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Total Variation Asymptotics for Refined Poisson Process Approximations of Random Logarithmic Assemblies

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Assemblies are decomposable combinatorial objects characterized by a sequence m_i that counts the number of possible components of size i. Permutations on n elements, mappings from a set containing n elements into itself, 2-regular graphs on n vertices, and set partitions on a set of size n are all assemblies with natural decompositions. Logarithmic assemblies are characterized by constants $\theta > 0$ and $\kappa_0 > 0$ such that $m_i \kappa_0^i/(i-1)! \to \theta$. Random mappings, permutations and 2-regular graphs are all logarithmic assemblies, but set partitions are not.

Given a logarithmic assembly, all representatives having total size n are chosen uniformly and a component counting process $\mathbf{C}(n) = (C_1(n), C_2(n), \ldots, C_n(n))$ is defined, where $C_i(n)$ is the number of components of size i. Our results also apply to $\mathbf{C}(n)$ distributed as the Ewens sampling formula with parameter θ. Denote the component counting process up to size at most b by $\mathbf{C}_b(n) = (C_1(n), C_2(n), \ldots, C_b(n))$. It is natural to approximate $\mathbf{C}_b$ by $\mathbf{Z}_b = (Z_1, Z_2, \ldots, Z_b)$, the b-dimensional process of independent Poisson variables Z_i for which the ith variable has expectation $\mathbb{E}Z_i = m_i \kappa_0^i \exp((1-\theta)i/n)/i!$. We find asymptotics for the total variation distance between $\mathbf{C}_b(n)$ and $\mathbf{Z}_b$.

1. Introduction

1.1. Random logarithmic assemblies

Combinatorial assemblies are labelled decomposable combinatorial objects. Typical examples of combinatorial assemblies are permutations of n elements, mappings of a finite set with n elements into itself, 2-regular graphs on n vertices and set partitions on n elements. For permutations the components are cycles, for mappings they are connected components, for 2-regular graphs they are undirected cycles of size at least 3, and for set partitions they are simply blocks with no additional structure added. Each assembly is characterized for our purposes by the number of possible components m_i on sets of size i. For permutations the number of cycles that may be formed from a set of size i is $m_i = (i-1)!$, for mappings $m_i = (i-1)!(1 + i + i^2/2 + \cdots + i^{i-1}/(i-1)!)$, for 2-regular graphs $m_i = \mathbf{1}(i > 2)\,(i-1)!/2$, and for set partitions $m_i = 1$ for all i.

For fixed m_i, given an assembly of size n, let $C_i(n)$ be the number of components of size

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i and let $\mathbf{C}(n)$ denote

$$\mathbf{C}(n) = (C_1(n), C_2(n), \ldots, C_n(n), 0, 0, \ldots).$$

Given $\mathbf{a} \in \mathbb{Z}_n^+$, define $N(n, \mathbf{a})$ to be the number of assemblies of size n for which $\mathbf{C}(n) = \mathbf{a}$. A formula for $N(n, \mathbf{a})$ generalizing Cauchy's formula for permutations is given by

$$N(n, \mathbf{a}) = \mathbf{1}(a_1 + 2a_2 + \cdots + na_n = n)\, n! \prod_{i=1}^{n} \left(\frac{m_i}{i!}\right)^{a_i} \frac{1}{a_i!}. \tag{1.1}$$

Let

$$p(n) = \sum_{\mathbf{a}} N(n, \mathbf{a}),$$

and suppose all assemblies of size n are chosen uniformly, so that

$$\mathbb{P}(C(n) = \mathbf{a}) = \frac{N(n, \mathbf{a})}{p(n)}.$$

Given any sequence of positive reals κ_n, if for each n we define $Z_i = Z_i(n)$, $i = 1, 2, \ldots, n$ to be independent Poisson variables with means given by

$$\mathbb{E}Z_i = \frac{m_i \kappa_n^i}{i!},$$

then Theorem 1 of [3] shows that

$$\mathbf{C}(n) \overset{\mathrm{d}}{=} \left(Z_1, Z_2, \ldots, Z_n, 0, 0, \ldots | Z_1 + 2Z_2 + \cdots + nZ_n = n\right). \tag{1.2}$$

The component process restricted to those components having size at most $b(n)$ for any sequence of integers $b(n) \leqslant n$ will be denoted by

$$\mathbf{C}_b(n) = (C_1(n), C_2(n), \ldots, C_b(n)).$$

For b/n small and a proper choice for κ_n the distribution of $\mathbf{C}_b(n)$ may be well approximated by

$$\mathbf{Z}_b = (Z_1, Z_2, \ldots, Z_b).$$

To measure how well the distribution of $\mathbf{Z}_b$ approximates that of $\mathbf{C}_b(n)$ total variation distance is used. The total variation distance between two random elements X and Y that take their values on a finite or countable space S is defined to be

$$d_{\mathrm{TV}}(X, Y) = \frac{1}{2} \sum_{s \in S} |\mathbb{P}(X = s) - \mathbb{P}(Y = s)|.$$

We define $d_b(n) = d_{\mathrm{TV}}(\mathbf{C}_b(n), \mathbf{Z}_b)$.

The Ewens sampling formula with parameter $\theta > 0$ from population genetics satisfies (1.2) with $\mathbb{E}Z_i = \theta/i$. The Ewens sampling formula with $\theta = 1$ is the distribution of $\mathbf{C}(n)$ for random permutations. More generally, the Ewens sampling formula having parameter θ is the distribution of $\mathbf{C}(n)$ when permutations are selected proportionally to $\theta^{\#\text{cycles}}$.

The logarithmic class of assemblies is closely related to the Ewens sampling formula. The main characteristic of logarithmic assemblies is that there exist constants $\theta > 0$ and $\kappa_0 > 0$ such that

$$m_i \kappa_0^i / (i-1)! \to \theta; \tag{1.3}$$

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see [3]. Permutations, mappings and 2-regular graphs are all logarithmic assemblies. For permutations $\theta = 1$ and $\kappa_0 = 1$; for mappings $\theta = 1/2$ and $\kappa_0 = e^{-1}$; and for 2-regular graphs $\theta = 1/2$ and $\kappa_0 = 1$. The Ewens sampling formula with parameter θ is distributed as a logarithmic assembly with $\kappa_0 = 1$.

Asymptotics for $d_b(n)$ when κ_n is set to $\kappa_n = \kappa_0$ for all n are derived in [2], the results of which are summarized in Section 1.3. In this paper we examine refined Poisson process approximations of $\mathbf{C}(n)$, in which κ_n is taken to be $\kappa_n = \kappa_0 \exp((1-\theta)/n)$. The refined Poisson process approximation was introduced in [3] and heuristics in that paper indicate that $d_b(n)$ should decrease like $(b/n)^2$. We show that this prediction is true under certain restrictions and we derive asymptotics for $d_b(n)$.

1.2. The exponential generating function

Various restrictive assumptions of logarithmic assemblies have appeared in the literature. Section 4 of [2] lists several and others are given in [1] and [11]. A Δ-domain is defined for parameters $\eta > 0$ and $0 < \phi < \pi/2$ to be

$$\Delta \equiv \Delta(\eta, \phi) = \{z \in \mathbb{C} : |z| \leqslant 1 + \eta, |\arg(z-1)| \geqslant \phi\}.$$

We require that there exists a constant $\kappa_0 > 0$ such that the exponential generating function

$$f(z) = \exp\left(\sum_{i=1}^{\infty} \frac{m_i \kappa_0^i}{i!} z^i\right) \tag{1.4}$$

is analytically continuable to a Δ-domain in $\mathbb{C}$. Moreover, we require that there are constants $\theta > 0$, $\delta > 0$, $K > 0$, K^* real, such that

$$f(z) = K(1-z)^{-\theta}\left\{1 + K^*(1-z)^{\theta} + O\left((1-z)^{\delta}\right)\right\} \tag{1.5}$$

as $z \to 1$ in Δ.

The term with K^* only adds a constant to f. It is useful for assemblies such as random mappings, for which δ would otherwise be set to $\delta = \theta$ and Theorem 2.1 would impose stronger conditions on b. If $\delta \leqslant \theta$, then the K^* term can be absorbed into the O term. Therefore, we might as well assume that $\delta > \theta$ whenever $K^* \neq 0$.

The benefit of (1.5) is that it allows us to use the contour integration estimates of [7] in our proofs. The following theorems appear as Theorem 1 and Corollary 2, respectively, in [7].

Theorem 1.1. *Let α be a real number. If g is analytic on $\Delta(\eta, \phi) \setminus \{1\}$ and $g(z) = O((1-z)^{-\alpha})$ as $z \to 1$ in Δ, then $[z^n]g(z) = O(n^{\alpha-1})$.*

Theorem 1.2. *Suppose $\alpha \notin \{0, -1, -2, -3, \ldots\}$. If g is analytic in $\Delta(\eta, \phi) \setminus \{1\}$ and $g(z) \sim K(1-z)^{-\alpha}$ as $z \to 1$ in Δ, then $[z^n]g(z) \sim \frac{K}{\Gamma(\alpha)} n^{\alpha-1}$.*

This expansion is also found in [7]: for $\theta \notin \{0, -1, -2, \ldots\}$,

$$[z^n](1-z)^{-\theta} \sim \frac{n^{\theta-1}}{\Gamma(\theta)}\left(1 - \frac{\theta(1-\theta)}{2n} - \frac{\theta(1-\theta)(2-\theta)(1-3\theta)}{24n^2}\right) + O(n^{\theta-4}). \tag{1.6}$$

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If $K^* = 0$ and one can take the logarithm of f in Δ, then

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{m_i \kappa_0^i}{i!} z^i &= \log K - \theta \log(1-z) + \log\left\{1 + O((1-z)^{\delta}\right\} \\ &= \log K - \theta \log(1-z) + O((1-z)^{\delta})\end{aligned}$$

as $z \to 1$ in Δ. An application of Theorem 1.1 gives

$$\frac{m_i \kappa_0^i}{i!} = \theta/i + O(i^{-\delta-1}) \sim \theta/i. \tag{1.7}$$

If $K^* \neq 0$ and one can take the logarithm of f in Δ, then

$$\sum_{i=1}^{\infty} \frac{m_i \kappa_0^i}{i!} z^i = \log K - \theta \log(1-z) + O((1-z)^{\theta})$$

as $z \to 1$ in Δ, and Theorem 1.1 gives

$$\frac{m_i \kappa_0^i}{i!} = \theta/i + O(i^{-\theta-1}) \sim \theta/i.$$

Our notion of logarithmic assembly is therefore consistent with (1.3).

It would be helpful to have simple sufficient conditions on the m_i implying the existence of an analytic continuation of $f(z)$ to a Δ-domain, but they may be quite hard to find. The Hadamard Gap Theorem (see page 88 of [9]) states that a series

$$u(z) = \sum_j u_j z^{n_j}$$

has its circle of convergence as a natural boundary provided there exists a fixed $r > 1$ such that, for all j,

$$\frac{n_{j+1}}{n_j} \geqslant r.$$

Suppose $f(z)$ is continuable to a Δ-domain. The Hadamard Gap Theorem ensures that there are functions $u(z) = \sum_i m_i' \kappa_0^i z^i / i!$, with $m_i' \in \mathbb{Z}_+$ such that $m_i' \kappa_0 / i!$ converges arbitrarily quickly to 0 and such that $\exp(u(z))$, hence $\exp(f(z) + u(z))$, is not analytically continuable across the unit disc. Thus, there are arbitrarily small perturbations of $f(z)$ that are not analytically continuable.

The logarithmic conditions must therefore be checked on a case-by-case basis. It may happen that $f(z)$ is easily summed and well known to be analytically continuable. An example is the Ewens sampling formula, which has $f(z)$ given by

$$f(z) = \exp\left(\frac{\theta}{i} z^i\right) = \exp\left(-\theta \log(1-z)\right) = (1-z)^{-\theta},$$

which satisfies (1.5) with δ arbitrarily large, $K = 1$, and $K^* = 0$. For 2-regular graphs,

$$\begin{aligned} f(z) &= (1-z)^{-1/2} \exp\left(-\frac{1}{2}z - \frac{1}{4}z^2\right) \\ &= e^{-3/4}(1-z)^{-1/2} + O((1-z)^{1/2}) \end{aligned}$$

and (1.5) is satisfied with $\theta = -1/2$, $\delta = 1$, $K = e^{-3/4}$, and $K^* = 0$. Another possibility is

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that $f(z)$ may be expressed in terms of functions that can be analysed through the implicit function theorem. For mappings of a finite set into itself, taking $\kappa_0 = e^{-1}$, we have

$$f(z) = \log \frac{1}{1 - t(\kappa_0 z)},$$

where $t(z) = \sum i^{i-1} z^i / i!$ is the exponential generating function of the number of rooted, labelled trees on i distinct elements. It is well known that $t(z)$ satisfies the equation $t(z) = z \exp(t(z))$, from which equation the implicit function theorem can be used to show $t(z)$ is extendable to a Δ-domain; see [8]. In fact,

$$t(z) = 1 - \sqrt{2}(1 - ez)^{1/2} - \frac{1}{3}(1 - ez) + O((1 - ez)^{3/2}),$$

hence

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2}(1-z)^{1/2} + \frac{1}{3}(1-z) + O((1-z)^{3/2})} \\ &= \frac{1}{\sqrt{2}(1-z)^{1/2}\left[1 + \frac{1}{3\sqrt{2}}(1-z)^{1/2} + O(1-z)\right]} \\ &= \frac{1}{\sqrt{2}}(1-z)^{-1/2}\left[1 - \frac{1}{3\sqrt{2}}(1-z)^{1/2} + O(1-z)\right], \end{aligned}$$

so that (1.5) is satisfied with $\theta = 1/2$, $\delta = 1$, $K = \frac{1}{\sqrt{2}}$, and $K^* = -\frac{1}{3\sqrt{2}}$. Here is an example for which $\delta < 1$. Let m_i and $f(z)$ be as for mappings and define $m_i^* = 2m_i$. The corresponding assembly could be mappings from a finite set into itself in which every connected component is coloured either red or blue. Let $\kappa_0 = e^{-1}$. The exponential generating function for the m_i^* is

$$g(z) := \exp\left(\sum_i \frac{m_i^* \kappa_0}{i!} z^i\right) = f^2(z) = \frac{1}{4}(1 - ez)^{-1} + O((1 - ez)^{-1/2}).$$

For this assembly, $\theta = -1$, $\delta = 1/2$, $K = 1/4$ and $K^* = 0$.

1.3. Unrefined Poisson process approximations

The asymptotics of $d_b(n)$ for logarithmic assemblies with $\kappa_n = \kappa_0$ are given in [2]. It is helpful to state some of those results for comparison with the refined Poisson process approximations of this paper. We start with Theorem 1.3 of [2]. The expectation of the ith unrefined Poisson variable Z_i is denoted by

$$\lambda_i = \mathbb{E}Z_i \equiv \frac{m_i \kappa_0^i}{i!}. \tag{1.8}$$

Theorem 1.3. *With $f(z)$ defined by* (1.4)*, assume that $f(z)$ is analytic on $\Delta(\eta, \phi)$ for some $\eta > 0$, $0 < \phi < \pi/2$. Moreover, assume that there are constants $\theta > 0$, $\delta > 0$, and $K > 0$ such that*

$$f(z) = K(1 - z)^{-\theta}\left\{1 + O\left((1 - z)^{\delta}\right)\right\}$$

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as $z \to 1$ within Δ. Assume further that $\sup i\lambda_i < \infty$. Let Z_i be mutually independent Poisson variables with means $\mathbb{E}Z_i = \lambda_i$ and define variables T_{lm} by

$$T_{lm} = \sum_{i=l+1}^{m} iZ_i.$$

Assume, moreover, that $b = b(n)$ is a sequence satisfying

$$b = o(n/\log n).$$

Under these assumptions,

$$d_{\mathrm{TV}}(\mathbf{C}_b(n), \mathbf{Z}_b) = \frac{|1-\theta|}{2n}\mathbb{E}|T_{0b} - \mathbb{E}T_{0b}| + o\left(\frac{b}{n}\right).$$

When $\theta \neq 1$, the asymptotics of $d_b(n)$ may be expressed in terms of a random variable X_θ having mean θ, variance $\theta/2$, and Laplace transform

$$\mathbb{E}e^{-tX_\theta} = \exp\left(-\int_0^1 \frac{1-\exp(-tx)}{x}\theta dx\right), \quad t > 0. \tag{1.9}$$

For sequences a_n and b_n we write $a_n \sim b_n$ if $a_n/b_n \to 1$. Corollary 1.4 summarizes Corollary 4, Lemma 6, and Corollary 5 of [2].

Corollary 1.4. *Under the assumptions of Theorem 1.3, if $\theta \neq 1$, then for fixed b*

$$d_{\mathrm{TV}}(\mathbf{C}_b(n), \mathbf{Z}_b) \sim \frac{|1-\theta|}{2n}\mathbb{E}|T_{0b} - \mathbb{E}T_{0b}|. \tag{1.10}$$

If, in addition,

$$\mathbb{E}\,|T_{0b} - \mathbb{E}T_{0b}| \asymp b, \tag{1.11}$$

then (1.10) *holds for all $b(n)$ satisfying $b = o(n/\log n)$. In particular,* (1.11) *holds when $\liminf i\lambda_i > 0$ and $\sup i\lambda_i < \infty$.*

Suppose, in addition, that $b^{-1}T_{0b} \Rightarrow X_\theta$ as $b \to \infty$. If $\theta \neq 1$, $b \to \infty$ and $b = o(n/\log n)$, then

$$d_{\mathrm{TV}}(\mathbf{C}_b(n), \mathbf{Z}_b) \sim \frac{|1-\theta|}{2}\mathbb{E}|X_\theta - \theta|\frac{b}{n},$$

where X_θ is a random variable having Laplace transform given by (1.9).

Related independent process approximations for logarithmic multisets and selections were obtained in [12].

Using methods different from ours, the monograph [1] studies $d_b(n)$ for logarithmic assemblies and, under certain conditions on the rate of convergence of $i\lambda_i$ to θ, finds that

$$d_b = \frac{|1-\theta|}{2n}\mathbb{E}|T_{0b} - \mathbb{E}T_{0b}| + O\left(\frac{b}{n}\left(\frac{b}{n} + n^{-C}\right)\right)$$

for some $C > 0$. In particular, that result does not require $b = o(n/\log n)$ or the analytic continuation of $f(z)$ across the unit disc. For that reason it is likely that the existence of conditions (2.1) and (2.2) in Theorem 2.1 in this paper should be attributed to the methods used, not to the essential nature of logarithmic assemblies.

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2. Main results

Recall that the variables Z_i are defined in such a way that (1.2) holds. We set

$$\kappa_n = \kappa_0 \exp((1-\theta)/n),$$

so the means of the Poisson variables Z_i are now given by

$$\mathbb{E}Z_i(n) = \mu_i \equiv \lambda_i \exp\left((1-\theta)i/n\right),$$

where λ_i is defined by (1.8). Note that the Z_i now depend on n, as opposed to the unrefined approximations of Section 1.3 where they were fixed. The variance of a random variable X is written as $\sigma^2(X)$.

Theorem 2.1. *With $f(z)$ given by* (1.4)*, assume that $f(z)$ is analytic on $\Delta(\eta, \phi)$ for some $\eta > 0$, $0 < \phi < \pi/2$. Moreover, assume that there are constants $\theta > 0$, $\delta > 0$, $K > 0$, and K^* real such that*

$$f(z) = K(1-z)^{-\theta}\left\{1 + K^*(1-z)^{\theta} + O\left((1-z)^{\delta}\right)\right\} \tag{2.1}$$

as $z \to 1$ within Δ. Assume further that $\sup i\lambda_i < \infty$ and, in addition, when $K^* \neq 0$ assume $\limsup i\lambda_i < 1$. Let Z_i be mutually independent Poisson variables with means $\mathbb{E}Z_i = \mu_i$ and define variables T_{lm} by

$$T_{lm} = \sum_{i=l+1}^{m} iZ_i.$$

Assume, moreover, that $b = b(n)$ is a sequence satisfying

$$b = o(n/\log n). \tag{2.2}$$

Under these assumptions, for every

$$u(n) = \min\left(\frac{\delta \log n - \log\log n - \omega(n)}{\log n - \log b(n)}, 1\right), \quad \omega(n) \to \infty, \tag{2.3}$$

we have

$$d_b(n) = \frac{|1-\theta|}{4n^2}\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| + o\left(\left(\frac{b}{n}\right)^{1+u}\right). \tag{2.4}$$

If $\delta > 1$, which is the case for the Ewens sampling formula, then, for all $b = o(n/\log n)$,

$$d_b(n) = \frac{|1-\theta|}{4n^2}\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| + o\left(\left(\frac{b}{n}\right)^{2}\right). \tag{2.5}$$

If $\delta = 1$, which is the case for many logarithmic assemblies, then, for every sequence $b(n)$ such that $b = o(n/\log n)$ and $b/\log n \to \infty$, (2.5) *holds.*

The condition $\limsup i\lambda_i < 1$ is satisfied by mappings, for which $i\lambda_i \to 1/2$. It is worth noting that, even when the logarithm of $f(z)$ may be taken, Tauberian theorems only show that $\sum_{i=1}^{n} \lambda_i \sim \theta \log n$, hence $\sup_i i\lambda_i < \infty$ does not automatically follow from (2.1), nor does condition (2.11) below.

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Corollary 2.2. *Assume the conditions of Theorem 2.1 hold. Suppose, in addition, that* $\theta = 1$. *Given* $b = b(n)$ *such that* $b = o(n/\log n)$, *for* u *defined as in* (2.3), *we have*

$$d_b(n) = o\left(\left(\frac{b}{n}\right)^{1+u}\right).$$

If $\delta > 1$, *then, for every sequence* $b(n) = o(n/\log n)$,

$$d_b(n) = o\left(\left(\frac{b}{n}\right)^{2}\right). \tag{2.6}$$

If $\delta = 1$, *then, for every sequence* $b = o(n/\log n)$ *such that* $b/\log n \to \infty$, (2.6) *holds.*

Corollary 2.3. *Assume the conditions of Theorem 2.1 hold. The condition* $\sup_i i\lambda_i < \infty$ *implies*

$$\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| = O(b^2).$$

Assume, in addition, that $\theta \neq 1$. *Given* $b = b(n)$ *such that* $b = o(n/\log n)$, *for* u *defined by* (2.3), *we have*

$$d_b(n) = O\left(\left(\frac{b}{n}\right)^{1+u}\right).$$

If $\delta > 1$, *then, for every* $b(n) = o(n/\log n)$,

$$d_b(n) = O\left(\left(\frac{b}{n}\right)^{2}\right). \tag{2.7}$$

If $\delta = 1$, *then, for every sequence* $b = o(n/\log n)$ *such that* $b/\log n \to \infty$, (2.7) *holds.*

Corollary 2.4. *Assume the conditions of Theorem 2.1 hold. Suppose, in addition, that* $\theta \neq 1$. *For* b *fixed, if* $\delta > 1$, *then*

$$d_b(n) \sim \frac{|1-\theta|}{4n^2}\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right|. \tag{2.8}$$

Now, assume

$$\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| \asymp b^2. \tag{2.9}$$

For any $\delta > 0$, *if* $b(n)$ *is such that* $b(n) = o(n/\log n)$ *and* $bn^{\delta-1}/\log n \to \infty$, *then* (2.8) *holds. In particular, when* $\delta > 1$, *if* $b(n)$ *is such that* $b(n) = o(n/\log n)$, *then* (2.8) *holds.*

Lemma 2.5 gives sufficient conditions for (2.9).

Lemma 2.5. *Assume that* $\liminf i\lambda_i = \underline{\theta} > 0$ *and*

$$\sup_i i\lambda_i = \bar{\theta} < \infty. \tag{2.10}$$

As $b \to \infty$,

$$\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| \asymp b^2.$$

The behaviour of $b^{-1}T_{0b}$ may be better understood with Lemma 2.6.

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Lemma 2.6. *Assume that* $\bar{\theta} = \sup i\lambda_i < \infty$, *and fix* $\theta \in (0, \infty)$. *The following are equivalent:*

$$\lim_{b\to\infty} \frac{1}{b} \sum_{i=1}^{b} i\lambda_i = \theta; \tag{2.11}$$

$$b^{-1} T_{0b} \Rightarrow U, \text{ and } \theta = \mathbb{E}U; \tag{2.12}$$

$$b^{-1} T_{0b} \Rightarrow X_\theta. \tag{2.13}$$

Corollary 2.7. *Assume the conditions of Theorem 2.1 hold. Suppose, in addition, that* $b^{-1}T_{0b} \Rightarrow X_\theta$ *as* $b \to \infty$. *If* $\theta \neq 1$, $b \to \infty$, $b = o(n/\log n)$, *and* $bn^{\delta-1}/\log n \to \infty$, *then*

$$d_b(n) \sim \frac{|1-\theta|}{4} \mathbb{E}\left|(X_\theta - \theta)^2 - \theta/2\right| \left(\frac{b}{n}\right)^2, \tag{2.14}$$

where X_θ *is a random variable having Laplace transform given by* (1.9). *In particular, if* $\delta > 1$, *then* (2.14) *holds for all* $b \to \infty$, $b = o(n/\log n)$.

Expressions for the density $g_\theta(x)$ of X_θ are given in [6]. In particular,

$$g_\theta(x) = \frac{e^{-\gamma\theta}}{\Gamma(\theta)} x^{\theta-1}, \quad 0 < x \leqslant 1,$$

where γ is Euler's constant. It follows from the identity $\mathbb{E}\left((X_\theta - \theta)^2 - \theta/2\right) = 0$ that

$$\mathbb{E}\left|(X_\theta - \theta)^2 - \theta/2\right| = 2\mathbb{E}\left((X_\theta - \theta)^2 - \theta/2\right)^-.$$

If $\theta \leqslant 1/2$, then the quadratic expression $(x-\theta)^2 - \theta/2$ is nonpositive if and only if $\theta - \sqrt{\theta/2} \leqslant x \leqslant \theta + \sqrt{\theta/2} \leqslant 1$, hence

$$\begin{aligned} \mathbb{E}\left|(X_\theta - \theta)^2 - \theta/2\right| &= 2\int_{\theta-\sqrt{\theta/2}}^{\theta+\sqrt{\theta/2}} \left(\theta/2 - (x-\theta)^2\right) g_\theta(x)dx \\ &= \frac{2e^{-\gamma\theta}(2^{1/2}\theta^{3/2} + 2^{1/2}\theta^{1/2} - 2\theta)\left(\theta + (\theta/2)^{1/2}\right)^{\theta+1}}{\Gamma(\theta+3)}. \end{aligned}$$

When $\theta = 1/2$, which is the case for random graphs,

$$d_b(n) \sim \frac{e^{-\gamma/2}}{15\sqrt{\pi}} \left(\frac{b}{n}\right)^2 \doteq 0.0282 \left(\frac{b}{n}\right)^2.$$

Note that the parameter δ appears in both the hypothesis and conclusion of Theorem 2.1, as opposed to the unrefined approximations of Theorem 1.3, where δ only appears in the hypothesis. To help in understanding the meaning of δ, note that (1.7) shows that $i\lambda_i \to \theta$ at least as fast as $i^{-\delta}$ when $K^* = 0$ and the logarithm of f can be taken in Δ. We hope to investigate the asymptotics of d_b without the restriction (3.28) in future work by making additional assumptions about the behaviour of $f(z)$.

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Results somewhat similar to those in this paper are obtained in [4] for the special case of the Ewens sampling formula. There the means $\mathbb{E}Z_i$ are taken exactly equal to $\mathbb{E}C_i(n)$:

$$\mathbb{E}Z_i = \frac{\theta(n-1)\cdots(n-i+1)}{i(\theta+n-1)\cdots(\theta+n-i)}. \tag{2.15}$$

Stein's method is used to show there exist positive constants $c_1(\theta)$ and $c_2(\theta)$ such that $c_1(\theta)\,(b/n)^2/\log^2(nb^{-1}) \leqslant d_b(n) \leqslant c_2(\theta)\,(b/n)^2$ for all $b \leqslant n$. That result finds the correct rate of $d_{\mathrm{TV}}(\mathbf{C}_b(n), \mathbf{Z}_b)$, but not asymptotics. Unfortunately, for $\mathbb{E}Z_i$ defined as (2.15) the relation (1.2) does not hold, and the methods in this paper are not applicable.

3. Structure of the proof

Proof of Theorem 2.1. It is shown in [3] that (1.2) implies

$$d_b(n) = \sum_{r=0}^{\infty} \mathbb{P}(T_{0b} = r)\left(1 - \frac{\mathbb{P}(T_{bn} = n-r)}{\mathbb{P}(T_{0n} = n)}\right)^+. \tag{3.1}$$

The expression (3.1) will be simplified in several steps, the first of which is to show that terms indexed by large r are insignificant. Introduce the sequence

$$L_n = \min\left(b\log n, b^{4/5}n^{1/5}\right). \tag{3.2}$$

Let d_b^* denote

$$d_b^* = \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left(1 - \frac{\mathbb{P}(T_{bn} = n-r)}{\mathbb{P}(T_{0n} = n)}\right)^+. \tag{3.3}$$

Lemma 3.1 and all the other lemmas and propositions stated during the course of this proof are proved in Section 4. The purpose of Lemma 3.1 is to help prove Lemma 3.2.

Lemma 3.1. *For any $b \geqslant 1$ and $w > 0$, $\mathbb{P}(T_{0b} \geqslant bw) \leqslant \left(\frac{\bar{\theta}e^2}{w}\right)^w$.*

Lemma 3.2. *For any k, as $n \to \infty$, with $1 \leqslant b = o(n)$,*

$$\sum_{r>L_n} \mathbb{P}(T_{0b} = r)\left(1 - \frac{\mathbb{P}(T_{bn} = n-r)}{\mathbb{P}(T_{0n} = n)}\right)^+ \leqslant \mathbb{P}(T_{0b} > L_n) = o((b/n)^k).$$

Lemma 3.2 shows that to prove Theorem 2.1 it suffices to prove

$$d_b^* = \frac{|1-\theta|}{4n^2}\mathbb{E}\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right| + o\left(\left(\frac{b}{n}\right)^{1+u}\right).$$

Let x_n denote $x_n = \exp((1-\theta)/n)$. It is useful to define functions $f_n(z)$ and $h_n(z)$ by

$$f_n(z) = f(x_n z) = \exp\left(\sum_{i=1}^{\infty} \mu_i z^i\right) \tag{3.4}$$

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and

$$h_n(z) = \exp\left(-\sum_{i=1}^{b(n)} \mu_i z^i\right), \tag{3.5}$$

so that (3.3) is

$$\begin{aligned} d_b^* &= \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left(1 - \frac{\exp(-\sum_{i=b+1}^{n} \mu_i)[z^{n-r}](h_n(z)f_n(z))}{\exp\left(-\sum_{i=1}^{n} \mu_i\right)[z^n]f_n(z)}\right)^+ \\ &= \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left(\frac{[z^n]f_n(z) - \exp(\sum_{i=1}^{b} \mu_i)[z^{n-r}](h_n(z)f_n(z))}{[z^n]f_n(z)}\right)^+. \end{aligned} \tag{3.6}$$

The method by which we simplify d_b^* consists of repeated applications of Lemma 3.3, which breaks sums such as the one in (3.3) into two parts, the second of which is asymptotically smaller than the first.

Lemma 3.3. *If*

$$d_b^* = \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)(S_{1,n,r} + S_{2,n,r})^+,$$

and

$$M_{b,n} = \max_{0 \leqslant r \leqslant L_n} |S_{2,n,r}|,$$

then

$$d_b^* = \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)(S_{1,n,r})^+ + E_n,$$

where $|E_n| \leqslant M_{b,n}$.

We will use propositions for overt simplifications of d_b^* and lemmas for auxiliary results. Our first simplification, Proposition 3.7, will require a technical lemma that is similar to Lemma 14 of [2].

Lemma 3.4. *Assume that* ψ *and* ξ *are analytic in* $\Delta(\eta, \phi) \setminus \{1\}$, *and* $|\psi(z)| \leqslant K_1|1 - z|^{-\alpha}$ *in* $\Delta(\eta, \phi) \setminus \{1\}$. *Then there is a universal constant* $W_{l,\phi}$ *such that, for all integers* $j \geqslant 0$ *such that* $|\alpha - j| \leqslant l$, *and nonnegative integers* m *and* n *such that* $0 \leqslant m \leqslant n - 1$ *and* $n - m \geqslant 2|\alpha - j| + 4$, *with the constant* $\tau_{\theta,\eta}$ *defined by*

$$\tau_{\theta,\eta} = 1 + e^{\theta}|1 - \theta|(1 + \eta),$$

we have

$$\begin{aligned} \left|[z^{n-m}]\xi(x_n z)\psi(x_n z)(1 - z)^j\right| &\leqslant eK_1 \sup_{|z| \leqslant 1+\eta} |\xi(z)| \cdot (n - m)^{\alpha-j-1}\tau_{\theta,\eta}^j \\ &\quad \times \left[W_{l,\phi} + \frac{\eta^{-\alpha+j}}{(1 + \eta)^{n-m}}(n - m)^{1-\alpha+j}\right] \end{aligned} \tag{3.7}$$

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and

$$W_{l,\phi} \leqslant A\left(1+\left(\frac{l}{\cos\phi}\right)^{l+2}\right) \tag{3.8}$$

where $A = (2+4/\cos\phi)^2$.

A major simplification of d_b^* is brought about by expanding the function $h_n(z)$ defined by (3.5) in a partial Taylor series with remainder $R_N(z)$ about $z=1$:

$$h_n(z) = \sum_{j=0}^{N} \frac{h_n{}^{(j)}(1)}{j!}(z-1)^j + R_N(z)(z-1)^{N+1}, \tag{3.9}$$

where N is defined by

$$N \equiv N(n) = \lceil \log n \rceil. \tag{3.10}$$

We will see in Proposition 3.9 that only the first three terms of (3.9) contribute to the asymptotics of d_b^*. The Taylor expansion of $h_n(z)$ also helps show the perturbation from the Ewens sampling formula is insignificant, a simplification of d_b^* accomplished in Proposition 3.8. We will need Lemma 3.5 and Lemma 3.6. Lemma 3.5 shows the Taylor coefficients of $h_n(z)$ are not too large and Lemma 3.6 shows $R_N(z)$ is small.

Lemma 3.5. *For* $k \geqslant 0$,

$$|h_n{}^{(k)}(1)| \leqslant \exp\left(-\sum_{i=1}^{b}\mu_i\right) B_k(\rho b)^k,$$

where B_k *is the* k*th Bell number, and*

$$\rho = \max\{1, e\bar{\theta}\}.$$

Moreover, define Λ *by*

$$\Lambda = \sup_{n\geqslant n_0}\sup_i\{i : i\lambda_i x_n^i > 1\} < \infty.$$

and suppose $\limsup i\lambda_i < 1$. *Then there is a* $n_0 \geqslant 1$ *such that, for every* $n > n_0$ *and* $k \geqslant b$,

$$\left|h_n^{(k)}(0)\right| \leqslant k!\, e^{\Lambda(\rho^2-1)/2+1} b\left(\frac{k}{eb}\right)^{-k/(2b)}.$$

Lemma 3.6. *On the disc*

$$D_\eta = \{z \in \mathbb{C} : |z| \leqslant 1+\eta\},$$

the following bound holds:

$$\sup_{z\in D_\eta}|R_N^{(n)}(z)| \leqslant \eta^{-(N+1)}\left((eb)^{\bar{\theta}e^{\eta b+1}} + \exp(e^{\rho\eta b}-1)\right).$$

Assumption (2.1) allows us to write

$$f(z) = K(1-z)^{-\theta} + \Psi(z), \tag{3.11}$$

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where $\Psi(z)$ is analytic in the punctured delta domain $\Delta(\eta,\phi)\setminus\{1\}$ and $\Psi(z)=KK^*+O((1-z)^{\delta-\theta})$ as $z\to 1$ in Δ. In Proposition 3.7 we use Lemma 3.3 to eliminate the $\Psi(x_nz)$ terms from $f_n(z)$ in the numerator of (3.6). The quantity $M_{b,n}$ of Lemma 3.3 is shown to be small by means of Lemma 3.4, Lemma 3.5 and Lemma 3.6.

Proposition 3.7.

$$\begin{aligned} d_b^* &= o\left(\left(\frac{b}{n}\right)^{1+u}\right)+ \\ &\quad \sum_{r=0}^{L_n}\mathbb{P}(T_{0b}=r)\left(\frac{[z^n]K(1-x_nz)^{-\theta}-\exp(\sum_{j=1}^{b}\mu_j)[z^{n-r}]h_n(z)K(1-x_nz)^{-\theta}}{[z^n]f_n(z)}\right)^+. \end{aligned}$$

Proposition 3.8 eliminates $\Psi(x_nz)$ terms from $f_n(z)$ in the denominator.

Proposition 3.8.

$$\begin{aligned} d_b^* &= o\left(\left(\frac{b}{n}\right)^{1+u}\right)+ \\ &\quad \sum_{r=0}^{L_n}\mathbb{P}(T_{0b}=r)\left(\frac{[z^n](1-x_nz)^{-\theta}-\exp(\sum_{j=1}^{b}\mu_j)[z^{n-r}](h_n(z)(1-x_nz)^{-\theta})}{[z^n](1-x_nz)^{-\theta}}\right)^+. \end{aligned}$$

In Proposition 3.9 we show that only the first three terms of the Taylor series of $h_n(z)$ about $z=1$ are significant. Define sequences A_n and B_n by

$$A_n=\sum_{i=1}^{b}i\mu_i=\mathbb{E}T_{0b} \tag{3.12}$$

and

$$\begin{aligned} B_n &= \frac{1}{2}\left(\left(\sum_{i=1}^{b}i\mu_i\right)^2-\sum_{i=1}^{b}i(i-1)\mu_i\right) \\ &= \frac{1}{2}\left((\mathbb{E}T_{0b})^2+\mathbb{E}T_{0b}-\sigma^2(T_{0b})\right). \end{aligned} \tag{3.13}$$

The value of $h_n(z)$ and its first two derivatives at $z=1$ are, respectively,

$$h_n(1) = \exp\left(-\sum_{i=1}^{b}\mu_i\right), \tag{3.14}$$

$$h_n'(1) = -A_n\exp\left(-\sum_{i=1}^{b}\mu_i\right), \tag{3.15}$$

$$h_n''(1) = 2B_n\exp\left(-\sum_{i=1}^{b}\mu_i\right). \tag{3.16}$$

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Proposition 3.9.

$$d_b^* = o\left(\left(\frac{b}{n}\right)^{1+u}\right) + \sum_{r=0}^{L_n}\left\{\mathbb{P}(T_{0b}=r) \times\left(\frac{[z^n](1-x_nz)^{-\theta} - [z^{n-r}]\left\{1+A_n(1-z)+B_n(1-z)^2\right\}(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right)^+\right\}.$$

Let H be defined by

$$H = [z^n](1-x_nz)^{-\theta} - [z^{n-r}]\left\{1+A_n(1-z)+B_n(1-z)^2\right\}(1-x_nz)^{-\theta}, \tag{3.17}$$

so that H is the numerator of the fraction under the positive part sign in Proposition 3.9. Note that

$$1-z = (1-x_nz)+(x_n-1)z$$

and

$$(1-z)^2 = (1-x_nz)^2 + 2(x_n-1)z(1-x_nz) + (x_n-1)^2z^2.$$

The substitution of these expressions into (3.17) results in

$$\begin{aligned} H = {} & [z^n](1-x_nz)^{-\theta} - [z^{n-r}](1-x_nz)^{-\theta} \\ & - A_n[z^{n-r}](1-x_nz)^{-\theta+1} - A_n(x_n-1)[z^{n-r-1}](1-x_nz)^{-\theta} \\ & - B_n[z^{n-r}](1-x_nz)^{-\theta+2} - 2B_n(x_n-1)[z^{n-r-1}](1-x_nz)^{-\theta+1} \\ & - B_n(x_n-1)^2[z^{n-r-2}](1-x_nz)^{-\theta}. \end{aligned}$$

Lemma 3.10 is a technical lemma used to expand each of the terms in H.

Lemma 3.10. *Let $k(\alpha)$ denote $k(\alpha) = \alpha(1-\alpha)(2-\alpha)(1-3\alpha)/24$. For $s = o(n)$ and v a fixed real such that $-\theta+v \notin \{0,1,2,\ldots\}$,*

$$[z^{n-s}](1-x_nz)^{-\theta+v} = \frac{e^{1-\theta}}{\Gamma(\theta-v)}\left(n^{\theta-v-1} - d_1(s,v)\,n^{\theta-v-2} + d_2(s,v)\,n^{\theta-v-3}\right) + O\left(s^3n^{\theta-v-4}\right), \tag{3.18}$$

where

$$d_1(s,v) = -vs + \frac{(\theta-v)(1-\theta+v)}{2}$$

and

$$\begin{aligned} d_2(s,v) = {} & \left(\frac{(1-\theta)^2}{2} + (1-\theta)(\theta-v-1) + \frac{(\theta-v-1)(\theta-v-2)}{2}\right)s^2 \\ & + \left(\frac{(1-\theta)(\theta-v)(1-\theta+v)}{2} + \frac{(\theta-v)(1-\theta+v)(\theta-v-2)}{2}\right)s \\ & - k(\theta-v). \end{aligned}$$

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When $v = 0$,

$$d_2(s,0) = \frac{1-\theta}{2}s^2 - \frac{\theta(1-\theta)}{2}s - k(\theta).$$

Lemma 3.11 simplifies H with the help of Lemma 3.10.

Lemma 3.11.

$$\begin{aligned} H &= \frac{(\theta-1)(n/e)^{\theta-1}}{n^2\Gamma(\theta)}\left\{(r^2/2 - \theta r/2) - A_n(r + (1-\theta)/2) + B_n\right\} \\ &\quad + O\left(r^3 n^{\theta-4} + A_n r^2 n^{\theta-4} + B_n r n^{\theta-4}\right), \end{aligned} \tag{3.19}$$

It is now easy to write d_b^* in a compact form.

Proposition 3.12.

$$\begin{aligned} d_b^* &= \frac{1}{2n^2}\sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\big((\theta-1)\left\{(r - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta(r - \mathbb{E}T_{0b})\right\}\big)^+ \\ &\quad + o\left(\left(\frac{b}{n}\right)^{1+u}\right). \end{aligned}$$

We will need Lemma 3.13 for the last steps.

Lemma 3.13. *For any* k, *as* $n \to \infty$, *the quantity*

$$\frac{|1-\theta|}{2n^2}\mathbb{E}\left(\left|(T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta(T_{0b} - \mathbb{E}T_{0b})\right| \mathbf{1}_{\{T_{0b} > L_n\}}\right)$$

is of order $o((b/n)^k)$.

Now, observe that

$$\begin{aligned} d_b &= d_b^* + o\left(\left(\frac{b}{n}\right)^{1+u}\right) \\ &= \frac{1}{2n^2}\sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left((\theta-1)\left\{(r - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta(r - \mathbb{E}T_{0b})\right\}\right)^+ \\ &\quad + o\left(\left(\frac{b}{n}\right)^{1+u}\right) \\ &= \frac{1}{2n^2}\sum_{r=0}^{\infty} \mathbb{P}(T_{0b} = r)\left((\theta-1)\left\{(r - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta(r - \mathbb{E}T_{0b})\right\}\right)^+ \\ &\quad + o\left(\left(\frac{b}{n}\right)^{1+u}\right) \\ &= \frac{1}{2n^2}\mathbb{E}\left((\theta-1)\left\{(T_{0b} - \mathbb{E}T_{0b})^2 - \theta(T_{0b} - \mathbb{E}T_{0b})\right\}\right)^+ + o\left(\left(\frac{b}{n}\right)^{1+u}\right). \end{aligned}$$

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The third equality follows from Lemma 3.13. Finally,

$$\mathbb{E}\left((\theta-1)\left\{(T_{0b}-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta(T_{0b}-\mathbb{E}T_{0b})\right\}\right)=0$$

implies

$$\begin{aligned}\mathbb{E}\big((\theta-1)\left\{(T_{0b}-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta(T_{0b}-\mathbb{E}T_{0b})\right\}\big)^+ \\ = \frac{|\theta-1|}{2}\mathbb{E}\left|(T_{0b}-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta(T_{0b}-\mathbb{E}T_{0b})\right|.\end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Corollary 2.3. First, note that, for all $1 \leqslant i \leqslant n$,

$$e^{-\theta} \leqslant \exp((1-\theta)i/n) \leqslant e. \tag{3.20}$$

Next, we will show

$$\mathbb{E}T_{0b}^j = O(b^j) \quad \text{for all} \quad j \geqslant 1, \tag{3.21}$$

a result which will also be useful in proving Lemma 3.13, by applying a simple observation. Given a sequence $b = b(n)$, $b(n) \leqslant n$, suppose $W_i(n)$, $i = 1, 2, \ldots, b(n)$, is an array of nonnegative independent random variables and that we want to prove that $\mathbb{E}(\sum_{i=1}^b W_i)^k = O(b^k)$ for all fixed $k \geqslant 1$. It is enough to show that there exists a positive constant $C \geqslant 1$ such that $\mathbb{E}W_i(n)^j \leqslant C$ for all $n \geqslant 1$, $1 \leqslant j \leqslant k$, and $1 \leqslant i \leqslant b$, because then, by the multinomial theorem,

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^b W_i(n)\right)^k &= \sum_{a_1+a_2+\cdots+a_b=k}\binom{k}{a_1,a_2,\ldots,a_b}\mathbb{E}\left(W_1^{a_1}\right)\mathbb{E}\left(W_2^{a_2}\right)\cdots\mathbb{E}\left(W_b^{a_b}\right) \\ &\leqslant \sum_{a_1+a_2+\cdots+a_b=k}\binom{k}{a_1,a_2,\ldots,a_b}C^{\mathbf{1}(a_1\neq 0)}C^{\mathbf{1}(a_2\neq 0)}\cdots C^{\mathbf{1}(a_b\neq 0)} \\ &\leqslant C^k\sum_{a_1+a_2+\cdots+a_b=k}\binom{k}{a_1,a_2,\ldots,a_b} \qquad (3.22) \\ &= (Cb)^k,\end{aligned}$$

where we have used the fact that at most k of the a_i can be nonzero at (3.22). Because Z_i is Poisson,

$$\mathbb{E}(Z_i)^j = O(\mu_i^j). \tag{3.23}$$

If we let $W_i(n) = iZ_i$, then (3.23), (2.10) and (3.20) show that, for any fixed j and any i, we have $(\mathbb{E}W_i(n))^j = O(i^j\mu_i^j(n)) = O\left(i^j\left(\bar{\theta}^j e^j i^{-j}\right)\right) = O(1)$. Thus, (3.21) is proved.

Applying (3.21) with $j = 1, 2$ gives the bound

$$\mathbb{E}\left|(T_{0b}-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta\,(T_{0b}-\mathbb{E}T_{0b})\right| = O(b^2). \qquad \square$$

Proof of Lemma 2.5. It was shown in Corollary 2.3 that

$$\mathbb{E}\left|(T_{0b}-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta\,(T_{0b}-\mathbb{E}T_{0b})\right| = O(b^2).$$

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Suppose that $\liminf_i i\lambda_i = \underline{\theta} > 0$ and $\sup_i i\lambda_i = \bar{\theta} < \infty$. Because of (3.20), $\liminf_i i\mu_i \geqslant e^{-\theta}\underline{\theta}$, hence

$$\mathbb{E}T_{0b} = \sum_{i=1}^{b} i\mu_i \geqslant e^{-\theta}\underline{\theta}b. \tag{3.24}$$

Let $S = \sum_{b/2\leqslant i\leqslant b} Z_i$. Note that S is Poisson, and if i_0 is such that $i\mu_i \geqslant e^{-\theta}\underline{\theta}/2$ for all $i \geqslant i_0$, then for $b \geqslant 2i_0$,

$$\mathbb{E}S \geqslant \frac{1}{2}e^{-\theta}\underline{\theta} \sum_{b/2\leqslant i\leqslant b} \frac{1}{i} \geqslant \frac{1}{2}e^{-\theta}\underline{\theta}\log(3/2) \equiv \xi.$$

At this point we note that

$$\mathbb{E}T_{0b} = \sum_{i=1}^{b} i\mu_i \leqslant \bar{\theta}eb \tag{3.25}$$

and

$$\sigma^2(T_{0b}) = \sum_{i=1}^{b} i^2\mu_i \leqslant \bar{\theta}eb(b+1)/2. \tag{3.26}$$

Let S' have Poisson distribution with mean ξ. Write v for $\mathbb{E}T_{0b}$. By the quadratic formula, (3.24), (3.25) and (3.26), for $b \geqslant 2i_0$, we have

$$\begin{aligned}
&\mathbb{P}\left((T_{0b}-v)^2 - \sigma^2 - \theta(T_{0b}-v) \geqslant b^2\right) \\
&\quad\geqslant \mathbb{P}\left(T_{0b} \geqslant \frac{(2v+\theta) + \sqrt{(2v+\theta)^2 - 4(v^2 - \sigma^2 + \theta v - b^2)}}{2}\right) \\
&\quad\geqslant \mathbb{P}\left(T_{0b} \geqslant e\bar{\theta}b + \frac{\theta}{2} + \frac{\sqrt{\theta^2 + 2e\bar{\theta}b(b+1) + 4b^2}}{2}\right) \\
&\quad\geqslant \mathbb{P}\left(S \geqslant 2e\bar{\theta} + \frac{\theta}{b} + \sqrt{\left(\frac{\theta}{b}\right)^2 + 2e\bar{\theta}\left(\frac{b+1}{b}\right) + 4}\right) \\
&\quad\geqslant \mathbb{P}\left(S \geqslant 2e\bar{\theta} + \theta + \sqrt{\theta^2 + 4e\bar{\theta} + 4}\right) \equiv c(\underline{\theta}, \bar{\theta}, \theta) > 0.
\end{aligned}$$

Therefore, for $b \geqslant 2i_0$,

$$\begin{aligned}
b^{-2}\mathbb{E}\left|(T_{0b}-v)^2 - \sigma^2 - \theta(T_{0b}-v)\right| \\
\quad\geqslant \mathbb{P}\left((T_{0b}-v)^2 - \sigma^2 - \theta(T_{0b}-v) \geqslant b^2\right) \\
\quad\geqslant c(\underline{\theta}, \bar{\theta}, \theta) > 0.
\end{aligned}$$

□

Proof of Lemma 2.6. Let Z_i^* be independent Poisson variables with means $\mathbb{E}Z_i^* = \lambda_i$ and let $T_{0b}^* = Z_1^* + 2Z_2^* + \cdots + bZ_b^*$. To prove this lemma it suffices to show that $d_{\mathrm{TV}}(T_{0b}, T_{0b}^*) = o(1)$ and apply Lemma 7 of [2] and Slutsky's theorem. It is well known that, if Y is a Poisson variable with mean $\mathbb{E}Y = \zeta$ and Y^* is a Poisson variable with

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mean $\mathbb{E}Y^* = \zeta^*$, then $d_{\mathrm{TV}}(Y, Y^*) \leqslant |\zeta - \zeta^*|$. Therefore,

$$\begin{aligned} d_{\mathrm{TV}}(T_{0b}, T_{0b}^*) &\leqslant \sum_{i=1}^{b} d_{\mathrm{TV}}(Z_i, Z_i^*) \\ &\leqslant \sum_{i=1}^{b} |\mu_i - \lambda_i| \\ &= \sum_{i=1}^{b} |\lambda_i \exp(i(1-\theta)/n) - \lambda_i| \\ &= O\left(n^{-1} \sum_{i=1}^{b} i\lambda_i\right) \qquad (3.27) \\ &= O(b/n) = o(1), \end{aligned}$$

where we used (3.20) at (3.27). □

Proof of Corollary 2.4. It is easy to show that

$$u = \min\left(\frac{\delta \log n - \log\log n - \omega(n)}{\log(n/b)}, 1\right)$$

if and only if

$$bn^{\delta/u-1}/\log^{1/u} n \to \infty. \qquad (3.28)$$

□

Proof of Corollary 2.7. With Y_b defined as

$$Y_b := b^{-2}\left((T_{0b} - \mathbb{E}T_{0b})^2 - \sigma^2(T_{0b}) - \theta\,(T_{0b} - \mathbb{E}T_{0b})\right), \qquad (3.29)$$

Schwartz's inequality gives

$$\mathbb{E}(Y_b; Y_b > M) \leqslant \sqrt{\mathbb{E}Y_b^2\,\mathbb{P}(Y_b > M)}.$$

The bound (3.21) for $j = 1, 2, 3, 4$ means that $\mathbb{E}Y_b^2 = O(1)$. To conclude the proof of uniform integrability of Y_b, note that $\mathbb{P}(Y_b > M) \leqslant M^{-1}\mathbb{E}Y_b = O(M^{-1})$.

The uniform integrability of Y_b, along with $\mathbb{E}X_\theta = \theta$ and $\sigma^2(X_\theta) = \theta/2$, gives us $\mathbb{E}Y_b \Rightarrow \mathbb{E}\left|(X_\theta - \mathbb{E}X_\theta)^2 - \sigma^2(X_\theta)\right|$. Note that $b^{-2}\theta\,(T_{0b} - \mathbb{E}T_{0b}) \Rightarrow 0$ as $b \to \infty$. □

4. Proofs of propositions and lemmas

Proof of Lemma 3.1. Using Chebyshev's inequality, for any $\beta \geqslant 0$,

$$\mathbb{P}(T_{0b} \geqslant bw) = \mathbb{P}(e^{\beta T_{0b}} \geqslant e^{\beta bw}) \leqslant \mathbb{E}e^{\beta T_{0b} - \beta bw}.$$

We calculate

$$\log \mathbb{E}e^{\beta T_{0b}} = \log \prod_{j=1}^{b} \mathbb{E}e^{j\beta Z_j}$$

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$$\begin{aligned}
&= \log \prod_{j=1}^{b} \exp(-\mu_j + \mu_j e^{j\beta}) \\
&= \sum_{j=1}^{b} (-\mu_j + \mu_j e^{j\beta}) \\
&= \sum_{j=1}^{b} \int_0^{\beta} j\mu_j e^{jt} dt \\
&\leqslant \sum_{j=1}^{b} \int_0^{\beta} e\bar{\theta} e^{jt} dt \\
&\leqslant e\bar{\theta} b \int_0^{\beta} e^{bt} dt \\
&= e\bar{\theta}(e^{\beta b} - 1) \\
&\leqslant e\bar{\theta} e^{\beta b}.
\end{aligned}$$

Thus, for any $\beta \geqslant 0$,

$$\log \mathbb{P}(T_{0b} \geqslant bw) \leqslant e\bar{\theta} e^{\beta b} - \beta bw.$$

Letting $\beta = b^{-1} \log(w/e\bar{\theta})$ finishes the proof. □

Proof of Lemma 3.2. The inequality is clear because the expression under the positive part symbol is bounded by 1.

It follows from the definition (3.2) of L_n that if $b \leqslant n/(\log n)^5$, then $L_n = b \log n$ and otherwise $L_n = b^{4/5} n^{1/5}$. In the first case, taking $w = \log n$ in Lemma 3.1 gives

$$\log \mathbb{P}(T_{0b} > L_n) \leqslant w \log(\bar{\theta} e^2 / w) \sim -w \log w = -(\log \log n) \log n,$$

which decreases to minus infinity faster than $\log n^{-k} = -k \log n$ for any fixed k. In the second case, taking $w = \left(\frac{n}{b}\right)^{1/5}$, we have

$$\log \mathbb{P}(T_{0b} > L_n) \leqslant w \log(\bar{\theta} e^2 / w) \sim -w \log w = -\frac{1}{5} w \log(n/b),$$

which, for any fixed k, decreases to minus infinity faster than $-k \log(n/b)$. □

Proof of Lemma 3.3. Since $|(x + y)^+ - x^+| \leqslant |y|$, we see that

$$|(S_{1,n,r} + S_{2,n,r})^+ - (S_{1,n,r})^+| \leqslant M_{b,n}.$$

Summing against weights $\mathbb{P}(T_{0b} = r)$, we get

$$|E_n| = \left| d_b^* - \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)(S_{1,n,r})^+ \right| \leqslant M_{b,n},$$

establishing the result. □

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Proof of Lemma 3.4. We define $c = 1 - \theta$ throughout this paper. Denote the sequence y_n by $y_n = e^{-c/n}$, so that

$$\left|[z^{n-m}]\xi(x_n z)\psi(x_n z)(1-z)^j\right| = x_n^{n-m}\left|[z^{n-m}]\xi(z)\psi(z)(1-y_n z)^j\right|.$$

Let $K = K_1 \sup_{|z|\leqslant 1+\eta}|\xi(z)|$ and define $\hat{\psi}(z) = \xi(z)\psi(z)(1-y_n z)^j$. The coefficient $[z^{n-m}]\hat{\psi}(z)$ may be expressed as a contour integral by using the Cauchy integral formula

$$[z^{n-m}]\hat{\psi}(z) = \frac{1}{2\pi i}\int_\gamma \hat{\psi}(z)\frac{dz}{z^{n-m+1}},$$

where γ is a positively oriented contour in Δ. Choose γ to be $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\begin{aligned}
\gamma_1 &= \left\{z : |z-1| = (n-m)^{-1},\ |\mathrm{Arg}(z-1)| \geqslant \phi\right\},\\
\gamma_2 &= \left\{z : (n-m)^{-1} \leqslant |z-1|,\ |z| \leqslant 1+\eta,\ \mathrm{Arg}(z-1) = \phi\right\},\\
\gamma_3 &= \left\{z : |z| = 1+\eta,\ |\mathrm{Arg}(z-1)| \geqslant \phi\right\},\\
\gamma_4 &= \left\{z : (n-m)^{-1} \leqslant |z-1|,\ |z| \leqslant 1+\eta,\ \mathrm{Arg}(z-1) = -\phi\right\}.
\end{aligned}$$

(Those readers who compare our definition of γ with the one in [7] should be aware that there is a typo in the definition of γ_3 in [7]; see page 1168 of [10]).

Taking absolute values inside the integral gives

$$[z^{n-m}]\hat{\psi}(z) \leqslant \frac{1}{2\pi}\int_\gamma |\hat{\psi}(z)|\frac{|dz|}{|z|^{n-m+1}}.$$

Note that

$$\begin{aligned}
\sup_{z\in\gamma}\frac{|1-y_n z|^j}{|1-z|^j} &= \sup_{z\in\gamma}\left|1+\frac{(1-y_n)z}{1-z}\right|^j\\
&\leqslant \left(1+|1-y_n|\sup_{z\in\gamma}\frac{|z|}{|1-z|}\right)^j\\
&\leqslant \left(1+\frac{e^\theta|c|}{n}\frac{1+\eta}{(n-m)^{-1}}\right)^j \qquad (4.1)\\
&\leqslant \left(1+e^\theta|c|(1+\eta)\right)^j := \tau_{\theta,\eta}^j,
\end{aligned}$$

where the bound at (4.1) holds by the mean value theorem. Therefore,

$$[z^{n-m}]\hat{\psi}(z) \leqslant \tau_{\theta,\eta}^j \cdot \frac{1}{2\pi}\int_\gamma K|1-z|^{-\alpha+j}\frac{|dz|}{|z|^{n-m+1}}.$$

At line (2.15) of the proof of Theorem 1 in [7] it is shown that, whenever $n-m \geqslant 2|\alpha-j|+4$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_\gamma K|1-z|^{-\alpha+j}\frac{|dz|}{|z|^{n-m+1}}\\
&\qquad\leqslant K(n-m)^{\alpha-j-1}\left[5+\frac{J(\alpha-j,\phi)}{\pi}+\frac{\eta^{-\alpha+j}}{(1+\eta)^{n-m}}(n-m)^{1-\alpha+j}\right],
\end{aligned}$$

where

$$J(\alpha-j,\phi)) := \int_1^\infty t^{-\alpha+j}\left(1+\frac{t\cos\phi}{2|\alpha-j|+4}\right)^{-2|\alpha-j|-4}dt.$$

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It was established in the proof of Theorem 14 of [2] that there is a universal constant $W_{l,\phi}$ satisfying (3.8) such that, for all $|\alpha - j| \leqslant l$,

$$5 + \frac{J(\alpha - j, \phi)}{\pi} \leqslant W_{l,\phi}.$$

The bound ${x_n}^{n-m} \leqslant e$ concludes the proof. □

Proof of Lemma 3.5. We use Faa di Bruno's formula

$$\frac{d^k(\phi \circ g)}{dz^k} = \sum_{a_1+2a_2+\cdots ka_k=k} \frac{k!}{a_1! \cdots a_k!} \frac{d^m \phi}{dz^m} \left(\frac{g'}{1!}\right)^{a_1} \cdots \left(\frac{g^{(k)}}{k!}\right)^{a_k},$$

where $m = a_1 + a_2 + \cdots + a_k$; either see [5] or else use (1.1) for set partitions to give a short proof. In our case $h = \phi \circ g$, with $\phi(z) = \exp(-z)$ and $g(z) = \sum_{j=1}^{b} \mu_j z^j$. Define $(t)_i = t(t-1)(t-2)\cdots(t-i+1)$, for $i > 0$. Then for $l > 0$

$$\begin{aligned}
\frac{d^l g}{dz^l}\,|_{z=1} &= \sum_{j=1}^{b} (j)_l \mu_j \\
&\leqslant \sum_{j=1}^{b} (j)_l \frac{e\bar{\theta}}{j} \\
&= e\bar{\theta} \sum_{j=1}^{b} (j-1)_{l-1} \\
&= e\bar{\theta}(b)_l / l \\
&\leqslant e\bar{\theta} b^l.
\end{aligned}$$

Observing that, for $m \equiv \sum a_i$, $(e\bar{\theta})^m \leqslant \rho^m \leqslant \rho^k$ when $m \leqslant k$, we obtain

$$\begin{aligned}
|h^{(k)}(1)| &\leqslant \sum_{a_1+2a_2+\cdots+ka_k=k} \frac{k!}{a_1! \cdots a_k! (1!)^{a_1} \cdots (k!)^{a_k}} e^{-\sum_{j=1}^{b} \mu_j} (e\bar{\theta} b)^{a_1} \cdots (e\bar{\theta} b^k)^{a_k} \\
&\leqslant e^{-\sum_{j=1}^{b} \mu_j} \rho^k b^k \sum_{a_1+2a_2+\cdots+ka_k=k} \frac{k!}{a_1! \cdots a_k! (1!)^{a_1} \cdots (k!)^{a_k}} \\
&= e^{-\sum_{j=1}^{b} \mu_j} B_k(\rho b)^k.
\end{aligned}$$

For the part about $h^{(k)}(0)$, we first show that n_0 can be chosen in such a way that $\Lambda < \infty$. Since we are assuming $\limsup i\lambda_i < 1$, we may choose $i_0 \geqslant 1$ and $\omega > 0$ such that $i\lambda_i \leqslant 1 - \omega$ for $i > i_0$. Since $b/n \to 0$, we may pick $n_0 \geqslant 1$ such that $e^{b/n} \leqslant (1-\omega)^{-1/2}$ for all $n \geqslant n_0$. For all $n \geqslant n_0$, therefore, if $i > i_0$, then $i\lambda_i x_n^i \leqslant i\lambda_i e^{b/n} \leqslant (1-\omega)^{1/2} < 1$. Therefore, for all $n \geqslant n_0$, we have $\sup\{i : i\lambda_i x_n^i > 1\} \leqslant i_0$ and for this choice of n_0, $\Lambda \leqslant i_0 < \infty$.

We now use Faa di Bruno's formula to bound $h^{(k)}(0)$. Use $l!\mu_l = (l-1)!l\mu_l$ to evaluate

$$\left.\frac{d^l g}{dz^l}\right|_{z=0} = l!\mu_l \mathbf{1}(l \leqslant b) \leqslant \begin{cases} (l-1)!e\bar{\theta}, & \text{if } 1 \leqslant l \leqslant \Lambda, \\ (l-1)!, & \text{if } \Lambda < l \leqslant b. \end{cases}$$

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Let Z_i^* be independent Poisson-distributed variables with means $\mathbb{E}Z_i^* = 1/i$. We get, for $n > n_0$ and any $k \geqslant b$,

$$\begin{aligned}
|h_n^{(k)}(0)| &\leqslant \sum_{a_1+2a_2+\cdots+ba_b=k} \frac{k!}{a_1!\cdots a_b!(1!)^{a_1}\cdots(b!)^{a_b}}(1!\mu_1)^{a_1}\cdots(b!\mu_b)^{a_b} \\
&\leqslant \sum_{a_1+2a_2+\cdots+ba_b=k} \frac{k!}{a_1!\cdots a_b!1^{a_1}\cdots b^{a_b}}(e\bar{\theta})^{a_1}\cdots(e\bar{\theta})^{a_{\Lambda\wedge b}} \\
&\leqslant k! \sum_{a_1+2a_2+\cdots+ba_b=k} \frac{1}{a_1!\cdots a_b!1^{a_1}\cdots b^{a_b}}\rho^{a_1+\cdots+a_{\Lambda\wedge b}} \\
&= k!\exp\left(\sum_{i=1}^{b}\frac{1}{i}\right)\sum_{a_1+2a_2+\cdots+ba_b=k}\prod_{i=1}^{b}\left\{\frac{e^{-1/i}}{a_i!}\left(\frac{1}{i}\right)^{a_i}\right\}\rho^{a_1+\cdots+a_{\Lambda\wedge b}} \\
&\leqslant k!eb\,\mathbb{E}(\rho^{Z_1^*+\cdots+Z_{\Lambda\wedge b}^*}\mathbf{1}(Z_1^*+2Z_2^*+\cdots+bZ_b^*=k)) \\
&\leqslant k!eb\,\mathbb{E}(\rho^{Z_1^*+\cdots+Z_{\Lambda}^*}\mathbf{1}(Z_1^*+2Z_2^*+\cdots+bZ_b^*\geqslant k)) \\
&\leqslant k!eb\sqrt{\mathbb{E}\rho^{2(Z_1^*+\cdots+Z_\Lambda^*)}\mathbb{P}\left(Z_1^*+2Z_2^*+\cdots+bZ_b^*\geqslant k\right)} \\
&\leqslant k!eb(\mathbb{E}\rho^{2Z_1^*})^{\Lambda/2}\sqrt{\mathbb{P}\left(Z_1^*+2Z_2^*+\cdots+bZ_b^*\geqslant k\right)} \\
&= k!eb(e^{\rho^2-1})^{\Lambda/2}\sqrt{\mathbb{P}\left(Z_1^*+2Z_2^*+\cdots+bZ_b^*\geqslant k\right)} \\
&\leqslant k!e^{\Lambda(\rho^2-1)/2+1}b\left(\frac{k}{eb}\right)^{-k/(2b)}, \qquad (4.2)
\end{aligned}$$

where we have used Lemma 8 of [2] at (4.2). □

Proof of Lemma 3.6. We use the maximum modulus principle. If g is a function defined and continuous on a closed bounded set D and analytic on the interior of D, then the maximum of $|g(z)|$ on D is assumed on the boundary of D.

$$\begin{aligned}
\sup_{z\in D_\eta}|R_N(z)| &= \sup_{|z|=1+\eta}|R_N(z)| \\
&= \sup_{|z|=1+\eta}\left|\frac{h(z)-\sum_{k=0}^{N}\frac{h^{(k)}(1)}{k!}(z-1)^k}{(z-1)^{N+1}}\right| \\
&\leqslant \eta^{-(N+1)}\left(\sup_{|z|=1+\eta}|h(z)|+\sum_{k=0}^{N}\exp\left(-\sum_{j=1}^{b}\mu_j\right)\frac{(\rho\eta b)^k B_k}{k!}\right) \\
&\leqslant \eta^{-(N+1)}\left(\sup_{|z|=1+\eta}|h(z)|+\sum_{k=0}^{\infty}\frac{(\rho\eta b)^k B_k}{k!}\right) \\
&= \eta^{-(N+1)}\left(\sup_{|z|=1+\eta}|h(z)|+\exp(e^{\rho\eta b}-1)\right). \qquad (4.3)
\end{aligned}$$

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Analysing the first term of (4.3), we see that

$$\begin{aligned}\sup_{|z|=1+\eta}|h(z)| &= \sup_{|z|=1+\eta}\left|\exp\left(-\sum_{j=1}^{b}\mu_j z^j\right)\right| \\ &\leqslant \exp\left(\sum_{j=1}^{b}\mu_j(1+\eta)^j\right) \\ &\leqslant \exp\left(\sum_{j=1}^{b}\frac{e\bar{\theta}}{j}e^{\eta b}\right) \\ &\leqslant \exp\left(\bar{\theta}e^{\eta b+1}(1+\log b)\right) \\ &= (eb)^{\bar{\theta}e^{\eta b+1}}.\end{aligned}$$

□

Proof of Proposition 3.7. Assume for now that $K^* = 0$. We are going to use Lemma 3.3 to show that

$$\max_{0\leqslant r\leqslant L_n}\left|\frac{[z^n]\Psi(x_n z) - \exp(\sum_{j=1}^{b}\mu_j)[z^{n-r}](h_n(z)\Psi(x_n z))}{[z^n]f_n(z)}\right| = o\left(\left(\frac{b}{n}\right)^{1+u}\right). \tag{4.4}$$

Use (3.9), (3.11) and (3.14) to conclude that (4.4) is dominated by the sum of three terms:

$$\max_{0\leqslant r\leqslant L_n}\left|\frac{[z^n]\Psi(x_n z) - [z^{n-r}]\Psi(x_n z)}{[z^n]f_n(z)}\right| \tag{4.5}$$

$$+\max_{0\leqslant r\leqslant L_n}\left|\frac{\exp(\sum_{j=1}^{b}\mu_j)[z^{n-r}]\{\Psi(x_n z)(\sum_{k=1}^{N}\{h_n^{(k)}(1)/k!\}(z-1)^k)\}}{[z^n]f_n(z)}\right| \tag{4.6}$$

$$+\max_{0\leqslant r\leqslant L_n}\left|\frac{\exp(\sum_{j=1}^{b}\mu_j)[z^{n-r}]\{\Psi(x_n z)R_N(z)(z-1)^{N+1}\}}{[z^n]f_n(z)}\right|. \tag{4.7}$$

We first look at (4.5). Let $\Phi_n(z) = \Psi(x_n z)(1-z)$. The identity $1 - z^r = (1-z)(1+z+\cdots+z^{r-1})$ implies that for all $r \in [0, L_n]$

$$\begin{aligned}|[z^n]\Psi(x_n z) - [z^{n-r}]\Psi(x_n z)| &= \left|[z^n]\Phi_n(z) + \cdots + [z^{n-r+1}]\Phi_n(z)\right| \\ &\leqslant r\max_{0\leqslant l\leqslant r-1}|[z^{n-l}]\Phi_n(z)| \\ &\leqslant L_n\max_{0\leqslant l\leqslant L_n}|[z^{n-l}]\Phi_n(z)|.\end{aligned}$$

Lemma 3.4 with $j = 1$ along with $n/(n-L_n) = O(1)$ gives

$$|[z^n]\Psi(x_n z) - [z^{n-r}]\Psi(x_n z)| = O(L_n n^{\theta-\delta-2}).$$

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Lemma 3.4, (3.4) and (1.6) give the asymptotics

$$\begin{aligned}[z^n]f_n(z) = [z^n]f(x_n z) = x_n^n\,[z^n]f(z) &= e^{1-\theta}Kn^{\theta-1}/\Gamma(\theta) + O(n^{\theta-\delta-1}) \\ &\sim e^{1-\theta}Kn^{\theta-1}/\Gamma(\theta),\end{aligned}$$

which, along with the condition $bn^{\delta/u-1}/\log^{1/u} n \to \infty$ from (3.28) and the definition of L_n,

$$L_n = \min\left(b\log n, b^{1/5}n^{4/5}\right)$$

from (3.2), show

$$\begin{aligned}\left(\frac{n}{b}\right)^{1+u}\max_{0\leqslant r\leqslant L_n}\left|\frac{[z^n]\Psi(x_n z) - [z^{n-r}]\Psi(x_n z)}{[z^n]f_n(z)}\right| &= O\left(\frac{n^{1+u}L_n n^{\theta-\delta-2}}{b^{1+u}n^{\theta-1}}\right) \\ &= O\left(\frac{L_n n^{u-\delta}}{b^{1+u}}\right) \to 0.\end{aligned}$$

Next we consider expression (4.6). Using the result of Lemma 3.5, the numerator is bounded above by

$$\max_{0\leqslant r\leqslant L_n}\sum_{k=1}^{N}\frac{\rho^k B_k b^k}{k!}\,|[z^{n-r}]\Psi(x_n z)(z-1)^k|. \tag{4.8}$$

We now apply the result of Lemma 3.4 to each term in the absolute value signs, taking $\psi(z) = \Psi(z)$, $\xi(z) \equiv 1, \alpha = \theta - \delta$, $j = k$, $m = r$, $l = k + \delta$. The second term in brackets on the right of (3.7) yields a O(1) bound, uniformly in r, which follows by taking η fixed, and noting that the exponential growth of $(1+\eta)^{n-L_n}$ relative to n dominates $\sup_{k\leqslant N} n^{-\alpha+1} = O(n^{\log n})$. For the first term, $W_{l,\phi}$, we take a bound depending on k: if $k \geqslant \theta - \delta$, then $|\alpha - j| = |\theta - \delta - k| \leqslant k + \delta \equiv l$, so that from (3.8)

$$W_{l,\phi} \leqslant A\left(1 + \left(\frac{k+\delta}{\cos\phi}\right)^{k+\delta+2}\right).$$

Using the fact that $B_k \leqslant k!$, we see that $\left(\frac{n}{b}\right)^{1+u}$ times expression (4.8) is at most

$$eK_1\rho\tau_{\theta,\eta}\frac{n^{\theta-\delta-1+u}}{b^u}\sum_{k=1}^{N}\left(\frac{\tau_{\theta,\eta}b\rho}{n}\right)^{k-1}\max_{0\leqslant r\leqslant L_n}\left(1-\frac{r}{n}\right)^{\theta-\delta-k-1}(W_{k+\delta,\phi} + O(1)).$$

Since $L_n \leqslant n/2$, it follows that $\max_{0\leqslant r\leqslant L_n}\left(1-\frac{r}{n}\right)^{\theta-\delta-k-1} \leqslant \max(1, 2^{k+1+\delta-\theta})$. Using this and the bound (3.8), we see that, for $k \geqslant \theta - \delta$, the kth term in the series above is bounded by

$$u_k = 2\left(\frac{2\tau_{\theta,\eta}\rho b}{n}\right)^{k-1}\left(\frac{k+\delta}{\cos\phi}\right)^{k+\delta+2},$$

which satisfy

$$\frac{u_{k+1}}{u_k} = O(Nb/n) \to 0$$

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using (2.2) and (3.10). We conclude that the $k = 1$ term of (4.8) dominates. Combining this bound with the asymptotic $[z^n]f_n(z) \sim Ke^{1-\theta}n^{\theta-1}/\Gamma(\theta)$, we see that indeed

$$\left(\frac{n}{b}\right)^{1+u}\{\text{Expression (4.6)}\} = O(n^{u-\delta}/b^u) \to 0.$$

Finally we consider the term (4.7). First,

$$\begin{aligned}\{\text{Expression (4.7)}\} &\leqslant \frac{\exp(\sum_{j=1}^{b} e\bar{\theta}/j)}{[z^n]f_n(z)} \max_{0\leqslant r\leqslant L_n} \left|[z^{n-r}]R_N(z)\,\Psi(x_n z)(1-z)^{N+1}\right| \\ &\leqslant \frac{(eb)^{e\bar{\theta}}}{[z^n]f_n(z)} \max_{0\leqslant r\leqslant L_n} \left|[z^{n-r}]R_N(z)\,\Psi(x_n z)(1-z)^{N+1}\right|. \qquad (4.9)\end{aligned}$$

We now use Lemma 3.4 with

$$\xi(z) = R_N(y_n z),\ \psi(z) = \Psi(z),$$

$\alpha = \theta - \delta$, $j = N + 1$, $m = r$, $l = \delta + N + 1$. Combining (4.9) with the results of Lemmas 3.6 and 3.4, we see that

$$\begin{aligned}\{\text{Expression (4.7)}\} \leqslant\ & \frac{(eb)^{e\bar{\theta}}}{[z^n]f_n(z)} \max_{0\leqslant r\leqslant L_n} eK_1\eta^{-(N+1)}\tau_{\theta,\eta}^{N+1}\left((eb)^{\bar{\theta}e^{\eta b+1}} + e^{e^{\rho\eta b}-1}\right) \\ & (n-r)^{\alpha-j-1}\left[W_{l,\phi} + \frac{\eta^{-\alpha+j}}{(1+\eta)^{n-r}}(n-r)^{1-\alpha+j}\right], \qquad (4.10)\end{aligned}$$

where $l = |\theta - \delta - N - 1|$. There are essentially two cases to consider: b fixed, and $b \to \infty$ as $n \to \infty$.

In the first case, we may take $\eta = \eta_0$, determined by the Δ-domain of f. As in the previous part of the proof, the rightmost term in brackets in (4.10) has a $O(1)$ bound uniform in $0 \leqslant r \leqslant L_n$. Since $n - L_n \geqslant n/2$, we may use (3.8) and the fact that $[z^n]f(z) \sim e^{1-\theta}Kn^{\theta-1}/\Gamma(\theta)$ to deduce that (4.10) is bounded by a term of the form

$$\left(\frac{CN}{n}\right)^N \times \text{ a polynomial in } n,$$

for some positive constant C. Since $N \asymp \log n$, we see that this term is $o(n^{-r})$ for any r. It follows that indeed

$$\left(\frac{n}{b}\right)^{1+u}\{\text{Expression (4.7)}\} = o(1).$$

In the case $b \to \infty$, we may take $\eta = 1/b$, and assume that n is so large that $\eta \leqslant \eta_0$. The second term in brackets in (4.10) is bounded by

$$\begin{aligned}\frac{\eta^{-\alpha+j}}{(1+\eta)^{n-r}}(n-r)^{1-\alpha+j} &\leqslant \frac{n^{1-\alpha+j}b^{\alpha-j}(1+\eta)^r}{(1+\eta)^n} \\ &\leqslant n(1+\eta)^{L_n}2^{-n/b}\left(\frac{b}{n}\right)^{\alpha-j} \\ &\leqslant n^2 2^{-n/b}\left(\frac{b}{n}\right)^{\alpha-j},\end{aligned}$$

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the last two inequalities following from the fact that $(1+\eta)^n \geqslant 2^{n/b}$, and $(1+\eta)^{L_n} \leqslant n$. Hence the right side of (4.10) is bounded above by

$$\frac{(en)^{e\bar{\theta}}}{[z^n]f_n(z)} eK_1 n^{\alpha-j-1} 2^{1-\alpha+j} (\tau_{\theta,\eta_0} b)^{N+1} \left((en)^{\bar{\theta}e^2} + e^{e^\rho - 1} \right) \left[W_{l,\phi} + n^2 2^{-n/b} \left(\frac{b}{n} \right)^{\alpha-j} \right],$$

since $n - L_n \geqslant n/2$. Using the bound for $W_{l,\phi}$ once more, the first term above is of the form

$$\left(\frac{CbN}{n} \right)^N \times \text{ a polynomial in } n,$$

which is $o(n^{-r})$ for any r since $bN/n \to 0$ and $N \asymp \log n$. The second term is of the form

$$2^{-n/b} \times \text{ a polynomial in } n,$$

which is $o(n^{-r})$ for any r, since $\log n = o(n/b)$. Thus, if $b \to \infty$,

$$\left(\frac{n}{b} \right)^{1+u} \{\text{Expression (4.7)}\} = o(1).$$

If we put $K^* \neq 0$, then $[z^n]\Psi(x_n z)$ does not change because that amounts to adding a constant term KK^* to $\Psi(x_n z)$. We will show that the additional term

$$\max_{0 \leqslant r \leqslant L_n} \left| KK^* \frac{\exp\left(\sum_{j=1}^b \mu_j \right) [z^{n-r}] h_n(z)}{[z^n] f_n(z)} \right| \tag{4.11}$$

that gets added to (4.4) is insignificant. We use $[z^{n-r}]h(z) = h^{(n-r)}(0)/(n-r)!$ and Lemma 3.5, obtaining, for each $n > n_0$ where n_0 is given by Lemma 3.5,

$$\begin{aligned} \{\text{Expression (4.11)}\} &\leqslant KK^* \sup_{0 \leqslant r \leqslant L_n} \frac{(eb)^{e\bar{\theta}} \left| h^{(n-r)}(0) \right| / (n-r)!}{[z^n] f_n(z)} \\ &\leqslant KK^* \frac{(eb)^{e\bar{\theta}}}{[z^n] f_n(z)} e^{\Lambda(\rho^2-1)/2+1} b \sup_{0 \leqslant r \leqslant L_n} \left(\frac{n-r}{eb} \right)^{-(n-r)/(2b)} \\ &= KK^* \frac{(eb)^{e\bar{\theta}}}{[z^n] f_n(z)} e^{\Lambda(\rho^2-1)/2+1} b \left(\frac{n-L_n}{eb} \right)^{-(n-L_n)/(2b)}, \end{aligned}$$

which is of the order

$$\left(\frac{n-L_n}{eb} \right)^{-(n-L_n)/(2b)} \times \text{ a polynomial in } n,$$

which is $o(n^{-r})$ for any r, since $L_n = o(n)$ and $\log n = o(n/b)$.

This concludes the proof of Proposition 3.7. □

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Proof of Proposition 3.8. Assume for now that $K^* = 0$ and rewrite Proposition 3.7 as $d_b^* = o\left(\left(\frac{b}{n}\right)^{1+u}\right) +$

$$\sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r) \left(\frac{[z^n]K(1-x_nz)^{-\theta} - \exp(\sum_{j=1}^b \mu_j)[z^{n-r}](h_n(z)K(1-x_nz)^{-\theta})}{[z^n]K(1-x_nz)^{-\theta}} \right.$$

$$\left. -[z^n]\Psi(x_nz)\frac{[z^n]K(1-x_nz)^{-\theta} - \exp(\sum_{j=1}^b \mu_j)[z^{n-r}](h_n(z)(1-z)^{-\theta})}{[z^n](1-x_nz)^{-\theta}[z^n]f_n(z)} \right)^+.$$

In this case, to use Lemma 3.3 we must show that

$$\max_{0\leqslant r\leqslant L_n} \left| [z^n]\Psi(x_nz)\frac{[z^n](1-x_nz)^{-\theta} - \exp(\sum_{j=1}^b \mu_j)[z^{n-r}](h_n(z)(1-x_nz)^{-\theta})}{[z^n](1-x_nz)^{-\theta}[z^n]f_n(z)} \right| \tag{4.12}$$

is $o\left(\left(\frac{b}{n}\right)^{1+u}\right)$. Expression (4.12) is dominated by the sum of three terms:

$$\max_{0\leqslant r\leqslant L_n} \left| [z^n]\Psi(x_nz)\frac{[z^n](1-x_nz)^{-\theta} - [z^{n-r}](1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}[z^n]f_n(z)} \right| \tag{4.13}$$

$$+ \max_{0\leqslant r\leqslant L_n} \left| [z^n]\Psi(x_nz)\frac{\sum_{k=1}^N [z^{n-r}](1-x_nz)^{-\theta}\left\{h_n^{(k)}(1)/k!\right\}(z-1)^k}{[z^n](1-x_nz)^{-\theta}[z^n]f_n(z)} \right| \tag{4.14}$$

$$+ \max_{0\leqslant r\leqslant L_n} \left| [z^n]\Psi(x_nz)\frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}](R_N(z)(1-x_nz)^{-\theta}(z-1)^{N+1})}{[z^n](1-x_nz)^{-\theta}[z^n]f_n(z)} \right|. \tag{4.15}$$

To bound (4.13) we use the binomial expansion: for all $r \in [0, L_n]$,

$$(1-r/n)^{\theta-1} = 1 + O(r/n).$$

Recall that $c := 1-\theta$. It follows from (1.6) that

$$\begin{aligned}
&[z^n](1-x_nz)^{-\theta} - [z^{n-r}](1-x_nz)^{-\theta} \\
&\quad = e^c[z^n](1-z)^{-\theta} - e^{c(1-r/n)}[z^{n-r}](1-z)^{-\theta} \\
&\quad = e^c\left([z^n](1-z)^{-\theta} - [z^{n-r}](1-z)^{-\theta}\right) + e^c(1-e^{-cr/n})[z^{n-r}](1-z)^{-\theta} \\
&\quad = \frac{e^c n^{\theta-1}}{\Gamma(\theta)}(1-(1-r/n)^{\theta-1}) + \frac{ce^c}{\Gamma(\theta)} r n^{\theta-2} + O(n^{\theta-2}) \\
&\quad = O(rn^{\theta-2}) = O(L_n n^{\theta-2}).
\end{aligned}$$

Hence, from Lemma 3.4 and (1.6),

$$\begin{aligned}
\left(\frac{n}{b}\right)^{1+u}\{\text{Expression (4.13)}\} &= O\left(\left(\frac{n}{b}\right)^{1+u} n^{\theta-\delta-1}\frac{L_n n^{\theta-2}}{n^{\theta-1}n^{\theta-1}}\right) \\
&= O\left(\frac{L_n n^{u-\delta}}{b^{1+u}}\right) \to 0.
\end{aligned}$$

The argument to show that expression (4.14) is $o\left(\left(\frac{b}{n}\right)^{1+u}\right)$ is similar to that used to analyse (4.6), using Lemma 3.4. The argument to show that expression (4.15) is $o\left(\left(\frac{b}{n}\right)^{1+u}\right)$ is analogous to that used to analyse (4.7). We omit further details.

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Finally, putting $K^* \neq 0$ makes no difference to our analysis because $[z^n]\Psi(x_n z)$, $n > 0$, does not change when a constant is added to $\Psi(x_n z)$. □

Proof of Proposition 3.9. Rewrite Proposition 3.8 as

$$d_b^* = o\left(\left(\frac{b}{n}\right)^{1+u}\right) + \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left(\left\{\frac{[z^n](1-x_nz)^{-\theta} - \exp(\sum_{j=1}^b \mu_j)[z^{n-r}]h_n(1)(1-x_nz)^{-\theta})}{[z^n](1-x_nz)^{-\theta}}\right.\right.$$
$$+ \frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]\{h_n'(1)\}(1-z)(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}$$
$$\left. - \frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]\{h_n''(1)/2\}(1-z)^2(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right\}$$
$$- \left\{\frac{(\exp(\sum_{j=1}^b \mu_j)\sum_{k=3}^N [z^{n-r}]\left\{h_n^{(k)}(1)/k!\right\}(z-1)^k(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right.$$
$$\left.\left. + \frac{(\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]R_N(z)(z-1)^{N+1}(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right\}\right)^+.$$

The expressions in braces correspond to $S_{1,n,r}$ and $S_{2,n,r}$ of Lemma 3.3. Clearly, the maximum $\max_{0\leqslant r\leqslant L_n}|S_{2,n,r}|$ is dominated by the sum

$$\max_{0\leqslant r\leqslant L_n}\left|\frac{\exp(\sum_{j=1}^b \mu_j)\sum_{k=3}^N [z^{n-r}](-1)^k\left\{h^{(k)}(1)/k!\right\}(1-z)^k(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right| \tag{4.16}$$
$$+ \max_{0\leqslant r\leqslant L_n}\left|\frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]R_N(z)(1-z)^{N+1}(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right|. \tag{4.17}$$

The magnitude of the kth term is roughly $(b/n)^k$, so the $k = 1, 2, 3$ terms can not be ignored. The argument that (4.16) is of order $o\left(\left(\frac{b}{n}\right)^{1+u}\right)$ is analogous to the bound on (4.6) in Proposition 3.7; the proof that (4.17) is $o(n^{-r})$ for any r is analogous to the way (4.7) is bounded.

We conclude from Lemma 3.3 that

$$d_b^* = o\left(\left(\frac{b}{n}\right)^{1+u}\right) + \sum_{r=0}^{L_n} \mathbb{P}(T_{0b} = r)\left(\frac{[z^n](1-z)^{-\theta} - \exp(\sum_{j=1}^b \mu_j)[z^{n-r}]h_n(1)(1-z)^{-\theta})}{[z^n](1-x_nz)^{-\theta}}\right.$$
$$+ \frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]\{h_n'(1)\}(1-z)(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}$$
$$\left. - \frac{\exp(\sum_{j=1}^b \mu_j)[z^{n-r}]\{h_n''(1)/2\}(1-z)^2(1-x_nz)^{-\theta}}{[z^n](1-x_nz)^{-\theta}}\right)^+.$$

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Substituting (3.14), (3.15) and (3.16) in the expression for d_b^* above completes the proof. □

Proof of Lemma 3.10. Let F denote $F = [z^{n-s}](1 - x_n z)^{-\theta+v}$. As a consequence of (1.6), we see that

$$\begin{aligned} F &= e^{c(1-s/n)}[z^{n-s}](1-z)^{-\theta+v} \\ &= \frac{e^c}{\Gamma(\theta-v)} e^{-cs/n} G, \end{aligned} \tag{4.18}$$

where

$$G = (n-s)^{\theta-v-1} - \frac{(\theta-v)(1-\theta+v)}{2}(n-s)^{\theta-v-2} - k(\theta-v)(n-s)^{\theta-v-3} + O\left(n^{\theta-v-4}\right).$$

We expand G with the binomial formula

$$\begin{aligned} G &= n^{\theta-v-1}\left(1-sn^{-1}\right)^{\theta-v-1} - \frac{(\theta-v)(1-\theta+v)}{2} n^{\theta-v-2}\left(1-sn^{-1}\right)^{\theta-v-2} \\ &\quad - k(\theta-v)n^{\theta-v-3}\left(1-sn^{-1}\right)^{\theta-v-3} + O(n^{\theta-v-4}) \\ &= n^{\theta-v-1}\left(1-(\theta-v-1)sn^{-1} + \frac{(\theta-v-1)(\theta-v-2)}{2}s^2n^{-2} + O(s^3n^{-3})\right) \\ &\quad - \frac{(\theta-v)(1-\theta+v)}{2} n^{\theta-v-2}\left(1-(\theta-v-2)sn^{-1} + O(s^2n^{-2})\right) \\ &\quad - k(\theta-v)n^{\theta-v-3}\left(1+O(sn^{-1})\right) + O(n^{\theta-v-4}) \\ &= n^{\theta-v-1} - n^{\theta-v-2}\left((\theta-v-1)s + \frac{(\theta-v)(1-\theta+v)}{2}\right) \\ &\quad + n^{\theta-v-3}\left(\frac{(\theta-v-1)(\theta-v-2)}{2}s^2 + \frac{(\theta-v)(1-\theta+v)(\theta-v-2)}{2}s - k(\theta-v)\right) \\ &\quad + O\left(s^3n^{\theta-v-4}\right) \end{aligned}$$

Next, expand $e^{-cs/n}$:

$$e^{-cs/n} = 1 - \frac{cs}{n} + \frac{c^2s^2}{2n^2} + O\left(s^3n^{-3}\right).$$

By substituting the expansions for G and $e^{-c/n}$ into (4.18), we find (3.18) holds with

$$d_1(s,v) = cs + \left((\theta-v-1)s + \frac{(\theta-v)(1-\theta+v)}{2}\right) = -vs + \frac{(\theta-v)(1-\theta+v)}{2},$$

and with

$$\begin{aligned} d_2(s,v) &= \frac{c^2s^2}{2} + cs\left((\theta-v-1)s + \frac{(\theta-v)(1-\theta+v)}{2}\right) \\ &\quad + \left(\frac{(\theta-v-1)(\theta-v-2)}{2}s^2 + \frac{(\theta-v)(1-\theta+v)(\theta-v-2)}{2}s - k(\theta-v)\right) \\ &= \left(\frac{c^2}{2} + c(\theta-v-1) + \frac{(\theta-v-1)(\theta-v-2)}{2}\right)s^2 \\ &\quad + \left(c\frac{(\theta-v)(1-\theta+v)}{2} + \frac{(\theta-v)(1-\theta+v)(\theta-v-2)}{2}\right)s - k(\theta-v). \end{aligned}$$

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It is easy to check that

$$d_2(s,0) = \frac{1-\theta}{2}s^2 - \frac{\theta(1-\theta)}{2}s - k(\theta). \qquad \square$$

Proof of Lemma 3.11. Assume for now that $\theta \notin \{1,2\}$ and use Lemma 3.10 and then the identity $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ to find

$$\begin{aligned}
H &= e^c\Bigg(\frac{1}{\Gamma(\theta)} \left\{ n^{\theta-1} - d_1(0,0)\, n^{\theta-2} + d_2(0,0)\, n^{\theta-3} + O(n^{\theta-4}) \right\} \\
&\quad - \frac{1}{\Gamma(\theta)} \left\{ n^{\theta-1} - d_1(r,0)\, n^{\theta-2} + d_2(r,0)\, n^{\theta-3} + O(r^3 n^{\theta-4}) \right\} \\
&\quad - \frac{A_n}{\Gamma(\theta-1)} \left\{ n^{\theta-2} - d_1(r,1)\, n^{\theta-3} + O(r^2 n^{\theta-4}) \right\} \qquad (4.19) \\
&\quad - \frac{A_n}{\Gamma(\theta)}(x_n - 1) \left\{ n^{\theta-1} - d_1(r+1,0)\, n^{\theta-2} + O(r^2 n^{\theta-3}) \right\} \\
&\quad - \frac{B_n}{\Gamma(\theta-2)} \left\{ n^{\theta-3} + O(r n^{\theta-4}) \right\} \qquad (4.20) \\
&\quad - \frac{2B_n}{\Gamma(\theta-1)}(x_n - 1) \left\{ n^{\theta-2} + O(r n^{\theta-3}) \right\} \qquad (4.21) \\
&\quad - \frac{B_n}{\Gamma(\theta)}(x_n - 1)^2 \left\{ n^{\theta-1} + O(r n^{\theta-2}) \right\} \Bigg) \\
&= \frac{e^c}{\Gamma(\theta)} \Bigg(\left\{ n^{\theta-1} - d_1(0,0)\, n^{\theta-2} + d_2(0,0)\, n^{\theta-3} + O(n^{\theta-4}) \right\} \\
&\quad - \left\{ n^{\theta-1} - d_1(r,0)\, n^{\theta-2} + d_2(r,0)\, n^{\theta-3} + O(r^3 n^{\theta-4}) \right\} \\
&\quad - A_n(\theta-1) \left\{ n^{\theta-2} - d_1(r,1)\, n^{\theta-3} + O(r^2 n^{\theta-4}) \right\} \\
&\quad - A_n(x_n - 1) \left\{ n^{\theta-1} - d_1(r+1,0)\, n^{\theta-2} + O(r^2 n^{\theta-3}) \right\} \\
&\quad - B_n(\theta-1)(\theta-2) \left\{ n^{\theta-3} + O(r n^{\theta-4}) \right\} \qquad (4.22) \\
&\quad - 2B_n(\theta-1)(x_n - 1) \left\{ n^{\theta-2} + O(r n^{\theta-3}) \right\} \\
&\quad - B_n(x_n - 1)^2 \left\{ n^{\theta-1} + O(r n^{\theta-2}) \right\} \Bigg).
\end{aligned}$$

The expansions

$$x_n - 1 = \frac{1-\theta}{n} + \frac{(1-\theta)^2}{2n^2} + O(n^{-3})$$

and

$$(x_n - 1)^2 = \frac{(1-\theta)^2}{n^2} + O(n^{-3})$$

show that

$$H = \frac{e^c}{\Gamma(\theta)} \left(D_1 n^{\theta-1} + D_2 n^{\theta-2} + D_3 n^{\theta-3} + O\left(r^3 n^{\theta-4} + A_n r^2 n^{\theta-4} + B_n r n^{\theta-4} \right) \right),$$

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where

$$\begin{aligned} D_1 &= 1-1=0, \\ D_2 &= -d_1(0,0)+d_1(r,0)-A_n(\theta-1)-A_n(1-\theta)=0, \end{aligned}$$

and

$$\begin{aligned} D_3 &= d_2(0,0)-d_2(r,0)+A_n(\theta-1)\,d_1(r,1)+A_n(1-\theta)d_1(r+1,0)-A_n\frac{(1-\theta)^2}{2} \\ &\quad -B_n(\theta-1)(\theta-2)-2B_n(\theta-1)(1-\theta)-B_n(1-\theta)^2 \\ &= -k(\theta)-\left(\frac{1-\theta}{2}r^2-\frac{\theta(1-\theta)}{2}r-k(\theta)\right) \\ &\quad -A_n(\theta-1)r+A_n\frac{(\theta-1)^2(2-\theta)}{2}+A_n\frac{\theta(1-\theta)^2}{2}-A_n\frac{(1-\theta)^2}{2} \\ &\quad +B_n(\theta-1) \\ &= \frac{\theta-1}{2}r^2-(\theta-1)\left(A_n+\frac{\theta}{2}\right)r+A_n\frac{(\theta-1)^2}{2}+B_n(\theta-1). \end{aligned}$$

When $\theta=2$ the term (4.20) must be replaced with 0, which is true of (4.22), as well. Similarly, when $\theta=1$, though each of the terms (4.19), (4.20), and (4.21) must be replaced with 0, the conclusion of the lemma is still valid. □

Proof of Proposition 3.12. By using (3.12) and (3.13), it is easy to check that (3.19) is equivalent to

$$\begin{aligned} H &= \frac{(\theta-1)(n/e)^{\theta-1}}{2n^2\Gamma(\theta)}\left\{(r-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta(r-\mathbb{E}T_{0b})\right\} \\ &\quad +O\left(r^3n^{\theta-4}+A_nr^2n^{\theta-4}+B_nrn^{\theta-4}\right). \end{aligned}$$

Lemma 3.3 will imply

$$\begin{aligned} d_b^* &= o\left(\left(\frac{b}{n}\right)^{1+u}\right)+ \\ &\quad \sum_{r=0}^{L_n}\mathbb{P}(T_{0b}=r)\left(\frac{(\theta-1)(n/e)^{\theta-1}\left\{(r-\mathbb{E}T_{0b})^2-\sigma^2(T_{0b})-\theta(r-\mathbb{E}T_{0b})\right\}}{2n^2\Gamma(\theta)\,[z^n](1-x_nz)^{-\theta}}\right)^+ \end{aligned}$$

when it is shown that

$$\left(\frac{n}{b}\right)^{1+u}\left(\frac{L_n^3+A_nL_n^2+B_nL_n}{n^3}\right)\to 0.$$

It is true that

$$\left(\frac{n}{b}\right)^{2}\left(\frac{L_n^3+A_nL_n^2+B_nL_n}{n^3}\right)\to 0. \tag{4.23}$$

We know $A_n=O(b)$ and $B_n=O(b^2)$ because of (3.21). Now (3.2) implies (4.23).

Finally, substituting $(n/e)^{\theta-1}/\Gamma(\theta)$ for $[z^n](1-x_nz)^{-\theta}=(n/e)^{\theta-1}/\Gamma(\theta)+O(n^{\theta-2})$ makes a difference of only $O(d_b^*/n)=O(n^{-1})$. □

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Proof of Lemma 3.13. Let Y_b be defined by (3.29). Since $\mathbb{E}Y_b^2 = O(1)$ because of (3.21), it follows that

$$n^{-2}\mathbb{E}\left(|Y_b|\mathbf{1}_{\{T_{0b}>L_n\}}\right) \leqslant n^{-2}\sqrt{\mathbb{E}Y_b^2\,\mathbb{P}(T_{0b} > L_n)} = O\left(n^{-2}\sqrt{\mathbb{P}(T_{0b} > L_n)}\right).$$

The bound from Lemma 3.2

$$\mathbb{P}(T_{0b} > L_n) = o\left(\left(\frac{b}{n}\right)^{2k}\right)$$

completes the proof. □

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