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Dimensions of harmonic measures on non-autonomous conformal Cantor sets

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Abstract. We consider non-autonomous conformal iterated function systems (NACIFSs) and their limit sets. Our main focus is on harmonic measure and its dimensions: Hausdorff and packing. We prove that these two dimensions are continuous under perturbations and that they satisfy Bowen's and Manning's type formulas. To achieve this, we establish general results about measures, and more broadly about positive functionals, defined on a symbolic space. We also develop tools from thermodynamic formalism in a non-autonomous setting.

Key words: dynamical systems, metric geometry, probability, potential theory
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1. *Introduction and statement of results*

Given a compact set in the plane defined dynamically by iteration of conformal contractions, this article focuses on the variations of Hausdorff dimension and packing dimension of the harmonic measure under perturbations of the conformal contractions. In a natural framework, which is precisely stated below, we demonstrate that these functions are continuous.

Let $\mathbb{K} \subset \mathbb{C}$ be a compact set and ω the harmonic measure of $\hat{\mathbb{C}} \setminus \mathbb{K}$ evaluated at ∞. Since the works of Makarov [15], and Jones and Wolff [13], it is known that the Hausdorff dimension of ω is less than or equal to 1. This result was generalized by Bourgain to $\mathbb{R}^d$ in [4], where he proved the existence of a constant $\ell(d) > 0$ such that the dimension of the harmonic measure of any open set in $\mathbb{R}^d$ is less than $d - \ell(d)$. According to Makarov, Jones, and Wolff, we know that $\ell(2) = 1$.

If $\mathbb{K}$ has Hausdorff dimension in $]1, 2]$, the previous results tell us that harmonic measure only 'sees' a small part of $\mathbb{K}$ of dimension less than or equal to 1. Let, for instance, $\mathbb{K}$ be the Mandelbrot set. This is a compact and connected set, so that $\mathbb{C} \setminus \mathbb{K}$,

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is simply connected, and we know by [15] that the support of harmonic measure, which by definition is a subset of the boundary of $\mathbb{K}$, has Hausdorff dimension equal to 1. However, Shishikura proved in [20] that any intersection of this boundary with an open set has Hausdorff dimension 2.

When the compact set $\mathbb{K}$ is fractal and has Hausdorff dimension less than one, it is still reasonable to believe that in many cases, the dimension of harmonic measure remains strictly less than the Hausdorff dimension of $\mathbb{K}$ (sometimes called the 'dimension drop' phenomenon). This has been proved to be true for several types of conformal Cantor sets [2, 8, 14, 22]. More recently, Batakis and Zdunik proved in [7] that this remains true for a class of non-autonomous Cantor sets generated by similitudes. Note that to the best of the authors' knowledge, the question whether this is true for all conformal Cantor sets is still open. Azzam [1] and Tolsa [21] have recently shown that for flat Ahlfors regular sets in $\mathbb{R}^{nd+1}$ of dimension ρ with $d - \frac{1}{2} < \rho < d$, we also have a dimension drop phenomenon. However, the first author exhibits in [2] examples of Cantor sets for which harmonic measure has full dimension, and very recently, David, Jeznach, and Julia in [9] give an example of an Alhfors-regular Cantor set in the plane, of small dimension $\rho < \frac{1}{2}$, for which harmonic measure is equivalent with the ρ-dimensional Hausdorff measure.

In this paper, the work of [3] is generalized in two distinct directions. First, the functions defining the Cantor set are no longer piecewise linear but rather piecewise conformal. Second, the number of pieces is also modified at each step of the construction. These generalizations, which would be impossible without precise control of certain quantities, required the development of a robust theoretical framework to handle tools such as entropy, Lyapunov exponents, pressure, and dimensions in a non-autonomous symbolic context. Most of the results are, in fact, proven in this symbolic setting in §2. While these results are of independent interest, the primary focus of this paper is harmonic measure and continuity properties of its packing and Hausdorff dimensions. Consequently, we do not explore their broader implications in further detail here.

To state our results precisely, we describe now our settings and we introduce some notation.

Following [19], given a Jordan domain U that we will assume to be the unit disc $\mathbb{D}$, a non-autonomous conformal iterated function system (NACIFS) is a sequence $\Psi = (\Psi_n)$ of finite collections of conformal contractions $\Psi_n = \{{}_n\psi_i : U \to U;\ \ i = 1, \dots, d_n\}$ verifying the following properties (1), (2), and (3) for some $\boldsymbol{\eta} > 0$. For the sake of simplicity, we have chosen to encode the geometric constraints we work with into this single constant $\boldsymbol{\eta} > 0$.

(1) *Conformality.* There is a fixed neighborhood $V \supset \mathbb{D}_{1+\eta} := \{|z| = 1 + \eta\}$ of $\mathbb{D}$ such that ${}_n\psi_i$ extends to a conformal diffeomorphism from V to V, for all $1 \le i \le d_n$ and all $n \in \mathbb{N}$.

(2) *Open set condition.* ${}_nV_i \cap {}_nV_j = \varnothing$ for all $1 \le i \ne j \le d_n$, where ${}_nV_i = {}_n\psi_i(V)$.
Annulus condition. Moreover, for any n and any $1 \le i \le d_n$, we have ${}_nV_i \subset (1 - \boldsymbol{\eta})U$.

(3) *Bounded contraction.* For all $n \in \mathbb{N}$, all $1 \le i \ne j \le d_n$, and all $x \in {}_nV_i$,

$$\boldsymbol{\eta} < \|{}_n\psi_i'(x)\| < 1 - \boldsymbol{\eta}.$$

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Combining the open set and the bounded contraction conditions, we get a uniform bound on d_n:

$$\text{there exists } D \text{ such that } d_n \leq D \text{ for all } n \in \mathbb{N}. \tag{1.1}$$

Notation.

- For any $E \subset V$, any $n \in \mathbb{N}$, and any $1 \leq i \leq d_n$, let ${}_nE_i = {}_n\psi_i(E)$. The index on the left stands for the *generation* of the map, while the one on the right stands for the chosen *branch* of the map.
- For $n \in \mathbb{N}$, we denote by $\mathcal{U}_n$ the collection $\{{}_nU_1, \ldots, {}_nU_{d_n}\}$. Let $\Phi_n = \sum_{i=1}^{d_n} {}_n\phi_i \mathbb{1}_{{}_nU_i}$, where ${}_n\phi_i$, defined on ${}_nU_i$, is the inverse conformal dilatation of ${}_n\psi_i$, i.e., ${}_n\phi_i : {}_nU_i \to U$ and ${}_n\phi_i \circ {}_n\psi_i = \mathrm{id}_U$.

In this way, we have introduced piecewise conformal maps denoted again by Φ_n defined on the union of topological discs: $\Phi_n : \bigcup_{{}_nU_i \in \mathcal{U}_n} {}_nU_i \to U$ by the formula

$$\Phi_n(z) = {}_n\phi_i(z) \quad \text{for } z \in {}_nU_i.$$

Remark 1.1. Note that the condition of bounded contraction is stronger than that of [19] (uniform contraction). This is due to the fact that dealing with harmonic measure requires estimates on capacity and, hence, on the size of the sub-components. Also, the conformality condition has been strengthened to ensure existence of conformal annuli with bounded from below moduli. This is crucial to be able to use Harnack's principle (cf. [8]).

Furthermore, without loss of generality, we can suppose that U is the unit disc.

We consider the following non-autonomous limit set.

Definition 1.1. Given a NACIFS Ψ_n on a Jordan domain U, we define the *limit set* $\mathbb{X} \subset \mathbb{C}$ by

$$\mathbb{X} = \bigcap_{n=1}^{\infty} (\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1)^{-1}(U),$$

where (Φ_n) is the sequence of piecewise conformal dilatations introduced above.

For any n, the map $T_n := \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_2 \circ \Phi_1$ is well defined on $\mathbb{X}$ and for any $x \in \mathbb{X}$, we have $T_n(x) \in U$. Moreover, there exists $\prod_{i=1}^{n} d_i$ conformal inverse branches of T_n defined on V. Each of these branches sends U onto a conformal disc included in U, and each point in $\mathbb{X}$ belongs to one and only one of these conformal discs. Remember that $\mathcal{U}_n$ is the collection of connected components of $T_n^{-1}(U)$.

There is an obvious way to code the situation.

For any integer n, let $\mathcal{A}_n := \{1 \cdots d_n\}$ and set $\mathbb{K} := \prod_{k=1}^{+\infty} \mathcal{A}_k$. There is a one-to-one correspondence from $\mathbb{K}$ to $\mathbb{X}$. More precisely, if $\mathtt{a} = (a_n)$ is an element of $\mathbb{K}$, then the sequence of subsets of V defined by $U_0 = \bar{U}$ and $U_n = {}_n\psi_{a_n}(U_{n-1})$ is a decreasing sequence of compact sets with diameters converging to 0, thus converging to a point x, which is obviously an element of $\mathbb{X}$. Reciprocally, if $x \in \mathbb{X}$, then for any n, we have: $\Phi_n \circ \cdots \circ \Phi_1(x) \in U$ and there exists $\mathtt{a} = (a_n) \in \mathbb{K}$ such that $x \in {}_1U_{a_1}$, $\Phi_1(x) \in {}_2U_{a_2}, \ldots, \Phi_n \circ \cdots \circ \Phi_1(x) \in {}_{n+1}U_{a_{n+1}}$.

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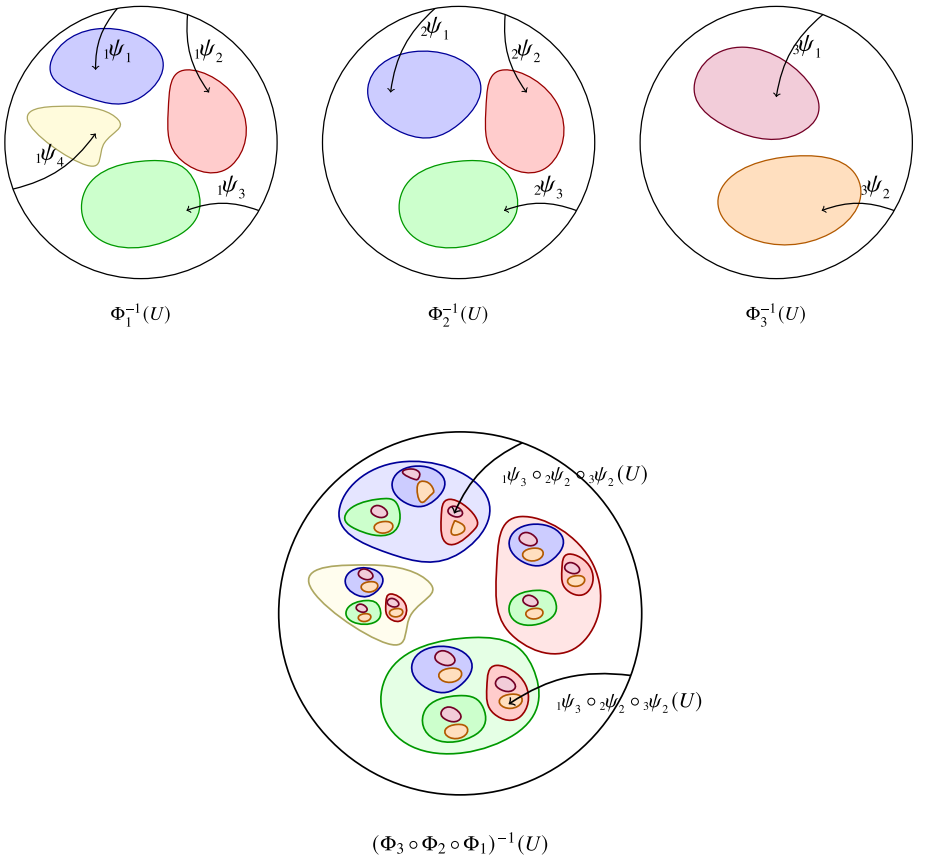


FIGURE 1. First steps of the construction in a given example (colour online).

If $x \in \mathbb{X}$ is associated with $\mathtt{a} \in \mathbb{K}$, then for any n, we have $x \in {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}(U)$, which is the conformal disc mentioned above. In the following, we use $X_n(x) := a_1 \cdots a_n$ to denote the set $\mathbb{X} \cap {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}(U)$, which we call a cylinder of size n, and $\Psi_{X_n} = \Psi^{-1}_{a_1 \cdots a_n}$ will be the map ${}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$.

Figure 1 illustrates the three first steps of the construction in a given example.

We briefly recall here a few definitions; further details and generalities can be found in [10, Ch. 2 and 10]. Given a measure μ in the plane, we define its Hausdorff dimension and packing dimension as follows:

$$HD(\mu) = \inf_{\{Z:\mu(Z)=1\}} HD(Z) \quad \text{and} \quad PD(\mu) = \inf_{\{Z:\mu(Z)=1\}} PD(Z),$$

the infimum and the supremum being taken over all Borel subsets Z of the support of μ.

We also define its lower and upper local dimension as follows:

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

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In our setting, since we work with conformal mappings, which ensure that cylinders are comparable to balls, it is straightforward to verify that for any measure μ supported on $\mathbb{K}$ and any $\mathtt{x} \in \mathbb{K}$,

$$\underline{d}_\mu(\mathtt{x}) = \liminf_{n\to+\infty} \frac{\log \mu(X_n(\mathtt{x}))}{\log \operatorname{diam}(X_n(\mathtt{x}))}, \quad \overline{d}_\mu(x) = \limsup_{n\to+\infty} \frac{\log \mu(X_n(\mathtt{x}))}{\log \operatorname{diam}(X_n(\mathtt{x}))}, \tag{1.2}$$

where $X_n(\mathtt{x})$ is the cylinder of size n containing $\mathtt{x}$.

When these local dimensions are μ-almost surely (a.s.) constant, we then have, see [**10**, Remark above Proposition 10.5],

$$\underline{d}_\mu(\cdot) \overset{\mu\text{-a.s.}}{=} HD(\mu) \quad \text{and} \quad \overline{d}_\mu(\cdot) \overset{\mu\text{-a.s.}}{=} PD(\mu).$$

While the Hausdorff dimension captures the minimal covering complexity of a set in the complex plane, the packing dimension reflects its maximal packing efficiency. The packing dimension of the harmonic measure is far less studied than its Hausdorff dimension. However, it is worth noting that, in the context of autonomous conformal dynamical systems, these two dimensions typically coincide.

Let ω be the harmonic measure on $\hat{\mathbb{C}} \setminus \mathbb{X}$ evaluated at ∞. By the celebrated results of Makarov [**15**] and Jones and Wolff [**13**], the Hausdorff dimension of ω is not larger than one. However, it is clear that the Hausdorff dimension of ω is at most $HD(\mathbb{X})$.

For any continuous function f defined on U and any set $E \subset U$, let $\|f\|_E := \sup_E |f|$, which is simply noted $\|f\|$ when $E = U$, where $|\cdot| : \mathbb{C} \to \mathbb{R}^+$ is the modulus.

Let $\mathbb{X}$ and $\tilde{\mathbb{X}}$ be two non-autonomous limit sets associated to the conformal iterated systems Ψ and $\tilde{\Psi}$, respectively. We define

$$d(\Psi, \tilde{\Psi}) = \sup_n \max_{1\le i\le d_n} (\|{}_n\psi_i - {}_n\tilde{\psi}_i\|).$$

Using general results on the symbolic space $\mathbb{K}$, and studied in the second and third part of this paper, we prove the following theorem.

THEOREM 1.2. *Le $\mathbb{X}$ and $(\mathbb{X}_k)_{k\in\mathbb{N}}$ be non-autonomous limit sets associated with the conformal iterated systems Ψ and $(\Psi_k)_{k\in\mathbb{N}}$, respectively. Let ω and ω_k be the harmonic measures of their complementaries. Assume that* $\lim_{k\to+\infty} d(\Psi, \Psi_k) = 0$*, then,*

$$\lim_{k\to+\infty} HD(\omega_k) = HD(\omega) \quad \textit{and} \quad \lim_{k\to+\infty} PD(\omega_k) = PD(\omega).$$

We also discuss the validity of well-known formulas to calculate these dimensions. In particular, we show that an adapted form of Manning's formula applies.

THEOREM 1.3. *Let ω be the harmonic measure of $\mathbb{C} \setminus \mathbb{X}$, where $\mathbb{X}$ is the limit set of a non-autonomous conformal iterated system Ψ.*

We have

$$HD(\omega) = \liminf_n \frac{H_{\omega,n}}{\chi_{\omega,n}} \quad \textit{and} \quad PD(\omega) = \limsup_n \frac{H_{\omega,n}}{\chi_{\omega,n}},$$

where $H_{\omega,n} = -(1/n)\sum_{X\in\mathcal{U}_n} \omega(X)\log\omega(X)$ *and* $\chi_{\omega,n} = (1/n)\sum_{X\in\mathcal{U}_n} \omega(X)\log\|T_n'\|_X$.

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Moreover, ω-a.s.

$$HD(\omega) = \liminf_n \frac{h_{\omega,n}(x)}{\chi_n(x)} \quad \text{and} \quad PD(\omega) = \limsup_n \frac{h_{\omega,n}(x)}{\chi_n(x)},$$

where for any $x \in \mathbb{X}_0$, $h_{\omega,n}(x) = -(1/n)\log \omega(X_n(x))$, and $\chi_n(x) = (1/n)\log |T_n'(x)|$.

2. *Symbolic point of view*

This section deals exclusively with symbolic Cantor sets. More precisely, for any integer n, let $\mathcal{A}_n$ be a finite alphabet with d_n symbols. Set $\mathbb{K}_n := \prod_{k=n}^{+\infty} \mathcal{A}_k$. We are mainly interested in $\mathbb{K}_1$ that we denote by $\mathbb{K}$.

All measures discussed in this section have their support that is one of the $\mathbb{K}_n$, and in most cases, $n = 1$.

Let $T : \mathbb{K}_n \to \mathbb{K}_{n+1}$ be the shift map sending a sequence $\mathtt{a} = (a_i)_{i\geq 0}$ to $T(\mathtt{a}) = (a_i)_{i\geq 1}$. It should be emphasized that T goes from $\mathbb{K}_n$ to $\mathbb{K}_{n+1}$. This is a bit ambiguous, since the same letter designates different maps, but it lightens considerably the notation. We let $D_n^p := \prod_{k=0}^{n-1} d_{k+p}$ be the degree of T^n on $\mathbb{K}_p$.

Given a sequence of symbols $\mathtt{a} = (a_i)_{i\in\mathbb{N}} \in \mathbb{K}$, we set

$$X_n(\mathtt{a}) := a_0 \cdots a_{n-1} := \{\mathtt{b} = (b_i) \in \mathbb{K} \mid a_i = b_i \ \text{ for } 0 \leq i \leq n-1\}.$$

The set $X_n(\mathtt{a})$ is the *cylinder* (word) of *length* (size) n to which $\mathtt{a}$ belongs. Given a cylinder X, we denote by $|X|$ its length and, for any n, by C_n the set of all cylinders of length $n \geq 1$, in which, for practical reasons, we add $\varnothing$ with the convention $\varnothing X = X\varnothing = X$ and $|\varnothing| = 0$. Note that C_n is a finite partition of $\mathbb{K}$ by D_n^1 cylinders.

Obviously, for any integer p, one can also define cylinders in $\mathbb{K}_p$ and we will denote T^pC their set.

Notation. To simplify notation with cylinders, we are a little bit imprecise, and instead of writing XY with $X \in C_n$ and $Y \in T^nC_{n+k}$, we write $XY \in C$ with $|X| = n$ and $|Y| = k$, the context reminding us that Y is not in C_k but in T^nC_{n+k}.

The set $\mathbb{K}$ endowed with the product topology is compact. Let $\mathtt{d}$ be any metric on $\mathbb{K}$ that generates this topology. If $E \subset \mathbb{K}$, we denote by $|E|_{\mathtt{d}} := \sup\{\mathtt{d}(x, y) \mid (x, y) \in E \times E\}$ its diameter and we assume that $|\mathbb{K}|_{\mathtt{d}} = 1$.

Actually, since the planar Cantor sets that we consider are conformal repellers, they are well described using the coding and the symbolic point of view. It is then natural to choose a metric which ensures that a cylinder in $\mathbb{K}$ and its 'image' in the plane have approximately the same diameters.

2.1. *Asymptotic siblings invariance.* Because of the strong intra-scalar-similarity of the Cantor sets with which we are dealing, one expects that natural geometric measures, or natural geometric objects, defined on them have nice scaling or invariance properties.

Note that in this section, we do not need to make any assumption on the finite numbers d_n.

To define precisely the properties we have in mind, we need to introduce some definitions.

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For any integer $p \geq 1$, let $\mathfrak{F}_p$ be the set of real positive maps defined on the set of cylinders $T^p(C)$. For $p = 0$, we denote by $\mathfrak{F} = \mathfrak{F}_0$ the set of maps defined on the set of cylinders $C := \bigcup_{n\geq 0} C_n$ with values in $]0, +\infty[$.

In this paper, we will mainly consider the following elements $\nu \in \mathfrak{F}_p$:

- $\nu(X) = \mu(X)$, where μ is a measure on $\mathbb{K}_p$;
- $\nu(X) = |X|_{\mathrm{d}}$.

For $\nu = \nu_\varnothing \in \mathfrak{F}$ and any $XY \in C$, we note $\nu_X(Y) := (\nu(XY)/\nu(X))$. Note that ν_X is an element of $\mathfrak{F}_{|X|}$.

A finite measure ν on $\mathbb{K}_p$ that gives positive mass to any open set is naturally an element of $\mathfrak{F}_p$. Reciprocally, we say that $\nu \in \mathfrak{F}_p$ is a (finite) measure if it projects as such on the compact set defined asymptotically by the cylinders, which is if and only if:

$$\text{for all } n \in \mathbb{N}, \quad \text{for all } X \in T^p(C), \qquad \frac{1}{\nu(X)} \sum_{|Y|=n} \nu(XY) = 1.$$

Definition 2.1.

- Given ν and ν' in $\mathfrak{F}_p$, we naturally define $\nu + \nu'$, $\lambda\nu$ for any $\lambda \in \mathbb{R}$, and $\nu.\nu'$.
- For any real function φ and any $\nu \in \mathfrak{F}_p$, we denote $\varphi(\nu)$ the map from T^pC to $\mathbb{R}$ defined by $\varphi(\nu)(X) = \varphi(\nu(X))$. Clearly, this only makes sense if φ is well defined on $\nu(T^pC)$.
- Given ν and ν' in $\mathfrak{F}_p$ and $n \geq 1$, let $\langle \nu, \nu' \rangle_n := \sum_{X\in T^pC_{n+p}} \nu(X)\nu'(X)$.
- For any $\nu \in \mathfrak{F}_p$, we define its *mass* on C_n (with $n \geq p$),

$$\mathfrak{m}_n(\nu) := \langle \nu, 1 \rangle_n.$$

  Note that ν is a measure on $\mathbb{K}_p$ if $\mathfrak{m}_n(\nu_X) = 1$ for any n and any $X \in C$.
- We say that ν and ν' in $\mathfrak{F}_p$ are equivalent if there exists $C > 1$ such that for any X, we have $1/C \leq (\nu(X)/\nu'(X)) \leq C$. We then note $\nu \sim \nu'$.

Note that we use the notation $\langle \cdot, \cdot \rangle_n$ with ν a probability measure and ν' of the form $\log s$, where s is an element of $\mathcal{F}$. For instance, if ν is a probability measure with $\nu' = \log(1/\nu)$, then

$$\frac{1}{n}\langle \nu, \nu' \rangle_n = -\frac{1}{n} \sum_{|X|=n} \nu(X) \log \nu(X),$$

which corresponds to an entropy.

We give some elementary properties that are used later on.

Properties. For any $s \in \mathfrak{F}$ and any $XYZ \in C_n$, we have

$$s_X(YZ) = s_X(Y)s_{XY}(Z) \quad \text{and} \quad s_{X_Y}(Z) = s_{XY}(Z). \tag{2.1}$$

For any finite measure $\nu \in \mathfrak{F}$, any cylinder $X \in C$, and any integers n and p, we have

$$\langle \nu_X, \log s_X \rangle_{n+p} = \langle \nu_X, \log s_X \rangle_n + \sum_{Y\in T^{|X|}C_{|X|+n}} \nu_X(Y)\langle \nu_{XY}, \log s_{XY} \rangle_p. \tag{2.2}$$

Proof. Since equality (2.2) will be used extensively in what follows, and since it allows one to become familiar with the introduced notation, we provide a proof.

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Starting from the definition, we have

$$\begin{aligned}\langle \nu_X, \log s_X\rangle_{n+p} &= \sum_{|Y|=n+p} \nu_X(Y)\log s_X(Y)\\ &= \sum_{|Y|=n,\,|Z|=p} \frac{\nu(XY)}{\nu(X)}\frac{\nu(XYZ)}{\nu(XY)}\log\left(\frac{s(XYZ)}{s(XY)}\frac{s(XY)}{s(X)}\right)\\ &= \sum_{|Y|=n,\,|Z|=p} \nu_X(Y)\nu_{XY}(Z)\log(s_{XY}(Z)s_X(Y))\\ &= \sum_{|Y|=n,\,|Z|=p} \nu_X(Y)[\nu_{XY}(Z)\log(s_{XY}(Z)) + \nu_{XY}(Z)\log(s_X(Y))]\\ &= \sum_{|Y|=n} \nu_X(Y)\Big[\sum_{|Z|=p}\nu_{XY}(Z)\log(s_{XY}(Z)) + \log(s_X(Y))\sum_{|Z|=p}\nu_{XY}(Z)\Big]\\ &= \sum_{|Y|=n} \nu_X(Y)[\langle \nu_{XY}, \log s_{XY}\rangle_p + \log(s_X(Y))\mathfrak{m}_p(\nu_{XY})].\end{aligned}$$

Since ν is a finite measure, we have $\mathfrak{m}_p(\nu_{XY}) = 1$ so that we get the desired equality:

$$\langle \nu_X, \log s_X\rangle_{n+p} = \langle \nu_X, \log s_X\rangle_n + \sum_{Y\in T^{|X|}C_{|X|+n}} \nu_X(Y)\langle \nu_{XY}, \log s_{XY}\rangle_p. \qquad \square$$

The following is an application of the Jensen inequality.

PROPOSITION 2.1. *Let ν and ν' be two elements of $\mathfrak{F}_p$. For any n, we have*

$$\langle \nu, \log\frac{\nu'}{\nu}\rangle_n \le \mathfrak{m}_n(\nu)\log\frac{\mathfrak{m}_n(\nu')}{\mathfrak{m}_n(\nu)}, \tag{2.3}$$

with equality if and only if there exists k such that $\nu(X) = k\nu'(X)$ for any $X \in T^pC_{n+p}$.

Proof. We make use of Jensen's inequality for the function log, evaluated at $(\nu'(Y)/\nu(Y))_{|Y|=n}$, and the convex combination defined by the coefficients $(\nu(Y)/\mathfrak{m}_n(\nu))_{|Y|=n}$. We get

$$\frac{1}{\mathfrak{m}_n(\nu)}\langle \nu, \log\frac{\nu'}{\nu}\rangle_n = \sum_{|Y|=n}\frac{\nu(Y)}{\mathfrak{m}_n(\nu)}\log\left(\frac{\nu'(Y)}{\nu(Y)}\right) \le \log\left(\sum_{|Y|=n}\frac{\nu'(Y)}{\mathfrak{m}_n(\nu)}\right) = \log\left(\frac{\mathfrak{m}_n(\nu')}{\mathfrak{m}_n(\nu)}\right).$$

Multiplying this inequality by $\mathfrak{m}_n(\nu)$, we obtain the result.

Since log is strictly concave, equality holds if and only if $\nu'(Y)/\nu(Y)$ does not depend on Y. $\square$

We introduce now our main assumption that we refer to as asymptotic sibling invariance (ASI). It is a kind of long-range Markov property of a functional that asserts that along cylinders which share a big common part of history, the relative functional does not depend on its origin.

Definition 2.2. Let ν be an element of $\mathfrak{F}$ and (β_n) a sequence of positive numbers converging to 0. We say that ν satisfies ASI with sequence (β_n) if for any integers n, k,

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and p, we have

$$\text{for all } (X, X') \in \mathcal{C}_n^2, \text{ for all } Y \in T^n C_{n+k}, \text{ for all } Z \in T^{n+k} C_{n+k+p}, \quad \left|\log\left(\frac{\nu_{XY}(Z)}{\nu_{X'Y}(Z)}\right)\right| \leq \beta_k. \tag{2.4}$$

Note that we sometimes use the following version of the inequality:

$$\left|\frac{\nu_{XY}(Z)}{\nu_{X'Y}(Z)} - 1\right| \leq \beta_k.$$

Definition 2.3. Given two sequences of positive real numbers (γ_n) and (β_n), we say that they are compatible if (γ_n) is increasing, (β_n) converges towards 0, and there exists a sequence of integers, (p_n), such that $p_n \leq n$ and $((1/n)\max(\gamma_{p_n}, \gamma_n\beta_{p_n}))$ converges towards 0, so that

$$c(\gamma_n, \beta_n) := \max(\gamma_{p_n}, \gamma_n\beta_{p_n}) \text{ is such that } \frac{c(\gamma_n, \beta_n)}{n} \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.5}$$

As an example, one may consider the situation that we will have to deal with in studying the harmonic measure and the diameters of the cylinders. In that case, we will have $\gamma_n = O(n)$, and it appears that such a sequence is compatible with any sequence (β_n) converging towards 0. As a matter of fact, we have $(1/n)\max(\gamma_{p_n}, \gamma_n\beta_{p_n}) = \max(O(p_n)/n, (O(n)/n)\beta_{p_n})$, and any $p_n = o(n)$, for instance, $p_n = \lfloor\sqrt{n}\rfloor$, fulfilling the desired growth condition.

This is the only case we will need to consider in the geometric setting, but since the proofs are given in the symbolic setting, it seemed appropriate to present them in full generality, as they may be useful in other contexts.

We recall that the notation $s \sim s'$ has been introduced in Definition 2.1

PROPOSITION 2.2. *Let (γ_n) and (β_n) be two compatible sequences.*

Assume that s and ν are two elements of $\mathfrak{F}$, that ν is a finite measure, and that they both satisfied ASI with (β_n). Assume moreover that one of these conditions is fulfilled:

$$\text{for all } XY \in C, \quad |\log s_X(Y)| \leq \gamma_{|Y|} \tag{2.5.1}$$

or

$$\text{for all } X \in C \text{ for all } n \in \mathbb{N}, \quad s(X) \geq 1 \quad \text{and} \quad \langle \nu_X, \log(s_X)\rangle_n \leq \gamma_n. \tag{2.5.2}$$

Then, for any $\tilde{s} \sim s$:

- *there exists $C > 0$ such that*

$$\alpha_n(\nu, \tilde{s}) := \sup_k \left\{ \max_{|X|=k} \langle \nu_X, \log \tilde{s}_X\rangle_n - \min_{|X|=k} \langle \nu_X, \log \tilde{s}_X\rangle_n \right\} \leq C c(\gamma_n, \beta_n); \tag{2.6}$$

- *for any $(X, X') \in \mathcal{C}^2$, we have*

$$\lim_{n\to+\infty} \left(\frac{1}{n}\langle \nu_{X'}, \log \tilde{s}_{X'}\rangle_n - \frac{1}{n}\langle \nu_X, \log \tilde{s}_X\rangle_n \right) = 0.$$

Proof. The constant C will change from line to line, but will stay independent of integers or cylinders, only depending on ν and s.

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For any $\tilde{s} \sim s$, we have, for any $XY \in C$, $-2\log C \leq \log \tilde{s}_X(Y) - \log s_X(Y) \leq 2\log C$.

We then deduce, because ν is a finite measure,

$$-2\log C \leq \langle \nu_X, \log \tilde{s}_X\rangle_n - \langle \nu_X, \log s_X\rangle_n \leq 2\log C.$$

This tells us that conclusions of the theorem are true for $\tilde{s}$ as soon as they are true for s, and in the following, we focus on s.

Let:

- for any $X \in C$, $g_n(X) := \langle \nu_X, \log s_X\rangle_n$, with $g_n := g_n(\varnothing)$;
- $\alpha_n := \alpha_n(\nu, s) = \sup_k\{\max_{|X|=k} g_n(X) - \min_{|X|=k} g_n(X)\}$.

We first note that (2.5.1) and the fact that ν is a finite measure imply that for any $X \in C$, we have

$$|g_n(X)| = |\langle \nu_X, \log s_X\rangle_n| \leq \gamma_n, \tag{2.7}$$

an inequality that is also true with (2.5.2) and leads to

$$\alpha_n \leq 2\gamma_n. \tag{2.8}$$

Note also that given any $X \in C$, and any integers n and p, we have from (2.2):

$$g_{n+p}(X) = g_n(X) + \sum_{|Y|=n} \nu_X(Y) g_p(XY). \tag{2.9}$$

From this, we deduce that for any $X' \in C_{|X|}$, we have

$$\begin{aligned} g_{n+p}(X) - g_{n+p}(X') = g_n(X) - g_n(X') &+ \sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y)) g_p(XY) \\ &+ \sum_{|Y|=n} \nu_{X'}(Y)(g_p(XY) - g_p(X'Y)). \end{aligned} \tag{2.10}$$

Let $\mathcal{B}^+$ be the subset of $T^{|X|}C_{|X|+n}$ for which we have $\nu_X(Y) - \nu_{X'}(Y) \geq 0$ and let $\mathcal{B}^-$ be its complementary. Then, we have

$$\begin{aligned} \sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y)) g_p(XY) \leq &\max_{|Z|=|X|+n} g_p(Z) \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) \\ &+ \min_{|Z|=|X|+n} g_p(Z) \sum_{Y\in\mathcal{B}^-} (\nu_X(Y) - \nu_{X'}(Y)). \end{aligned}$$

Since ν is a finite measure, we have $\sum_{|Y|=n} \nu_X(Y) = \sum_{|Y|=n} \nu_{X'}(Y) = 1$.

This implies that $\sum_{|Y|=n}(\nu_X(Y) - \nu_{X'}(Y)) = 0$.

Therefore, $\sum_{Y\in\mathcal{B}^+}(\nu_X(Y) - \nu_{X'}(Y)) = -\sum_{Y\in\mathcal{B}^-}(\nu_X(Y) - \nu_{X'}(Y))$, which leads to

$$\begin{aligned} \sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y)) g_p(XY) &\leq \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) \left(\max_{|Z|=|X|+n} g_p(Z) - \min_{|Z|=|X|+n} g_p(Z) \right) \\ &\leq \alpha_p \sum_{Y\in\mathcal{B}^+} (\nu_X(Y) - \nu_{X'}(Y)) \\ &\leq \alpha_p \sum_{Y\in\mathcal{B}^+} \nu_{X'}(Y) \left(\frac{\nu_X(Y)}{\nu_{X'}(Y)} - 1 \right) \end{aligned}$$

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$$\leq \alpha_p \beta_0 \sum_{Y \in \mathcal{B}^+} \nu_{X'}(Y) \quad \text{(because of ASI)}$$
$$\leq \alpha_p \beta_0 \quad \text{(because } \nu \text{ is a finite measure).}$$

We thus have

$$\sum_{|Y|=n} (\nu_X(Y) - \nu_{X'}(Y)) g_p(XY) \leq \lambda \alpha_p. \tag{2.11}$$

From ASI, we deduce that

$$\nu_{XY}(Z) \leq (1 + \beta_{|Y|}) \nu_{X'Y}(Z),$$

and also

$$\log s_{XY}(Z) \leq \log s_{X'Y}(Z) + \beta_{|Y|}.$$

This implies the following.

- If (2.5.1) is fulfilled, then we have $|\log s_{X'Y}(Z)| \leq \gamma_{|Z|}$ so that

$$\begin{aligned} &\nu_{XY}(Z) \log s_{XY}(Z) - \nu_{X'Y}(Z) \log s_{X'Y}(Z) \\ &\quad = (\nu_{XY}(Z) - \nu_{X'Y}(Z)) \log s_{XY}(Z) + \nu_{X'Y}(Z) \log \frac{s_{XY}(Z)}{s_{X'Y}(Z)} \\ &\quad \leq |\nu_{XY}(Z) - \nu_{X'Y}(Z)| \gamma_{|Z|} + \nu_{X'Y}(Z) \left| \log \frac{s_{XY}(Z)}{s_{X'Y}(Z)} \right| \end{aligned}$$

  and since ν and s fulfill ASI, we get

$$\begin{aligned} &\leq \beta_{|Y|} \nu_{X'Y}(Z) \gamma_{|Z|} + \nu_{X'Y}(Z) \beta_{|Y|} \\ &\leq \beta_{|Y|} (1 + \gamma_{|Z|}) \nu_{X'Y}(Z). \end{aligned}$$

  Summing over all cylinders Z of length p and using once again the fact that ν is a finite measure, we get

$$g_p(XY) - g_p(X'Y) \leq (1 + \gamma_p) \beta_{|Y|} \leq C \gamma_p \beta_{|Y|}. \tag{2.12}$$

- If (2.5.2) is fulfilled, then we have

$$\begin{aligned} &\nu_{XY}(Z) \log s_{XY}(Z) - \nu_{X'Y}(Z) \log s_{X'Y}(Z) \\ &\quad \leq \beta_{|Y|} (\nu_{X'Y}(Z) \log s_{X'Y}(Z) + C \nu_{X'Y}(Z)). \end{aligned}$$

  Summing over all cylinders Z of length p and using (2.7), we also get (2.12).

Injecting (2.12) and (2.11) into (2.10) leads to

$$g_{n+p}(X) - g_{n+p}(X') \leq g_n(X) - g_n(X') + C \gamma_p \beta_n + \lambda \alpha_p,$$

from which we get that

$$\alpha_{n+p} \leq \alpha_n + C \gamma_p \beta_n + \lambda \alpha_p,$$

which might be rewritten, for any $p \leq n$, in the following form:

$$\alpha_n \leq \alpha_p + C \gamma_{n-p} \beta_p + \lambda \alpha_{n-p} \leq \alpha_p + C \gamma_n \beta_p + \lambda \alpha_{n-p}, \quad (\gamma_n) \text{ being increasing.}$$

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Using (2.8), we get

$$\alpha_n \leq 2\gamma_p + C\gamma_n\beta_p + \lambda\alpha_{n-p} \leq C\max(\gamma_p, \gamma_n\beta_p) + \lambda\alpha_{n-p}.$$

Let q and $r \leq p$ be the quotient and the remainder of the Euclidean division of n by p. By induction, we get

$$\begin{aligned}\alpha_n &\leq C\max(\gamma_p, \gamma_n\beta_p)\sum_{k=0}^{q-1}\lambda^k + \lambda^q\alpha_r\\ &\leq C\max(\gamma_p, \gamma_n\beta_p) + 2\lambda^q\gamma_r\\ &\leq C\max(\gamma_p, \gamma_n\beta_p) + 2\lambda^q\gamma_p\\ &\leq C\max(\gamma_p, \gamma_n\beta_p).\end{aligned}$$

Since (γ_n) and (β_n) are compatible, one may take $p = p_n$ to get

$$\alpha_n \leq Cc(\gamma_n, \beta_n),$$

which is the first part of the proposition and tells us that for any X and X' in C, with $|X| = |X'|$, we have

$$\frac{1}{n}|g_{n(X)} - g_{n(X')}| \leq \frac{\alpha_n}{n} \leq C\frac{c(\gamma_n, \beta_n)}{n} \to_{n\to+\infty} 0.$$

We show now that we can get rid of the condition $|X| = |X'|$.

Let $X \in C$ and let a be a cylinder of length 1. Applying (2.9) with $n = 1$ and $p = q$,

$$\begin{aligned}g_{q+1}(X) &= g_1(X) + \sum_{|b|=1}\nu_X(b)g_q(Xb)\\ &= g_q(Xa) + g_1(X) + \sum_{|b|=1}\nu_X(b)(g_q(Xb) - g_q(Xa)).\end{aligned}$$

Apply now (2.9) with $n = q$ and $p = 1$:

$$g_{q+1}(X) = g_q(X) + \sum_{|Y|=q}\nu_X(Y)g_1(XY).$$

We thus have

$$g_q(X) - g_q(Xa) = g_1(X) + \sum_{|b|=1}\nu_X(b)(g_q(Xb) - g_q(Xa)) - \sum_{|Y|=q}\nu_X(Y)g_1(XY).$$

By hypothesis on ν and s, there exists $C > 0$ such that for any cylinder $Z \in C$, we have $g_1(Z) \leq C$. This leads to

$$|g_q(X) - g_q(Xa)| \leq C + \alpha_q,$$

and this easily leads, for any Y with $|Y| = p$, to

$$|g_q(X) - g_q(XY)| \leq (C + \alpha_q)p \leq C\alpha_q p. \tag{2.13}$$

Let $X' \in C_{|X|}$ and $XY \in C$. We have

$$|g_q(X') - g_q(XY)| \leq |g_q(X') - g_q(X)| + |g_q(X) - g_q(XY)| \leq \alpha_q + Cp\alpha_q \leq C'p\alpha_q$$

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and then,

$$\frac{1}{q}|g_q(X') - g_q(XY)| \leq C' p \frac{\alpha_q}{q},$$

which finishes the proof. □

2.2. *A continuity result.* Let ν and ν' be two elements of $\mathfrak{F}$. We define

$$\mathcal{D}(\nu, \nu') = \sup_{p \geq 0} \max \left\{ \left| \log \left(\frac{\nu_X(a)}{\nu'_X(a)} \right) \right|, \ Xa \in C_{p+1} \text{ and } |a| = 1 \right\}. \tag{2.14}$$

Note that we obviously have $\mathcal{D}(\nu, \nu') = \mathcal{D}(\nu', \nu)$ and $\mathcal{D}(\nu, \nu') = 0$ if and only if $\nu = \nu'$. Moreover, it is also clear that we have

$$\mathcal{D}(\nu, \nu'') \leq \mathcal{D}(\nu, \nu') + \mathcal{D}(\nu', \nu'').$$

Therefore, $\mathcal{D}$ could be seen as a metric on $\mathfrak{F}$ if $\mathcal{D}(\nu, \nu') < +\infty$ for any ν and ν', which is not the case. Nevertheless, if $\mathcal{D}(\nu, \nu_k)$ converges to 0, when $k \to +\infty$, this tells us that (ν_k) is 'converging' to ν.

Let $Y = y_1 \cdots y_p \in T^n C$, $Y_0 = \varnothing$ and, for any $1 \leq k \leq p$, $Y_k = y_1 \cdots y_k$. Since

$$\nu(Y) = \prod_{k=1}^{p} \nu_{Y_{i-1}}(y_i),$$

we conclude that for any $X \in C$ and any Y, we have

$$\left| \log \frac{\nu_X(Y)}{\nu'_X(Y)} \right| \leq |Y| \mathcal{D}(\nu, \nu').$$

This also implies, for any $XY \in C_{n+p}$, with $|Y| = n$, that we have

$$|\nu_X(Y) - \nu'_X(Y)| \leq \nu_X(Y) \max(e^{n\mathcal{D}(\nu,\nu')} - 1, 1 - e^{-n\mathcal{D}(\nu,\nu')}) \leq 2\nu_X(Y) \operatorname{sh}(n\mathcal{D}(\nu, \nu')).$$

Let $g_n(X) := \langle \nu_X, \log s_X \rangle_n$ and $g'_n(X) := \langle \nu'_X, \log s'_X \rangle_n$.

We have

$$g_n(X) - g'_n(X) = \left\langle \left(1 - \frac{\nu'_X}{\nu_X}\right) \nu_X, \log s_X \right\rangle_n + \left\langle \nu'_X, \log \frac{s'_X}{s_X} \right\rangle_n. \tag{2.15}$$

Assume that ν and ν' are both finite measures so that $\langle \nu_X, 1 \rangle_n = \langle \nu'_X, 1 \rangle_n = 1$.

Assume moreover that, for any n, we have

$$\sup_{X \in C} \max_{|Y|=n} |\log S_X(Y)| = \gamma_n < +\infty.$$

Then, from (2.15), we deduce

$$|g_n(X) - g'_n(X)| \leq 2 \operatorname{sh}(n\mathcal{D}(\nu, \nu'))\gamma_n + n\mathcal{D}(s, s')$$

so that

$$\frac{1}{n}|g_n(X) - g'_n(X)| \leq \operatorname{sh}(n\mathcal{D}(\nu, \nu'))\frac{\gamma_n}{n} + \mathcal{D}(s, s'). \tag{2.16}$$

THEOREM 2.3. *Let ν and s be two elements of $\mathfrak{F}$, and let (ν_k) and (s_k) be two sequences of elements in $\mathfrak{F}$.*

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For any $n \in \mathbb{N}$, set $g_n = \langle \nu, \log s \rangle_n$ and $g_{k,n} = \langle \nu_k, \log s_k \rangle_n$.
Assume that:

- *ν and ν_n are finite measures;*
- *all those elements of $\mathfrak{F}$ verify ASI, with a same sequence (β_n);*
- *there exists a sequence (γ_n), compatible with (β_n), such that for any $s' \in \{s,\ s_n\}_{n\in\mathbb{N}}$ and any $XY \in C$,*

$$|\log s'_X(Y)| \leq \gamma_{|Y|};$$

- $\lim_{k\to+\infty} \mathcal{D}(\nu, \nu_k) = \lim_{k\to+\infty} \mathcal{D}(s, s_k) = 0$.

Then, we have

$$\lim_{k\to+\infty} \limsup_n \left| \frac{g_{k,n}}{n} - \frac{g_n}{n} \right| = 0,$$

and, in particular, we have

$$\lim_{k\to+\infty} \liminf_n \frac{g_{k,n}}{n} = \liminf_n \frac{1}{n} g_n \quad \text{and} \quad \lim_{k\to+\infty} \limsup_n \frac{g_{k,n}}{n} = \limsup_n \frac{1}{n} g_n.$$

Moreover, the conclusions are still valid for any $\tilde{s}$ and $(\tilde{s_n})$ such that $\tilde{s} \sim s$ and $\tilde{s}_n \sim s_n$ for any n.

Proof. By hypothesis, we are in a position to apply Proposition 2.2 with ν and s, and ν_n and s_n, and we use the notation therein.

Let $p \leq n$, with $q \in \mathbb{N}$ such that $qp < n \leq (q+1)p$. Using (2.9), we get

$$g_n(\varnothing) := g_n = g_{pq+r} = g_{pq} + \sum_{|Y|=pq} \nu(Y) g_r(Y),$$

from which we get

$$g_n - g_{k,n} = g_{pq} - g_{k,pq} + \sum_{|Y|=pq} [\nu(Y) g_r(Y) - \nu_k(Y) g_{k,r}(Y)]$$

and also

$$\frac{1}{n}|g_n - g_{k,n}| \leq \frac{1}{n}|g_{pq} - g_{k,pq}| + \frac{1}{n}\left| \sum_{|Y|=pq} [\nu(Y) g_r(Y) - \nu_k(Y) g_{k,r}(Y)] \right|. \tag{2.17}$$

By hypothesis, we have $|g_r(Y)| \leq \gamma_r \leq \gamma_p$ and $|g_{k,r}(Y)| \leq \gamma_r \leq \gamma_p$, so that

$$\frac{1}{n}\left| \sum_{|Y|=pq} [\nu(Y) g_r(Y) - \nu_k(Y) g_{k,r}(Y)] \right| \leq 2\frac{\gamma_p}{n}. \tag{2.18}$$

Using once again (2.9), we get

$$g_{pq} = g_{p(q-1)} + \sum_{|Y|=p(q-1)} \nu(Y) g_p(Y),$$

so that

$$g_{pq} = \sum_{i=0}^{q-1} \sum_{|Y|=ip} \nu(Y) g_p(Y)$$

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and, thus,

$$g_{pq} - g_{k,pq} = \sum_{i=0}^{q-1} \sum_{|Y|=ip} [\nu_k(Y)(g_p(Y) - g_{k,p}(Y)) + g_p(Y)(\nu(Y) - \nu_k(Y))].$$

Using (2.16), we get

$$|g_p(Y) - g_{k,p}(Y)| \leq 2 \text{ sh } (p\mathcal{D}(\nu, \nu_k))\gamma_p + p\mathcal{D}(s, s_k),$$

which leads to

$$\left|\sum_{i=0}^{q-1} \sum_{|Y|=ip} \nu_k(Y)(g_p(Y) - g_{k,p}(Y))\right| \leq 2q \text{ sh } (p\mathcal{D}(\nu, \nu_k))\gamma_p + pq\mathcal{D}(s, s_k)$$
$$\leq n\left(2 \text{ sh } (p\mathcal{D}(\nu, \nu_k))\frac{\gamma_p}{p} + \mathcal{D}(s, s_k)\right). \quad (2.19)$$

However, we have

$$\sum_{i=0}^{q-1} \sum_{|Y|=ip} g_p(Y)(\nu(Y) - \nu_k(Y)) \leq \sum_{i=0}^{q-1} \sum_{|Y|=ip} \left(\nu(Y) \max_{|Z|=ip} g_p(Z) - \nu_k(Y) \min_{|Z|=ip} g_p(Z)\right)$$
$$\leq q\left(\max_{|Z|=ip} g_p(Z) - \min_{|Z|=ip} g_p(Z)\right)$$
$$\leq Cqc(\gamma_p, \beta_p) \quad \text{(by Proposition 2.2, (2.6)).} \quad (2.20)$$

Inequalities (2.19) and (2.20) lead to

$$\frac{1}{n}|g_{pq} - g_{k,pq}| \leq 2 \text{ sh } (p\mathcal{D}(\nu, \nu_k))\frac{\gamma_p}{p} + \mathcal{D}(s, s_k) + \frac{\alpha_p(\nu, s)}{p}. \quad (2.21)$$

Using now (2.18), we get

$$\frac{1}{n}|g_n - g_{k,n}| \leq 2 \text{ sh } (p\mathcal{D}(\nu, \nu_k))\frac{\gamma_p}{p} + \mathcal{D}(s, s_k) + C\frac{c(\gamma_p, \beta_p)}{p} + 2\frac{\gamma_p}{n}.$$

Let $\varepsilon > 0$.

Fix p big enough to ensure that $C(c(\gamma_p, \beta_p)/p) < \varepsilon/3$ and let $n_0 \in \mathbb{N}$ such that $2(\gamma_p/n_0) < \varepsilon/3$.

Finally, let k_0 such that for any $k \geq k_0$, we have

$$2 \text{ sh } (p\mathcal{D}(\nu, \nu_k))\frac{\gamma_p}{p} < \frac{\varepsilon}{3}.$$

Then, for any $k \geq k_0$ and any $n \geq n_0$, we have $|g_n/n - g_{k,n}/n| < \varepsilon$, from where the result follows. □

2.3. *Almost sure convergence.* In the previous section, we work with a probability measure μ supported on $\mathbb{K}$. We are interested in almost sure asymptotic values of sequences of the form: $Z_n(\mathrm{x}) := \log s(X_n(\mathrm{x}))$, where $s \in \mathfrak{F}$, with $s \geq 1$, and for any $\mathrm{x} \in \mathbb{K}$, $X_n(\mathrm{x})$ is the unique element of C_n containing x. We will use the following version of the strong large numbers law, see for instance [**12**, Theorem 2.19].

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THEOREM 2.4. *Let $\{Y_n, n \geq 1\}$ be a sequence of random variables in a probability space. Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing sequence of σ-fields with Y_n measurable with respect to $\mathcal{F}_n$ for each n. Assume that (Y_n) is uniformly bounded. Then, a.s.,*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mathop{\mathbb{E}}_{\mu}[Y_i|\mathcal{F}_{i-1}]) = 0.$$

THEOREM 2.5. *Let ν be a probability measure and let $s \in \mathfrak{F}$.*

Assume that ν and s fulfilled ASI with sequence (β_n), that there exists a sequence (γ_n), compatible with (β_n) and such that

$$\text{for all } XY \in C, \quad |\log s_X(Y)| \leq \gamma_{|Y|}.$$

Then, for any $\tilde{s} \sim s$, there is a set $\mathcal{U} \subset \mathbb{K}$ of full ν-measure such that for any $\mathtt{a} \in \mathcal{U}$, we have

$$\lim_{n\to+\infty} \left(\frac{1}{n} \sum_{|Y|=n} \nu(Y) \log \tilde{s}(Y) - \frac{1}{n} \log \tilde{s}(X_n(\mathtt{a})) \right) = 0.$$

Proof. Let us first observe that if the result holds for s, then it also holds for every $\tilde{s} \sim s$. Indeed,

$$\frac{1}{n} \left| \sum_{|Y|=n} \nu(Y)(\log(\tilde{s}) - \log(s)) \right| \leq \frac{\log(c)}{n}.$$

Given any $n \in \mathbb{N}$, let $X_n : \mathbb{K} \to C_n$, $\mathtt{x} \mapsto X_n(\mathtt{x})$ be the cylinder of size n containing $\mathtt{x}$. We define $Z_n(x) := \log s(X_n(x))$.

If we adopt the notation introduced in Proposition 2.2, with $g_n = \langle \nu, \log(s) \rangle_n$, the theorem asserts that the sequence $(1/n)(g_n - Z_n(\cdot))$ converges to 0ν-a.s.

For any integer $p > 0$, let $Y_{n,p} := (1/p)(Z_{np} - Z_{(n-1)p})$. Let also $\mathcal{F}_n$ be the σ-algebra generated by cylinders in C_n. The idea, once again taken from [3], is to apply Theorem 2.4 on $(Y_{n,p})_n$, with respect to $(\mathcal{F}_{np})_n$, and to prove that the conditional expectations are asymptotically, on n and p, independent of $\mathtt{x} \in \mathbb{K}$.

Fix $p > 0$, and $n > p$ and let $n = pq + r$ be the Euclidian division of n by p. We define

$$① = \frac{1}{n} Z_n - \frac{pq}{n} \frac{1}{q} \sum_{k=1}^{q} Y_{k+1,p}$$

$$② = \frac{1}{q} \sum_{k=1}^{q} [Y_{k+1,p} - \mathop{\mathbb{E}}_{\nu}[Y_{k+1,p}|\mathcal{F}_{kp}]],$$

$$③ = \frac{1}{q} \sum_{k=1}^{q} \mathop{\mathbb{E}}_{\nu}[Y_{k+1,p}|\mathcal{F}_{kp}] - \frac{1}{qp} g_{qp}(X_p),$$

$$④ = \frac{1}{n} (g_{qp}(X_p) - g_n(X_p)),$$

$$⑤ = \frac{1}{n} (g_n(X_p) - g_n),$$

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so that

$$\frac{1}{n}(Z_n - g_n) = ① + \frac{pq}{n}② + \frac{pq}{n}③ + ④ + ⑤.$$

We treat each of the five terms in the sum separately.

Term ①. By definition, we have

$$\frac{pq}{n}\frac{1}{q}\sum_{k=1}^{q} Y_{k+1,p} = \frac{1}{n}Z_{(q+1)p}.$$

Therefore,

$$① = \frac{1}{n}(Z_n - Z_{(q+1)p}) = \frac{1}{n}\log\left(\frac{s(X_n)}{s(X_{(q+1)p})}\right).$$

There exists a cylinder Y, with $|Y| \le p$, such that $X_{(q+1)p} = X_n Y$. By hypothesis on s, we thus have

$$\left|\log\left(\frac{s(X_n)}{s(X_{(q+1)p})}\right)\right| = |\log(s_{X_n}(Y))| \le \gamma_p.$$

This tells us that

$$|①| \le \frac{\gamma_p}{n}. \tag{2.22}$$

Terms ④ and ⑤. Recall that by (2.9), we have

$$g_n(X_p) = g_{pq+r}(X_p) = g_{pq}(X_p) + \sum_{|Y|=pq} \nu_{X_p}(Y) g_r(X_p Y).$$

Therefore,

$$|④| = \frac{1}{n}\left|\sum_{|Y|=pq} \nu_{X_p}(Y) g_r(X_p Y)\right| \le \frac{\gamma_r}{n} \le \frac{\gamma_p}{n}. \tag{2.23}$$

However, by (2.13) and Proposition 2.2, we have

$$|g_n(X_p) - g_n| \le Cpc(\gamma_n, \beta_n),$$

which allows us to conclude that

$$|⑤| \le Cp\frac{c(\gamma_n, \beta_n)}{n}. \tag{2.24}$$

Term ③. By definition, the conditional expectation $\mathbb{E}_\nu[Y_{k+1,p}|\mathcal{F}_{kp}]$ is constant on any cylinder of size kp. More precisely, for any $\mathrm{x} \in X_{kp} \in C_{kp}$, it is equal to the average of $Y_{k+1,p}$ on X_{kp}, so that we have

$$\mathop{\mathbb{E}}_{\nu}\left[Y_{k+1,p}|\mathcal{F}_{kp}\right](\mathrm{x}) = \frac{1}{p}\sum_{|Y|=p}\frac{\nu(X_{kp}Y)}{\nu(X_{kp})}\log\frac{s(X_{kp}Y)}{s(X_{kp})} = \frac{1}{p}g_p(X_{kp}).$$

Recall that by (2.9), for any integer k, we have

$$g_{kp}(X_p) = g_{(k-1)p}(X_p) + \sum_{|Y|=(k-1)p} \nu_{X_p}(Y) g_p(X_p Y).$$

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Therefore,

$$g_{kp}(X_p) - g_{(k-1)p}(X_p) - g_p(X_{kp}) = \sum_{|Y|=(k-1)p} \nu_{X_p}(Y)(g_p(X_pY) - g_p(X_{kp})).$$

Summing from 1 to q and dividing by pq, we get

$$\frac{1}{pq} g_{qp}(X_p) - \frac{1}{q}\sum_{k=1}^{q} \frac{g_p(X_{kp})}{p} = \frac{1}{pq}\sum_{k=1}^{q} \sum_{|Y|=(k-1)p} \nu_{X_p}(Y)(g_p(X_pY) - g_p(X_{kp})).$$

Since by Proposition 2.2 for any Y of length $(k-1)p$, we have $|g_p(X_pY) - g_pX_{kp}| \leq Cc(\gamma_p, \beta_p)$, we get

$$|③| = \left|\frac{1}{q}\sum_{k=1}^{q} \mathbb{E}_{\nu}[Y_{k+1,p}|\mathcal{F}_{kp}] - \frac{1}{qp} g_{qp}(X_p)\right| \leq C\frac{c(\gamma_p, \beta_p)}{p}. \tag{2.25}$$

Term ②. Note that for any n, we have by hypothesis that

$$|Y_{k,p}| = \left|\log \frac{s(X_{kp})}{s(X_{(k-1)p})}\right| = |\log s_{X_{(k-1)p}}(X_{kp})| \leq \gamma_p.$$

We may thus apply Theorem 2.4 with $(Y_{n,p})_n$ to conclude that p being fixed, $|②|\nu$-a.s. converges towards 0 as $q \to +\infty$, and we can easily conclude that there exists a set of full measure, denoted by $\mathcal{U}$, such that it occurs for any p.

We may now conclude the proof.

Let $\mathrm{x} \in \mathcal{U}$ and $\varepsilon > 0$. Choose p big enough to have $C(c(\gamma_p, \beta_p)/p) < \varepsilon/5$.

Let $n_0 \geq p$ be such that for any $n \geq n_0$, we have $\gamma_p/n < \varepsilon/5$.

Since p is fixed, q goes to infinity with n and we may choose n_0 big enough to have $|②| < \varepsilon/5$. Combining this with (2.22), (2.25), (2.23), and (2.24), we deduce that

$$\frac{1}{n}|Z_n(\mathrm{x}) - g_n| \leq |①| + |②| + |③| + |④| + |⑤| < \varepsilon.$$

□

3. *Pressures, entropies, Lyapunov exponents, and dimensions*

In this section, using results from the previous section, we show how natural generalizations of notions coming from thermodynamical formalism are possible.

3.1. *Lyapunov exponents and entropies.*

Given a sequence (u_n) of real numbers, we let $\overline{u_n}$ be its limsup and $\underline{u_n}$ its liminf.

For any positive integer n and any probability measure on $\mathbb{K}$, μ, we define

$$H_{\mu,n} := -\frac{1}{n}\sum_{|X|=n} \mu(X)\log\mu(X) \quad \text{and} \quad \chi_{\mu,n} := -\frac{1}{n}\sum_{|X|=n} \mu(X)\log|X|_{\mathrm{d}},$$

where the sum is taken over all cylinders $X \in C_n$, with the usual convention $0\ln(0) = 0$.

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We set $\overline{h_\mu} := \overline{H_{\mu,n}}$, $\underline{h_\mu} := \underline{H_{\mu,n}}$, $\overline{\chi_\mu} := \overline{\chi_{\mu,n}}$, and $\underline{\chi_\mu} := \underline{\chi_{\mu,n}}$.
Given any $\mathtt{a} \in \mathbb{K}$ and any positive number n, we define

$$\chi_n(\mathtt{a}) := -\frac{1}{n}\log|X_n(\mathtt{a})|_{\mathsf{d}} \quad \text{and} \quad h_{\mu,n}(\mathtt{a}) := -\frac{1}{n}\log\mu(X_n(\mathtt{a})).$$

As previously, let us note $\overline{h_\mu}(\mathtt{a}) := \overline{h_{\mu,n}(\mathtt{a})}$, $\underline{h_\mu}(\mathtt{a}) := \underline{h_{\mu,n}}(\mathtt{a})$, $\overline{\chi}(\mathtt{a}) := \overline{\chi_n}(\mathtt{a})$, and $\underline{\chi}(\mathtt{a}) := \underline{\chi_n}(\mathtt{a})$. Remember that $\mathcal{F}_n$ is the σ-algebra generated by cylinders of length less or equal to n. We have

$$H_{\mu,n} = \mathop{\mathbb{E}}_{\mu}[h_{\mu,n} \mid \mathcal{F}_n] \quad \text{and} \quad \mathcal{X}_{\mu,n} = \mathop{\mathbb{E}}_{\mu}[\chi_n \mid \mathcal{F}_n].$$

Note also that with $s : X \mapsto 1/|X|_{\mathsf{d}}$, we have

$$H_{\mu,n} = \frac{1}{n}\langle\mu, \log\frac{1}{\mu}\rangle_n \quad \text{and} \quad \mathcal{X}_{\mu,n} = \frac{1}{n}\langle\mu, \log s\rangle_n.$$

Remark 3.1. In this section, we work within a symbolic framework and do not specify a metric. In certain contexts, the choice of metric is a delicate issue. For instance, in [5], the pressure of a potential is studied using a metric tailored to the potential itself. It should be noted that in the remainder of this paper, when applying our results to measures in the complex plane, we shall adopt the 'symbolic metric' to ensure that the 'symbolic' and 'geometric' diameters coincide.

Applying Proposition 2.1 to that setting, remembering that $D_n = d_0 \cdots d_{n-1}$ is the degree of T^n, which is also the number of cylinder in C_n, we get the following proposition.

PROPOSITION 3.2. *For any n and any $t \in \mathbb{R}$,*

$$H_{\mu,n} - t\mathcal{X}_{\mu,n} \le \frac{1}{n}\log \mathsf{m}_n(|\cdot|_{\mathsf{d}}^t) = \frac{1}{n}\sum_{|X|=n}|X|_{\mathsf{d}}^t.$$

In particular, for $t = 0$,

$$0 \le H_{\mu,n} \le \frac{1}{n}\log D_n.$$

Moreover, if X_m and X_M in C_n are such that for all $X \in C_n$, we have $|X_m|_{\mathsf{d}} \le |X|_{\mathsf{d}} \le |X_M|_{\mathsf{d}}$, then

$$-\frac{1}{n}\log|X_M|_{\mathsf{d}} \le \mathcal{X}_{\mu,n} \le -\frac{1}{n}\log|X_m|_{\mathsf{d}}.$$

Remark 3.3. It is worth noting that this inequality is akin to a variational principle, whereby, the metric pressure of a potential, with respect to any probability measure, is bounded above by its topological pressure. The non-autonomous setting would require a precise definition of these objects; in particular, the notion of a potential must be replaced by that of a 'sequence of potentials'. While these tools are of intrinsic interest, they are not essential to the present discussion and will be addressed in future work.

Proof. We use Proposition 2.1 with $\nu = \mu$ and $\nu' = |\cdot|_{\mathsf{d}}^t$. We thus have for any n,

$$H_{\mu,n} - t\mathcal{X}_{\mu,n} = \frac{1}{n}\left\langle\nu, \log\frac{1}{\nu}\right\rangle_n - \frac{1}{n}\langle\nu, \log\nu'\rangle_n = \frac{1}{n}\left\langle\nu, \log\frac{\nu'}{\nu}\right\rangle_n \le \frac{1}{n}\log \mathsf{m}_n(\nu'). \qquad \square$$

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THEOREM 3.4. *Let μ be a probability measure that fulfilled ASI with sequence (β_n) and assume that there exists a sequence (γ_n), compatible with (β_n), and such that*

$$\text{for all } XY \in C, \quad |\log \mu_X(Y)| \leq \gamma_{|Y|}.$$

Then, there exists a set $\mathcal{U}$ of full μ-measure such that for any $\mathtt{a} \in \mathcal{U}$,

$$\lim_{n\to+\infty} (h_{\mu,n}(\mathtt{a}) - H_{\mu,n}) = 0,$$

and, in particular, we have μ-a.s.

$$\overline{h_\mu}(\cdot) = \overline{h_\mu} \quad \text{and} \quad \underline{h_\mu}(\cdot) = \underline{h_\mu}.$$

Assume moreover that $s : X \mapsto 1/|X|_{\mathsf{d}}$ fulfilled ASI and that there exist $0 < \delta < 1$ such that for any $XY \in C$, we have

$$\delta^{\gamma_{|Y|}} \leq \frac{|XY|_{\mathsf{d}}}{|X|_{\mathsf{d}}} \leq 1, \tag{3.1}$$

then there exists a set of full μ-measure, $\mathcal{V}$, such that for any $\mathtt{a} \in \mathcal{V}$,

$$\lim_{n\to+\infty} (\chi_n(\mathtt{a}) - \mathcal{X}_{\mu,n}) = 0,$$

and, in particular, we have μ-a.s.

$$\underline{\mathcal{X}} = \underline{\chi}(\cdot) \leq \overline{\chi}(\cdot) = \overline{\mathcal{X}}.$$

Proof. The hypothesis allows us to apply Theorem 2.5, with $\nu(X) = \mu(X)$ and $s(X) = 1/\mu(X)$, to conclude that

$$\lim_{n\to+\infty} (h_{\mu,n}(\mathtt{a}) - H_{\mu,n}) = 0.$$

Let now $s : X \mapsto 1/|X|_{\mathsf{d}}$.

From the assumption $\delta^{\gamma_{|Y|}} \leq |XY|_{\mathsf{d}}/|X|_{\mathsf{d}} \leq 1$, we deduce that $1 \leq s_X(Y) \leq \delta^{-\gamma_{|Y|}}$, and we easily conclude that for any $XY \in C$, with $|Y| = n$, we have

$$|\log s_X(Y)| \leq -\gamma_n \log d.$$

We may once again apply Theorem 2.5 to conclude. □

3.2. *Pressure of a measure and dimensions.* In the previous subsection, we assume uniform bounds for the diameters, which is a very natural hypothesis in our context: there exists $\delta \in]0, 1[$ such that for any $n \in \mathbb{N}$, any cylinder XY with $|Y| = n$, we have

$$\delta^n \leq \frac{|XY|_{\mathsf{d}}}{|X|_{\mathsf{d}}} \leq (1-\delta)^n. \tag{3.2}$$

In particular, for any n, we have

$$0 < -\log(1-\delta) \leq \mathcal{X}_{\mu,n} \leq -\log \delta$$

and also for any $\mathtt{a} \in \mathbb{K}$,

$$0 < -\log(1-\delta) \leq \chi_n(\mathtt{a}) \leq -\log \delta.$$

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Definition 3.1. For any real number t and any integer n, let $P_{\mu,n}(t) := H_{\mu,n} - t\mathcal{X}_{\mu,n}$.

We define $\underline{P_\mu}(t) = \underline{P_{\mu,n}}(t)$ and $\overline{P_\mu}(t) = \overline{P_{\mu,n}}(t)$.

PROPOSITION 3.5. *Assume that $|\cdot|_{\mathsf{d}}$ satisfies condition (3.2), and that μ and $|\cdot|_{\mathsf{d}}$ both fulfill ASI with sequence (β_n) compatible with $(\gamma_n = n)$, then the functions $t \mapsto \underline{P_\mu}(t)$ and $t \mapsto \overline{P_\mu}(t)$ are strictly decreasing and bi-Lipschitz.*

We have:

(1) $0 \leq \underline{P_\mu}(0) = \underline{h_\mu} \leq \overline{P_\mu}(0) = \overline{h_\mu}$;

(2) *for all* $t \in \mathbb{R}$, $\underline{h_\mu} - t\log(1-\delta) \leq \underline{P_\mu}(t) \leq \overline{P_\mu}(t) \leq \overline{h_\mu} - t\log\delta$;

(3) *let $HD(\mu)$ be the Hausdorff dimension of μ and $PD(\mu)$ its packing-dimension. Then,*

$$\underline{P_\mu}(HD(\mu)) = 0 = \overline{P_\mu}(PD(\mu));$$

(4) *for any $\mathtt{a}$ in a set of full μ-measure, we have*

$$\liminf \frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}} = \liminf \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = HD(\mu)$$

and

$$\limsup \frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}} = \limsup \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = PD(\mu).$$

Proof. From (3.2), we easily get for any $t < t+h$ and any $n \in \mathbb{N}$,

$$-h\log(1-\delta) + P_{\mu,n}(t+h) \leq P_{\mu,n}(t) \leq P_{\mu,n}(t+h) - h\log\delta,$$

from which we deduce, with P standing either for $\underline{\P_\mu}$ or $\overline{\P_\mu}$, that we have

$$0 < -h\log(1-\delta) \leq P(t) - P(t+h) \leq -h\log\delta,$$

which tells us that $\underline{P_\mu}$ and $\overline{P_\mu}$ are strictly decreasing and bi-Lipschitz.

Points (1) and (2) are obvious from definitions and hypotheses.

We turn to results about dimensions.

Hypotheses on μ and $|\cdot|_{\mathsf{d}}$ allow us to apply Theorem 2.5 with, for any $X \in C$, $\nu(X) = \mu(X)$ and $s(X) = |X|_{\mathsf{d}}^t/\mu(X)$, as well as Theorem 3.4. In particular, μ-a.s. we have

$$\lim_{k\to+\infty} h_{\mu,n_k}(\cdot) = h \iff \lim_{k\to+\infty} H_{\mu,n_k} = h, \tag{E}$$

$$\lim_{k\to+\infty} \chi_{n_k}(\cdot) = \chi \iff \lim_{k\to+\infty} \mathcal{X}_{\mu,n_k} = \chi. \tag{L}$$

Since $\underline{P_\mu}$ and $\overline{P_\mu}$ are strictly decreasing, continuous with positive and negative values, there exists a unique $(\underline{t}, \bar{t}) \in \mathbb{R}^2$, with $0 \leq \underline{t} \leq \bar{t} \in \mathbb{R}$ such that $\underline{P_\mu}(\underline{t}) = 0 = \overline{P_\mu}(\bar{t})$.

We focus on the result with liminf.

Take a point $\mathtt{a} \in \mathbb{K}$ such that items (E) and (L) are true.

Let (n_k) be a sequence of integers such that

$$\lim_{k\to+\infty} \frac{h_{\mu,n_k}(\mathtt{a})}{\chi_{n_k}(\mathtt{a})} = \liminf \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = \underline{d}_\mu(\mathtt{a}),$$

see (1.2) for the definition of $\underline{d}_\mu(\mathtt{a})$.

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Using items (E) and (L), and extracting again if necessary, one may suppose that

$$\lim_{k\to+\infty} H_{\mu,n_k} = \lim_{k\to+\infty} h_{\mu,n_k}(\mathtt{a}) = h \text{ and } \lim_{k\to+\infty} \mathcal{X}_{\mu,n_k} = \lim_{k\to+\infty} \chi_{n_k}(\mathtt{a}) = \chi. \tag{3.3}$$

By hypothesis on $|\cdot|_{\mathtt{d}}$, we have $0 < \chi < +\infty$, thus $\underline{d}_\mu(\mathtt{a}) = \frac{h}{\chi}$, and we have

$$\underline{P_\mu}(\underline{d}_\mu(\mathtt{a})) \le \lim_{k\to+\infty} (H_{\mu,n_k} - \underline{d}_\mu(\mathtt{a})\mathcal{X}_{\mu,n_k}) = h - \underline{d}_\mu(\mathtt{a})\chi = 0,$$

from which we deduce that $\underline{d}_\mu(\mathtt{a}) \ge \underline{t}$.

Fix $t > 0$ and assume that n_k is such that

$$\lim_{k\to+\infty} (H_{\mu,n_k} - t\mathcal{X}_{\mu,n_k}) = \underline{P_\mu}(t).$$

As before, using items (E) and (L) and extracting a further subsequence if necessary, we may assume that (3.3) is fulfilled. Note that h and χ denote limits that are different from the previous case. In particular, we can only assert that $h/\chi \ge \underline{d}_\mu(\mathtt{a})$, since

$$\frac{h}{\chi} = \lim_{k\to+\infty} \frac{h_{\mu,n_k}(\mathtt{a})}{\chi_{n_k}(\mathtt{a})} \ge \liminf \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = \underline{d}_\mu(\mathtt{a}).$$

We have

$$\underline{P_\mu}(t) = h - t\chi = \chi\left(\frac{h}{\chi} - t\right) \ge \chi(\underline{d}_\mu(\mathtt{a}) - t).$$

We deduce that $\underline{P_\mu}(\underline{d}_\mu(\mathtt{a})) \ge 0$ so that $\underline{d}_\mu(\mathtt{a}) \le \underline{t}$.

This tells us that μ-a.s. $\underline{d}_\mu(\cdot)$ is constant equal to $\underline{t}$. By classical results on Hausdorff dimension of measures, see [**10**, Remark above Proposition 10.5], we can conclude that μ-a.s.

$$HD(\mu) = \underline{d}_\mu(\mathtt{a}) = \underline{t}.$$

Consider a sequence of integers (n_k) such that

$$\lim_{k\to+\infty} (H_{\mu,n_k} - HD(\mu)\mathcal{X}_{\mu,n_k}) = 0.$$

Since $(\mathcal{X}_{\mu,n_k})$ is bounded away from zero, we have

$$\liminf \left(\frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}}\right) \le \lim_{k\to+\infty} \frac{H_{\mu,n_k}}{\mathcal{X}_{\mu,n_k}} = HD(\mu).$$

We now conclude by considering a sequence of integers (n_k) such that

$$\liminf \left(\frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}}\right) = \lim_{k\to+\infty} \frac{H_{\mu,n_k}}{\mathcal{X}_{\mu,n_k}}.$$

Using the fact that $(\mathcal{X}_{\mu,n_k})$ is bounded, we deduce that

$$\underline{P_\mu}\left(\liminf \left(\frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}}\right)\right) \le \lim_{k\to+\infty} \left(H_{\mu,n_k} - \liminf \left(\frac{H_{\mu,n}}{\mathcal{X}_{\mu,n}}\right)\mathcal{X}_{\mu,n_k}\right) = 0,$$

which implies that $HD(\mu) = \underline{t} \le \liminf(H_{\mu,n}/\mathcal{X}_{\mu,n})$.

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We thus have proved that for any $\mathtt{a}$ in a set of full μ-measure, we have

$$\liminf \frac{H_{\mu,n}}{\chi_{\mu,n}} = \liminf \frac{h_{\mu,n}(\mathtt{a})}{\chi_n(\mathtt{a})} = HD(\mu). \qquad \square$$

3.3. *Continuity of dimension.*

THEOREM 3.6. *Let μ be a probability measure on $\mathbb{K}$ and (μ_k) be a sequence of probability measures.*

Let d be a metric on $\mathbb{K}$ and d_k a sequence of metrics. For any $X \in C$, we let $\tilde{s}(X) = |X|_{\mathsf{d}}$ and $\tilde{s}_k(X) = |X|_{\mathsf{d}_k}$.

Assume that:

- *the metrics $|\cdot|_{\mathsf{d}}$ and $|\cdot|_{\mathsf{d}_k}$, $k \in \mathbb{N}$, verify condition (3.2) with a common constant δ;*
- *μ, s, and for any k, s_k, and μ_k, fulfilled ASI with a common sequence (β_n);*
- *there exists a sequence (γ_n), compatible with (β_n), such that for any $XY \in C$ and any k,*

$$\log \frac{\mu(X)}{\mu(XY)} \le \gamma_{|Y|} \quad \text{and} \quad \log \frac{\mu_k(X)}{\mu_k(XY)} \le \gamma_{|Y|};$$

- *we have $\lim_{k\to+\infty} \mathcal{D}(\mu, \mu_k) = \lim_{k\to+\infty} \mathcal{D}(s, s_k) = 0$;*
- *(t_k) is a real sequence converging towards t.*

Then,

$$\lim_{k\to+\infty} \limsup_n ((H_{\mu,n} - t\chi_{\mu,n}) - (H_{\mu_k,n} - t_k\chi_{\mu_k,n})) = 0.$$

In particular,

$$\lim_{k\to+\infty} \underline{P}_{\mu_k}(t_k) = \underline{P}_{\mu}(t) \quad \text{and} \quad \lim_{k\to+\infty} \overline{P}_{\mu_k}(t_k) = \overline{P}_{\mu}(t),$$

$$\lim_{k\to+\infty} {}_kHD\,(\mu_k) = HD(\mu) \quad \text{and} \quad \lim_{k\to+\infty} {}_kPD\,(\mu_k) = PD(\mu).$$

(We would like to emphasize the fact that when the metrics are changing with k, those dimensions are also changing, since Hausdorff and packing dimensions rely on the metric used.)

Proof. For any $X \in C$, let

$$p(X) = \frac{|X|_{\mathsf{d}}^t}{\mu(X)} = \frac{s(X)^t}{\mu(X)} \quad \text{and for any } k, \quad p_k(X) = \frac{|X|_{\mathsf{d}_k}^{t_k}}{\mu_k(X)} = \frac{s_k(X)^{t_k}}{\mu_k(X)}.$$

Let $C = (1 + \sup_k\{|t|, |t_k|\})$. For any $XY \in C$, we have

$$|\log p_X(Y)| \le |t|\log s_X(Y) + \log \mu_X(Y) \le C\gamma_{|Y|}.$$

In the same way, we check that $|\log p_{k_X}(Y)| \le C\gamma_{|Y|}$. Thus, one may take $\gamma'_n = C\gamma_n$.

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We also have for any $XYZ \in C$ and any $X' \in C_{|X|}$,

$$\begin{aligned}\left|\frac{p_{XY}(Z)}{p_{X'Y}(Z)} - 1\right| &= \left|\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)}\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right| \\ &= \left|\left(\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right)\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} + \frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} - 1\right| \\ &\leq (1+\beta_0)\left|\left(\left(\frac{s_{XY}(Z)}{s_{X'Y}(Z)}\right)^t - 1\right)\right| + \left|\frac{\mu_{X'Y}(Z)}{\mu_{XY}(Z)} - 1\right| \\ &\leq 2(1+\beta_0)^{1+|t|}\beta_{|Y|} \\ &\leq C\beta_{|Y|}.\end{aligned}$$

We may thus take $\beta'_n = C\beta_n$. It is easy to prove that (γ'_n) and (β'_n) are compatible. We also have that $\mathcal{D}(p, p_k) \to 0$, so that we can apply Theorem 2.3. Using notation therein, we have

$$\frac{1}{n}g_n = H_{\mu,n} - t\chi_{\mu,n} = P_{\mu,n}(t) \quad \text{and} \quad \frac{1}{n}g_{k,n} = H_{\mu_k,n} - t_k\chi_{\mu_k,n} = P_{\mu_k,n}(t_k),$$

and Theorem 2.3 asserts that

$$\lim_{k\to+\infty} \limsup_n (P_{\mu,n}(t) - P_{\mu_k,n}(t_k)) = 0.$$

From this, we conclude that

$$\lim_{k\to+\infty} \underline{P}_{\mu_k}(t_k) = \underline{P}_\mu(t) \quad \text{and} \quad \lim_{k\to+\infty} \overline{P}_{\mu_k}(t_k) = \overline{P}_\mu(t).$$

With $t_k = {}_kHD\,(\mu_k)$ and δ any limit value of (t_k), we conclude, using Proposition 3.5, that

$$0 = \lim_{k\to+\infty} \underline{P}_{\mu_k}({}_kHD\,(\mu_k)) = \underline{P}_\mu(\delta),$$

which, by Proposition 3.5, tells us that $\delta = HD(\mu)$, so that $({}_kHD\,(\mu_k))$ converges towards $HD(\mu)$. □

4. *Harmonic measure of non-autonomous conformal repellers*

In this last section, we apply results of the previous sections to study the harmonic measure of the complementary of non-autonomous conformal repellers.

In the first subsection, we study geometric aspects of a given NACIFS Ψ, proving that diameter and harmonic measure fulfilled the desired conditions.

Then, we deal with sequence of such iterated function systems proving continuity of Hausdorff and Packing dimensions of harmonic measures.

Let $\Psi = (\Psi_n) = \{{}_n\psi_i : U = \mathbb{D} \to U = \mathbb{D};\ i = 1, \ldots, d_n\}$ be a NACIFS satisfying our assumptions with a constant $\boldsymbol{\eta}$ controlling contraction and modulus of the annulus, and a fixed neighborhood of $\mathbb{D}$, V. We note $\mathbb{X}$ its limit set. In this last section, we adopt the following notation.

- For any symbolic cylinder $X = a_1 \cdots a_n$, we note ψ_X the contraction ${}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$, and $\phi_X = {}_n\varphi_{a_n} \circ \cdots \circ {}_1\varphi_{a_1}$ its inverse.

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- If X is a symbolic cylinder, we also denote its geometric realization by X. That is, $X = \psi_X(\mathbb{D}) \cap \mathbb{X}$.
- If $E \subset V$ and X is a cylinder, then $E_X := \psi_X(E)$. In particular, we denote $\mathbb{D}_X = \Psi_X(\mathbb{D})$ and $\mathbb{D}_{\gamma X} = \Psi_X(\mathbb{D}_\gamma)$, where $\mathbb{D}_\gamma = \{|z| \leq \gamma\}$.
- For precise definitions and detailed proofs about harmonic measure, we refer to the online book of Prats and Tolsa [18].

  Given a smooth open set $\Omega \subset \mathbb{C}$, a point $x \in \Omega$, and a Borel set F, we will denote $\omega(x, F, \Omega)$ the harmonic measure assigned to the domain Ω of F at x. Throughout this paper, Ω will always take the form $\mathbb{C} \setminus X$, or $\mathbb{D}_X \setminus XY$, with XY a given cylinder.
- In particular, we will note $\omega := \omega(\infty, \cdot, \hat{\mathbb{C}} \setminus \mathbb{X})$, and for any k, $\omega_k := \omega(\infty, \cdot, \hat{\mathbb{C}} \setminus \mathbb{X}_k)$, where $\mathbb{X}_k = T^k(\mathbb{X})$.

4.1. *Controlling diameters of cylinders.* As one may expect, we use the Kœbe distortion theorem and, more precisely, its quantified versions to cope with diameters.

PROPOSITION 4.1. *For any $0 < \delta < 1$ fixed, there exists $C > 1$ such that for any conformal map, $f : \mathbb{D} \to \mathbb{C}$, any compact set $\Bbbk \subset \{|z| \leq 1 - \delta\}$, and any $(x, y) \in \Bbbk \times \Bbbk$,*

$$(1 - C \operatorname{diam}(\Bbbk)) \leq \frac{|f'(x)|}{|f'(y)|} \leq (1 + C \operatorname{diam}(\Bbbk)) \tag{4.1}$$

and

$$(1 - C \operatorname{diam}(\Bbbk))|f'(x)| \leq \frac{\operatorname{diam} f(\Bbbk)}{\operatorname{diam} \Bbbk} \leq |f'(x)|(1 + C \operatorname{diam}(\Bbbk)). \tag{4.2}$$

Proof. First, note that relations (4.1) and (4.2) are relevant only if $\operatorname{diam}(\Bbbk)$ is small. Otherwise, it is nothing else than the standard Kœbe distortion theorem. In addition, we assume from now on that $\operatorname{diam}(\Bbbk) \leq \delta/2$.

Let x and y be two points in $\Bbbk$. We apply [17, Corollary 1.5], with $z = (x - y)/(1 - \bar{y}x)$, we have

$$\frac{|f'(x)|}{|f'(y)|} \leq \left(\frac{1 + |z|}{1 - |z|}\right)^3.$$

Since $\Bbbk \subset \{|z| \leq 1 - \delta\}$, we have $|z| \leq 1/\delta|x - y| \leq 1/\delta \operatorname{diam} \Bbbk$. We also have $|z| < 1$ and we can deduce that we have

$$\frac{|f'(x)|}{|f'(y)|} \leq 1 + C \operatorname{diam}(\Bbbk). \tag{4.3}$$

By exchanging x and y, we also get the opposite inequality.

Let $Z = 1 - x\bar{y}$. Then, [11, Theorem 1] asserts that we have

$$\frac{|Z|}{1 - |x|^2} \frac{|Z| - |x - y|}{|Z| + |x - y|} \leq \frac{|f(x) - f(y)|}{|f'(x)||x - y|} \leq \frac{|Z|}{1 - |x|^2} \frac{|Z| + |x - y|}{|Z| - |x - y|}. \tag{4.4}$$

Since $Z = 1 - |x|^2 - x(\overline{y - x})$, we have

$$1 - |x|^2 - |x - y| \leq |Z| \leq 1 - |x|^2 + |x - y|,$$

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which leads to

$$1-\frac{|x-y|}{1-|x|^2}\le\frac{|Z|}{1-|x|^2}\le 1+\frac{|x-y|}{1-|x|^2}.$$

However, $|x|\le 1-\delta$, so $\delta\le 1-|x|^2\le 1$ and $|x-y|\le \operatorname{diam}\mathbb{k}$, which gives

$$1-\frac{\operatorname{diam}\mathbb{k}}{\delta}\le\frac{|Z|}{1-|x|^2}\le 1+\frac{\operatorname{diam}\mathbb{k}}{\delta}. \tag{4.5}$$

We have assumed that $\operatorname{diam}\mathbb{k}\le\delta/2$; thus,

$$\frac{|Z|+|x-y|}{|Z|-|x-y|}=1+2\frac{|x-y|}{|Z|-|x-y|}\le 1+2\frac{|x-y|}{1-|x|^2-2|x-y|}\le 1+\frac{2}{\delta(\delta-1)}\operatorname{diam}\mathbb{k},$$

and we prove in the same way that

$$1-\frac{2}{\delta}\operatorname{diam}\mathbb{k}\le\frac{|Z|-|x-y|}{|Z|+|x-y|}.$$

Using those two last inequalities and (4.5) in (4.4), we end up with

$$\left(1-\frac{2}{\delta}\operatorname{diam}\mathbb{k}\right)^2\le\frac{|f(x)-f(y)|}{|f'(x)||x-y|}\le\left(1+\frac{2}{\delta(\delta-1)}\operatorname{diam}\mathbb{k}\right)^2.$$

Using (4.3), with an adjustment of the constant, we may change $|f'(x)|$ with $|f'(z)|$ for any $z\in\mathbb{k}$, so that for any x, y and z in $\mathbb{k}$, we have

$$1-C\operatorname{diam}\mathbb{k}\le\frac{|f(x)-f(y)|}{|f'(z)||x-y|}\le 1+C\operatorname{diam}\mathbb{k}.$$

We easily conclude that for any $z\in\mathbb{k}$,

$$(1-C\operatorname{diam}\mathbb{k})|f'(z)|\le\frac{\operatorname{diam} f(\mathbb{k})}{\operatorname{diam}\mathbb{k}}\le(1+C\operatorname{diam}\mathbb{k})|f'(z)|.$$

□

We use this results to verify that the functionals encountered in the course of the proof satisfy the conditions required to apply the general results from §§2 and 3. We are particularly interested in the ASI property.

It should be noted that, in what follows, we identify symbolic cylinders with their geometric realizations, as the context allows us to avoid any ambiguity. In particular, if XYZ is a 'geometric' cylinder in $\mathbb{X}$, with $|X|=k$, then YZ is a 'geometric' cylinder in $\mathbb{X}_k$.

For any $X\in\mathcal{C}$, let $s(X):=\operatorname{diam}(X)$, where diam is the Euclidean diameter of the geometric cylinder X, and let $\tilde{s}:=|(\psi_X)'(0)|$. We recall that $U=\mathbb{D}$ so that $\tilde{s}$ is well defined.

COROLLARY 4.2. *There exists $C>1$ and $q\in\,]0,1[$ such that for any cylinder $XYZ\in\mathcal{C}$, we have*

$$1-Cq^{|Y|}\le\tilde{s}_{XY}(Z)\left|\frac{(\psi_Y)'(0)}{(\psi_{YZ})'(0)}\right|\le 1+Cq^{|Y|}$$

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and

$$1 - Cq^{|Y|} \leq s_{XY}(Z)\left|\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)}\right| \leq 1 + Cq^{|Y|}.$$

We have $s \sim \tilde{s}$ and they both verify ASI with ($\beta_n = Cq^n$).

Proof. If we apply classical Kœbe distortion estimates with $f = \psi_X$, $\mathbb{k} = \mathbb{X}_k$, and $x = 0$, we get

$$\frac{1}{C} \leq \frac{\operatorname{diam}(X)}{|(\psi_X)'(0)|} \leq C.$$

Thus, s and $\tilde{s}$ are equivalent.

We have

$$\left|\frac{(\psi_{XYZ})'(0)}{(\psi_{XY})'(0)}\right| = \left|\frac{(\psi_X)'(\psi_{YZ}(0))}{(\psi_X)'(\psi_Y(0))}\right|\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right|.$$

Since $\psi_{YZ}(0)$ and $\psi_Y(0)$ both lie in Y, by (4.1), we have

$$(1 - C\operatorname{diam}(Y)) \leq \left|\frac{(\psi_X)'(\psi_{YZ}(0))}{(\psi_X)'(\psi_Y(0))}\right| \leq (1 + C\operatorname{diam}(Y)),$$

from where we deduce, using also the fact that by hypothesis, we have $\operatorname{diam}(Y) \leq Cq^{|Y|}$, that

$$(1 - Cq^{|Y|})\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right| \leq \left|\frac{(\psi_{XYZ})'(0)}{(\psi_{XY})'(0)}\right| \leq (1 + Cq^{|Y|})\left|\frac{(\psi_{YZ})'(0)}{(\psi_Y)'(0)}\right|.$$

This can be rewritten as

$$1 - Cq^{|Y|} \leq \tilde{s}_{XY}(Z)\left|\frac{(\psi_Y)'(0)}{(\psi_{YZ})'(0)}\right| \leq 1 + Cq^{|Y|}.$$

The same is also true with X' instead of X and this leads to

$$\left|\frac{\tilde{s}_{XY}(Z)}{\tilde{s}_{X'Y}(Z)} - 1\right| \leq Cq^{|Y|}.$$

One can apply relation (4.2) with $f = \psi_X$, $\mathbb{k} = YZ$, and also $\mathbb{k} = Y$ and $x \in YZ$, to get

$$\frac{1 - C\operatorname{diam}(YZ)}{1 + C\operatorname{diam}(Y)} \leq \frac{\operatorname{diam}(XYZ)}{\operatorname{diam}(XY)}\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leq \frac{1 + C\operatorname{diam}(YZ)}{1 - C\operatorname{diam}(Y)}.$$

We have $\operatorname{diam}(YZ) < \operatorname{diam}(Y)$, and by hypothesis, we know that $\operatorname{diam}(Y) \leq Cq^{|Y|}$. Adjusting the constant C, we easily conclude that we have

$$1 - Cq^{|Y|} \leq \frac{\operatorname{diam}(XYZ)}{\operatorname{diam}(XY)}\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leq 1 + Cq^{|Y|},$$

which is exactly

$$1 - Cq^{|Y|} \leq s_{XY}(Z)\frac{\operatorname{diam}(Y)}{\operatorname{diam}(YZ)} \leq 1 + Cq^{|Y|}.$$

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As a matter of fact, this implies that for any other geometric cylinder $X' \subset \mathbb{X}$, with $|X'| = |X|$, we have

$$\left| \frac{s_{XY}(Z)}{s_{X'Y}(Z)} - 1 \right| \leq Cq^{|Y|}. \qquad \square$$

4.2. *Controlling diameters under perturbation.* We say that two NACIFSs Ψ and $\tilde{\Psi}$ are compatible if for any n:

- the degree of Φ_n and the degree of $\tilde{\Phi}_n$ are the same;
- for any i and j, ${}_n\psi_i(V) \cap {}_n\tilde{\psi}_j(V) \neq \varnothing$ if and only if $i = j$.

For two compatible NACIFSs Ψ and $\tilde{\Psi}$, we define

$$d(\Psi, \tilde{\Psi}) = \sup_n \max_{1 \leq i \leq d_n} \{\| {}_n\psi_i - {}_n\tilde{\psi}_i \|_\infty\},$$

with $\|f\|_\infty = \sup_{z \in \bar{\mathbb{D}}_\gamma} |f(z)|$. Let us also recall that $\mathcal{D}(s, \tilde{s})$ is introduced in the beginning of §2.2, (2.14).

PROPOSITION 4.3. *We denote* $\mathbb{D}_X = \Psi_X(\mathbb{D})$ *and* $\mathbb{D}_{\gamma X} = \Psi_X(\mathbb{D}_\gamma)$, *where* $\mathbb{D}_\gamma = \{|z| \leq \gamma\}$.

Assume that Ψ *and* $\tilde{\Psi}$ *are two compatible NACIFSs and that* $0 < \boldsymbol{\eta} < 1$ *is the constant controlling their annulus and their contractions. Let* $\alpha = (\log(1 - \boldsymbol{\eta}))/(\log(\boldsymbol{\eta}(1 - \boldsymbol{\eta})))$. *For any symbolic cylinder X, let* $s(X) = |\psi'_X(0)|$ *and* $\tilde{s}(X) = |\tilde{\psi}'_X(0)|$. *There exists* $C > 0$ *such that*

$$\mathcal{D}(s, \tilde{s}) \leq C d(\Psi, \tilde{\Psi})^\alpha.$$

Proof. Let $A_n = sup_{|X|=n} \|\psi_X - \tilde{\psi}_X\|_\infty$. Note that we have $A_1 = d(\Psi, \tilde{\Psi})$.

Let Xa be a cylinder with $|X| = n$ and $|a| = 1$. For any $z \in \mathbb{D}_\gamma$, we have

$$\begin{aligned} |\psi_{Xa}(z) - \tilde{\psi}_{Xa}(z)| &\leq |\psi_X(\psi_a(z)) - \psi_X(\tilde{\psi}_a(z))| + |\psi_X(\tilde{\psi}_a(z)) - \tilde{\psi}_X(\tilde{\psi}_a(z))| \\ &\leq (1 - \boldsymbol{\eta})^n |\psi_a(z) - \tilde{\psi}_a(z)| + A_n \\ &\leq (1 - \boldsymbol{\eta})^n A_1 + A_n, \end{aligned}$$

from where we deduce that

$$A_n \leq \frac{1}{\boldsymbol{\eta}} A_1 = \frac{1}{\boldsymbol{\eta}}\, d(\Psi, \tilde{\Psi}). \tag{4.6}$$

Using Cauchy representation theorem, for any cylinder X with $|X| = k$, we have

$$|\psi'_X(0) - \tilde{\psi}'_X(0)| = \frac{1}{2\pi} \left| \int_{\partial \mathbb{D}} \frac{\psi_X(z) - \tilde{\psi}_X(z)}{z^2}\, dz \right| \leq A_k.$$

Since, by assumption, $\boldsymbol{\eta}^k \leq |\tilde{\psi}'_X(0)|$, we deduce that

$$\left| \frac{\psi'_X(0)}{\tilde{\psi}'_X(0)} \right| \leq 1 + \frac{1}{\boldsymbol{\eta}^{k+1}}\, d(\Psi, \tilde{\Psi}). \tag{4.7}$$

Inequality is true for any cylinder X such that $|X| \leq k$, and stays true if we exchange the role played by ψ and $\tilde{\psi}$.

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Let now $X = YX_k$ be any cylinder of length n, with $X_k = \varnothing$ if $n \le k$, and $|X_k| = k$ otherwise.

By the chain rule, we have

$$\psi'_{YX_k}(0) = \psi'_Y(\psi_{X_k}(0))\psi'_{X_k}(0).$$

Therefore, for any cylinder a of length 1, we have

$$\frac{s_X(a)}{\tilde{s}_X(a)} = \left|\frac{\psi'_Y(\psi_{X_k a}(0))}{\psi'_Y(\psi_{X_k}(0))}\right| \left|\frac{\tilde{\psi}'_Y(\tilde{\psi}_{X_k}(0))}{\tilde{\psi}'_Y(\tilde{\psi}_{X_k a}(0))}\right| \left|\frac{\psi'_{X_k a}(0)}{\tilde{\psi}'_{X_k a}(0)}\right| \left|\frac{\tilde{\psi}'_{X_k}(0)}{\psi'_{X_k}(0)}\right|.$$

From (4.1) in Proposition 4.1, there exists $C > 0$ such that

$$\left|\frac{\psi'_Y(\psi_{X_k a}(0))}{\psi'_Y(\psi_{X_k}(0))}\right| \le 1 + C\mathrm{diam}X_k \quad \text{and} \quad \left|\frac{\tilde{\psi}'_Y(\tilde{\psi}_{X_k}(0))}{\tilde{\psi}'_Y(\tilde{\psi}_{X_k a}(0))}\right| \le 1 + C\mathrm{diam}\tilde{X}_k.$$

In addition, from (4.7),

$$\left|\frac{\psi'_{X_k a}(0)}{\tilde{\psi}'_{X_k a}(0)}\right| \left|\frac{\tilde{\psi}'_{X_k}(0)}{\psi'_{X_k}(0)}\right| \le \left(1 + \frac{1}{\eta^{k+1}} d(\Psi, \tilde{\Psi})\right)^2.$$

Using the fact that $\mathrm{diam}X_k$ and $\mathrm{diam}\tilde{X}_k$ are both smaller than $(1-\eta)^k$, and adjusting the constant C, we end up with

$$\frac{s_X(a)}{\tilde{s}_X(a)} \le 1 + C\left((1-\eta)^k + \frac{1}{\eta^k} d(\Psi, \tilde{\Psi})\right) = 1 + \frac{C}{\eta^k}((\eta(1-\eta))^k + d(\Psi, \tilde{\Psi})\Big). \quad (4.8)$$

Let k be the integer part of $\log(d(\Psi, \tilde{\Psi}))/\log(\eta(1-\eta)) + 1$, then $k \ge \log(d(\Psi, \tilde{\Psi}))/\log(\eta(1-\eta))$ and $(\eta(1-\eta))^k \le d(\Psi, \tilde{\Psi})$, so that we have

$$\frac{s_X(a)}{\tilde{s}_X(a)} \le 1 + 2C\frac{d(\Psi, \tilde{\Psi})}{\eta^k}.$$

Alternatively, we have $k \le \log(d(\Psi, \tilde{\Psi}))/\log(\eta(1-\eta)) + 1$ so that

$$\frac{1}{\eta^k} \le \frac{1}{\eta} \exp\left(\frac{\log(d(\Psi, \tilde{\Psi}))}{\log(\eta(1-\eta))} \log\frac{1}{\eta}\right) = \frac{1}{\eta}\left(\frac{1}{d(\Psi, \tilde{\Psi})}\right)^{(\log\eta)/\log(\eta(1-\eta))},$$

which leads, with $\alpha = \log(1-\eta)/\log(\eta(1-\eta))$, to

$$\frac{s_X(a)}{\tilde{s}_X(a)} \le 1 + Cd(\Psi, \tilde{\Psi})^\alpha,$$

and gives

$$\left|\log\left(\frac{s_X(a)}{\tilde{s}_X(a)}\right)\right| \le Cd(\Psi, \tilde{\Psi})^\alpha.$$

This being true for any cylinder X, we conclude that

$$\mathcal{D}(s, \tilde{s}) \le Cd(\Psi, \tilde{\Psi})^\alpha. \qquad \square$$

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4.3. *Controlling harmonic measure.* In this section, we extend well-known properties of harmonic measure of usual IFS to NACIFSs. We have tried to give all the ideas without drowning the reader with the total amount of technicalities.

Let Ω be a domain in $\mathbb{R}^2$ and $F \subset \partial\Omega$. For $x \in \Omega$, we denote $\omega(x, F, \Omega)$ the harmonic measure of F in Ω evaluated at x. If Ω^c is bounded, we denote $\omega(F, \Omega)$ the harmonic measure of F in Ω evaluated at infinity.

We frequently make use of the following (see, for instance, [6]). If $\Omega \subset \Omega'$ are two domains, $F \subset \partial\Omega \cap \partial\Omega'$ and $x \in \Omega$, then

$$\omega(x, F, \Omega) = \omega(x, F, \Omega') - \int_{z \in \partial\Omega \cap \Omega'} \omega(z, F, \Omega')\omega(x, dz, \Omega).$$

For α, β positive real functions, we write $\alpha \overset{c}{\sim} \beta$ if $1/c\alpha \leq \beta \leq c\alpha$ with $c > 0$ constant. Finally, with no loss of generality, we take $U = \mathbb{D}$ the unit disc.

Remember that for any symbolic cylinder $X = a_1 \cdots a_n$, we note $\Psi_X = {}_1\psi_{a_1} \circ \cdots \circ {}_n\psi_{a_n}$, and that we also denote X the geometric cylinder $\mathbb{X} \cap \Psi_X(\mathbb{D})$.

In the following, we deal with two NACIFSs Ψ and $\tilde{\Psi}$, and we denote $\tilde{X} = \tilde{\mathbb{X}} \cap \tilde{\Psi}_X(\mathbb{D})$ the geometric cylinder for $\tilde{\Psi}$.

To apply the previous machinery, we need to prove the following result.

THEOREM 4.4. *Let Ψ and $\tilde{\Psi}$ be two NACIFSs, $\mathbb{X}$, $\tilde{\mathbb{X}}$ the corresponding limit (Cantor) sets, and ω, $\tilde{\omega}$ the harmonic measures on these sets.*

We have

$$\lim_{d(\Psi,\tilde{\Psi})\to 0} \mathcal{D}(\omega, \tilde{\omega}) = 0. \tag{4.9}$$

The proof follows the ideas of [3] and heavily relies on a repeated use of the following well-known results.

PROPOSITION 4.5. [8, 16] *Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \cdots \subset A_n \subset B_n \subset \Omega$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω for $1 \leq i \leq n$. If the modules of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$, then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$, and for all $x \in \Omega \setminus B_n$, we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \leq Kq^n, \tag{4.10}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the modules of the annuli.

We recall the notation introduced at the beginning of §4: $\mathbb{D}_X = \Psi_X(\mathbb{D})$ and $\mathbb{D}_{\gamma X} = \Psi_X(\mathbb{D}_\gamma)$, where $\mathbb{D}_\gamma = \{|z| \leq \gamma\}$.

LEMMA 4.6. *Let Ψ be a NACIFS, $\mathbb{X}$ the corresponding limit (Cantor) set, and ω the harmonic measures on this set. There exists $c > 0$ depending only on $1 + \boldsymbol{\eta} = \gamma > 1$ such that $\omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}) \geq c$ whenever $|x| = 1$.*

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Proof. The proof relies on standard techniques involving capacity theory and harmonic measure in the context of Cantor-type sets: we hence leave some details to the reader (see [**18**, §6] and also [**2**]). In particular, we use the fact that every point in $\mathbb{X}$ is Wiener regular so that any harmonic functions is uniquely determined by its boundary values.

Let $\Omega = \mathbb{D}_\gamma \setminus \mathbb{X}$. Recall from (1.1) that for any n, we have $2 \leq d_n \leq D$.

Let μ be the uniform probability measure on $\mathbb{X}$ so that for any cylinder $X \in C_n$, we have $\mu(X) = 1/(d_1 \cdot \cdots \cdot d_n)$.

For each $x \in \Omega$, let $G^x : \Omega \to \mathbb{R}$ denote the Green function of $\mathbb{D}_\gamma$ with pole at x. There exists C_1 and C_2, only depending on $\boldsymbol{\eta}$ such that for any $y \in \mathbb{D}$, we have

$$\ln\left(\frac{C_1}{|x-y|}\right) \leq G^x(y) \leq \ln\left(\frac{C_2}{|x-y|}\right).$$

Let $h : \mathbb{D}_\gamma \to \mathbb{R}$, $x \mapsto \int_{\mathbb{X}} G^x(y)\mu(dy)$.

For $x \in \mathbb{X}$, if (X_n), with $|X_n| = n$, is the sequence of cylinders converging to x, we have

$$h(x) = \sum_{n=1}^{+\infty} \int_{X_n \setminus X_{n+1}} G^x(y)\mu(dy). \tag{4.11}$$

For any n, we have $\mu(X_n \setminus X_{n+1}) = (d_{n+1}-1)/(d_1 \cdots d_{n+1})$, from which we deduce

$$\frac{1}{2}D^{-n} \leq \mu(X_n \setminus X_{n+1}) \leq D2^{-n}.$$

However, for $y \in X_n \setminus X_{n+1}$, we have $(1-\boldsymbol{\eta})^{-n} \leq |x-y| \leq \boldsymbol{\eta}^{-n}$. Consequently, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \leq \ln\left(\frac{C_1}{(1-\boldsymbol{\eta})^n}\right) \leq G^x(y) \leq \ln\left(\frac{C_2}{\boldsymbol{\eta}^n}\right) \leq c_2 n.$$

Injecting those inequalities in (4.11) leads to

$$0 < c := \frac{1}{2}c_1 \sum_{n=1}^{+\infty} nD^{-n} \leq h(x) \leq c_2 D \sum_{n=1}^{+\infty} n2^{-n} := C < +\infty.$$

Since the function $x \mapsto \omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus X)$ is constant and equal to 1 on $\mathbb{X}$, and since by definition $h(x) = \omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus X) = 0$ for any $x \in \partial\mathbb{D}_\gamma$, we have proved, for any $x \in \mathbb{X} \cup \partial\mathbb{D}_\gamma$, that we have

$$c\omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}) \leq h(x) \leq C\omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}).$$

By the maximum principle, those inequalities are true for any $x \in \Omega$.

Clearly, $h(x) \geq c > 0$ for $x \in \partial\mathbb{D}$ and hence the result. □

The following lemma is a first application of this result. It will subsequently be used several times to obtain 'localization' results for harmonic measure.

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LEMMA 4.7. *Let Ψ be a NACIFS and $\mathbb{X}$ its limit Cantor set. There exists $C > 0$ depending only on $1 + \boldsymbol{\eta} = \gamma$, so that for any cylinder $X \in C$ and any $x \in \partial\mathbb{D}_X$, we have*

$$1 \leq \frac{\omega(x, X, \hat{\mathbb{C}} \setminus \mathbb{X})}{\omega(x, X, \mathbb{D}_{\gamma X} \setminus X)} \leq C.$$

Proof. Let $\Gamma := \{|x| = \gamma\}$, $u : x \mapsto \omega(x, X, \hat{\mathbb{C}} \setminus \mathbb{X})$, $\tilde{u} : x \mapsto \omega(x, X, \mathbb{D}_{\gamma X} \setminus X)$, and $h = u - \tilde{u}$. The function h is harmonic on $\mathbb{D}_{\gamma X} \setminus X$, with $h = 0$ on X and $h = u$ on $\partial\mathbb{D}_{\gamma X}$ so that

$$h(x) = \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x, dz, \mathbb{D}_{\gamma X} \setminus \mathbb{X}).$$

Let $x_\gamma \in \partial\mathbb{D}_X$ be such that for any $x \in \partial\mathbb{D}_X$, $u(x) \leq u(x_\gamma)$, and $x_m \in \partial\mathbb{D}_{\gamma X}$ be such that for any $x \in \partial\mathbb{D}_{\gamma X}$, we have $u(x) \leq u(x_m)$.

The function $x \mapsto u(x) - u(x_\gamma)$ is harmonic outside $\mathbb{D}_X \cup \mathbb{X}$, non-positive on $\partial\mathbb{D}_X \cup \mathbb{X} \setminus X$, and it is thus negative on $\mathbb{C} \setminus \mathbb{D}_X$ and, in particular, we have $u(x_m) < u(x_\gamma)$.

We deduce the following:

$$\begin{aligned}\tilde{u}(x_\gamma) &= u(x_\gamma) - \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x_\gamma, dz, \mathbb{D}_{\gamma X} \setminus \mathbb{X}) \\ &\geq u(x_\gamma)(1 - \omega(x_\gamma, \partial\mathbb{D}_{\gamma X}, \mathbb{D}_X \setminus \mathbb{X})) \\ &= u(x_\gamma)\omega(x_\gamma, X, \mathbb{D}_{\gamma X} \setminus X).\end{aligned}$$

By conformal invariance, with $n = |X|$, we have

$$\omega(x_\gamma, X, \mathbb{D}_{\gamma X} \setminus \mathbb{X}) = \omega(\Phi_X(x_\gamma), \mathbb{X}_n, \mathbb{D}_\gamma \setminus \mathbb{X}_n),$$

with, by definition, $|\Phi_X(x_\gamma)| = 1$. Lemma 4.6 leads to $\tilde{u}(x_\gamma) \geq u(x_{\boldsymbol{\eta}})c$, since $\mathbb{X}_n$ is the limit Cantor set associated to a NACIFS with the same constant than $\mathbb{X}$.

We conclude that we have $1 \leq u(x_\gamma)/\tilde{u}(x_\gamma) \leq C$.

By Harnack inequality on $\partial\mathbb{D}_X$, we conclude that $1 \leq u(x)/\tilde{u}(x) \leq C$ is true for any $x \in \partial\mathbb{D}_X$, with a different C, still uniquely depending on γ. □

We also need to establish the following lemma that restrains the number of generations necessary to establish an estimate on distribution of harmonic measure.

Remember that ω_k was introduced at the beginning of §4 as the harmonic measure of $\mathbb{C} \setminus \mathbb{X}_k$.

LEMMA 4.8. *Let Ψ be a NACIFS, $\mathbb{X}$ its limit Cantor set, and for any $\ell \geq 1$, let $\mathbb{X}_\ell = T^\ell(\mathbb{X})$.*

There exist $q < 1$ and $K > 0$ depending only on $\boldsymbol{\eta}$ such that for any $XYZ \in C$ with $|X| = k$, $|Y| = n$, we have

$$\left|\log\left(\frac{\omega(XYZ)}{\omega(XY)}\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| < Kq^n. \tag{4.12}$$

The proof of this lemma follows the lines of that [3]; therefore, we will only give the highlights and main ideas of proof to facilitate the reading.

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Proof. Let $u : x \mapsto (\omega(x, XYZ, \hat{\mathbb{C}} \setminus \mathbb{X})/\omega(XYZ))$ and $v : x \mapsto (\omega(x, XY, \hat{\mathbb{C}} \setminus \mathbb{X})/\omega(XY))$. Note that $u(\infty) = v(\infty) = 1$.

Assume that $Y = y_1 \cdots y_n$ and for any $\ell \in [1, n]$, let $X_0 = X$, $X_\ell = Xy_1 \cdots y_\ell$, $A_\ell = \mathbb{D}_{X_{n-\ell}}$, and $B_\ell = \mathbb{D}_{\gamma X_{n-\ell}}$.

We have $A_0 \subset B_0 \subset \cdots \subset A_{n-1} \subset B_{n-1} \subset \hat{\mathbb{C}} \setminus \mathbb{X}$, and each annulus $B_i \setminus A_i$ is conformally equal to $\mathbb{D}_\gamma \setminus \mathbb{D}$.

The functions u and v are both harmonic in $\hat{\mathbb{C}} \setminus \mathbb{X}$, and they vanish on $\mathbb{X} \setminus XY$.

We are thus in a position to apply Proposition 4.5 to conclude that for any $x \notin \mathbb{X} \cup \mathbb{D}_{\gamma X y_1}$, and in particular for any $x \in \partial\mathbb{D}_X \cup \partial\mathbb{D}_{\gamma X}$,

$$\left|\log\left(\frac{\omega(x, XYZ, \hat{\mathbb{C}} \setminus \mathbb{X})}{\omega(x, XY, \hat{\mathbb{C}} \setminus \mathbb{X})}\frac{\omega(XY)}{\omega(XYZ)}\right)\right| = \left|\log\left(\frac{u(x)}{v(x)}\frac{v(\infty)}{u(\infty)}\right)\right| \le Cq^n. \tag{4.13}$$

Let now $\tilde{u} : x \mapsto (\omega(x, XYZ, \mathbb{D}_{\gamma X} \setminus X)/\omega(XYZ))$ and $\tilde{v} : x \mapsto (\omega(x, XY, \mathbb{D}_{\gamma X} \setminus X)/\omega(XY))$. We have for any $x \in \mathbb{D}_{\gamma X} \setminus X$,

$$\tilde{u}(x) = u(x) - \int_{\partial\mathbb{D}_{\gamma X}} u(z)\omega(x, dz, \mathbb{D}_{\gamma X} \setminus X),$$

and also

$$\tilde{v}(x) = v(x) - \int_{\partial\mathbb{D}_{\gamma X}} v(z)\omega(x, dz, \mathbb{D}_{\gamma X} \setminus X).$$

We thus have

$$\tilde{u}(x) - \tilde{v}(x) = u(x) - v(x) - \int_{\partial\mathbb{D}_{\gamma X}} (u(z) - v(z))\omega(x, dz, \mathbb{D}_{\gamma X} \setminus X),$$

which may be written as

$$\frac{\tilde{u}(x)}{\tilde{v}(x)} - 1 = \frac{v(x)}{\tilde{v}(x)}\left(\left(\frac{u(x)}{v(x)} - 1\right) - \int_{\partial\mathbb{D}_{\gamma X}} \frac{v(z)}{v(x)}\left(\frac{u(z)}{v(z)} - 1\right)\omega(x, dz, \mathbb{D}_{\gamma X} \setminus X)\right).$$

Since (4.13) is true for any $x \in \partial\mathbb{D}_X \cup \partial\mathbb{D}_{\gamma X}$, and by Lemma 4.7, we conclude that there exist $C > 0$ and $q < 1$ such that for any $x \in \partial\mathbb{D}_X$,

$$\left|\frac{\tilde{u}(x)}{\tilde{v}(x)} - 1\right| \le Cq^n\left(1 + \int_{\partial\mathbb{D}_{\gamma X}} \frac{v(z)}{v(x)}\omega(x, dz, \mathbb{D}_{\gamma X} \setminus X)\right) = Cq^n\left(1 + \frac{v(x) - \tilde{v}(x)}{v(x)}\right) \le 2Cq^n.$$

From this, we conclude that for any $x \in \partial\mathbb{D}_X$, we have

$$\left|\log\left(\frac{\omega(x, XYZ, \mathbb{D}_{\gamma X} \setminus X)}{\omega(x, XY, \mathbb{D}_{\gamma X} \setminus X)}\frac{\omega(XY)}{\omega(XYZ)}\right)\right| = \left|\log\left(\frac{\tilde{u}(x)}{\tilde{v}(x)}\right)\right| \le Cq^n. \tag{4.14}$$

By conformal invariance of harmonic measure, we conclude that for any $x \in \partial\mathbb{D}$,

$$\left|\log\left(\frac{\omega(x, YZ, \mathbb{D}_\gamma \setminus \mathbb{X}_k)}{\omega(x, Y, \mathbb{D}_\gamma \setminus \mathbb{X}_k)}\frac{\omega(XY)}{\omega(XYZ)}\right)\right| \le Cq^n.$$

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Applying (4.13) with $\mathbb{X}_k$ and ω_k, which is possible since the NACIFS giving birth to $\mathbb{X}_k$ is comparable to that giving birth to $\mathbb{X}$, we get for any $x \in \partial\mathbb{D}$,

$$\left|\log\left(\frac{\omega(x, YZ, \mathbb{D}_\gamma \setminus \mathbb{X}_k)}{\omega(x, Y, \mathbb{D}_\gamma \setminus \mathbb{X}_k)}\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \leq Cq^n. \tag{4.15}$$

Combining (4.13), (4.14), and (4.15), we end up with

$$\left|\log\left(\frac{\omega(XYZ)}{\omega(XY)}\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \leq Cq^n. \tag{4.16}$$

□

From this result, we easily deduce that harmonic measure verifies the two assumptions needed to apply the main results of the previous section: ASI and uniform control on the measure of cylinders.

COROLLARY 4.9. *Let Ψ be a NACIFS, $\mathbb{X}$ its limit set, and ω the harmonic measure on $\mathbb{C} \setminus \mathbb{X}$ evaluated at ∞. Under our geometric assumptions on the NACIFS, there exists $q \in]0, 1[$ and $C > 0$ such that for any geometric cylinders $XYZ \in C$ and $X'YZ \in C$, with $|X| = |X'|$ and $|Y| = n$,*

$$\left|\log\left(\frac{\omega_{XY}(Z)}{\omega_{X'Y}(Z)}\right)\right| \leq Cq^n, \tag{4.17}$$

$$|\log(\omega_X(Y))| \leq Cn. \tag{4.18}$$

Proof. Relation (4.16) asserts that for any $XYZ \in C$ with $|X| = k$ and $|Y| = n$,

$$\left|\log\left(\omega_{XY}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \leq Cq^n,$$

from which we get, for any $X' \in C$ with $|X'| = |X|$,

$$\begin{aligned}\left|\log\left(\frac{\omega_{XY}(Z)}{\omega_{X'Y}(Z)}\right)\right| &= \left|\log\left(\omega_{XY}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right) - \log\left(\omega_{X'Y}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \\ &\leq \left|\log\left(\omega_{XY}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| + \left|\log(\omega_{X'Y}(Z)\frac{\omega_k(Y)}{\omega_k(YZ)}\right)\right| \\ &\leq Cq^n.\end{aligned}$$

From (4.16), we deduce that there exists $C > 1$ such that

$$\frac{1}{C}\omega_k(Z) \leq \frac{\omega(XZ)}{\omega(X)} \leq 1.$$

Assume that $Z = z_0 \cdots z_{n-1}$ and for any $i \in [\![0, n-1]\!]$, let $Z_i = z_0 \cdots z_i$, then

$$\omega_k(Z) = \omega_k(z_0)\prod_{i=1}^{n-1}\frac{\omega_k(Z_i)}{\omega_k(Z_{i-1})},$$

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since for any $i \in [\![1, n-1]\!]$, (4.16) tells us that $1/C\omega_{k+i}(z_i) \leq (\omega_k(Z_i)/\omega_k(Z_{i-1})) \leq 1$, and we conclude that

$$\frac{1}{C^n} \prod_{i=1}^{n-1} \omega_{k+i}(z_i) \leq \frac{\omega(XZ)}{\omega(X)} \leq 1.$$

From Lemma 4.6, we can deduce that given a NACIFS Ψ, there exists a constant $\alpha > 0$, depending only the geometric constants that constrain the geometry of the NACIFS such that for any $\ell \geq 0$ and any $a \in C_\ell$ with $|a| = 1$, we have $\omega_\ell(a) \geq \alpha$, so we conclude that

$$\left(\frac{\alpha}{C}\right)^n \leq \frac{\omega(XZ)}{\omega(X)} \leq 1,$$

and the result follows. □

4.4. *Controlling harmonic measure under perturbation.* We turn to a comparison of harmonic measures of the limit sets $\mathbb{X}$ and $\tilde{\mathbb{X}}$.

LEMMA 4.10. *Let $\gamma = 1 + \boldsymbol{\eta} > 1$ such that $\mathbb{D}_\gamma \subset V$ appearing in the definition of the NACIFS Ψ. There exist $K > 0$ and $\alpha > 0$ such that for any $x \in \mathbb{D} \setminus \mathbb{X}$,*

$$\omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}) \geq 1 - K\, dist(x, \mathbb{X})^\alpha.$$

Proof. Given any $x \in \mathbb{D} \setminus \mathbb{X}$, let $n(x) = \max\{k \in \mathbb{N} \mid \text{there exists } X \in C_n,\ x \in \mathbb{D}_X\}$. Note that $n(x)$ is such that there exists $X \in C_n$ with $x \in \mathbb{D}_X$ and $x \notin \mathbb{D}_{Xa}$ for any cylinder $Xa \in C_{n+1}$.

Using open set and bounded contraction hypothesis, we deduce from the Kœbe distortion theorem that there exist three constants $K' > 0$ and $0 < \beta' < \beta$ such that

$$\frac{1}{K'} \operatorname{dist}(x, \mathbb{X})^\beta \leq e^{-n} \leq K' \operatorname{dist}(x, \mathbb{X})^{\beta'}. \tag{4.19}$$

It is important to note that K', β', and β only depend on the constant $\boldsymbol{\eta}$ used to control the geometry of the NACIFS.

Let $x \in \mathbb{D}_X$, with $X = a_1 \cdots a_n$ and $|X| = n(x) := n$. Let $X_0 = \mathbb{X}$ and, for any $1 \leq k \leq n$, $X_k = a_1 \cdots a_k$ and set $u_k : y \mapsto \omega(y, X_k, \mathbb{D}_{\gamma X_k} \setminus X_k)$. The function u_k is harmonic on $\mathbb{D}_{\gamma X_k} \setminus X_k$ and equal 1 on X_k. Our goal is to prove that

$$u_0(x) = \omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}) \geq 1 - K \operatorname{dist}(x, \mathbb{X})^\alpha.$$

First, we note using Lemma 4.6, conformal invariance, and maximum principle that there exists $c > 0$ such that for any $x \in \mathbb{D}$,

$$u_k(x) \geq c. \tag{4.20}$$

For any $1 \leq k \leq n$, we have

$$u_{k-1}(x) - u_k(x) = \int_{\partial \mathbb{D}_{\gamma X_k}} u_{k-1}(z)\omega(x, dz, \mathbb{D}_{\gamma X_k} \setminus X_k).$$

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Because of our geometric assumptions on the NACIFS and Harnack inequalities, there exists a constant $\lambda > 0$ such that for any $Y \in C$, any continuous function $h > 0$, defined on $\mathbb{D}_Y$ and harmonic on $\mathbb{D}_Y \setminus Y$, and any y and y' in $\mathbb{D}_Y \setminus Y$, $\lambda h(y) \leq h(y')$.

From this, we deduce that

$$u_{k-1}(x) - u_k(x) \geq \lambda u_{k-1}(x)\omega(x, \partial\mathbb{D}_{\gamma X_k}, D_{\gamma X_k} \setminus X_k) = \lambda u_{k-1}(x)(1 - u_k(x)).$$

Let $v_k = 1 - u_k$. We drop x to lighten notation.

Using (4.20), we get $\lambda c v_k \leq \lambda u_k v_k \leq v_k - v_{k-1}$ and

$$c\lambda \leq \frac{v_k - v_{k-1}}{v_k} \leq \int_{v_{k-1}}^{v_k} \frac{1}{t}\, dt = \log(v_k) - \log(v_{k-1}).$$

Summing for $1 \leq k \leq n$,

$$c\lambda n \leq \log v_n - \log v_0,$$

so that

$$1 - u_0 = v_0 \leq e^{-c\lambda n}.$$

We use (4.19) to conclude that $u_0(x) = \omega(x, \mathbb{X}, \mathbb{D}_\gamma \setminus \mathbb{X}) \geq 1 - (1/K' \operatorname{dist}(x, \mathbb{X})^\beta)^{c\lambda}$. This is the desired result with $K = 1/K'^{c\lambda}$ and $\alpha = c\lambda\beta$.

This corollary happens to be crucial to prove continuity of dimensions.

COROLLARY 4.11. *Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that for any $XY \in C$ with $|Y| = n$ and any $x \in \mathbb{D}_{XY}$,*

$$\omega(x, X, \mathbb{C} \setminus \mathbb{X}) \geq 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$.

With $K > 0$ and $\alpha > 0$ appearing in Lemma 4.10, let λ be such that $K\lambda^\alpha < \varepsilon$.

Let n big enough so that for any $k \in \mathbb{N}$ and any $Y \in C$, with $|Y| = n$, we have $\operatorname{diam} \mathbb{D}_{\gamma Y} < \lambda$. Let $X \in C$ and $x \in \partial\mathbb{D}_{XY}$, with $|Y| = n$. We have

$$\omega(x, X, \mathbb{C} \setminus \mathbb{X}) \geq \omega(x, X, \mathbb{D}_{\gamma X} \setminus \mathbb{X}).$$

By conformal invariance, with $k = |X|$, we have

$$\omega(x, X, \mathbb{D}_{\gamma X} \setminus \mathbb{X}) = \omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \setminus \mathbb{X}_k).$$

Since $x \in \partial\mathbb{D}_{XY}$, we have $\varphi_X(x) \in \partial\mathbb{D}_Y$, so that $\operatorname{dist}(\varphi_X(x), \mathbb{X}_k) \leq \operatorname{diam} \mathbb{D}_{\gamma Y} < \lambda$ and Lemma 4.10 gives $\omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \setminus \mathbb{X}_k) > 1 - \varepsilon$.

We have proved that for any $x \in \partial\mathbb{D}_{XY}$, we have

$$\omega(x, X, \mathbb{C} \setminus \mathbb{X}) \geq \omega(x, X, \mathbb{D}_{\gamma XY} \setminus \mathbb{X}) = \omega_k(\varphi_X(x), \mathbb{X}_k, \mathbb{D}_\gamma \setminus \mathbb{X}_k) \geq 1 - \varepsilon.$$

This inequality extends to $\mathbb{D}_{XY}$ by maximum principle.

Proof of Theorem 4.4. Let $\varepsilon > 0$ and $N_1 \in \mathbb{N}$ be big enough so that Kq^{N_1} in Corollary 4.8 is strictly less than ε.

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We use this first assumption to show that the proof of continuity may be restricted to cylinders of length less than or equal to N_1. More precisely, let $X \in C_n$, with $n > N_1$. One may write $X = YZ$, with $|Y| = k > N_1$ and $|Z| = \ell < N_1$. Let a, with $|a| = 1$, we have

$$\omega_X(a) = \frac{\omega(YZa)}{\omega(YZ)} = \frac{\omega_k(Za)}{\omega_k(Z)} \frac{\omega(YZa)\omega_k(Z)}{\omega(YZ)\omega_k(Za)}.$$

The choice made for N_1, together with Proposition 4.8, allows us to conclude that

$$\left| \log \left(\frac{\omega(YZa)\omega_k(Z)}{\omega(YZ)\omega_k(Za)} \right) \right| \leq \varepsilon.$$

The same being true for $\tilde{\omega}$, we deduce that

$$\left| \log \left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)} \right) \right| \leq \left| \log \left(\frac{\omega_k(Za)}{\tilde{\omega}_k(Za)} \right) \right| + \left| \log \left(\frac{\omega_k(Z)}{\tilde{\omega}_k(Z)} \right) \right| + 2\varepsilon. \tag{4.21}$$

From Corollary 4.9, we know that $cq^{N_1} \leq \omega_k(Z)$ for any $k \in \mathbb{N}$, any $Z \in T^k(C)$ with $|Z| \leq N_1$. Let $\varepsilon' = cq^{N_1}\varepsilon$. For any $k \in \mathbb{N}$, any $Z \in T^k(C)$ with $|Z| \leq N_1$, we have

$$\frac{\varepsilon'}{\omega(X)} \leq \varepsilon. \tag{4.22}$$

Let N_2 be given by Corollary 4.11 with $\varepsilon' > 0$ so that for any $XY \in C$ with $|Y| = N_2$ and any $x \in \partial\mathbb{D}_{XY}$, we have $\omega(x, X, \mathbb{C} \setminus \mathbb{X}) \geq 1 - \varepsilon'$.

Let now $d(\Psi, \tilde{\Psi})$ be as small as necessary to ensure that for any $k \in \mathbb{N}$, any $Y \in T^k(C)$ with $|Y| \leq N_1 + N_2$, we have $\tilde{Y} \subset \mathbb{D}_Y$, where $\tilde{Y}$ is the geometric cylinder associated to $\tilde{\Psi}$ with coding Y. This means that, geometrically speaking, cylinders of length less than or equal to $N_1 + N_2$ are very close to one another.

By (4.21), we are led to prove that $\log(\omega_k(X)/\tilde{\omega}_k(X))$ is small for any cylinder X with $|X| \leq N_1$ and any k.

To simplify the notation, we omit the index k. Recall that our estimates for the harmonic measure depend only on the geometric constants appearing in our assumptions, namely, the Bounded Contraction and Annulus conditions, and are therefore uniform in k.

Let $|X| = k \leq N_1$, $F = \bigcup_{|Z|=N_1+N_2} \mathbb{D}_Z$, and $\Omega = \mathbb{C} \setminus F$. Our choices imply that for any Z with $|Z| = N_1 + N_2$, we have $\tilde{Z} \subset \mathbb{D}_Z$ so that $\tilde{\mathbb{X}} \subset F$.

The functions $u : x \mapsto \omega(x, X, \mathbb{C} \setminus \mathbb{X})$ and $v : x \mapsto \omega(x, \tilde{X}, \mathbb{C} \setminus \tilde{\mathbb{X}})$ are harmonic on Ω.

Let $|Z| = N_1 + N_2$ with $Z \subset X$, then from Corollary 4.11, we deduce, for any $x \in \partial_Z$, that $u(x) \geq 1 - \varepsilon$ as well as $v(x) \geq 1 - \varepsilon'$, which implies that $|u(x) - v(x)| \leq \varepsilon'$ for any $x \in \partial_Z$.

Let now $Z = YW$, with $|Y| = |X|$, $|Z| = N_1 + N_2$, and $Z \not\subset X$. By Corollary 4.11, we have $\omega(x, Y, \mathbb{C} \setminus \mathbb{X}) > 1 - \varepsilon'$ and also $\omega(x, \tilde{Y}, \mathbb{C} \setminus \tilde{\mathbb{X}}) > 1 - \varepsilon'$ for any $x \in \partial\mathbb{D}_Z$.

Since

$$1 = \sum_{|V|=|X|} \omega(x, V, \mathbb{C} \setminus \mathbb{X}) = \sum_{|V|=|X|} \omega(x, \tilde{V}, \mathbb{C} \setminus \tilde{\mathbb{X}}),$$

we deduce that $u(x) \leq 1 - \omega(x, Y, \mathbb{C} \setminus \mathbb{X}) \leq \varepsilon'$ and $v(x) \leq 1 - \omega(x, \tilde{Y}, \mathbb{C} \setminus \tilde{\mathbb{X}}) \leq \varepsilon'$. We thus have, for any $x \in \partial\mathbb{D}_Z$, $|u(x) - v(x)| \leq 2\varepsilon'$.

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The functions u and v are both harmonic on Ω and for any $x \in \partial\Omega$, we have $|u(x) - v(x)| \leq 2\varepsilon'$. By maximum principle, we have $|u(x) - v(x)| \leq 2\varepsilon'$ for any $x \in \Omega$ and in particular for $x = \infty$, so that $|\omega(X) - \tilde{\omega}(X)| \leq 2\varepsilon'$.

We deduce from (4.22) that for any cylinder X with $|X| \leq N_1$, we have

$$\left|\frac{\tilde{\omega}(X)}{\omega(X)} - 1\right| \leq 2\varepsilon' \frac{1}{\omega(X)} \leq 2\varepsilon.$$

From here, it is easy to deduce that we have, for such cylinders,

$$\left|\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right)\right| = \left|\log\left(\frac{\omega(Xa)}{\tilde{\omega}(Xa)}\right) + \log\left(\frac{\tilde{\omega}(X)}{\omega(X)}\right)\right| \leq 4\varepsilon.$$

With inequality (4.21), we can conclude that for any cylinder X of any length, we have

$$\left|\log\left(\frac{\omega_X(a)}{\tilde{\omega}_X(a)}\right)\right| \leq 6\varepsilon,$$

which tells us that $\mathcal{D}(\omega, \tilde{\omega}) \leq 6\varepsilon$ and concludes the proof. □

4.5. *Back to the complex plane.* We are now in a position to give a proof of the main results of this paper about harmonic measure on the plane: Theorems 1.2 and 1.3.

Let $\mathbb{X}$ be a non-autonomous Cantor set associated with the conformal iterated systems Ψ and let ω be the harmonic measures of its complementary in $\mathbb{C}$.

We defined on the symbolic space $\mathbb{K}$ the metric d. For any $\mathtt{a} = (a_n) \in \mathbb{K}$ and $\mathtt{b} = (b_n) \in \mathbb{K}$, let $d(\mathtt{a}, \mathtt{b}) = |x(\mathtt{a}) - x(\mathtt{b})|$, which is the Euclidian distance in $\mathbb{C}$ of the points encoded by $\mathtt{a}$ and $\mathtt{b}$. In that way, symbolic cylinders and geometric cylinders have the same diameters.

From the discussions in §4.1, we easily get

$$\frac{|XY|_{\mathtt{d}}}{|X|_{\mathtt{d}}} \sim \frac{|\Psi'_{XY}(0)|}{|\Psi'_X(0)|} = \frac{|\Psi'_X(\Psi_Y(0))|}{|\Psi'_X(0)|}|\Psi'_Y(0)| \sim |\Psi'_Y(0)|.$$

Hypothesis bounded contraction implies that there exists a constant $\eta \in]0, 1[$ such that $(1 - \eta)^{|Y|} \leq |\Psi'_Y(0)| \leq \eta^{|Y|}$.

In addition, we can conclude that d satisfies condition (3.2).

By Corollary 4.2, we also know that $s(X) := |X|_{\mathtt{d}}$ satisfies ASI with sequence $(\beta_n = Cq^n)$ with $0 < q < 1$.

Corollary 4.9 expresses the fact that ω also fulfills ASI with sequence $(\beta_n = Cq^n)$, with $0 < q < 1$, with an adjustment of the constant q, as well as condition (2.5.1) with $\gamma_n \sim Cn$.

We are thus in a position to apply Theorem 3.5 with $\mu = \omega$ to conclude that Theorem 1.3 is true.

Let now $(\mathbb{X}^k)_{k\in\mathbb{N}}$ be a sequence of non-autonomous Cantor sets associated with the conformal iterated systems $(\Psi^k)_{k\in\mathbb{N}}$, and let $(\omega^k)_{k\in\mathbb{N}}$ be the harmonic measures of their complementaries. Assume that $\lim_k d(\Psi, \Psi^k) = 0$.

We define for any k a metric d_k such that geometric cylinders for Ψ^k and symbolic cylinders for d_k have the same diameter.

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We know that d_k and ω^k fulfill the same properties as d and ω, with the same constants, because of our hypothesis: annulus condition and bounded contraction, and the fact that (Ψ^k) is converging towards Ψ.

Let $s_k(X) = |X|_{\mathsf{d}_k}$. Since $\lim_k d(\Psi, \Psi^k) = 0$, by Proposition 4.3, we know that $\lim_k \mathcal{D}(s, s_k) = 0$, and by Theorem 4.4 that $\lim_k \mathcal{D}(\omega, \omega^k) = 0$. We may thus apply Theorem 3.6 to conclude for the continuity of the dimensions of harmonic measure.

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