J. Plasma Physics (2008), *vol.* 74, *part* 2, *pp.* 207–214. © 2007 Cambridge University Press
doi:10.1017/S0022377807006708 First published online 26 July 2007 Printed in the United Kingdom

Rankine–Hugoniot relations for shocks with demagnetized ions

M. GEDALIN[1] and M. BALIKHIN[2]

[1] Department of Physics, Ben-Gurion University, Beer-Sheva, Israel
(gedalin@bgu.ac.il)

[2] ACSE, University of Sheffield, Sheffield, UK

(Received 26 February 2007 and in revised form 21 May 2007)

Abstract. The width of a quasi-perpendicular collisionless shock front is smaller than the convective ion gyroradius so that ions become demagnetized in the ramp. An approach is proposed for derivation of approximate expressions for the magnetic compression ratio and cross-shock potential from the analysis of the ion motion across the ramp and pressure balance condition, without making assumptions about the ion equation of state. The cross-shock potential and magnetic compression ratio are found as functions of the Mach number for low-Mach-number perpendicular shocks.

1. Introduction

Rankine–Hugoniot (RH) relations (Tidman and Krall 1971) relate the upstream shock parameters (Mach number, angle between the shock normal and upstream magnetic field, kinetic-to-magnetic pressure ratio) to the corresponding downstream parameters. The traditional approach is to apply the mass, momentum, and energy conservation within the magneto-hydrodynamical (single fluid) approximation, adopting phenomenologically a kind of state equation (usually polytropic). Such an approach has proved to be very useful in estimating the far downstream compression, yet it suffers from a lack of justification, it cannot be applied to the transition layer itself, and it does not allow one to estimate the cross-shock potential. The latter plays an important role in ion reflection and electron heating (Moiseev and Sagdeev 1963; Sagdeev 1966; Forslund and Shonk 1970; Forslund and Freidberg 1971; Schamel 1972; Leroy 1983; Sckopke et al. 1983; Goodrich and Scudder 1984; Schwartz 1987; Balikhin et al. 1993; Gedalin 1996). We present here a direct derivation of the approximate relations for the magnetic compression ratio and cross-shock potential, based on the non-adiabaticity of the ion motion in the ramp. For simplicity, the derivation is carried out for perpendicular low-β shock conditions without ion reflection taken into account, yet the basic principles also remain valid for high-Mach-number shocks. Corresponding modifications will be considered elsewhere.

2. Pressure balance with demagnetized ions

Observations of the heliospheric shocks, especially the Earth bow shock, show that the quasi-perpendicular shock front is well structured (Scudder et al. 1986), the

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

thinnest part being the ramp where the main magnetic field increase occurs, most of the cross-shock potential is applied, and most of the ion deceleration happens. The ramp width is substantially smaller than the convective ion gyroradius (Balikhin et al. 1995) and becomes thinner with the increase of the Mach number (Newbury et al. 1998). Owing to the narrowness of the ramp, the ion deflection inside it is weak (Gedalin 1997) and can be neglected in the lowest order of approximation for the bulk of the ion distribution (although this deflection has to be taken into account for the analysis of ion reflection (Gedalin 1996)). Therefore, we may consider the ramp as a discontinuity for ions where the only effect on the ion motion is produced by the cross-shock electric potential ϕ. If the incident ion distribution is not too hot, the ions form a quasi-monoenergetic gyrating beam behind the ramp. On the other hand, electrons are considered as completely magnetized and experiencing $\mathbf{E}\times\mathbf{B}$ drift in the ramp, thus producing the necessary current. The pressure balance is maintained mainly by ions. Having outlined the approach, we shall proceed step by step. Let the upstream plasma velocity and density be $v_{\rm u}$ and $n_{\rm u}$, respectively. We define the Mach number as follows: $M = v_{\rm u}/v_{\rm A}$ where $v_{\rm A} = B_{\rm u}/\sqrt{4\pi n_{\rm u} m_{\rm i}}$ is the upstream Alfvén velocity. The upstream ion and electron pressures are $p_{\rm e,i} \equiv (\beta_{\rm e,i}/2M^2) n_{\rm u} m_{\rm i} v_{\rm u}^2$. Here we consider the case of low kinetic pressure $\beta \ll 1$.

In the ramp of a stationary one-dimensional perpendicular shock the fields are $B_z = B(x)$, $E_y =$ constant, and $E_x = -d\phi(x)/dx$. We follow the approach by Gedalin (1997). An ion with incident (upstream) velocity v_x, v_y leaves the ramp with a velocity

$$v_x' = \sqrt{v_x^2 - 2e\phi/m_{\rm i}}, \quad v_y' = v_y, \tag{1}$$

where we have neglected the ion deflection in the y-direction, and ϕ is the cross-ramp potential. For $\beta_{\rm i} \ll 1$ the incident ion distribution is almost a cold beam with $v_x = v_{\rm u}(1+w_x)$, $v_y = v_{\rm u} w_y$, $|\sqrt{w_x^2 + w_y^2}| \ll 1$, so that (Gedalin 1997)

$$v_x' = \sqrt{v_x^2 - 2e\phi/m_{\rm i}} \approx v_{\rm u}\sqrt{1-s}\left[1 + \frac{w_x}{1-s} - \frac{s w_x^2}{2(1-s)^2}\right], \tag{2}$$

and $v_y' = v_{\rm u} w_y$, where $s = 2e\phi/m_{\rm i} v_{\rm u}^2$. The ion density upon exit from the ramp is obtained as

$$n' = n_{\rm u}\int |v_x/v_x'|\, dv_x\, dv_y = (1-s)^{-1/2}\left[1 + \frac{3s}{2(1-s)^2}\frac{\beta_{\rm i}}{2M^2}\right]. \tag{3}$$

Since quasi-neutrality is maintained throughout, electrons should have the same density. The pressure balance condition reads

$$T_{xx} = \sum_{\rm e,i}\langle p_x v_x\rangle + \frac{B^2}{8\pi} = \text{constant} \tag{4}$$

where $\langle\cdots\rangle$ denotes proper averaging multiplied by the density for ions and electrons as well. The pressure tensor $T_{xx} = \langle p_x v_x\rangle$ includes the ram pressure and the kinetic pressure of the species. Assuming that the ramp width is substantially large for electrons to be adiabatic, we have approximately $T_{\rm e} \propto B$ and $p_{\rm e} \propto nB$. For the incident plasma,

$$T_{xx,{\rm u}} = n_{\rm u} m_{\rm i} v_{\rm u}^2 + p_{\rm e} + p_{\rm i} = n_{\rm u} m_{\rm i} v_{\rm u}^2\left(1 + \frac{\beta_{\rm e} + \beta_{\rm i}}{2M^2}\right). \tag{5}$$

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

Upon exit from the ramp

$$T_{xx,\mathrm{i}} = n_{\mathrm{u}} m_{\mathrm{i}} \int |v_x' v_x| \, dv_x \, dv_y = n_{\mathrm{u}} m_{\mathrm{i}} v_{\mathrm{u}}^2 \sqrt{1-s} \left[1 + \frac{\beta_{\mathrm{i}}(2-3s)}{2M^2(1-s)^2} \right], \tag{6}$$

and eventually the pressure balance equation takes the form

$$1 + \frac{1+\beta_{\mathrm{e}}+\beta_{\mathrm{i}}}{2M^2} = T_{xx,\mathrm{i}} + \frac{(n'/n_{\mathrm{u}})\beta_{\mathrm{e}} R}{2M^2} + \frac{R^2}{2M^2}, \tag{7}$$

where R is the magnetic compression ratio. This expression provides the relation between the cross-shock (cross-ramp) potential and the magnetic compression at the ramp. The latter does not always have to be equal to the far downstream compression ratio since shocks often exhibit overshoots at the downstream edge of the ramp (Mellott and Livesey 1987). However, in the derivation above we have not taken into account the reflected ions which play an important role in high-Mach-number shocks, so that certain modifications should be done to (7) before applying it to high-Mach-number shocks. Thus, in what follows R is also the magnetic compression downstream.

We describe the region behind the ramp with a uniform magnetic field $B_{\mathrm{d}} = RB_{\mathrm{u}}$. An ion drifts and gyrates (Gedalin 1997),

$$V_x = v_{\mathrm{d}} + v_\perp \cos(\Omega_{\mathrm{d}} t), \tag{8}$$

$$V_y = -v_\perp \sin(\Omega_{\mathrm{d}} t), \tag{9}$$

$$x = v_{\mathrm{d}} t + r_{\mathrm{g}} \sin(\Omega_{\mathrm{d}} t), \tag{10}$$

$$v_{\mathrm{d}} = \frac{v_{\mathrm{u}}}{R}, \quad \Omega_{\mathrm{d}} = \frac{eB_{\mathrm{d}}}{m_{\mathrm{i}} c} = R\Omega_{\mathrm{u}}, \quad r_{\mathrm{g}} = \frac{v_\perp}{\Omega_{\mathrm{d}}}, \tag{11}$$

$$v_\perp = \sqrt{(v_x' - v_{\mathrm{d}})^2 + v_y'^{\,2}}, \tag{12}$$

where t is time for a single ion motion and serves as a parameter for the parametric representation of the beam. We choose the initial x and phase arbitrarily since we are going to spatially average T_{xx} assuming gyrophase bunching well behind the ramp.

It is easy to derive

$$T_{xx,\mathrm{i}} = \langle p_x v_x \rangle = \int n_{\mathrm{u}} m_{\mathrm{i}} v_x \langle |v_{\mathrm{d}} + v_\perp \cos(\Omega_{\mathrm{d}} t)| \rangle_x \, dv_x \, dv_x, \tag{13}$$

where the spatial averaging is substituted by temporal averaging as follows:

$$\langle |v_{\mathrm{d}} + v_\perp \cos(\Omega_{\mathrm{d}} t)| \rangle_x = \langle |v_{\mathrm{d}} + v_\perp \cos(\Omega_{\mathrm{d}} t)| |V_x| / v_{\mathrm{d}} \rangle_t. \tag{14}$$

The last average is easily evaluated and eventually one has

$$\frac{T_{xx,\mathrm{i}}}{n_{\mathrm{u}} m_{\mathrm{i}} v_{\mathrm{u}}^2} = \frac{1}{R} + \frac{R}{2} \int (1 + w_x) U_\perp^2 \, dw_x \, dw_y, \tag{15}$$

where

$$U_\perp^2 = \left\{ \sqrt{1-s} \left[1 + \frac{w_x}{1-s} - \frac{s w_x^2}{2(1-s)^2} \right] - 1/R \right\}^2 + w_y^2. \tag{16}$$

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

Expanding up to w^2 one has

$$\frac{T_{xx,\mathrm{id}}}{n_\mathrm{u}m_\mathrm{i}v_\mathrm{u}^2}=\frac{1}{R}+\frac{R}{2}\left(\sqrt{1-s}-\frac{1}{R}\right)^2 + \frac{\beta_\mathrm{i}R}{4M^2}\left[\frac{6-5s}{2(1-s)}-\frac{4-5s}{2R(1-s)^{3/2}}+1\right] \tag{17}$$

The downstream electron pressure is $p_{\mathrm{e,d}}=n_\mathrm{d}p_{\mathrm{e,u}}R$, or

$$\frac{p_{\mathrm{e,d}}}{n_\mathrm{u}m_\mathrm{i}v_\mathrm{u}^2}=\frac{R^2\beta_\mathrm{e}}{2M^2}. \tag{18}$$

The pressure balance requires

$$1+\frac{1+\beta_\mathrm{e}+\beta_\mathrm{i}}{2M^2}=\frac{T_{xx,\mathrm{id}}}{n_\mathrm{u}m_\mathrm{i}v_\mathrm{u}^2}+\frac{R^2\beta_\mathrm{e}}{2M^2}+\frac{R^2}{2M^2}. \tag{19}$$

Equations (7) and (19) allow one, in principle, to find R and s as functions of the Mach number M. Here we shall analyze the simplest version of these equations for $\beta_\mathrm{i}=0$, when they are reduced to

$$1+\frac{1+\beta_\mathrm{e}}{2M^2}=\sqrt{1-s}+\frac{\beta_\mathrm{e}R}{2M^2\sqrt{1-s}}+\frac{R^2}{2M^2}, \tag{20}$$

$$1+\frac{1+\beta_\mathrm{e}}{2M^2}=\frac{1}{R}+\frac{R}{2}\left(\sqrt{1-s}-\frac{1}{R}\right)^2+\frac{\beta_\mathrm{e}R^2}{2M^2}+\frac{R^2}{2M^2}. \tag{21}$$

The obvious solution $\sqrt{1-s}=1/R$ corresponds to the cold drifting downstream ion beam.

It is easy to derive the other solution in the form

$$\sqrt{1-s}=2+\frac{1+\beta_\mathrm{e}-R^2}{M^2}-\frac{3}{R}, \tag{22}$$

$$2(1+\beta_\mathrm{e}-R^2)^2Y^2+\left[\left(4-\frac{9}{R}\right)(1+\beta_\mathrm{e}-R^2)+\beta_\mathrm{e}R\right]Y+\left(1-\frac{3}{R}\right)\left(2-\frac{3}{R}\right)=0, \tag{23}$$

where $Y=1/2M^2$. Figure 1 shows M versus R, M versus s, and R versus s for $\beta_\mathrm{e}=0.1$. There are two branches of solutions, shown by diamonds and pluses, respectively. The low-Mach-number branch has higher cross-shock potentials. There are no solutions for Mach numbers that are too low, which is reasonable since the lower the shock Mach number, the wider the shock, and the approximation of ion deceleration by the cross-shock potential only becomes invalid. For high Mach numbers and compression ratios we expect the appearance of reflected ions, which change the picture. Therefore, the present analysis is, most probably, not applicable to the higher-Mach-number branch. The lower-Mach-number branch (diamonds) is characterized by rather high values of the cross-shock potential, which decreases with increasing Mach number and increasing compression ratio.

Figure 2 shows M versus R, M versus s, and R versus s for $\beta_\mathrm{e}=0.4$. This value of the upstream β_e together with the assumption of cold ions is unrealistic and is used only to illustrate the influence of electron heating on the shock structure. It is seen

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

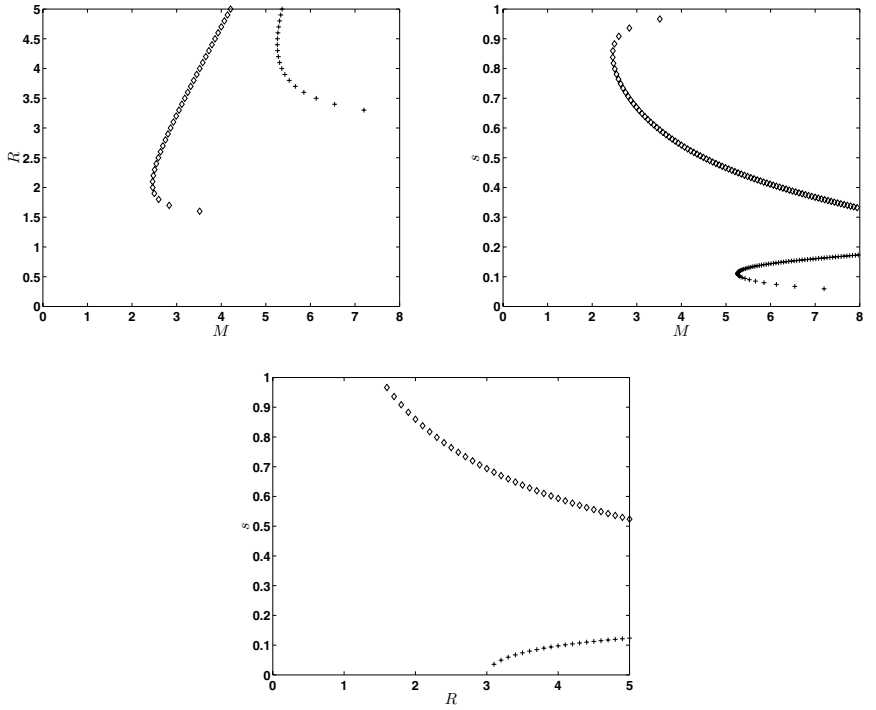


Figure 1. Relation between the Mach number, magnetic compression, and cross-shock potential for $\beta_e = 0.1$. The two branches are shown by diamonds and pluses, respectively.

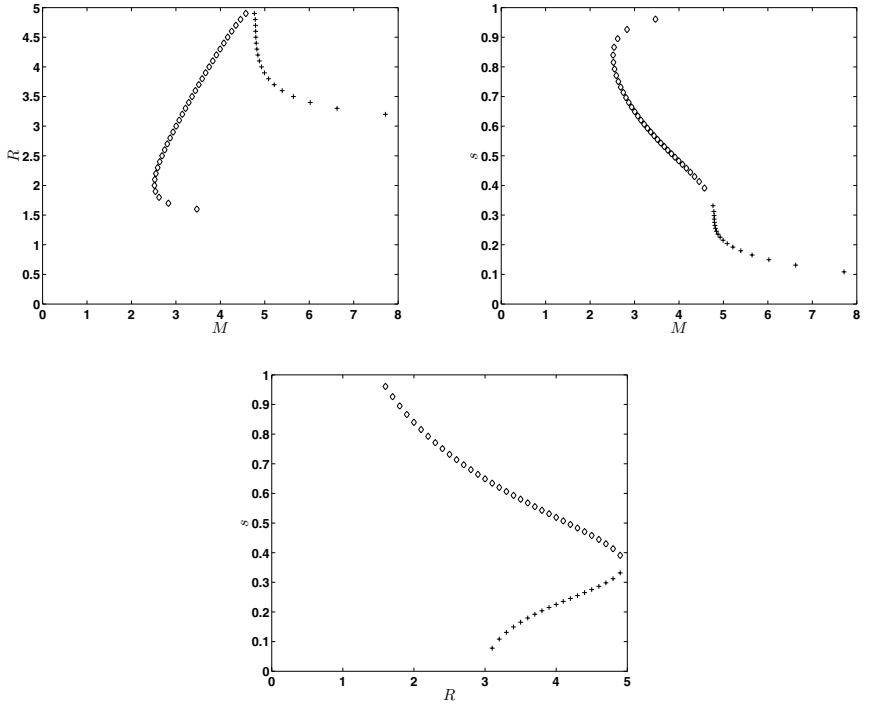


Figure 2. Relation between the Mach number, magnetic compression, and cross-shock potential for $\beta_e = 0.4$. The two branches are shown by diamonds and pluses, respectively.

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

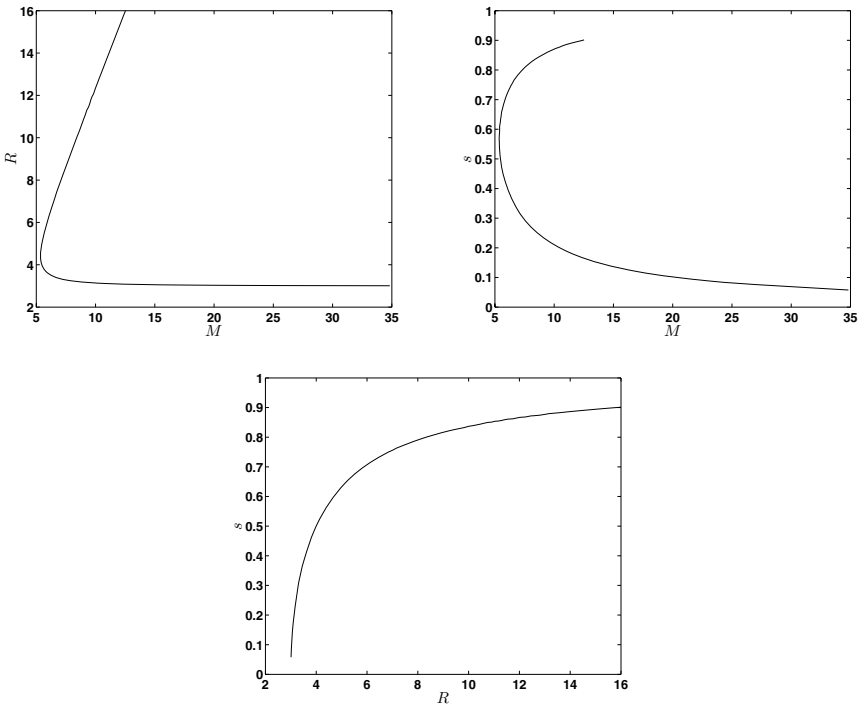


Figure 3. Relation between the Mach number, magnetic compression, and cross-shock potential for $\beta_e = 0$. The two branches are shown by diamonds and pluses, respectively.

that the maximum magnetic compression is reduced and the lower-Mach-number branch extends only up to $M \approx 4$.

In the simplest case $\beta_e = 0$, the solution reduces to the following very simple form:

$$\sqrt{1-s} = \frac{3}{R}, \tag{24}$$

$$2M^2 = \frac{R(R^2-1)}{R-3}. \tag{25}$$

The corresponding relations are shown in Figure 3. The magnetic compression ratio is now limited from below by $R_{\min} = 3$, since cold electrons cannot be responsible for a sufficiently large fraction of the downstream pressure and the magnetic field has to assume the role of decelerated ions. As can be seen, such a transition cannot exist for $M < 5$, which emphasizes the role of electron heating in the shock formation.

3. Discussion and conclusions

The above analysis is, in its present form, inapplicable to shocks where ion reflection begins to play a noticeable role. More general analysis with inclusion of reflected ions will be presented elsewhere. Here we only briefly outline the general principles that were proposed and these should remain unchanged even in high-Mach-number shocks. A reflected ion crosses the ramp three times, each time approximately

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

satisfying (1). Since the motion is collisionless the phase-space density is conserved along the trajectory, which allows one to estimate the contribution to the density and pressure in a way similar to what has been done here (see also Gedalin 1997). Thus, inclusion of reflected ions at the ramp is straightforward (although somewhat more difficult technically). Similarly, reflected ions gyrate downstream as a high-energy beam and the contribution to the downstream pressure can be calculated in the same style as has been done in (13)–(15). Inclusion of reflected ions in the RH relations (within the above approach), as well as a dissipation mechanism for high-Mach-number shocks (Sagdeev 1966) is still an open problem.

To summarize, in the present paper we have derived, from first principles, the RH relations for a low-Mach-number quasi-perpendicular shock, establishing the relation between the downstream magnetic compression, cross-shock potential, and ion heating. The derivation assumes that the shock width is substantially smaller than the ion gyroradius, this assumption being supported by observations in space plasmas (Balikhin et al. 1995). We have been able to establish the relation between the Mach number and compression ratio without invoking a phenomenological polytropic state equation (Tidman and Krall 1971). The dependencies obtained may be applied to the observed bow shock crossings. The analysis has shown that there is no one-to-one correspondence between the Mach number and downstream state. It also shows the importance of electron heating for the shock structure.

References

Balikhin, M., Gedalin, M. and Petrukovich, A. 1993 New mechanism for electron heating in shocks. *Phys. Rev. Lett.* **70**, 1259–1262.

Balikhin, M., Krasnosselskikh, V. and Gedalin, M. 1995 The scales in quasiperpendicular shocks. *Adv. Space Res.* **15**, 247–260.

Forslund, D. W. and Freidberg, J. P. 1971 Theory of laminar collisionless shocks. *Phys. Rev. Lett.* **27**, 1189–1192.

Forslund, D. W. and Shonk, C. R. 1970 Formation and structure of electrostatic collisionless shocks. *Phys. Rev. Lett.* **25**, 1699–1702.

Gedalin, M. 1996 Ion reflection at the shock front revisited. *J. Geophys. Res.* **101**, 4871–4878.

Gedalin, M. 1997 Ion heating in oblique low-Mach number shocks. *Geophys. Res. Lett.* **24**, 2511–2514.

Goodrich, C. C. and Scudder, J. D. 1984 The adiabatic energy change of plasma electrons and the frame dependence of the cross-shock potential at collisionless magnetosonic shock waves. *J. Geophys. Res.* **89**, 6654–6662.

Leroy, M. M. 1983 Structure of perpendicular shocks in collisionless plasma. *Phys. Fluids* **26**, 2742–2753.

Mellott, M. M. and Livesey, W. A. 1987 Shock overshoots revisited. *J. Geophys. Res.* **92**, 13 661–13 665.

Moiseev, S. S. and Sagdeev, R. Z. 1963 Collisionless shock waves in a plasma in a weak magnetic field. *J. Nucl. Energy* C **5**, 43–47.

Newbury, J. A., Russell, C. T. and Gedalin, M. 1998 The ramp widths of high-Mach-number, quasi-perpendicular collisionless shocks. *J. Geophys. Res.* **103**, 29 581–29 594.

Sagdeev, R. Z. 1966 Cooperative phenomena and shock waves in collisionless plasmas. *Rev. Plasma Phys.* **4**, 23.

Schamel, H. 1972 Stationary solitary, snoidal and sinusoidal ion acoustic waves. *Plasma Phys.* **14**, 905–924.

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press

Schwartz, S. J. 1987 Comment on "the resolved layer of a collisionless, high β, supercritical, quasi-perpendicular shock wave, 2, dissipative fluid electrodynamics" by J. D. Scudder, A. Mangeney, C. Lacombe, C. C. Harvey, and T. L. Aggson. *J. Geophys. Res.* **92**, 6168–6168.

Sckopke, N., Paschmann, G., Bame, S. J., Gosling, J. T. and Russell, C. T. 1983 Evolution of ion distributions across the nearly perpendicular bow shock: specularly and non-specularly reflected-gyrating ions. *J. Geophys. Res.* A **88**, 6121–6136.

Scudder, J. D., Aggson, T. L., Mangeney, A., Lacombe, C. and Harvey, C. C. 1986 The resolved layer of a collisionless, high beta, supercritical, quasi-perpendicular shock wave. I—Rankine–Hugoniot geometry, currents, and stationarity. *J. Geophys. Res.* **91**, 11 019–11 052.

Tidman, D. A. and Krall, N. A. 1971 *Shock Waves in Collisionless Plasmas* (New York: Wiley-Interscience).

https://doi.org/10.1017/S0022377807006708 Published online by Cambridge University Press