{\infty}^{h} u := |Du|^{h-3} \Delta_{\infty} u = |Du|^{h-3} \sum_{i,j=1}^{n} D_i u D_{ij} u D_j u, \quad h > 1.$$

Infinity Laplacians Δ_{∞} originated from Aronsson's work on absolutely minimizing Lipschitz extensions in the 1960s [6–10]. Since then, infinity Laplace equations have received much attention, such as Armstrong and Smart [5], Aronsson *et al.* [11], Barron *et al.* [12, 13], Crandall [15], Crandall *et al.* [16], Evans and Savin [24], Koch *et al.* [26, 27], Lu and Wang [33, 34], Peres *et al.* [35], and the references therein.

Reaction-diffusion models involving strong absorption terms have a wide range of applications in biology, physics, chemical engineering, and so forth. A typical example was given in [36] in studying the steady-state reaction-diffusion model

$$\Delta u(x) = \lambda (u(x)_+)^{\gamma}, \quad 0 \le \gamma < 1, \quad x \in \Omega,$$

where the constant $\lambda > 0$ is called the Thiele modulus to control the ratio of reaction rate to diffusion rate. If $0 \le \gamma < 1$, the lack of the strong minimum principle may cause the solution to present a dead-core region, that is, a region where the nonnegative solution vanishes. The solution $u(x)$ usually denotes the density of a certain reactant, such as a chemical gas or a solvent. In this case, the dead-core region presents an area where the reaction stops completely and the reaction changes significantly as it crosses the free boundary $\partial\{u > 0\}$. For the fully nonlinear uniformly elliptic equation, Teixeira [37] considered the strong absorption problem

$$\mathcal{F}(x, D^2 u(x)) = (u(x)_+)^{\gamma}, \quad 0 \le \gamma < 1.$$

da Silva *et al.* [20] established geometric regularity results for second-order nonlinear degenerate or singular reaction-diffusion equations. For p-Laplacians, da Silva *et al.* [21] established improved $C^{p/(p-1-q)}$-regularity across the free boundary in the model

$$-\Delta_p u(x) + \lambda_0(x)(u(x)_+)^q = 0 \quad \text{in } \Omega,$$

where $\lambda_0(x) > 0$ is a bounded function, $2 \le p < +\infty$, and $0 \le q < p - 1$. Under the assumption that $l = \lim_{p \to +\infty}(q(p)/p) \in [0, 1)$, they considered the limit equation

$$\max\{-\Delta_{\infty} u(x), -|Du(x)| + u(x)^l\} = 0 \quad \text{in } \Omega \cap \{u > 0\}$$

and proved sharp $C^{1/(1-l)}$-regularity along the free boundary. da Sliva and Salort [22] obtained geometric regularity for the p-Laplace type elliptic equations with strong

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absorptions. Ara újo *et al.* [1] studied the infinity Laplace equation with strong absorptions

$$\Delta_\infty u(x) = (u(x)_+)^\gamma \quad \text{in } \Omega,$$

where $0 \le \gamma < 3$. In [23], Diehl and Teymurazyan studied the reaction-diffusion problem

$$\Delta_\infty u(x) = f(u(x)) \quad \text{in } \Omega$$

under some suitable conditions of $f(\cdot) \in C(\mathbb{R}^+)$. Ara újo and S á [2] investigated the singular absorption problem

$$\Delta_\infty u(x) = u(x)^{-\gamma} \quad \text{in } \Omega \cap \{u > 0\},$$

where $0 \le \gamma < 1$. In [29, 30], inhomogeneous reaction-diffusion equations were considered when the inhomogeneous terms satisfy certain growth and monotonicity conditions. Ara újo and Teymurazyan [3] investigated the regularity of the solutions for fully nonlinear dead-core systems. Ara újo and Teixeira [4] investigated the optimal regularity of the solutions for fully nonlinear uniformly elliptic equations with singular absorption terms.

Motivated by these works, we study the geometric and analytic properties of (1-1).

THEOREM 1.1 (Existence and uniqueness). *Let $\phi(x) \in C(\partial\Omega)$ with $\phi(x) \ge 0$, a nonnegative function $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, and $f(x,t,p) = 0$ in $\Omega \times (-\infty, 0] \times \mathbb{R}^n$. Assume that $f(x,t,p)$ is nondecreasing with respect to t and $\iota \longmapsto f(x,t,\iota p)$ is nonincreasing in a small interval $[1, 1+\eta)$ for some $\eta > 0$. If $f(x,t,p) > 0$ for each $(x,t,p) \in \Omega \times (0,+\infty) \times \mathbb{R}^n$ and $f(x,t,p)$ satisfies ($\mathbf{f}_b$), then (1-1) admits a nonnegative unique viscosity solution $u(x) \in C(\overline{\Omega})$.*

The uniqueness is based on the comparison principle and the existence is derived by the standard Perron's method. The crucial point is to construct a viscosity subsolution. For the case of $\inf f > 0$ or $\sup f < 0$, we refer the interested reader to [31]. For $1 \le h \le 3$, Liu and Yang [32] studied the existence and uniqueness of the Dirichlet problem of Δ_∞^h for $\inf_{x\in\Omega} f(x) > 0$ or $\sup_{x\in\Omega} f(x) < 0$.

THEOREM 1.2 (Geometric regularity along the free boundary). *Suppose that $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is nonnegative and satisfies Conditions ($\mathbf{f}_b$) and ($\mathbf{f}_\Lambda$). Let $u(x)$ be a nonnegative bounded viscosity solution to (1-1) in Ω. If $x_0 \in \partial\{u > 0\} \cap \Omega$, then there exists a positive constant C depending on $\gamma, \|u\|_{L^\infty(\Omega)}, \operatorname{dist}(x_0, \partial\Omega)$, and the dimension such that*

$$u(x) \le C|x - x_0|^{(h+1)/(h-\gamma)}$$

for $x \in \{u > 0\}$ near x_0.

Theorem 1.2 shows that the nonnegative viscosity solution touches the dead-core region with the smoothness of $C^{(h+1)/(h-\gamma)}$. To establish Theorem 1.2, we first establish

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a flattening estimate and then use iterative techniques. The flattening estimate is a useful method in solving free boundary problems. Here, Δ_∞^h is singular and degenerate if $1 < h < 3$, and the degeneracy increases with respect to the index h. Due to the dependence of gradient Du of $f(x, u, Du)$, we must impose appropriate structural conditions and make subtle analysis. Since $u(x_1, x_2) = (h^{(h+1)/h})/(h+1)(x_1^2 + x_2^2)^{(h+1)/2h}$ is a solution of $\Delta_\infty^h u(x) = 1$ in $\mathbb{R}^2 \setminus \{0\}$, we remark that the regularity of the viscosity solution of the strong absorption problem (1-1) is superior to $C^{(h+1)/h}$. Hence, such an improved regularity estimate cannot be extended to the whole region. This means that it is interesting to study the fully nonlinear equation (1-1).

To give the rate of the solution leaving the free boundary, we need the following growth condition:

$(\mathbf{f}_\lambda)$ $0 \le \lambda(x)\delta^\gamma f(x, t, p) \le f(x, \delta t, p)$, where the continuous function $\lambda(x)$ satisfies $\inf_{x\in\Omega} \lambda(x) > 0$, $\inf_{(x,p)\in\Omega\times\mathbb{R}^n} f(x, 1, p) > 0$, $\gamma \in [0, h)$, $t > 0$, and $\delta > 0$ is small enough.

THEOREM 1.3 (Nondegeneracy). *Let $f(x, t, p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative. Suppose that $f(x, t, p)$ is nondecreasing with respect to t in $[0, +\infty)$ and $\iota \longmapsto f(x, t, \iota p)$ is nonincreasing in a small interval $[1, 1 + \eta)$ for some $\eta > 0$. Let $u(x)$ be a nonnegative bounded viscosity solution to (1-1) in Ω. If $f(x, t, p) > 0$ for each $(x, t, p) \in \Omega \times (0, +\infty) \times \mathbb{R}^n$ and satisfies Condition $(\mathbf{f}_\lambda)$, then there exists a universal constant $c > 0$ such that*

$$\sup_{B_r(x_0)} u(x) \ge c r^{(h+1)/(h-\gamma)} \tag{1-2}$$

for any $x_0 \in \overline{\{u > 0\}} \cap \Omega$ and $0 < r < \operatorname{dist}(x_0, \partial\Omega)$.

Theorem 1.3 shows that the solution leaves the dead-core region with the $(h+1)/(h-\gamma)$-order of dist $(x, \partial\{u > 0\})$. The key to the proof is to construct an appropriate viscosity supersolution to compare with $u(x)$. Thanks to the good structure of Δ_∞^h, we can construct an $(h+1)/(h-\gamma)$-power cone as a viscosity supersolution and then use the comparison principle. It is clear that Theorems 1.2 and 1.3 show that the geometric regularity $C^{(h+1)/(h-\gamma)}$ is sharp. The optimal regularity estimate shows the free boundary is a porous set and then has Lebesgue measure zero. Using the sharp regularity, we can also give a quantitative estimate at infinity for the blow-up solution (see Theorem 4.4 for details).

Liouville-type theorems play a vital part in the development of modern mathematical analysis and applications. We establish two different Liouville-type theorems for dead-core problems. One is based on the comparison principle via a suitable radical-barrier function (see Theorem 5.1). The other comes from the improved $C^{(h+1)/(h-\gamma)}$-regularity estimate (see Theorem 5.3). The key is to compare the growth rate of solutions at infinity with $|x|^{(h+1)/(h-\gamma)}$. It is worth highlighting that the constant in Theorem 5.1 is optimal in the sense that we cannot remove the strict inequality in (5-2). For the special case $f(x, t, p) = t_+^\gamma$, we remark that the conditions in Theorem 5.1

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are weaker than those in Theorem 5.3. For $1 \le h \le 3$, Biswas and Vo [14] established Liouville theorems for the infinity Laplacian with gradient and KPP type equations.

To consider the limit case $\gamma = h$, we propose the extra growth condition:

($\mathbf{f}_h$) $0 \le f(x, \delta t, p) \le \Lambda(x)\delta^h f(x, t, p)$, where $\Lambda(x) \in C(\overline{\Omega})$ is positive, $t > 0$ and $\delta > 0$ small enough.

THEOREM 1.4 (The limit case $\gamma = h$). *Let $f(x, t, p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative. Assume that $f(x, t, p)$ is nondecreasing with respect to t in $[0, +\infty)$ and $\iota \longmapsto f(x, t, \iota p)$ is nonincreasing in a small interval $[1, 1 + \eta)$ for some $\eta > 0$. Suppose that $f(x, t, p) > 0$ for each $(x, t, p) \in \Omega \times (0, +\infty) \times \mathbb{R}^n$ and satisfies Conditions ($\mathbf{f}_b$) and ($\mathbf{f}_h$). Let $u(x)$ be a nonnegative bounded viscosity solution to (1-1) in Ω. If $u(x)$ vanishes at some point in Ω, then $u(x) \equiv 0$ in Ω.*

For the limit case $\gamma = h$, it is reasonable to conjecture that a nonnegative nonconstant solution is prevented from vanishing at an interior point. In fact, Theorem 1.4 shows that the dead-core region must disappear. To overcome the high degeneracy of Δ_∞^h, the crucial point is to construct a viscosity subsolution with strictly positive gradient modulus and then we can adopt Jensen's idea in [25]. By Theorem 1.2, the regularity of the solution is $C^{(h+1)/(h-\gamma)}$. Clearly, $(h + 1)/(h - \gamma) \to +\infty$ as $\gamma \to h^-$. Hence, Theorem 1.4 is in fact a strong unique continuation principle.

The paper is organized as follows. In Section 2, we give the definition of a viscosity solution, and establish the comparison principle and existence theorem. In Section 3, we establish a flattening estimate for the viscosity solution and then give the regularity results along its free boundary. In Section 4, we establish the nondegeneracy estimate and blow-up analysis. In Section 5, we prove two different Liouville-type theorems. In Section 6, we show that the dead-core region vanishes in the limit case $\gamma = h$.

2. Existence and uniqueness

In this section, we first give the definition of a viscosity solution for the equation

$$\Delta_\infty^h u(x) = f(x, u(x), Du(x)) \quad \text{in } \Omega. \tag{2-1}$$

Since Δ_∞^h is singular at the points where the gradient vanishes, we adopt the viscosity solutions based on the semi-continuous extension [17–19]. Let

$$F(p, M) = |p|^{h-3}\langle Mp, p\rangle, \quad p \in \mathbb{R}^n, M \in \mathcal{S}_{n\times n},$$

where $\mathcal{S}_{n\times n}$ denotes the set of real symmetric matrices. Equation (2-1) can be rewritten as

$$F(Du(x), D^2u(x)) = f(x, u(x), Du(x)), \quad x \in \Omega.$$

Observing $\lim_{p\to 0} F(p, M) = 0$, we define the continuous extension

$$\overline{F}(p, M) := \begin{cases} F(p, M) & \text{if } p \ne 0, \\ 0 & \text{if } p = 0. \end{cases}$$

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DEFINITION 2.1. For $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, an upper semi-continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution to (2-1) in Ω if for any $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x, there holds

$$\overline{F}(D\varphi(x), D^2\varphi(x)) \geq f(x, \varphi(x), D\varphi(x)).$$

Similarly, a lower semi-continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution to (2-1) in Ω if for any $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x, there holds

$$\overline{F}(D\varphi(x), D^2\varphi(x)) \leq f(x, \varphi(x), D\varphi(x)).$$

We say that $u(x)$ is a viscosity solution to (2-1) in Ω if $u(x)$ is both a viscosity subsolution and supersolution to (2-1) in Ω.

Let us recall some useful lemmas.

LEMMA 2.2 [11]. *If $u(x)$ is a viscosity subsolution (supersolution) of $\Delta_\infty u(x) = 0$ in Ω, then $u(x)$ attains its maximum (minimum) only on the boundary unless $u(x)$ is a constant.*

Since $f(x,u,Du)$ is nonnegative, Lemma 2.2 implies that the viscosity supersolutions of (1-1) are nonnegative in Ω.

LEMMA 2.3 [31]. *Let α be a constant. If $u(x) \in C(\Omega) \cap L^\infty(\Omega)$ satisfies $\Delta_\infty^h u(x) \geq \alpha$ in the viscosity sense, then $u(x)$ is locally Lipschitz continuous in Ω. Moreover, for any given $x_0 \in \Omega$, there exists a constant C such that*

$$|u(x) - u(y)| \leq C|x - y| \quad \text{for all } x, y \in B_{\operatorname{dist}(x_0, \partial\Omega)/3}(x_0),$$

where C depends on x_0, $\operatorname{diam}(\Omega)$, $|\alpha|$, and $\|u\|_{L^\infty(\Omega)}$.

LEMMA 2.4 [31, Theorem 1.2]. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\phi(x) \in C(\partial\Omega)$. Assume that $f(x,t,p) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and satisfies*

$$\sup_{\Omega \times I \times \mathbb{R}^n} |f(x,t,p)| < +\infty$$

for every compact interval $I \subseteq \mathbb{R}$. Suppose that $\underline{u}(x) \in C(\overline{\Omega})$ is a viscosity subsolution and $\overline{u}(x) \in C(\overline{\Omega})$ is a viscosity supersolution of

$$\begin{cases} \Delta_\infty^h u(x) = f(x, u(x), Du(x)) & \text{in } \Omega, \\ u(x) = \phi(x) & \text{on } \partial\Omega. \end{cases} \tag{2-2}$$

If $\underline{u}(x) \leq \overline{u}(x)$ in Ω, then (2-2) admits a viscosity solution $u(x) \in C(\overline{\Omega})$ such that $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ in Ω.

Now, we establish the comparison principle.

THEOREM 2.5. *Let $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative. Suppose that $f(x,t,p)$ is nondecreasing with respect to t in $[0, +\infty)$ and $\iota \longmapsto f(x,t,\iota p)$ is nonincreasing in*

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a small interval $[1, 1+\eta)$ *for some* $\eta > 0$. *Suppose also that* $f(x,t,p) > 0$ *for each* $(x,t,p) \in \Omega \times (0,+\infty) \times \mathbb{R}^n$. *Let* $u(x)$ *and* $v(x) \in C(\overline{\Omega})$ *satisfy*

$$\Delta_\infty^h u(x) \geq f(x, u(x), Du(x)) \quad \text{in } \Omega$$

and

$$\Delta_\infty^h v(x) \leq f(x, v(x), Dv(x)) \quad \text{in } \Omega$$

in the viscosity sense. If $u(x) \leq v(x)$ *on* $\partial\Omega$ *and* $v(x) \geq 0$ *in* Ω, *then* $u(x) \leq v(x)$ *in* Ω.

PROOF. Consider

$$u_\tau(x) := (1+\tau)u(x) - \tau \sup_{\bar{\Omega}} u(x), \quad 0 < \tau < \eta.$$

Since $u_\tau(x) \leq u(x)$, $f(x,t,p) \geq 0$ is nondecreasing in t, and $\iota \longmapsto f(x,t,\iota p)$ is nonincreasing in $[1, 1+\eta)$, we have

$$\Delta_\infty^h u_\tau(x) \geq (1+\tau)^h f(x, u(x), Du(x)) \geq (1+\tau)^h f(x, u_\tau(x), Du_\tau(x)) \tag{2-3}$$

in the viscosity sense. That is, $u_\tau(x)$ is a viscosity subsolution of $\Delta_\infty^h u(x) = (1+\tau)^h f(x, u(x), Du(x))$ in Ω.

We argue by contradiction that there exists $M_0 > 0$ such that

$$M_0 := \sup_{\overline{\Omega}} (u_\tau - v)(x) = u_\tau(x_0) - v(x_0), \quad x_0 \in \overline{\Omega}.$$

For each $\epsilon > 0$ small enough, define

$$\mathfrak{m}_\epsilon(x,y) := u_\tau(x) - v(y) - \frac{1}{4\epsilon}|x-y|^4$$

and

$$M_\epsilon := \sup_{\overline{\Omega}\times\overline{\Omega}} \mathfrak{m}_\epsilon(x,y) < +\infty.$$

Let $(x_\epsilon, y_\epsilon) \in \overline{\Omega} \times \overline{\Omega}$ be the point where $\mathfrak{m}_\epsilon(x,y)$ attains its maximum.

By [18, Proposition 3.7], we have

$$\lim_{\epsilon\to 0^+} \frac{1}{4\epsilon}|x_\epsilon - y_\epsilon|^4 = 0 \quad \text{and} \quad \lim_{\epsilon\to 0^+} M_\epsilon = M_0.$$

Obviously,

$$\lim_{\epsilon\to 0^+} x_\epsilon = \lim_{\epsilon\to 0^+} y_\epsilon = x_0.$$

Due to $M_0 > 0 \geq \sup_{\partial\Omega}(u_\tau - v)(x)$, we conclude that $x_\epsilon \in \widetilde{\Omega} \Subset \Omega$.

Applying [18, Theorem 3.2], there exist X_ϵ and $Y_\epsilon \in \mathcal{S}_{n\times n}$ such that

$$(p_\epsilon, X_\epsilon) \in \overline{J}_\Omega^{2,+} u_\tau(x_\epsilon), \quad (p_\epsilon, Y_\epsilon) \in \overline{J}_\Omega^{2,-} v(y_\epsilon), \tag{2-4}$$

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and

$$\begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix} \le \frac{1}{\epsilon}(|z_\epsilon|^2 + 2|z_\epsilon|^4)\begin{pmatrix} \mathrm{Id}_{n\times n} & -\mathrm{Id}_{n\times n} \\ -\mathrm{Id}_{n\times n} & \mathrm{Id}_{n\times n} \end{pmatrix} + 16\begin{pmatrix} z_\epsilon\otimes & z_\epsilon & -z_\epsilon\otimes & z_\epsilon \\ -z_\epsilon\otimes & z_\epsilon & z_\epsilon\otimes & z_\epsilon \end{pmatrix}, \quad (2\text{-}5)$$

where $p_\epsilon = (|z_\epsilon|^2 z_\epsilon)/(\epsilon)$, $z_\epsilon = x_\epsilon - y_\epsilon$, and $\mathrm{Id}_{n\times n}$ is the $n \times n$ identity matrix. By (2-5), we have $X_\epsilon \le Y_\epsilon$.

If $x_\epsilon = y_\epsilon$ for any $\epsilon > 0$, then we have $p_\epsilon = 0$. Thus, the continuous extension $\overline{F}$ equals zero. By (2-3),

$$(1+\tau)^h f(y_\epsilon, v(y_\epsilon) + M_\epsilon, 0) \le 0.$$

Letting $\epsilon \to 0^+$,

$$(1+\tau)^h f(x_0, v(x_0) + M_0, 0) \le 0,$$

which is impossible due to $M_0 > 0$ and $v(x) \ge 0$ in Ω.

If there exists a subsequence such that $x_\epsilon \ne y_\epsilon$, then $p_\epsilon \ne 0$. We have

$$\begin{aligned} (1+\tau)^h f(x_\epsilon, u_\tau(x_\epsilon), p_\epsilon) &= (1+\tau)^h f\Big(x_\epsilon, v(y_\epsilon) + \frac{1}{4\epsilon}|x_\epsilon - y_\epsilon|^4 + M_\epsilon, p_\epsilon\Big) \\ &\le |p_\epsilon|^{h-3} \cdot X_\epsilon(p_\epsilon) \cdot (p_\epsilon) \\ &\le |p_\epsilon|^{h-3} \cdot Y_\epsilon(p_\epsilon) \cdot (p_\epsilon) \\ &\le f(y_\epsilon, v(y_\epsilon), p_\epsilon) \end{aligned} \quad (2\text{-}6)$$

by (2-4) and (2-5). Since

$$u_\tau(y_\epsilon) - v(y_\epsilon) - \frac{1}{4\epsilon}|y_\epsilon - y_\epsilon|^4 \le u_\tau(x_\epsilon) - v(y_\epsilon) - \frac{1}{4\epsilon}|x_\epsilon - y_\epsilon|^4,$$

one has

$$\frac{1}{4\epsilon}|x_\epsilon - y_\epsilon|^4 \le u_\tau(x_\epsilon) - u_\tau(y_\epsilon).$$

By the Lipschitz continuity of $u_\tau(x)$, we see that

$$\frac{1}{4\epsilon}|x_\epsilon - y_\epsilon|^4 \le u_\tau(x_\epsilon) - u_\tau(y_\epsilon) \le L|x_\epsilon - y_\epsilon|,$$

where L is the Lipschitz constant of $u_\tau(x)$. Therefore,

$$|p_\epsilon| = \frac{|x_\epsilon - y_\epsilon|^3}{\epsilon} \le 4L.$$

Without loss of generality, we can assume

$$p_\epsilon \to \widetilde{p} \in \overline{B}_{4L}, \quad \epsilon \to 0^+,$$

where B_{4L} denotes the ball of radius $4L$ centered at the origin. By (2-6),

$$(1+\tau)^h f(x_0, v(x_0) + M_0, \widetilde{p}) \le f(x_0, v(x_0), \widetilde{p}).$$

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Since $f(x,t,p)$ is nondecreasing with respect to t and $f(x,t,p) > 0$ for each $(x,t,p) \in \Omega \times (0,+\infty) \times \mathbb{R}^n$,

$$0 < (1+\tau)^h f(x_0, v(x_0) + M_0, \widetilde{p}) \le f(x_0, v(x_0), \widetilde{p}) \le f(x_0, v(x_0) + M_0, \widetilde{p}),$$

which contradicts to $f(x_0, v(x_0) + M_0, \widetilde{p}) > 0$. Thus, we have proven that $u_\tau(x) \le v(x)$ in Ω. Letting $\tau \to 0^+$, we have $u(x) \le v(x)$ in Ω. □

Note that if $\inf_{(x,t,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n} f(x,t,p) > 0$, the comparison principle was proven in [31].

PROOF OF THEOREM 1.1. The uniqueness follows by the comparison principle. The existence reduces to finding a viscosity subsolution and a supersolution based on Lemma 2.4. Let

$$L_1 := \inf_{x\in\partial\Omega} \phi(x).$$

We can choose constants K_1 and d_1 such that

$$K_1^h \ge \max\left\{ \sup_{(x,p)\in\Omega\times\mathbb{R}^n} f(x, L_1, p), 1 \right\} \quad \text{and} \quad d_1 \le \frac{L_1}{K_1} - \frac{1}{h+1} h^{(h+1)/h} (\operatorname{diam}(\Omega))^{(h+1)/h}.$$

Construct

$$\underline{u}(x) := K_1 \left(\frac{1}{h+1} h^{(h+1)/h} |x - z|^{(h+1)/h} + d_1 \right), \quad z \in \partial\Omega.$$

We can demonstrate that $\underline{u}(x) \in C(\overline{\Omega})$, $\Delta_\infty^h \underline{u}(x) = K_1^h \ge \sup_{(x,p)\in\Omega\times\mathbb{R}^n} f(x, L_1, p)$ and $\underline{u}(x) \le L_1$ by the choice of K_1 and d_1. Since $f(x,t,p)$ is nondecreasing with respect to t, we have

$$\begin{cases} \Delta_\infty^h \underline{u}(x) \ge f(x, \underline{u}(x), D\underline{u}(x)) & \text{in } \Omega, \\ \underline{u}(x) \le \phi(x) & \text{on } \partial\Omega. \end{cases}$$

Hence, $\underline{u}(x)$ is a viscosity subsolution of (1-1) in Ω. Let

$$\overline{u}(x) := \sup_{x\in\partial\Omega} \phi(x) + 1.$$

It is obvious that $\overline{u}(x)$ satisfies

$$\begin{cases} \Delta_\infty^h \overline{u}(x) \le 0 & \text{in } \Omega, \\ \overline{u}(x) \ge \phi(x) & \text{on } \partial\Omega. \end{cases}$$

That is, $\overline{u}(x)$ is a viscosity supersolution of (1-1) in Ω. By Lemma 2.4, we obtain the existence of a viscosity solution to (1-1). □

3. Geometric regularity results across the free boundary

In this section, our goal is to establish an improved geometric regularity of viscosity solutions for (1-1) along the free boundary $\partial\{u > 0\}$. We give a flattening estimate

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and use iteration techniques to prove the $C^{(h+1)/(h-\gamma)}$-regularity of viscosity solutions for (1-1) along the free boundary. The flattening estimate plays a crucial role in characterizing the decay of geometric properties, which was established for the fully nonlinear uniformly elliptic operator in [4, 37]. We extend the idea in [1, 23] to obtain the flattening estimate for the operator Δ_∞^h.

LEMMA 3.1 (Flattening estimate). *Let $f(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative and satisfy Conditions ($\mathbf{f}_b$) and ($\mathbf{f}_\Lambda$). Suppose that $G(x,\tilde{t},p) \in C(\Omega \times [0,1] \times \mathbb{R}^n)$ and*

$$0 \le G(x,\tilde{t},p) \le \max\{1, \|\Lambda(x)\|_{L^\infty(\Omega)}\} \sup_{(x,t,p)\in\Omega\times[0,\|u\|_{L^\infty(\Omega)}]\times\mathbb{R}^n} f(x,t,p).$$

For any $x_0 \in \Omega$ and $\mu > 0$, there exists a constant $\kappa_{\mu,n} = \kappa(\mu,n) > 0$ such that if $\xi(x) \in C(B_1(x_0))$ satisfies

$$\xi(x_0) = 0, \quad 0 \le \xi(x) \le 1, \quad x \in B_1(x_0) \cap \Omega$$

and

$$\Delta_\infty^h \xi(x) = \kappa^{h+1} G(x, \xi(x), D\xi(x)), \quad x \in B_1(x_0) \cap \Omega$$

in the viscosity sense for $0 < \kappa \le \kappa_{\mu,n}$, then

$$\sup_{B_{(1/2)}(x_0)\cap\Omega} \xi(x) \le \mu.$$

PROOF. We argue by contradiction assuming that there exist $\mu_* > 0$ and sequences $\{\kappa_i\}$, $\{\xi_i\}$ ($i \in \mathbb{N}$) with $\kappa_i = o(1)$,

$$\xi_i(x_0) = 0, \quad 0 \le \xi_i(x) \le 1, \quad x \in B_1(x_0) \cap \Omega,$$

and

$$\Delta_\infty^h \xi_i(x) = |D\xi_i(x)|^{h-3} \Delta_\infty \xi_i(x) = \kappa_i^{h+1} G(x, \xi_i(x), D\xi_i(x)), \quad x \in B_1(x_0) \cap \Omega$$

such that

$$\sup_{B_{(1/2)}(x_0)\cap\Omega} \xi_i(x) > \mu_*.$$

Noting that $\{\xi_i\}$ is uniformly bounded and satisfies

$$\Delta_\infty^h \xi_i(x) = \kappa_i^{h+1} G(x, \xi_i(x), D\xi_i(x)) \ge 0, \quad x \in B_1(x_0) \cap \Omega$$

in the viscosity sense, Lemma 2.3 demonstrates that $\{\xi_i\}$ is a family of locally Lipschitz continuous functions in Ω. By the Arzela–Ascoli compact theorem, up to a subsequence, $\xi_i(x)$ converges locally uniformly to a function $\xi_\infty(x)$ in $\overline{B_{2/3}(x_0)} \cap \Omega$. Moreover, by the stability of viscosity solutions, $\xi_\infty(x)$ satisfies that $\xi_\infty(x_0) = 0$, $0 \le \xi_\infty(x) \le 1$, and

$$|D\xi_\infty(x)|^{h-3} \Delta_\infty \xi_\infty(x) = 0, \quad x \in \overline{B_{2/3}(x_0)} \cap \Omega$$

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in the viscosity sense. By the maximum principle of infinity harmonic functions, we conclude that $\xi_\infty(x) \equiv 0$. This is impossible for i large enough. □

We are ready to prove Theorem 1.2, which reveals that the regularity along the free boundary is of class $C^{(h+1)/(h-\gamma)}$.

PROOF OF THEOREM 1.2. Set $\tau := \min\{1, \|u\|^{-1}_{L^\infty(\Omega)}\}$, $\rho := \overline{\kappa}\tau^{-(h-\gamma)/(h+1)}$ and

$$w_1(x) := \tau u(\rho x + x_0), \quad x \in B_1 \cap \Omega,$$

where $\overline{\kappa}$ is to be determined later. Since $u(\cdot)$ is a viscosity solution of (1-1) in Ω,

$$\Delta^h_\infty w_1(x) = \overline{\kappa}^{h+1}\tau^\gamma f(\rho x + x_0, \tau^{-1}w_1(x), \tau^{-1}\rho^{-1}Dw_1(x)) =: \overline{\kappa}^{h+1}G_1(x, w_1(x), Dw_1(x))$$

in the viscosity sense. Moreover, by the selection of τ, we find $w_1(0) = 0$, $0 \le w_1(x) \le 1$, and

$$\begin{aligned} 0 \le G_1(x, w_1(x), Dw_1(x)) &\le f(\rho x + x_0, \tau^{-1}w_1(x), \tau^{-1}\rho^{-1}Dw_1(x)) \\ &\le \sup_{(x,t,p)\in\Omega\times[0,\|u\|_{L^\infty(\Omega)}]\times\mathbb{R}^n} f(x,t,p). \end{aligned}$$

If we take $\mu = 2^{-(h+1)/(h-\gamma)}$ and get the corresponding constant $\kappa_{\mu,n} > 0$ from Lemma 3.1, then we can verify that $w_1(x)$ satisfies the conditions of Lemma 3.1 by choosing $\overline{\kappa} \le \min\{1, \kappa_{\mu,n}, \tau^{(h-\gamma)/(h+1)}\}$. This means

$$\sup_{B_{(1/2)}\cap\Omega} w_1(x) \le 2^{-(h+1)/(h-\gamma)}.$$

Let

$$w_2(x) := 2^{(h+1)/(h-\gamma)}w_1(2^{-1}x), \quad x \in B_1 \cap \Omega.$$

By direct computations, we observe that

$$\begin{aligned} &\Delta^h_\infty w_2(x) \\ &\quad = \kappa_a^{h+1}2^{(h+1)\gamma/(h-\gamma)}\tau^\gamma f(2^{-1}\rho x + x_0, 2^{-(h+1)/(h-\gamma)}\tau^{-1}w_2(x), \tau^{-1}\rho^{-1}2^{-(2h+1-\gamma)/(h-\gamma)}Dw_2(x)) \\ &\quad =: \kappa_a^{h+1}G_2(x, w_2(x), Dw_2(x)) \end{aligned}$$

in the viscosity sense. Since $\tau \le 1$, $w_2(0) = 0$, $0 \le w_2(x) \le 1$, and

$$\begin{aligned} &0 \le G_2(x, w_2(x), Dw_2(x)) \\ &\quad = 2^{(h+1)\gamma/(h-\gamma)}\tau^\gamma f(2^{-1}\rho x + x_0, 2^{-(h+1)/(h-\gamma)}\tau^{-1}w_2(x), \tau^{-1}\rho^{-1}2^{-(2h+1-\gamma)/(h-\gamma)}Dw_2(x)) \\ &\quad \le \Lambda(2^{-1}\rho x + x_0)\tau^\gamma f(2^{-1}\rho x + x_0, \tau^{-1}w_2(x), \tau^{-1}\rho^{-1}2^{-(2h+1-\gamma)/(h-\gamma)}Dw_2(x)) \\ &\quad \le \|\Lambda(x)\|_{L^\infty(\Omega)} \sup_{(x,t,p)\in\Omega\times[0,\|u\|_{L^\infty(\Omega)}]\times\mathbb{R}^n} f(x,t,p), \end{aligned}$$

applying Lemma 3.1 again,

$$\sup_{B_{(1/2)}\cap\Omega} w_2(x) \le 2^{-(h+1)/(h-\gamma)}.$$

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For each $k \in \mathbb{N},\ k \geq 2$, we construct a series of iterative functions

$$w_k(x) := 2^{(h+1)/(h-\gamma)} w_{k-1}(2^{-1}x), \quad x \in B_1 \cap \Omega.$$

Similarly, we can verify that $w_k(x)$ satisfies the conditions of Lemma 3.1 and then

$$\sup_{B_{(1/2)}\cap\Omega} w_k(x) \leq 2^{-(h+1)/(h-\gamma)}, \quad k = 2, 3, \ldots.$$

By the iterative relationship of $\{w_k\}$, we obtain

$$\sup_{B_{2^{-k}}\cap\Omega} w_1(x) \leq 2^{-((h+1)k)/(h-\gamma)}, \quad k = 1, 2, 3, \ldots. \tag{3-1}$$

For any $0 < r \leq (\rho/2)$, we can select some $k \in \mathbb{N}$ such that

$$\frac{r}{\rho} \in [2^{-(k+1)}, 2^{-k}].$$

Therefore, by the construction of $w_1(x)$ and (3-1),

$$\begin{aligned}\sup_{B_r(x_0)\cap\Omega} u(x) &\leq \sup_{B_{\rho 2^{-k}}(x_0)\cap\Omega} u(x) \leq \tau^{-1} \sup_{B_{2^{-k}}\cap\Omega} w_1(x) \leq \tau^{-1} 2^{(h+1)/(h-\gamma)} 2^{-(h+1)(k+1)/(h-\gamma)} \\ &\leq \tau^{-1}(2\rho^{-1})^{(h+1)/(h-\gamma)} r^{(h+1)/(h-\gamma)} := C r^{(h+1)/(h-\gamma)}.\end{aligned}$$

This completes the proof of Theorem 1.2. □

REMARK 3.2. Theorem 1.2 implies that if $x_0 \in \partial\{u > 0\}$, then $Du(x_0) = 0$.

4. Nondegeneracy estimate and blow-up analysis

In this section, we establish a nondegeneracy estimate for the nonnegative solution on the free boundary $\partial\{u > 0\}$. Nondegeneracy is an important topic in the dead-core problem since it can provide the minimal growth rate of solutions away from the free boundary. In fact, we can prove the optimal regularity is $C^{(h+1)/(h-\gamma)}$. Based on the nondegeneracy estimate, we can give a characterization of the weak geometric properties of the free boundary including the Hausdorff dimension and so on.

We first give the proof of the nondegeneracy estimate.

PROOF OF THEOREM 1.3. By continuity, we only need to prove (1-2) for the point $x_0 \in \{u > 0\} \cap \Omega'$.

Consider a function

$$\varphi(x) := c\,|x - x_0|^{(h+1)/(h-\gamma)}, \quad x \in B_r(x_0),$$

where the constant $c > 0$ is to be determined later. For $x \neq x_0$,

$$D\varphi(x) = c\left(\frac{h+1}{h-\gamma}\right)|x - x_0|^{(1-h+2\gamma)/(h-\gamma)}(x - x_0),$$

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$$D^2\varphi(x) = c\left(\frac{h+1}{h-\gamma}\right)\left(\frac{1+\gamma}{h-\gamma}\right)|x-x_0|^{(1-h+2\gamma)/(h-\gamma)}\frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2}$$
$$+ c\left(\frac{h+1}{h-\gamma}\right)|x-x_0|^{(1-h+2\gamma)/(h-\gamma)}\left(\mathrm{Id}_{n\times n} - \frac{(x-x_0)\otimes(x-x_0)}{|x-x_0|^2}\right).$$

We can verify

$$\Delta_\infty^h\varphi(x) - f(x,\varphi(x),D\varphi(x)) = c^h\left(\frac{h+1}{h-\gamma}\right)^h\left(\frac{1+\gamma}{h-\gamma}\right)|x-x_0|^{((h+1)\gamma)/(h-\gamma)} - f(x,\varphi(x),D\varphi(x))$$
$$\le -\left[c^\gamma\lambda(x)f(x,1,D\varphi(x)) - c^h\left(\frac{h+1}{h-\gamma}\right)^h\left(\frac{1+\gamma}{h-\gamma}\right)\right]|x-x_0|^{(h+1)\gamma/h-\gamma}$$
$$\le -\left[\inf_{x\in\Omega}\lambda(x)\inf_{(x,p)\in\Omega\times\mathbb{R}^n} f(x,1,p) - c^{h-\gamma}\left(\frac{h+1}{h-\gamma}\right)^h\left(\frac{1+\gamma}{h-\gamma}\right)\right]c^\gamma\,|x-x_0|^{((h+1)\gamma)/(h-\gamma)}$$
$$\le 0,$$

where we have chosen

$$0 < c < \left[\left(\inf_{x\in\Omega}\lambda(x)\inf_{(x,p)\in\Omega\times\mathbb{R}^n} f(x,1,p)(h-\gamma)^{h+1}\right)/((h+1)^h(1+\gamma))\right]^{1/(h-\gamma)}.$$

For $x = x_0$, we can also derive $\Delta_\infty^h\varphi(x) - f(x,\varphi(x),D\varphi(x)) \le 0$. Hence, $\varphi(x)$ is a viscosity supersolution of $\Delta_\infty^h u(x) = f(x,u(x),Du(x))$ in $B_r(x_0)$.

Now, we illustrate that there must exist a point $\bar{x} \in \partial B_r(x_0)$ such that $u(\bar{x}) > \varphi(\bar{x})$. In fact, if $u(x) \le \varphi(x)$ on $\partial B_r(x_0)$, we can conclude $u(x) \le \varphi(x)$ in $B_r(x_0)$ by the comparison principle. This contradicts $u(x_0) > 0 = \varphi(x_0)$. Therefore,

$$\sup_{B_r(x_0)} u(x) \ge \sup_{\partial B_r(x_0)} u(x) \ge \varphi(\bar{x}) \ge cr^{(h+1)/(h-\gamma)}. \qquad \square$$

Theorems 1.2 and 1.3 show that the sharp regularity of $u(x)$ at the free boundary is $C^{(h+1)/(h-\gamma)}$.

In the following, we demonstrate that the free boundary is a porous set. We first present the definition of porosity.

DEFINITION 4.1. The set $P \subset \mathbb{R}^n$ is porous with porosity σ if there exists a radius $R > 0$ such that for any $x \in P$ and $0 < r < R$, there exists a point $y \in \mathbb{R}^n$ such that $B_{\sigma r}(y) \subset B_r(x) \setminus P$.

REMARK 4.2 [28]. The Hausdorff dimension of a porous set with porosity σ is at most $n - c\sigma^n$, where c is a positive dimensional constant.

COROLLARY 4.3. *Let $f(x,t,p) \in C(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ be nonnegative and satisfy Conditions ($\mathbf{f}_b$), ($\mathbf{f}_\Lambda$), and ($\mathbf{f}_\lambda$). Assume that $f(x,t,p)$ is nondecreasing with respect to t in $[0,+\infty)$ and $\iota \longmapsto f(x,t,\iota p)$ is nonincreasing in a small interval $[1,1+\eta)$ for some $\eta > 0$. Suppose that $f(x,t,p) > 0$ for each $(x,t,p) \in \Omega\times(0,+\infty)\times\mathbb{R}^n$. If $u(x)$ is a nonnegative bounded viscosity solution to (1-1) in Ω, then $\partial\{u > 0\}$ is a porous set and has Lebesgue measure zero.*

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PROOF. For any $r > 0$ and $x \in \partial\{u > 0\}$, we can select some $\xi \in \overline{B_r(x)}$ such that

$$u(\xi) = \sup_{B_r(x)} u(x).$$

According to Theorem 1.3, $u(\xi)$ satisfies the following nondegeneracy estimate:

$$u(\xi) \geq cr^{(h+1)/(h-\gamma)}. \tag{4-1}$$

By Theorem 1.2,

$$u(\xi) \leq C[d(\xi)]^{(h+1)/(h-\gamma)}, \tag{4-2}$$

where $d(\xi) := \operatorname{dist}(\xi, \partial\{u > 0\}) > 0$. By (4-1) and (4-2),

$$\left(\frac{c}{C}\right)^{(h-\gamma)/(h+1)} \leq \frac{d(\xi)}{r}.$$

Selecting $\sigma = \frac{1}{2}(c/C)^{(h-\gamma)/(h+1)}$,

$$B_{2\sigma r}(\xi) \subset \{u > 0\}.$$

For $\overline{\eta} \in (0, 1)$, choosing a point $z = \overline{\eta}\xi + (1 - \overline{\eta})x$ such that $|\xi - z| = \sigma r$,

$$B_{\sigma r}(z) \subset B_{2\sigma r}(\xi) \cap B_r(x) \subset B_r(x) \setminus \partial\{u > 0\},$$

which means that $\partial\{u > 0\}$ is porous with porosity σ.

Using Remark 4.2, we can immediately demonstrate that the Lebesgue measure of $\partial\{u > 0\}$ is zero. □

For any $x_0 \in \partial\{u > 0\}$ and $r > 0$ such that $B_r(x_0) \subset \Omega$, we consider the blow-up sequence $\{u_\varsigma\}$,

$$u_\varsigma(\tilde{x}) := \frac{u(\varsigma\tilde{x} + x_0)}{\varsigma^{(h+1)/(h-\gamma)}}, \quad \tilde{x} \in B_{(r/3)\varsigma^{-1}}.$$

Note that $u_\varsigma(\tilde{x})$ stands for a ς-zoom-in of u in the sense of appropriate re-scaling. By the scaling invariance, we can apply Theorem 1.2 to obtain

$$u_\varsigma(\tilde{x}) \leq C\,|\tilde{x}|^{(h+1)/(h-\gamma)}, \quad \tilde{x} \in B_{(r/3)\varsigma^{-1}},$$

where the constant C is independent of ς. Hence, $u_\varsigma(\tilde{x})$ is locally bounded in $B_{(r/3)\varsigma^{-1}}$. Since $\Delta^h_\infty u_\varsigma(\tilde{x}) \geq 0$ in the viscosity sense, we have that $u_\varsigma(\tilde{x})$ is locally Lipschitz equicontinuous, based on Lemma 2.3. By the Arzela–Ascoli compact theorem, up to a subsequence if necessary, we can assume that $u_\varsigma(\tilde{x})$ locally uniformly converges to an entire function $u_0(\tilde{x}) : \mathbb{R}^n \to \mathbb{R}$ as $\varsigma \to 0^+$. Since u_0 intrinsically reflects the limit case of the blow-up sequence u_ς, we call u_0 the blow-up solution at point x_0.

THEOREM 4.4 (Blow-up solution). *Let $f(x, t, p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative and satisfy Conditions ($\mathbf{f}_b$), ($\mathbf{f}_\Lambda$), and ($\mathbf{f}_\lambda$). Assume that $f(x, t, p) > 0$ for each $(x, t, p) \in \Omega \times (0, +\infty) \times \mathbb{R}^n$. Suppose that $f(x, t, p)$ is nondecreasing with respect to t in $[0, +\infty)$ and $\iota \longmapsto f(x, t, \iota p)$ is nonincreasing in a small interval $[1, 1 + \eta)$ for some $\eta > 0$. If $u(x)$ is*

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a nonnegative bounded viscosity solution of (1-1) and $u_0(\tilde{x})$ is a blow-up solution at $x_0 \in \partial\{u > 0\}$, then $u_0(0) = 0$ and $u_0(\tilde{x})$ satisfies

$$\lambda(x_0) f(x_0, u_0(\tilde{x}), 0) \le \Delta_\infty^h u_0(\tilde{x}) \le \Lambda(x_0) f(x_0, u_0(\tilde{x}), 0) \tag{4-3}$$

in the viscosity sense. Furthermore, there exists a universal constant $\overline{C}$ such that

$$\frac{1}{\overline{C}} \le \liminf_{|\tilde{x}|\to+\infty} (|\tilde{x}|^{(-h-1)/(h-\gamma)} u_0(\tilde{x})) \le \limsup_{|\tilde{x}|\to+\infty} (|\tilde{x}|^{(-h-1)/(h-\gamma)} u_0(\tilde{x})) \le \overline{C}. \tag{4-4}$$

PROOF. For each $\varsigma > 0$, we have $u_\varsigma(0) = (u(x_0))/(\varsigma^{(h+1)/(h-\gamma)}) = 0$ and

$$\Delta_\infty^h u_\varsigma(\tilde{x}) = \varsigma^{-((h+1)\gamma)/(h-\gamma)} f(x_0 + \varsigma\tilde{x}, \varsigma^{(h+1)/(h-\gamma)} u_\varsigma(\tilde{x}), \varsigma^{(1+\gamma)/(h-\gamma)} Du_\varsigma(\tilde{x}))$$

in the viscosity sense. By $(\mathbf{f}_\Lambda)$,

$$\begin{aligned}\Delta_\infty^h u_\varsigma(\tilde{x}) &= \varsigma^{-((h+1)\gamma)/(h-\gamma)} f(x_0 + \varsigma\tilde{x}, \varsigma^{(h+1)/(h-\gamma)} u_\varsigma(\tilde{x}), \varsigma^{(1+\gamma)/(h-\gamma)} Du_\varsigma(\tilde{x})) \\ &\le \Lambda(x_0 + \varsigma\tilde{x}) f(x_0 + \varsigma\tilde{x}, u_\varsigma(\tilde{x}), \varsigma^{(1+\gamma)/(h-\gamma)} Du_\varsigma(\tilde{x})).\end{aligned}$$

We can also conclude

$$\begin{aligned}\Delta_\infty^h u_\varsigma(\tilde{x}) &= \varsigma^{-((h+1)\gamma)/(h-\gamma)} f(x_0 + \varsigma\tilde{x}, \varsigma^{(h+1)/(h-\gamma)} u_\varsigma(\tilde{x}), \varsigma^{(1+\gamma)/(h-\gamma)} Du_\varsigma(\tilde{x})) \\ &\ge \lambda(x_0 + \varsigma\tilde{x}) f(x_0 + \varsigma\tilde{x}, u_\varsigma(\tilde{x}), \varsigma^{(1+\gamma)/(h-\gamma)} Du_\varsigma(\tilde{x}))\end{aligned}$$

according to Condition $(\mathbf{f}_\lambda)$. By Remark 3.2 and the stability of viscosity solutions under uniform limits, we have (4-3). The lower and upper estimates of (4-4) follow immediately by Theorems 1.3 and 1.2, respectively. □

It should be pointed out that Theorem 4.4 provides a growth estimate at infinity for the blow-up solution u_0, which is a specific entire solution to the dead-core problem. In the following section, we give two Liouville-type theorems for any entire solution.

5. Liouville-type theorems

In this section, we give Liouville-type theorems. For any $x_0 \in \mathbb{R}^n$, we first consider the strong absorption problem in $B_R(x_0)$:

$$\begin{cases}\Delta_\infty^h v(x) = b(v(x)_+)^\gamma & \text{in } B_R(x_0), \\ v(x) = a & \text{on } \partial B_R(x_0),\end{cases} \tag{5-1}$$

where $a > 0$ and $b > 0$ are constant. Let $v(x) \ge 0$ be the unique solution of (5-1), where the existence of solution $v(x)$ is guaranteed by Theorem 1.1.

To find the radial solution of (5-1), we solve the following ordinary differential equation (ODE):

$$|g'(r)|^{h-3}(g''(r))(g'(r))^2 = b(g(r)_+)^\gamma \quad \text{in } [0, R],$$

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where $g(r)$ satisfies the boundary conditions $g(0) = 0$ and $g(R) = a$. If $(a/\rho(b,\gamma))^{(h-\gamma)/(h+1)} < R$,

$$g(r) = \rho(b,\gamma) \cdot \left(r - R + \left(\frac{a}{\rho(b,\gamma)} \right)^{(h-\gamma)/(h+1)} \right)_+^{(h+1)/(h-\gamma)},$$

where

$$\rho(b,\gamma) = \left(\frac{b(h-\gamma)^{h+1}}{(\gamma+1)(h+1)^h} \right)^{1/(h-\gamma)}.$$

Hence, we obtain the solution

$$v(x) = \rho(b,\gamma) \left(|x - x_0| - R + \left(\frac{a}{\rho(b,\gamma)} \right)^{(h-\gamma)/(h+1)} \right)_+^{(h+1/h-\gamma)}$$

and the dead-core region is $B_{\overline{r}}(x_0)$, where $\overline{r} = R - (a/\rho(b,\gamma))^{(h-\gamma)/(h+1)}$.

Based on the comparison principle, we establish the following Liouville-type theorem. The key is to construct a suitable barrier function via the radical solution. Now, we give the following growth condition:

($\mathbf{f}_{\lambda'}$) $0 \le \lambda'(x)\delta^\gamma f(x,t,p) \le f(x,\delta t,p)$, where the continuous function $\lambda'(x)$ satisfies $\inf_{x\in\mathbb{R}^n} \lambda'(x) > 0$, $\inf_{(x,p)\in\mathbb{R}^n\times\mathbb{R}^n} f(x,1,p) > 0$, $\gamma \in [0,h)$, $t > 0$, and $\delta > 0$ is small enough.

THEOREM 5.1 (Liouville type theorem). *Let $f(x,t,p) \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ be nonnegative and satisfy Condition ($\mathbf{f}_{\lambda'}$). Suppose that $f(x,t,p)$ is nondecreasing with respect to t in $[0,+\infty)$ and $f(x,t,p) > 0$ for each $(x,t,p) \in \mathbb{R}^n \times (0,+\infty) \times \mathbb{R}^n$. Assume that $\iota \longmapsto f(x,t,\iota p)$ is nonincreasing in a small interval $[1,1+\eta)$ for some $\eta > 0$. Let $u(x)$ be a nonnegative bounded viscosity solution to*

$$\Delta_\infty^h u(x) = f(x,u(x),Du(x)) \quad \text{in } \mathbb{R}^n$$

and $b = \tilde{\eta}^{-\gamma} \inf_{x\in\mathbb{R}^n} \lambda'(x) \inf_{(x,p)\in\mathbb{R}^n\times\mathbb{R}^n} f(x,1,p) > 0$, where $\tilde{\eta} > 0$ is large enough. If

$$\limsup_{|x|\to+\infty} \frac{u(x)}{|x-x_0|^{(h+1)/(h-\gamma)}} < \left(\frac{b(h-\gamma)^{h+1}}{(\gamma+1)(h+1)^h} \right)^{1/(h-\gamma)} \tag{5-2}$$

for some point $x_0 \in \mathbb{R}^n$, then $u(x) \equiv 0$ in $\mathbb{R}^n$.

PROOF. By (5-2), taking $R \gg 1$ sufficiently large,

$$\sup_{\partial B_R(x_0)} \frac{u(x)}{|x-x_0|^{(h+1)/(h-\gamma)}} < \theta \left(\frac{b(h-\gamma)^{h+1}}{(\gamma+1)(h+1)^h} \right)^{1/(h-\gamma)} \tag{5-3}$$

for some $0 < \theta < 1$. Choosing $a = \sup_{\partial B_R(x_0)} u(x)$ in (5-1), we get that

$$v(x) = \rho(b,\gamma) \left(|x-x_0| - R + \left(\frac{\sup_{\partial B_R(x_0)} u}{\rho(b,\gamma)} \right)^{(h-\gamma)/(h+1)} \right)_+^{(h+1)/(h-\gamma)}$$

is the unique viscosity solution of (5-1) in $B_R(x_0)$.

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The selection of b implies that $u(x)$ is a viscosity subsolution of (5-1) in $B_R(x_0)$. Since $u(x) \le \sup_{\partial B_R(x_0)} u(x)$ on $\partial B_R(x_0)$, by the comparison principle, we conclude $u(x) \le v(x)$ in $B_R(x_0)$. Combining (5-3),

$$u(x) \le \rho(b,\gamma)(|x-x_0| - (1-\theta^{(h-\gamma)/(h+1)})R)_+^{(h+1)/(h-\gamma)}.$$

Letting $R \to +\infty$, we observe $u(x) \equiv 0$, which completes the proof. □

REMARK 5.2. Inequality (5-2) is sharp. Indeed, if we choose $f(x,u(x),Du(x)) = b(u(x)_+)^\gamma$, then $f(x,t,p)$ satisfies the conditions in Theorem 5.1. Setting

$$T(x) := \left(\frac{b(h-\gamma)^{h+1}}{(\gamma+1)(h+1)^h}\right)^{1/(h-\gamma)} |x-x_0|^{(h+1)/(h-\gamma)},$$

we can verify that $T(x)$ is a viscosity solution of $\Delta_\infty^h u(x) = b(u(x)_+)^\gamma$ in $\mathbb{R}^n$ and

$$\limsup_{|x|\to+\infty} \frac{T(x)}{|x-x_0|^{(h+1)/(h-\gamma)}} = \left(\frac{b(h-\gamma)^{h+1}}{(\gamma+1)(h+1)^h}\right)^{1/(h-\gamma)}.$$

Therefore, this means that (5-2) cannot be weakened.

If the solution satisfies some growth condition at infinity, we can also establish a Liouville-type theorem by Theorem 1.2.

THEOREM 5.3. *Suppose that $f(x,t,p) \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ is nonnegative and satisfies Conditions ($\mathbf{f}_b$) and ($\mathbf{f}_\Lambda$). Let $u(x)$ be a nonnegative bounded viscosity solution to*

$$\Delta_\infty^h u(x) = f(x,u(x),Du(x)) \quad \text{in } \mathbb{R}^n.$$

If $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$ with

$$u(x) = o(|x|^{(h+1)/(h-\gamma)}) \quad \text{as } |x| \to +\infty, \tag{5-4}$$

then $u(x) \equiv 0$ in $\mathbb{R}^n$.

PROOF. For $k \in \mathbb{N}$ and $0 < r \ll 1$, let

$$u_k(x) := k^{-(h+1)/(h-\gamma)} u(x_0 + kx), \quad x \in B_r.$$

By computations, it is easy to check

$$\begin{aligned}
&\Delta_\infty^h u_k(x) \\
&\quad = |Du_k(x)|^{h-3}\Delta_\infty u_k(x) = k^{-(h+1)\gamma/(h-\gamma)}|Du(x_0+kx)|^{h-3}\Delta_\infty u(x_0+kx) \\
&\quad = k^{-(h+1)\gamma/(h-\gamma)} f(x_0+kx, k^{(h+1)/(h-\gamma)}u_k(x), k^{(1+\gamma)/(h-\gamma)}Du_k(x)) =: \hat{G}(x,u_k(x),Du_k(x)),
\end{aligned}$$

and $\hat{G}$ satisfies Conditions ($\mathbf{f}_b$) and ($\mathbf{f}_\Lambda$).

We are going to prove

$$\|u_k\|_{L^\infty(\overline{B_r})} \to 0^+ \quad \text{as } k \to +\infty.$$

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Let x_k be the maximum point of $u_k(x)$ in $\overline{B_r}$. Now, we consider two cases of $|kx_k|$. If $|kx_k|$ remains bounded as $k \to +\infty$, then

$$u_k(x_k) \to 0^+ \quad \text{as } k \to +\infty.$$

If $|kx_k|$ goes to infinity as $k \to +\infty$, we can use (5-4) to get

$$u_k(x_k) \to 0^+ \quad \text{as } k \to +\infty.$$

Applying Theorem 1.2,

$$u_k(x) \le C_k\, |x|^{(h+1)/(h-\gamma)}, \tag{5-5}$$

where $C_k > 0$ and $C_k \to 0^+$ for $k \to +\infty$.

We complete the rest of the proof by contradiction. Suppose that there exists some point $y \in \mathbb{R}^n$ such that $u(x_0 + y) > 0$. Choosing k large enough so that $|y| \le kr$ and

$$\sup_{B_r} \frac{u_k(x)}{|x|^{(h+1)/(h-\gamma)}} \le \frac{u(x_0+y)}{20\,|y|^{(h+1)/(h-\gamma)}}, \tag{5-6}$$

where (5-6) follows from (5-5), we can show

$$\frac{u(x_0+y)}{|y|^{(h+1)/(h-\gamma)}} \le \sup_{B_{kr}} \frac{u(x_0+x)}{|x|^{(h+1)/(h-\gamma)}} \le \sup_{B_r} \frac{u_k(x)}{|x|^{(h+1)/(h-\gamma)}} \le \frac{u(x_0+y)}{20|y|^{(h+1)/(h-\gamma)}}.$$

That is, $u(x_0 + y) = 0$, which is a contradiction. □

REMARK 5.4. If $f(x,t,p) = t_+^\gamma$, one can see that the conditions in Theorem 5.1 are weaker than those in Theorem 5.3.

6. The limit case $\gamma = h$

In this section, we prove that the dead-core region vanishes for the limit case $\gamma = h$. Due to the dependence of gradient Du on $f(x,u,Du)$, the construction of the barrier function is more complex. The method is based on the scaling technique and comparison principle. The cases of $f(x,t,p) = t^3$ and $f(x,t,p) = f(t)$ were respectively demonstrated in [1, 23] for infinity Laplace equations.

PROOF OF THEOREM 1.4. We argue by contradiction. Assume that there exists a point $y_0 \in \Omega$ such that $u(y_0) > 0$ and

$$r := \operatorname{dist}(y_0, \{u = 0\}) < \tfrac{1}{10}\operatorname{dist}(y_0, \partial\Omega).$$

Now, we aim to construct a suitable barrier function as a viscosity subsolution to (1-1). Let

$$\Gamma_\xi(x) := \begin{cases} e^{-\xi(r/2)^2} - e^{-\xi r^2} & \text{in } B_{(r/2)}(y_0), \\ e^{-\xi|x-y_0|^2} - e^{-\xi r^2} & \text{in } B_r(y_0)\backslash B_{(r/2)}(y_0), \\ 0 & \text{in } \Omega\backslash B_r(y_0), \end{cases}$$

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where ξ is a positive constant to be determined later. Direct computations show that there exists some $\varepsilon_0 > 0$ such that

$$\inf_{B_r(y_0)\setminus B_{(r/2)}(y_0)} |D\Gamma_\xi(x)| \geq \varepsilon_0.$$

Choosing $0 < \beta \ll 1$ and $\xi \gg 1$, we can verify

$$\beta\Gamma_\xi(x) \leq u(x) \quad \text{on } \partial B_r(y_0) \cup \partial B_{(r/2)}(y_0)$$

and

$$\begin{aligned}\Delta_\infty^h(\beta\Gamma_\xi(x)) - f(x,\beta\Gamma_\xi(x),\beta D\Gamma_\xi(x)) &\geq \beta^h(\Delta_\infty^h\Gamma_\xi(x) - \Lambda(x)f(x,\Gamma_\xi(x),\beta D\Gamma_\xi(x)))\\ &\geq 0 \quad \text{in } B_r(y_0)\setminus B_{(r/2)}(y_0).\end{aligned}$$

Thus, $\beta\Gamma_\xi(x)$ is a viscosity subsolution to (1-1) in $B_r(y_0)\setminus B_{(r/2)}(y_0)$ and then

$$\beta\Gamma_\xi(x) \leq u(x) \quad \text{in } B_r(y_0)\setminus B_{(r/2)}(y_0)$$

by the comparison principle.

Based on Condition ($\mathbf{f}_h$),

$$0 \leq f(x,\delta t,p) \leq \Lambda(x)\delta^{h-1}\delta f(x,t,p) = \Lambda'(x)\delta f(x,t,p),$$

where $\Lambda'(x) = \Lambda(x)\delta^{h-1} > 0$, $t > 0$, and $\delta > 0$ is small enough. By Theorem 1.2,

$$\sup_{B_d(z)} u(x) \leq Cd^{(h+1)/(h-1)}$$

for $z \in \partial B_r(y_0) \cap \partial\{u > 0\}$ and $0 < d \ll (\beta\varepsilon_0/8C)^{(h-1)/2}$. Hence,

$$0 < \tfrac{1}{2}\beta\varepsilon_0 d \leq \sup_{B_d(z)} \beta|\Gamma_\xi(x) - \Gamma_\xi(z)| \leq \sup_{B_d(z)} \beta\Gamma_\xi(x) \leq \sup_{B_d(z)} u(x) \leq Cd^{(h+1)/(h-1)} \leq \tfrac{1}{8}\beta\varepsilon_0 d,$$

which provides us with a contradiction. □

REMARK 6.1. Theorem 1.4 shows that the viscosity solution $u(x)$ of (1-1) has the strong unique continuation property.

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References

[1] D. J. Araújo, R. Leitão and E. V. Teixeira, 'Infinity Laplacian equation with strong absorptions', *J. Funct. Anal.* **270** (2016), 2249–2267.

[2] D. J. Araújo and G. S. Sá, 'Infinity Laplacian equations with singular absorptions', *Calc. Var. Partial Differential Equations* **61** (2022), Paper no. 132, 16 pages.

[3] D. J. Araújo and E. V. Teixeira, 'Geometric approach to nonvariational singular elliptic equations', *Arch. Rational Mech. Anal.* **209** (2013), 1019–1054.

https://doi.org/10.1017/S1446788726101499 Published online by Cambridge University Press

[4] D. J. Araújo and R. Teymurazyan, 'Fully nonlinear dead-core systems', *J. Funct. Anal.* **287** (2024), Paper no. 110586, 23 pages.
[5] S. N. Armstrong and C. K. Smart, 'A finite difference approach to the infinity Laplace equation and tug-of-war games', *Trans. Amer. Math. Soc.* **364** (2012), 595–636.
[6] G. Aronsson, 'Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$', *Ark. Mat.* **6** (1965), 33–53.
[7] G. Aronsson, 'Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. (II)', *Ark. Mat.* **6** (1966), 409–431.
[8] G. Aronsson, 'Extension of functions satisfying Lipschitz conditions', *Ark. Mat.* **6** (1967), 551–561.
[9] G. Aronsson, 'On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$', *Ark. Mat.* **7** (1968), 395–425.
[10] G. Aronsson, 'Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. (III)', *Ark. Mat.* **7** (1969), 509–512.
[11] G. Aronsson, M. G. Crandall and P. Juutinen, 'A tour of the theory of absolutely minimizing functions', *Bull. Amer. Math. Soc.* **41** (2004), 439–505.
[12] E. N. Barron, L. C. Evans and R. Jensen, 'The infinity Laplacian, Aronsson's equation and their generalizations', *Trans. Amer. Math. Soc.* **360** (2008), 77–101.
[13] E. N. Barron, R. Jensen and C. Wang, 'The Euler equation and absolute minimizers of L^∞ functionals', *Arch. Rational Mech. Anal.* **157** (2001), 255–283.
[14] A. Biswas and H. H. Vo, 'Liouville theorems for infinity Laplacian with gradient and KPP type equation', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **24** (2023), 1223–1256.
[15] M. G. Crandall, 'An efficient derivation of the Aronsson equation', *Arch. Ration. Mech. Anal.* **167** (2003), 271–279.
[16] M. G. Crandall, L. C. Evans and R. F. Gariepy, 'Optimal Lipschitz extensions and the infinity Laplacian', *Calc. Var. Partial Differential Equations* **13** (2001), 123–139.
[17] M. G. Crandall, L. C. Evans and P. L. Lions, 'Some properties of viscosity solutions of Hamilton Jacobi equations', *Trans. Amer. Math. Soc.* **282** (1984), 487–502.
[18] M. G. Crandall, H. Ishii and P. L. Lions, 'User's guide to viscosity solutions of second-order partial differential equations', *Bull. Amer. Math. Soc.* **27** (1992), 1–67.
[19] M. G. Crandall and P. L. Lions, 'Viscosity solutions and Hamilton–Jacobi equations', *Trans. Amer. Math. Soc.* **277** (1983), 1–42.
[20] J. V. da Silva, R. A. Leitão Júnior and G. C. Ricarte, 'Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries', *Math. Nachr.* **294** (2021), 38–55.
[21] J. V. da Silva, D. J. Rossi and A. M. Salort, 'Regularity properties for p-dead-core problems and their asymptotic limit as $p \to \infty$', *J. London Math. Soc.* **99** (2019), 69–96.
[22] J. V. da Silva and A. M. Salort, 'Sharp regularity estimates for quasi-linear elliptic dead-core problems and applications', *Calc. Var. Partial Differential Equations* **57** (2018), Paper no. 83, 24 pages.
[23] N. M. L. Diehl and R. Teymurazyan, 'Reaction-diffusion equations for the infinity Laplacian', *Nonlinear Anal.* **199** (2020), 111956, 12 pages.
[24] L. C. Evans and O. Savin, ' $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions', *Calc. Var. Partial Differential Equations* **32** (2008), 325–347.
[25] R. Jensen, 'Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient', *Arch. Ration. Mech. Anal.* **123** (1993), 51–74.
[26] H. Koch, Y. R. Zhang and Y. Zhou, 'An asymptotic sharp Sobolev regularity for planar infinity harmonic functions', *J. Math. Pures Appl. (9)* **132** (2019), 457–482.
[27] H. Koch, Y. R. Zhang and Y. Zhou, 'Some sharp Sobolev regularity for inhomogeneous ∞-Laplace equation in plane', *J. Math. Pures Appl. (9)* **132** (2019), 483–521.
[28] P. Koskela and S. Rohde, 'Hausdorff dimension and mean porosity', *Math. Ann.* **309** (1997), 593–609.

https://doi.org/10.1017/S1446788726101499 Published online by Cambridge University Press

[29] T. Lin and F. Liu, 'Viscosity solutions to the infinity Laplacian equation with strong absorptions', *Commun. Pure Appl. Anal.* **21** (2022), 4251–4267.
[30] T. Lin and F. Liu, 'Viscosity solutions to the inhomogeneous reaction-diffusion equation involving the infinity Laplacian', *Anal. Appl.* **23** (2025), 263–286.
[31] F. Liu and H. Sun, 'Viscosity solutions to the infinity Laplacian equation with singular nonlinear terms', *J. Aust. Math. Soc.* **117** (2024), 345–374.
[32] F. Liu and X. Yang, 'Solutions to an inhomogeneous equation involving infinity Laplacian', *Nonlinear Anal.* **75** (2012), 5693–5701.
[33] G. Lu and P. Wang, 'Inhomogeneous infinity Laplace equation', *Adv. Math.* **217** (2008), 1838–1868.
[34] G. Lu and P. Wang, 'A PDE perspective of the normalized infinity Laplacian', *Comm. Partial Differential Equations* **33** (2008), 1788–1817.
[35] Y. Peres, O. Schramm, S. Sheffield and D. B. Wilson, 'Tug-of-war and the infinity Laplacian', *J. Amer. Math. Soc.* **22** (2009), 167–210.
[36] I. Stakgold, 'Reaction-diffusion problems in chemical engineering', in *Nonlinear Diffusion Problems*, Lecture Notes in Mathematics, 1224 (ed. A. Fasano and M. Primicerio) (Springer, Berlin, 1986), 119–152.
[37] E. V. Teixeira, 'Regularity for the fully nonlinear dead-core problem', *Math. Ann.* **364** (2016), 1121–1134.

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