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The relative Mayer–Vietoris sequence

BY RICHARD STEINER
Department of Mathematics, University of Glasgow, Glasgow G12 8QW

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The object of this paper is to generalize the relative Mayer–Vietoris sequence of Eilenberg and Steenrod ([3], p. 42) by allowing both the containing spaces and the subspaces to vary. Fix a generalized homology theory h_*. We call a couple of spaces $\{X_1, X_2\}$ *excisive* (for h_*) if they are subspaces of some larger space and the inclusion $(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ induces an isomorphism under h_* (see [4], pp. 208–9 for equivalent definitions).

THEOREM. *Let* (X_1, A_1) *and* (X_2, A_2) *be pairs of spaces such that* $\{X_1, X_2\}$ *and* $\{A_1, A_2\}$ *are excisive couples. Then there is a natural exact sequence*

$$\dots \longrightarrow h_{q+1}(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{\Delta} h_q(X_1 \cap X_2, A_1 \cap A_2) \xrightarrow{(i_*, j_*)} h_q(X_1, A_1) \oplus h_q(X_2, A_2)$$
$$\xrightarrow{(k_*, -l_*)} h_q(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{\Delta} h_{q-1}(X_1 \cap X_2, A_1 \cap A_2) \longrightarrow \dots, \qquad (1)$$

where i, j, k *and* l *are inclusions.*

There is a similar sequence in cohomology with reversed arrows, as usual.

For ordinary singular homology the Theorem is given by Spanier ([4], p. 190), and there are other special cases in various textbooks. But the Theorem in general seems to be unknown, although it would be useful (see [1], p. 262 and [5], pp. 310, 315), and its proof is not quite obvious.

In fact the proof uses the homology of maps and squares, following the ideas of Eckmann and Hilton ([2]). Let I be the closed interval $[0, 1]$. If $\alpha: A \to B$ is a map, then its *mapping cylinder* Z_α is the space got from $A \times I$ and B by identifying $(a, 1) \in A \times I$ with $\alpha(a) \in B$. The homology of α is defined as $h_*(\alpha) = h_*(Z_\alpha, A \times 0)$. The point of this is the natural exact sequence

$$\dots \longrightarrow h_q(A) \xrightarrow{\alpha_*} h_q(B) \longrightarrow h_q(\alpha) \longrightarrow h_{q-1}(A) \xrightarrow{\alpha_*} h_{q-1}(B) \longrightarrow \dots,$$

which is got from the exact sequence of the pair $(Z_\alpha, A \times 0)$, since B is evidently a deformation retract of Z_α. What we need is the observation that, if $\alpha: A \to B$ is an inclusion, then the evident retraction from Z_α to B gives a natural isomorphism

$$h_*(\alpha) \cong h_*(B, A), \qquad (2)$$

by the five-lemma.

Next, let Q be a commutative square of spaces and maps

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & \overset{\delta}{\searrow} & \downarrow{\scriptstyle\gamma} \\ C & \longrightarrow & D. \end{array}$$

This can be regarded as a map of maps in two ways: either $(\gamma, \beta): \alpha \to \delta$ or $(\delta, \alpha): \beta \to \gamma$.

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We shall define a *single* group $h_*(Q)$ to act as the homology of either map; that is, there will be natural exact sequences

$$\cdots \longrightarrow h_q(\alpha) \xrightarrow{(\gamma,\beta)_*} h_q(\delta) \longrightarrow h_q(Q) \longrightarrow h_{q-1}(\alpha) \xrightarrow{(\gamma,\beta)_*} h_{q-1}(\delta) \longrightarrow \cdots \tag{3}$$

and

$$\cdots \longrightarrow h_q(\beta) \xrightarrow{(\delta,\alpha)_*} h_q(\gamma) \longrightarrow h_q(Q) \longrightarrow h_{q-1}(\beta) \xrightarrow{(\delta,\alpha)_*} h_{q-1}(\gamma) \longrightarrow \cdots. \tag{4}$$

To define $h_*(Q)$, let L_Q, the *homotopy colimit* of Q, be the space got from $A \times I^2$, $B \times I$, $C \times I$ and D by identifying $(a, 1, t) \in A \times I^2$ with $(\alpha(a), t) \in B \times I$, $(a, s, 1) \in A \times I^2$ with $(\beta(a), s) \in C \times I$, $(b, 1) \in B \times I$ with $\gamma(b) \in D$, and $(c, 1) \in C \times I$ with $\delta(c) \in D$. Evidently L_Q contains copies of Z_α (got from $A \times I \times 0$ and $B \times 0$), Z_β (from $A \times 0 \times I$ and $C \times 0$), Z_γ (from $B \times I$ and D) and Z_δ (from $C \times I$ and D). Let M_Q be the subspace $Z_\alpha \cup Z_\beta$, and define the homology of Q as $h_*(Q) = h_*(L_Q, M_Q)$.

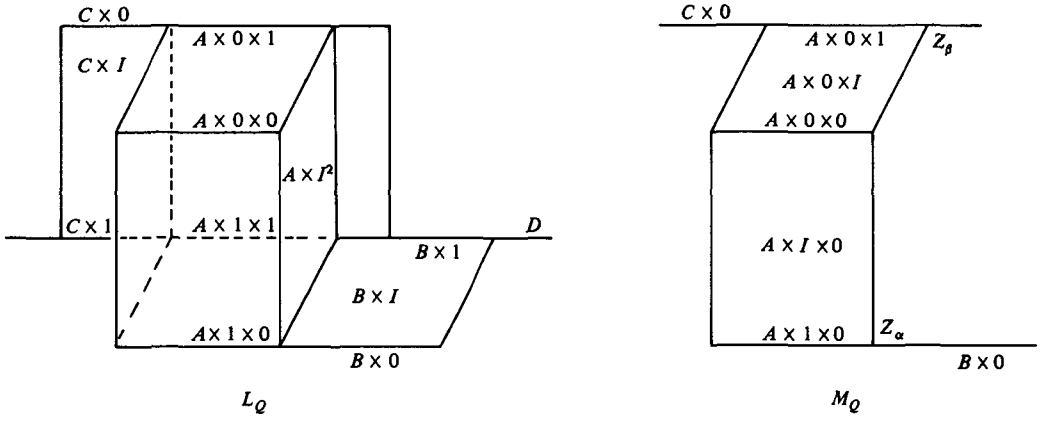


Fig. 1

We now get the exact sequence (3) as the exact sequence of the triple (L_Q, M_Q, Z_β). For $h_*(L_Q, M_Q) = h_*(Q)$ by definition, $h_*(L_Q, Z_\beta) \cong h_*(Z_\delta, C \times 0) = h_*(\delta)$ because of the evident deformation retraction $(L_Q, Z_\beta) \to (Z_\delta, C \times 0)$, and

$$h_*(M_Q, Z_\beta) \cong h_*(Z_\alpha, A \times 0 \times 0) = h_*(\alpha)$$

because the first of the inclusions

$$(Z_\alpha, A \times 0 \times 0) \to (Z_\alpha \cup A \times 0 \times [0, \tfrac{1}{2}], A \times 0 \times [0, \tfrac{1}{2}]) \to (M_Q, Z_\beta),$$

is a homotopy equivalence and the second is an excision. Similarly (4) is got from the triple (L_Q, M_Q, Z_α).

Proof of the Theorem. We have commutative squares of inclusions

$$Q_1: \begin{array}{ccc} A_1 \cap A_2 & \to & A_1, \\ \downarrow & & \downarrow \\ X_1 \cap X_2 & \to & X_1 \end{array} \qquad Q_2: \begin{array}{ccc} A_2 & \to & A_1 \cup A_2, \\ \downarrow & & \downarrow \\ X_2 & \to & X_1 \cup X_2 \end{array}$$

and there is an inclusion map from Q_1 to Q_2. In the induced map of exact sequences (3) we have, in view of (2), the homomorphisms

$$h_*(A_1, A_1 \cap A_2) \to h_*(A_1 \cup A_2, A_2) \quad \text{and} \quad h_*(X_1, X_1 \cap X_2) \to h_*(X_1 \cup X_2, X_2).$$

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By the excision hypothesis, they are isomorphisms, so $h_*(Q_1) \cong h_*(Q_2)$ by the five-lemma. The exact sequences (4) therefore give a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & h_{q+1}(Q_1) & \xrightarrow{\lambda} & h_q(X_1 \cap X_2, A_1 \cap A_2) & \xrightarrow{i_*} & h_q(X_1, A_1) & \longrightarrow & h_q(Q_1) \xrightarrow{\lambda} h_{q-1}(X_1 \cap X_2, A_1 \cap A_2) \\
 & & \mu \downarrow \cong & & j_* \downarrow & & k_* \downarrow & & \mu \downarrow \cong \\
h_{q+1}(X_1 \cup X_2, A_1 \cup A_2) & \xrightarrow{\nu} & h_{q+1}(Q_2) & \longrightarrow & h_q(X_2, A_2) & \xrightarrow{l_*} & h_q(X_1 \cup X_2, A_1 \cup A_2) & \xrightarrow{\nu} & h_q(Q_2).
\end{array}$$

A standard diagram-chase now shows that there is an exact sequence (1), where Δ is defined to be $\lambda\mu^{-1}\nu$.

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