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Cherry flow: physical measures and perturbation theory

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Abstract. In this article we consider Cherry flows on the torus which have two singularities, a source and a saddle, and no periodic orbits. We show that every Cherry flow admits a unique physical measure, whose basin has full volume. This proves a conjecture given by Saghin and Vargas [Invariant measures for Cherry flows. *Comm. Math. Phys.* **317**(1) (2013), 55–67]. We also show that the perturbation of Cherry flows depends on the divergence at the saddle: when the divergence is negative, this flow admits a neighborhood, such that any flow in this neighborhood belongs to one of the following three cases: it has a saddle connection; it is a Cherry flow; it is a Morse–Smale flow whose non-wandering set consists of two singularities and one periodic sink. In contrast, when the divergence is non-negative, this flow can be approximated by a non-hyperbolic flow with an arbitrarily large number of periodic sinks.

1. *Introduction*

In this article we consider Cherry flows on the torus. Examples were first given by Cherry in [**4**] (see also [**19**]).

Definition 1.1. We say that a flow has *non-trivial recurrence* if there is a point x which belongs to its ω set and is not a critical element, i.e. x is not a singularity and its orbit is not periodic. A flow on a torus $\mathbb{T}^2$ is called a Cherry flow if:

- it has singularities and non-trivial recurrences;
- all the singularities are hyperbolic;
- it has no closed orbits.

The classification of general Cherry flows was treated in [**16**]. However, in this article we will only consider the class of Cherry flows which has been studied in [**23**]. For simplicity, from now on, we will make use of the following definition for Cherry flows.

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Definition 1.2. A flow on $\mathbb{T}^2$ is called a *Cherry flow* if its non-wandering set consists of a source b and an attractor Λ which is transitive, contains a saddle a, and has neither periodic orbits nor saddle connections.

Remark 1.3. We say that a compact invariant set of a flow is *quasi-minimal* if it is transitive, contains a saddle, and has neither periodic orbits nor saddle connections.

We are interested in the physical measures and perturbation theory for Cherry flows. An invariant probability μ of a flow ϕ is a *physical measure* if its *basin*

$$B(\mu) = \left\{ x \in \mathbb{T}^2 : \lim_{t\to+\infty} \frac{1}{t} \int_0^t f(\phi_s(x))\, ds = \int f\, d\mu \text{ for any continuous function } f \right\}$$

has positive volume.

We prove the following conjecture given by Saghin and Vargas [**23**].

THEOREM A. *Every* C^∞ *Cherry flow admits a unique physical measure, whose basin has full volume.*

Throughout this paper, we fix a vector field X which generates a Cherry flow ϕ. Denote by $\mathcal{D}(a)$ the divergence at the saddle a of X. It was observed by [**23**] that the behavior of physical measures for ϕ depends on the divergence at the saddle a. And in general it is necessary to assume this flow to be C^∞, because C^∞ regularity implies that this flow is C^{1+bv} linearizable at a neighborhood of a (see [**16**, Appendix]).

With the C^∞ hypothesis, the existence of physical measures for Cherry flows with non-positive divergence at a was proved in [**23**, Theorem 1.1]. Then Theorem A is an immediate corollary of the following result on the case of positive divergence at the saddle. We observe that the assumption on regularity for the flows is removed here.

THEOREM B. *Suppose that* ϕ *is* C^1 *and* $\mathcal{D}(a) > 0$. *Then* ϕ *admits a non-trivial physical measure supported on* Λ. *This physical measure is ergodic, non-hyperbolic, and its basin has full volume.*

An ergodic measure being *non-trivial* means that it is not supported on critical elements, i.e. on singularities or on periodic orbits. Theorem B generalizes the results of [**23**, **21**]. Unlike most dynamical systems with physical measures, where the physical measures in general are either hyperbolic (see [**3**, **1**]) or atomic (see [**9**, **22**]), the physical measures in Theorem B are non-hyperbolic and non-trivial. An interesting and important fact is that the proof of Theorem B depends on the non-hyperbolicity of this measure. Moreover, we will show that every non-trivial ergodic measure for a two-dimensional flow is always non-hyperbolic.

In [**17**, **18**], a different kind of Cherry flows was considered, where b is a sink. This sink automatically supports a physical measure, and we are not going to consider this case here.

We also consider the perturbation of Cherry flow. It turns out that the perturbation depends also on the divergence at the saddle. For a vector Y sufficiently close to X, we denote by ϕ^Y and ϕ^X the generated flows, and by a^Y and a^X the corresponding hyperbolic saddles for Y and X, respectively.

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Definition 1.4. A flow is called a *star flow* if it admits a C^1 neighborhood $\mathcal{U}$, such that for every flow in $\mathcal{U}$, its critical elements are all hyperbolic.

THEOREM C. *Suppose that ϕ^X is C^∞ and $\mathcal{D}_X(a) < 0$. Then ϕ^X is a star flow. Moreover, there is a C^1 neighborhood $\mathcal{U}$ of ϕ^X, such that every flow $Y \in \mathcal{U}$ belongs to one of the following three cases:*

- *it has a saddle connection;*
- *it is a Morse–Smale flow, whose non-wandering set consists of two singularities and one periodic sink;*
- *it is a Cherry flow, which admits a unique physical measure δ_{a^Y} supported on a^Y with basin $B(\delta_{a^Y}) = \mathbb{T}^2 \backslash b^Y$.*

THEOREM D. *Suppose that ϕ^X is C^1 and $\mathcal{D}_X(a^X) \geq 0$. Then ϕ^X is C^1 approximated by non-hyperbolic flows with an arbitrarily large number of periodic sinks.*

2. *Preliminaries*

2.1. *Ergodic measures for Cherry flows.* One can always take a circle $\mathcal{S}_0 \subset \mathbb{T}^2$ which does not bound a disk and is everywhere transverse to X; see [**16**, Proposition 7.1]. The inverse of the Cherry flow is a suspension of a continuous circle map $g : \mathcal{S}_0 \to \mathcal{S}_0$, where g is a monotone map and constant on an interval. Such circle maps were well studied in [**16**, **8**, **7**, **20**]. Note that g has no periodic point, which implies that g has irrational rotation number $\rho(g)$ and is semi-conjugate to the rotation $R_{\rho(g)} : x \to x + \rho(g) \bmod 1$. The semi-conjugacy h is continuous, monotone, has degree one and maps orbits of g to orbits of $R_{\rho(g)}$. Because the h pre-image of every point in the circle is either a single point or a non-trivial connected interval, and by the fact that a circle contains at most countably many disjoint open intervals, there are at most countably many points that have non-trivial h pre-images. In particular, this implies that Lebesgue almost every point has a unique h pre-image. Then $\nu = (h)^* \mathrm{vol}|_{S^1}$ is well defined, and is the unique ergodic measure of g. For every $\theta \in \mathcal{S}_0^1$, denote by $\tau_X(\theta)$ the first return time of θ for ϕ. It is well known that $\int \tau_X(\theta)\, d\nu(\theta) < \infty$ if and only if

$$\mu = \int_{\mathcal{S}^1} \int_{t \in [0, \tau_X(\theta))} \delta_{\phi_t(\theta)}\, d\nu(\theta)$$

is an ergodic measure of ϕ. In summary, we have the following result.

PROPOSITION 2.1. *The Cherry flow ϕ has three ergodic measures δ_a, δ_b and μ if and only if $\int \tau_X(\theta)\, d\nu(\theta) < \infty$.*

THEOREM 2.2. [**23**, Theorems 1.1, 1.2] *Suppose that ϕ is a C^∞ Cherry flow.*

(a) *If $\mathcal{D}(a) \leq 0$, then δ_b and δ_a are the only two ergodic measures for ϕ. Consequently, δ_a is a physical measure with basin equal to $\mathbb{T}^2 \backslash b$.*

(b) *If $\mathcal{D}(a) > 0$, then there exist exactly three ergodic measures for ϕ: δ_b, δ_a and an ergodic probability measure μ which is supported on the quasi-minimal set Λ.*

Remark 2.3. Let us observe that, although the flows in [**23**] are always supposed to be C^∞, the proof of [**23**, Theorem 1.2] (which corresponds to Theorem 2.2(b) above) also works for C^1 flows; for a different proof, see Lemma 2.16 below.

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2.2. *Ergodic theory for flows.* We recall the ergodic theory for two-dimensional flows. Let ϕ^Y be a flow over a surface M which is generated by a vector field Y, and ν a non-trivial ergodic measure of ϕ^Y. Denote by $\mathrm{Sing}(Y)$ the set of singularities of Y, and by $\Phi_t = d\phi_t^Y : TM \to TM$ the induced tangent flow.

Denote the normal bundle of ϕ^Y over M by

$$\mathcal{N}_M = \bigcup_{x \in M \setminus \mathrm{Sing}(Y)} \mathcal{N}_x,$$

where $\mathcal{N}_x$ is the orthogonal complement of the flow direction $Y(x)$, i.e.

$$\mathcal{N}_x = \{v \in T_x M : v \perp Y(x)\},$$

and the orthogonal projection from $T_x M$ to $\mathcal{N}_x$ by π_x.

Definition 2.4. The *linear Poincaré flow* ψ_t on $\mathcal{N}_M$ is defined as follows: for any $x \in \Lambda \setminus \sigma$ and $v \in \mathcal{N}_x$,

$$\psi_t(v) = \pi_{\phi_t^Y(x)}(\Phi_t(v)) = \Phi_t(v) - \frac{\langle \Phi_t(v),\, Y(\phi_t^Y(x))\rangle}{\|Y(\phi_t^Y(x))\|^2} Y(\phi_t^Y(x))$$

is the orthogonal projection of $\Phi_t(v)$ on $\mathcal{N}_{\phi_t^Y(x)}$, where $\langle \cdot\,, \cdot \rangle$ is the inner product on $T_x M$ given by the Riemannian metric.

2.2.1. *Oseledets's theorem for flows.* We now state the flow version of Oseledets's theorem.

PROPOSITION 2.5. *There exists a real number λ such that, for ν-almost every x,*

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|\psi_t(v)\| = \lambda \quad \textit{for every non-zero } v \in \mathcal{N}_x \setminus \{0\}.$$

Definition 2.6. The value λ in the above proposition is called the *Lyapunov exponent* of ν for the flow ϕ.

The definition of Lyapunov exponents for diffeomorphisms is a little different. For simplicity, we assume f to be a two-dimensional diffeomorphism. We need the version of Oseledets's theorem for diffeomorphisms:

THEOREM 2.7. *Let ν be an ergodic invariant measure of f. Then:*

- *either there is λ_1 such that, for ν-almost every x, and for any $0 \neq v \in T_x M$,*
$$\lim \frac{1}{n} \|Df^n(v)\| = \lambda_1;$$
- *or there are $\lambda_1 < \lambda_2$ and a measurable splitting $T_x M = E_1(x) \oplus E_2(x)$ for μ-almost every x, such that, for any $0 \neq v \in E_i(x)$ $(i = 1, 2)$,*
$$\lim \frac{1}{n} \|Df^n(v)\| = \lambda_i.$$

Definition 2.8. The value λ_i ($i = 1$ or 2) in the above proposition is called the *Lyapunov exponent* of ν for the diffeomorphism f. And the splitting $T_x M = E_1(x) \oplus E_2(x)$ defined for ν-almost every x is called the *Oseledets splitting* (in the first case, we assume that $E_2(x) = \emptyset$).

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When the time $t_0 > 0$ is fixed, $\phi^Y_{t_0}$ is a diffeomorphism and ν is still an invariant measure of this diffeomorphism. Suppose that $\hat{\nu}$ is an ergodic decomposition of ν for $\phi^Y_{t_0}$ (see [**24**, Theorem 5.1.3] for a statement of the ergodic decomposition theorem). Then

$$\nu = \int_{t\in[0,t_0)} d(\phi^Y_t)_*\hat{\nu}.$$

We may always choose t_0 such that ν is ergodic for $\phi^Y_{t_0}$. For such a time t_0, denote $f = \phi^Y_{t_0}$. We now state the relation between Lyapunov exponent(s) of ν for ϕ^Y_t and for f; the proof is simple and is therefore omitted.

THEOREM 2.9. *The Lyapunov exponents of ν for f coincide with $\{t_0\lambda\}\cup\{0\}$, and for ν-almost every $x \in M\backslash\mathrm{Sing}(Y)$, the flow direction $\langle Y(x)\rangle$ is contained in the subbundle of the Oseledets splitting of f corresponding to zero exponent. Moreover, we have* $\lim (1/n)\log\det(Df^n(x)) = t_0\lambda$.

Remark 2.10. The above theorem also implies that, for ν-almost every x,

$$\lim_{t\to\infty}\frac{1}{t}\det\Phi_t|_{T_xM} = \lambda.$$

Definition 2.11. A non-trivial ergodic measure ν of flow ϕ^Y is a *hyperbolic measure* if its exponent is non-vanishing.

2.2.2. *Divergence.* We need the following version of Liouville's theorem which has been proved in [**15**, Ch. 1, §8].

LEMMA 2.12. *For any $x \in M$ and $t > 0$,*

$$\ln\det\Phi^Y_t|_{T_xM} = \int_0^t \mathrm{div}_Y(\phi^Y_s(x))\,ds.$$

The definition of divergence depends on the Riemmannian metric. For simplicity, in the Appendix we provide a short proof of the above lemma with the assumption that $M = \mathbb{T}^2$ and the divergence is defined by the flat metric on the torus. More precisely, every small open set of $\mathbb{T}^2$ can be seen as a subset of $\mathbb{R}^2$; choosing local coordinates (x, y), we may write $Y = (Y_1, Y_2)$ and $\phi^Y_t = (f_t(x, y), g_t(x, y))$. Then for any $x \in \mathbb{T}^2$, the *divergence of Y at x* is defined by $\mathrm{div}_Y(x) = \partial Y_1/\partial x + \partial Y_2/\partial y$.

Definition 2.13.

- Let $\mathrm{Orb}(p)$ be a periodic orbit of ϕ^Y with period $\tau_Y(p)$. The *divergence of* $\mathrm{Orb}(p)$ is defined by
$$\mathcal{D}_Y(\mathrm{Orb}(p)) = \frac{1}{\tau_Y(p)}\int_0^{\tau_Y(p)}\mathrm{div}_Y(\phi^Y_s(p))\,ds.$$
- Let $\tilde{\nu}$ be a probability measure which is not necessary invariant. The *divergence of* $\tilde{\nu}$ is defined by
$$\mathcal{D}_Y(\tilde{\nu}) = \int \mathrm{div}_Y(x)\,d\tilde{\nu}(x).$$

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Remark 2.14. For an atomic measure δ_σ supported on a singularity $\sigma \in \mathrm{Sing}(Y)$, $\mathcal{D}_Y(\delta_\sigma)$ coincides with the classical definition of divergence at σ, $\mathcal{D}_Y(\sigma)$. And when $\tilde{\nu}$ is supported on a periodic orbit $\mathrm{Orb}(p)$, the above two definitions coincide and, by Lemma 2.12, are equal to $\ln\det \Phi_{\tau_Y(p)}|_{T_pM} = \ln\|\psi_{\tau_Y(p)}|_{\mathcal{N}_p}\|$.

As a corollary of Remark 2.10 and Lemma 2.12, by Birkhoff's ergodic theorem, we obtain the following result.

COROLLARY 2.15. *$\mathcal{D}_Y(\nu)$ coincides with the Lyapunov exponent of ν for the flow ϕ^Y.*

The following lemma implies [**23**, Theorem 1.2].

LEMMA 2.16. *There exists an ergodic measure ν_0 of Y such that $\mathcal{D}_Y(\nu_0) \le 0$.*

Proof. Suppose by contradiction that the divergence of every ergodic measure of ϕ^Y is positive. Denote by $\mathcal{M}_Y$ the space of invariant measures of ϕ^Y. Then, by the ergodic decomposition theorem, there is $a > 0$ such that $\mathcal{D}_Y|_{\mathcal{M}_Y} > 2a > 0$. Take $\mathcal{V}$ a small neighborhood of $\mathcal{M}_Y$ in the probability space over M, such that for any $\tilde{\nu} \in \mathcal{V}$,

$$\int \mathrm{div}_Y(x)\, d\tilde{\nu}(x) > a.$$

It is well known that for any $x \in M$, there is $t_x > 0$ such that $(1/t)\int_0^t \delta_{\phi_s^Y(x)}\, ds \in \mathcal{V}$ for any $t \ge t_x$. A short proof is as follows. Suppose that this is false, then there are $t_n \to \infty$ such that $\mu_n = (1/t_n)\int_0^{t_n} \delta_{\phi_s^Y(x)}\, ds \notin \mathcal{V}$. Because the probability space over any compact manifold is compact, we can take a converging point μ_0 of $\{\mu_n\}_{n\in\mathbb{N}}$. Then μ_0 is an invariant measure but not contained in $\mathcal{V}$, which is a contradiction.

Let us continue the proof. For $n > 0$, denote $M_n = \{x;\, t_x \le n\}$. Because $\bigcup_n M_n = M$, we can choose N sufficiently large such that $\mathrm{vol}(M_N) > 0$. Observe that for any $x \in M_N$ and $n > N$,

$$\int \mathrm{div}_Y(y)\, d\left(\frac{1}{n}\int_0^n \delta_{\phi_s^Y(x)}\, ds\right)(y) = \frac{1}{n}\int_0^n \mathrm{div}_Y(\delta_{\phi_s^Y(x)})\, ds > a.$$

Then by Lemma 2.12,

$$\det \Phi_n|_{T_xM} = \exp\left(\int_0^n \mathrm{div}_Y(\delta_{\phi_s^Y(x)})\, ds\right) > \exp(na),$$

which implies that $\lim_n \mathrm{vol}(\phi_n^Y(M_N)) = \infty$, in contradiction to the fact that the volume of M is finite. □

2.2.3. *Star flow.* We now give a criterion for a two-dimensional flow Y to be a star flow.

THEOREM 2.17. *Suppose that all the singularities of Y are hyperbolic, and that all the invariant measures of ϕ^Y have negative divergence. Then Y is a star flow.*

Example 2.18. Consider a two-dimensional flow Y_0 on $\mathbb{R}^2$ with a saddle σ of negative divergence, and each branch of the unstable manifold of σ connected to a branch of its stable manifold—this set looks like a figure '8'. Denote by Γ the set consisting of σ and its two saddle connections. It is easy to see that Γ is an attractor: for each branch of

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the unstable manifold of σ, we take a transverse section near to σ and then show that the holonomy maps between these sections induced by flows are uniformly contracting. Take a small contracting neighborhood U of Γ. Then δ_σ is the unique invariant measure supported in U. By Theorem 2.17, Y_0 is a star flow in U.

Proof of Theorem 2.17. Because all the singularities of Y are hyperbolic, ϕ^Y has only finitely many singularities. And by the continuation of hyperbolic singularities, the singularities for flows in a small neighborhood of Y are all hyperbolic. Then, in order to prove this theorem, it suffices to show that the periodic orbits of nearby flows are all periodic sinks.

Denote by $\mathcal{M}_Z$ the space of invariant measures of a vector field Z. Then, by the assumption, $\mathcal{D}_Y(\cdot)|_{\mathcal{M}_Y} < 0$. Because $\mathcal{M}_Y$ is a compact space and $\mathcal{D}_Y(\cdot)$ is a continuous function, there is $a < 0$ such that $\mathcal{D}_Y(\cdot)|_{\mathcal{M}_Y} < a$. Moreover, by the upper semi-continuation of the invariant measure space, there is a C^1 neighborhood $\mathcal{U}$ of Y such that for any vector field $Z \in \mathcal{U}$, $\mathcal{D}_Z(\cdot)|_{\mathcal{M}_Z} < a/2 < 0$.

Suppose that $\mathrm{Orb}_Z(p)$ is a periodic orbit of Z. Denote by $\mu_{\mathrm{Orb}_Z(p)}$ the ϕ^Z ergodic measure supported on $\mathrm{Orb}_Z(p)$. Then, from the definition of divergence for measures,

$$\mathcal{D}_Z(\mathrm{Orb}_Z(p)) = \mathcal{D}_Z(\mu_{\mathrm{Orb}_Z(p)}) < \frac{a}{2} < 0.$$

By Remark 2.14, $\mathrm{Orb}_Z(p)$ is a periodic sink. The proof is complete. □

As an immediate corollary of Theorems 2.2(a) and 2.17, we can show the following result for the Cherry flow X.

LEMMA 2.19. *Suppose that ϕ is a C^∞ Cherry flow and $\mathcal{D}_X(a) < 0$. Then ϕ is a star flow. Moreover, there is a C^1 neighborhood $\mathcal{U}$ of X such that, for every flow $Y \in \mathcal{U}$, the periodic orbits of Y are all periodic sinks.*

We also need the following general description of ergodic measures for flows, whose proof depends on Liao theory, and is the most technical part of this paper. We postpone its proof to §5, and for simplicity we will only consider the two-dimensional case, which is all we need. The idea of the proof for the general situation is the same. The main difficulty in the proof arises from the existence of singularities. A similar statement for diffeomorphisms can be deduced by considering the corresponding suspension flows, or by using the C^1 version of Pesin theory for diffeomorphisms introduced in [**25**].

LEMMA 2.20. *Suppose that ν is a non-trivial ergodic measure of a C^1 flow. Then ν has at least one non-negative Lyapunov exponent.*

By considering again the inverse of this flow, we show the following corollary.

COROLLARY 2.21. *Every non-trivial ergodic measure of a two-dimensional flow is non-hyperbolic, i.e. the Lyapunov exponent and the divergence of this measure are both vanishing.*

Proof. Let ν be a non-trivial ergodic measure of a C^1 flow ϕ_t^X, and λ the Lyapunov exponent of ν. Then, by Lemma 2.20, $\lambda \geq 0$.

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Now observe that ν is also a non-trivial ergodic measure for the inverse of the flow: ϕ_t^{-X}. Moreover, $-\lambda$ is the Lyapunov exponent of ν for the flow ϕ_t^{-X}. By Lemma 2.20, $-\lambda \geq 0$, which implies that $\lambda = 0$. □

2.3. *Perturbation of flows.* We need the flow version of Franks's lemma, proved by Liao in [**12**, Proposition 3.4], to modify the linear Poincaré flow along a periodic orbit. The following statement comes from [**2**, Theorem A.1].

THEOREM 2.22. *Given a* C^1 *neighborhood* $\mathcal{U}$ *of* Y*, there is a neighborhood* $\mathcal{V} \subset \mathcal{U}$ *of* Y *and* $\varepsilon > 0$ *such that, for any* $Z \in \mathcal{V}$*, for any periodic orbit* $\mathrm{Orb}_Z(p)$ *of* Z *with period* $\tau(p) \geq 1$*, any neighborhood* U *of* $\mathrm{Orb}_Z(p)$ *and any partition of* $[0, \tau_Z(p)]$*,*

$$0 = t_0 < t_1 < \cdots < t_l = \tau_Z(p), \quad 1 \leq t_{i+1} - t_i \leq 2,\ i = 0, 1, \ldots, l-1,$$

and for any linear isomorphisms $L_i : \mathcal{N}_{\phi_{t_i}^Z(p)} \to \mathcal{N}_{\phi_{t_{i+1}}^Z(p)}$, $i = 0, 1, \ldots, l-1$*, satisfying* $\|L_i - \psi^Z_{t_{i+1}-t_i}|_{\mathcal{N}_{\phi_{t_i}(p)}}\| \leq \varepsilon$*, there exists* $\tilde{Z} \in \mathcal{U}$ *such that* $\psi^{\tilde{Z}}_{t_{i+1}-t_i}|_{\mathcal{N}_{\phi_{t_i}(p)}} = L_i$ *and* $\tilde{Z} = Z$ *on* $(M \backslash U) \cup \mathrm{Orb}_Z(p)$.

Remark 2.23. By the same proof, one can show that for any $Z \in \mathcal{V}$ and any finite periodic orbits of Z, $\mathrm{Orb}(p_1), \ldots, \mathrm{Orb}(p_k)$, we can perturb this flow near all these k periodic orbits in the above manner simultaneously.

We also need the closing lemma.

THEOREM 2.24. *Suppose that* $x \in M$ *is regular point of flow* Y *and its* ω *set,* $\omega(x)$*, contains regular points. Then. for any* C^1 *neighborhood* $\mathcal{U}$ *of* Y *and any neighborhood* U *of* $\omega(x)$*, there is a flow* $Z \in \mathcal{U}$ *such that* $Z = Y|_{M\backslash U}$ *and* Z *has a periodic orbit contained in* U.

A deep result of Liao [**14**] shows that in a small neighborhood of a star flow, the number of periodic sinks is uniformly bounded. The following version of Liao's result can be obtained directly from the original proof, and it is the main motivation of our Theorem C. Because it will not be used in this paper, we do not provide a proof.

Before the statement, let us introduce some necessary notation. For a flow Y, denote by $\mathrm{Sing}_1(Y)$ the set of hyperbolic singularities which are contained in non-trivial chain recurrent classes, and whose tangent bundle admits a codimension-one dominated splitting $E^{cs} \oplus E^u_1$, where E^u_1 is a one-dimensional strong unstable bundle. Note that every $\sigma \in \mathrm{Sing}_1(Y)$ admits a one-dimensional strong unstable manifold. We also write $S(Y)$ for the number of periodic sinks of ϕ^Y and $K(Y)$ for the number of singularities contained in $\mathrm{Sing}_1(Y)$.

THEOREM 2.25. *Suppose that* Y *is a star flow. Then* $S(Y)$ *is finite, and there is a* C^1 *neighborhood* $\mathcal{U}$ *of* Y *such that, for any* $Z \in \mathcal{U}$*,*

$$S(Z) \leq S(Y) + 2K(Y).$$

Moreover, the basin of every 'new' periodic sink of Z *intersects one branch of the one-dimensional strong unstable manifold of a singularity of* Z*; this singularity is the analytic continuation of a singularity* $\sigma \in \mathrm{Sing}_1(Y)$ *of* Y.

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Example 2.26. Applying the above theorem on the star flow Y_0 in Example 2.18, there is an open neighborhood $\mathcal{U}$ of Y_0 such that any flow $Z \in \mathcal{U}$ has at most two periodic sinks, and one saddle connection is broken when one periodic sink appears.

3. *Proof of Theorem B*

By Theorem 2.2(b) and Remark 2.3, Λ admits two ergodic measures, δ_a and μ. Then, by Corollaries 2.21 and 2.15, μ is non-hyperbolic and $\mathcal{D}_X(\mu) = 0$.

For every point x different from b, its forward orbit converges to the quasi-minimal set Λ. In particular, any accumulated point of $(1/t)\int_0^t \delta_{\phi_s(x)}\,ds$ is an invariant measure supported on Λ, and can be written as $p\delta_a + (1-p)\mu$ for some $0 \le p \le 1$.

For $0 \le k \le 1$, let $K_k = \{p\delta_a + (1-p)\mu;\ k \le p \le 1\}$ and

$$\Gamma_k = \left\{x \in \mathbb{T}^2;\ \text{there is } t_i \to \infty \text{ such that } \lim_i \frac{1}{t_i}\int_0^{t_i} \delta_{\phi_s(x)}\,ds \in K_k\right\}.$$

By the definition, $\Gamma_{k_1} \supset \Gamma_{k_2}$ for $0 \le k_1 < k_2 \le 1$.

We claim the following.

CLAIM. $\mathrm{vol}(\Gamma_k) = 0$ *for every* $k > 0$.

Let us explain how to deduce Theorem B from this claim. The claim implies that $\mathrm{vol}(\bigcup_{k>0}\Gamma_k) = 0$ and the complement of the set $\{b\} \cup (\bigcup_{k>0}\Gamma_k)$ has full Lebesgue measure. Then Theorem B follows from the observation that for any x in this full-volume subset, any accumulated point of $(1/t)\int_0^t \delta_{\phi_s(x)}\,ds$ coincides with δ_a, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \delta_{\phi_s(x)}\,ds = \delta_a.$$

It remains to prove the claim.

Proof of the Claim. Because $\mathcal{D}_X(a) > 0$ and $\mathcal{D}_X(\mu) = 0$, there is $c > 0$ such that $\mathcal{D}_X|_{K_k} > 2c$. Take $\mathcal{V}$ a small neighborhood of K_k in the probability space of $\mathbb{T}^2$, such that, for any measure $\nu \in \mathcal{V}$, we have $\mathcal{D}_X(\nu) = \int \mathrm{div}_X(x)\,d\nu(x) > c$.

Denote

$$\Gamma_{k,n} = \left\{x;\ \text{there is } t_x \in [n, n+1) \text{ such that } \frac{1}{t_x}\int_0^{t_x} \delta_{\phi_s(x)}\,ds \in \mathcal{V}\right\}.$$

Then, for every $x \in \Gamma_{k,n}$,

$$\int \mathrm{div}_X(y)\,d\left(\frac{1}{t_x}\int_0^{t_x}\delta_{\phi_s(x)}\,ds\right)(y) = \frac{1}{t_x}\int_0^{t_x}\mathrm{div}(\phi_s(x))\,ds > c.$$

Hence, by Lemma 2.12,

$$\det \Phi_{t_x}|_{T_x\mathbb{T}^2} = \exp\left(\int_0^{t_x}\mathrm{div}_X(\phi_s(x))\,ds\right) > \exp(ct_x).$$

Write $C = \max_{x\in\mathbb{T}^2, t\in[0,1)}\{\det \Phi_t|_{T_x\mathbb{T}^2}\}$. Then, for any $x \in \Gamma_{k,n}$,

$$\det \Phi_n|_{T_x\mathbb{T}^2} > \frac{1}{C}\exp(cn),$$

which implies that $\mathrm{vol}(\Gamma_{k,n}) \le C\exp(-cn)$.

Because $\Gamma_k \subset \bigcap_n \bigcup_{i>n}\Gamma_{k,i}$, the above argument shows that $\mathrm{vol}(\Gamma_k) = 0$. The proof of the claim is complete. □

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4. *Proof of Theorems C and D*

In this section, we build the perturbation theory for Cherry flows: in §4.1, we analyze the relation between the periodic orbits of a perturbation flow and of its return map to a fixed transverse section. The proofs of Theorems C and D are given in §4.2.

4.1. *Perturbation of Cherry flow.*

As in §2.1, $\mathcal{S}_0$ denotes a circle transverse to X. Then there is a C^1 neighborhood $\mathcal{U}_0$ of X such that $\mathcal{S}_0$ is transverse to every flow $Y \in \mathcal{U}_0$. The inverse of each flow ϕ^Y is a suspension of a continuous circle map $g^Y : \mathcal{S}_0 \to \mathcal{S}_0$, where g^Y is monotone and constant on an interval I^Y. The intersection between a periodic orbit of ϕ^Y with $\mathcal{S}_0$ corresponds to a periodic orbit of g^Y. In contrast, a periodic orbit of g^Y which does not intersect I^Y corresponds to a periodic orbit of ϕ^Y. Denote by $\mathrm{Per}(Y)$ and $\mathrm{Per}(g^Y)$ the set of periodic orbits for Y and for g^Y, respectively.

Definition 4.1. A segment $I = [p, q] \subset \mathcal{S}_0$ is called a *g^Y-periodic segment* if both p and q are periodic points of g^Y.

The proof of the following lemma is the same as for homeomorphisms over a circle.

LEMMA 4.2. *Suppose that g^Y has a periodic point with period k. Then its rotation number $\rho(g^Y)$ is rational, all the periodic points have the same period, and every periodic segment is fixed by $(g^Y)^k$. Moreover, for every $\theta \in \mathcal{S}_0$, its ω-limit set $\omega(\theta)$ is a periodic orbit of g^Y.*

The following theorem is the main result of this subsection, which reveals the relation between the periodic orbits of the flow ϕ^Y and of the circle map g^Y.

THEOREM 4.3. $\#\,\mathrm{Per}(Y) \geq \#\,\mathrm{Per}(g^Y) - 1$. *Moreover, suppose that Y contains n periodic sources. Then it has at least n other periodic orbits which are not periodic sources, and $\#\,\mathrm{Per}(Y) \geq 2n$. In contrast, when Y contains n periodic sinks, then it has at least $n - 1$ other periodic orbits which are not periodic sinks, and $\#\,\mathrm{Per}(Y) \geq 2n - 1$.*

Proof. Because the g^Y image of I^Y is a point, I^Y intersects at most one g^Y periodic orbit. If the intersection is not empty, then I^Y belongs to the stable set of a periodic orbit which is not a periodic source.

Suppose that $\mathrm{Orb}_Y(p_1), \ldots, \mathrm{Orb}_Y(p_n)$ are periodic orbits of ϕ^Y and $p_1, \ldots, p_n \in \mathcal{S}_0$.

When $\{\mathrm{Orb}_Y(p_i)\}_{i=1}^n$ are periodic sinks, for every $1 \leq i \leq n$, p_i belongs to a periodic source of g^Y. Take q_i the nearest periodic point of g^Y to p_i in the counterclockwise direction. Then q_i has a half neighborhood which is topologically contracting by g^Y. There are at least $n - 1$ distinct such periodic orbits which do not intersect I^Y; these g^Y periodic orbits correspond to the intersections of ϕ^Y periodic orbits with $\mathcal{S}_0$. Therefore, ϕ^Y has at least $n - 1$ periodic orbits which are not periodic sinks. As a conclusion, ϕ^Y has at least $2n - 1$ periodic orbits.

When $\{\mathrm{Orb}_Y(p_i)\}_{i=1}^n$ are periodic sources, for every $1 \leq i \leq n$, p_i belongs to a periodic sink of g^Y. We first consider the case where I^Y does not intersect any periodic orbit of g^Y. Taking q_i the nearest periodic point of g^Y to p_i in the counterclockwise direction, then q_i has at least a half neighborhood which is expanding by g^Y. By the assumption, $\{\mathrm{Orb}_{g^Y}(q_1), \ldots, \mathrm{Orb}_{g^Y}(q_n)\}$ do not intersect I^Y, they correspond to the intersection of

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periodic orbits of ϕ^Y with $\mathcal{S}_0$, and these periodic orbits of ϕ^Y are not periodic sources. Hence, ϕ^Y has at least $2n$ periodic orbits.

In the second case, we suppose that $q \in I^Y$ is a periodic point of g^Y. We may further suppose that $I\backslash q$ contains a segment in the clockwise direction of q; the proof of other case is similar. Then q has a half neighborhood in the clockwise direction which belongs to its stable set. Take q_i, $i = 1, \ldots, n$, the nearest periodic point of g^Y in the counterclockwise direction of p_i. Then each q_i has a half (topological) expanding neighborhood in the clockwise direction, which justifies that $q_i \neq q$ for $i = 1, \ldots, n$. Hence, $\{\mathrm{Orb}_{g^Y}(q_1), \ldots, \mathrm{Orb}_{g^Y}(q_n)\}$ are the intersection of periodic orbits of Y with $\mathcal{S}_0$, and these ϕ^Y periodic orbits are not periodic sources. This implies that Y has at least $2n$ periodic orbits.

The proof of this theorem is complete. □

4.2. *Proof of Theorems C and D.*

Proof of Theorem C. Fix $\mathcal{U}_0$ the neighborhood of X given in §4.1. Let U be any attracting neighborhood of Λ^X. Then there is a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of X such that, for any flow $Y \in \mathcal{U}$, the non-wandering set of Y can be written as $b^Y \cup \Omega_0^Y$, where $\Omega_0^Y \subset U$. In particular, every periodic orbit of Y is contained in U.

Observe that we may shrink $\mathcal{U}$ such that every $Y \in \mathcal{U}$ has exactly two singularities: a source b^Y and a saddle a^Y, both hyperbolic. A quick proof is as follows. Take small neighborhoods V_a and V_b of a^X and b^X, respectively. By continuation, Y contains only one singularity in each neighborhood. Moreover, because $\|X(\cdot)\|_{\mathbb{T}^2\backslash(V_1\cup V_2)}$ is uniformly larger than zero, for Y close to X, $\|Y(\cdot)\|_{\mathbb{T}^2\backslash(V_1\cup V_2)}$ is also positive, i.e. Y does not contain a singularity in $\mathbb{T}^2\backslash(V_1 \cup V_2)$.

By the hypothesis $D_X(a^X) < 0$ and Theorem 2.2(a), ϕ^X admits a unique invariant measure δ_{a^X} in U. Applying Lemma 2.19 on $\phi^X|U$ and shrinking $\mathcal{U}$ if necessary, for every flow $Y \in \mathcal{U}$, its periodic orbit has to be periodic sink, and this verifies that ϕ^X is a star flow. Moreover, by Theorem 4.3, every flow $Y \in \mathcal{U}$ has at most one periodic sink, otherwise Y has at least one periodic orbit which is not a periodic sink, which contradicts our previous discussion.

Therefore any flow $Y \in \mathcal{U}$ belongs to one of the following three cases:

(a) it has a saddle connection;
(b) it has a unique periodic orbit, which is a periodic sink;
(c) it has neither saddle connections nor periodic orbits.

From now on, we assume that Y has no saddle connection.

We first consider case (c). By the definition of Cherry flow (Definition 1.2), Y is automatically a C^1 Cherry flow.

Denote by Λ^Y the quasi-minimal attractor and a^Y the singularity contained in Λ^Y. By Theorem 2.2(b) and Remark 2.3, Λ^Y admits at most two ergodic measures. We claim the following.

CLAIM. *Λ^Y admits a unique ergodic measure δ_{a^Y}.*

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Proof of the Claim. Suppose that Λ^Y admits another non-trivial ergodic measure μ^Y of ϕ^Y. Because Λ^X only supports a unique invariant measure δ_{a^X} of ϕ^X with negative divergence, by the upper-semi continuation of the spaces of invariant measures, we have that $\mathcal{D}_Y(\mu^Y) = \int \mathrm{div}_Y(x)\, d\mu^Y(x) < 0$. This contradicts Corollary 2.21. □

The above claim implies that δ_{a^Y} is a physical measure whose basin coincides with $\mathbb{T}^2 \backslash b^Y$, where b^Y is the unique source of Y. In fact, for any $x \in \mathbb{T}^2 \backslash b^Y$, the positive orbit of x converges to Λ^Y, and hence, $(1/t) \int \delta_{\phi_s^Y(x)}\, ds$ converges to the unique ϕ_t invariant measure δ_{a^Y} supported on Λ^Y.

In the following we consider case (b). Denote the periodic sink of Y by $\mathrm{Orb}_Y(p)$. To show that ϕ^Y is a Morse–Smale flow, we will prove that every point $x \in \mathbb{T}^2 \backslash (b^Y \cup W^s(a^Y))$ belongs to the attracting basin of the periodic sink $\mathrm{Orb}_Y(p)$. Suppose that this is false. Then there is $x \in \mathbb{T}^2 \backslash (b^Y \cup W^s(a^Y))$ such that the ω set of x, $\omega(x)$, is not a critical element. Take a small neighborhood V of $\omega(x)$ such that $V \cap \mathrm{Orb}_Y(p) = \emptyset$. Then by the closing lemma (Theorem 2.24), there is another flow $Z \in \mathcal{U}$ such that $Z|_{V^c} = Y|_{V^c}$ and ϕ^Z has a periodic orbit $\mathrm{Orb}_Z(q) \subset V$. Then ϕ^Z has two periodic orbits, in contradiction to the above discussion that ϕ^Z has at most one periodic orbit. The proof of Theorem C is complete. □

Proof of Theorem D. We first prove that ϕ^X can be approximated by flows with an arbitrarily large number of periodic sinks. Suppose by contradiction that there exist $l > 0$ and $\mathcal{U} \subset \mathcal{U}_0$ a neighborhood of X such that every flow $Y \in \mathcal{U}$ has at most l periodic sinks. We claim the following.

CLAIM. $l \geq 1$.

Proof of the Claim. Suppose by contradiction that $l = 0$. By the closing lemma, there is a flow ϕ^Y arbitrarily close to ϕ^X such that ϕ^Y has at least one periodic orbit. Then, by Theorem 4.3, ϕ^Y has at least a periodic orbit $\mathrm{Orb}_Y(p)$ which is not a periodic source. We can apply Franks's lemma (Theorem 2.22) to show that there is a flow Z arbitrarily close to Y such that $\mathrm{Orb}_Z(p)$ is a periodic sink. This contradicts our previous assumption. □

Let ϕ^Y be a flow which is arbitrarily close to ϕ^X and contains l periodic sinks $\mathrm{Orb}_Y(p_1), \ldots, \mathrm{Orb}_Y(p_l)$. By Theorem 4.3, ϕ^Y contains at least $l-1$ other periodic orbits which are not periodic sinks. Replacing ϕ^Y an arbitrary small perturbation, we may suppose that the $l-1$ periodic orbits $\mathrm{Orb}_Y(q_1), \ldots, \mathrm{Orb}_Y(q_{l-1})$ are all periodic sources. As an application of Proposition 2.1, ϕ^X has three ergodic measures: δ_{a^X}, δ_{b^X} and μ^X. From the hypothesis and Corollary 2.21, all the invariant measures of ϕ^X have non-negative divergence. Because the invariant measure spaces vary upper semi-continuously with respect to the flows, there is ε close to zero such that for any $1 \leq i \leq l$, $\mathcal{D}_Y(\mathrm{Orb}_Y(p_i)) \geq -\varepsilon$. Applying Franks's lemma as observed in Remark 2.23, there is a flow ϕ^Z which is close to ϕ^Y such that:

- $\{\mathrm{Orb}_Z(p_i);\ i = 1, \ldots, l\}$ are all periodic sources with divergence smaller than ε;
- $\phi^Z = \phi^Y$ outside a small neighborhood of $\bigcup_{i=1,\ldots,l} \mathrm{Orb}_Y(p_i)$.

In particular, ϕ^Z has $2l-1$ periodic sources. By Theorem 4.3, ϕ^Z has at least another $2l-1$ periodic orbits $\mathrm{Orb}_Z(a_1), \ldots, \mathrm{Orb}_Z(a_{2l-1})$ which are not periodic sources.

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Because $\{\mathrm{Orb}_Z(p_i)\}_{i=1}^{l}$ all have small divergence, using Franks's lemma again, we obtain a flow $\phi^{\hat{Z}}$ such that

$$\mathrm{Orb}_{\hat{Z}}(p_1), \ldots, \mathrm{Orb}_{\hat{Z}}(p_l), \mathrm{Orb}_{\hat{Z}}(a_1), \ldots, \mathrm{Orb}_{\hat{Z}}(a_{2l-1})$$

are all periodic sinks. Hence $\phi^{\hat{Z}}$ has at least $3l-1 > l$ periodic sinks, in contradiction to the assumption that $\phi^{\hat{Z}}$ has at most l periodic sinks. This contradiction shows that ϕ^X can be approached by a flow ϕ^Y with arbitrarily large number of periodic sinks.

For any $l > 0$, take ϕ^Y sufficiently close to ϕ^X such that ϕ^Y has at least l periodic sinks $\{\mathrm{Orb}_Y(p_1), \ldots, \mathrm{Orb}_Y(p_l)\}$. As explained above, all these periodic sinks have negative divergence close to zero. Applying Franks's lemma on $\mathrm{Orb}_Y(p_1)$, there is a flow ϕ^Z close to ϕ^Y such that:

- $\mathrm{Orb}_Z(p_1)$ is non-hyperbolic;
- $\mathrm{Orb}_Z(p_2), \ldots, \mathrm{Orb}_Z(p_l)$ are still periodic sinks of ϕ^Z.

The proof is complete. □

5. *Proof of Lemma 2.20*

Throughout this section we suppose that ϕ^Y is a two-dimensional flow on the surface M generated by the vector field Y, and ν a non-trivial ergodic measure of ϕ^Y. Suppose by contradiction that the Lyapunov exponent of ν is negative. We will prove Lemma 2.20 by showing that ϕ^Y has infinite many periodic sinks, and the periodic sinks admit uniform size of basin.

In the proof we need to use the linear Poincaré flow and the corresponding shadowing lemma from Liao theory to build the periodic sinks. Readers unfamiliar with the Liao theory are invited to think of the similar problem for a C^{1+} diffeomorphism f, i.e. every non-trivial ergodic measure of f has a non-negative Lyapunov exponent. The proof comes from Katok's closing lemma in [**10**]: one will obtain the contradiction by showing that there exist an infinite number of sinks and all have uniform basin size.

The proof of Lemma 2.20 depends on the following result of [**25**, Theorem 5.1], which plays a similar role to Katok's closing lemma in Pesin theory. (Note that the 'F bundle' of the original statement is taken to be empty here.) For completeness, we outline its proof in §5.1.

THEOREM 5.1. *There is a compact, positive ν measure subset $\Lambda \subset \mathrm{supp}(\nu)\backslash\mathrm{Sing}(Y)$ satisfying the following property. For any $\varepsilon > 0$, there are L, L', $\delta > 0$ such that, for any x and $\phi_T^Y(x) \in \Lambda$ with $T > L'$ and $d(x, \phi_T^Y(x)) < \delta$, there exist a point $p \in M$ and a C^1 strictly increasing function $\theta : [0, T] \to \mathbb{R}$ such that:*

(a) $\theta(0) = 0$ *and* $1 - \varepsilon < \theta'(t) < 1 + \varepsilon$;

(b) p *is a periodic sink of* ϕ^Y *with period* $\tau(p) = \theta(T)$;

(c) $\mathrm{Orb}_Y(p)$ *has uniform size of stable manifold at* p.

Now let us conclude the proof of Lemma 2.20.

Proof of Lemma 2.20. Fix Λ, ε, δ, L, $L' > 0$ as in Theorem 5.1. By Birkhoff's ergodic theorem, there are $x \in \Lambda$ and $L' < t_1 < t_2 < \cdots$ such that:

(i) $\phi_{t_i}^Y(x) \in \Lambda$ for any $i > 0$ and $\phi_{t_i}^Y(x) \to x$;

(ii) $t_{i+1}(1-\varepsilon) > t_i(1+\varepsilon)$.

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Applying Theorem 5.1(a), (b), each pseudo-orbit $(\phi_t^Y(x))_{t\in[0,t_i]}$ is shadowed by a periodic sink p_i with period $(1-\varepsilon)t_i < \tau(p_i) < (1+\varepsilon)t_i$. Then by choice (ii) above, all the periods are different, which implies that all these periodic sinks are distinct. Moreover, by Theorem 5.1(c), every periodic sink $\mathrm{Orb}(p_i)$ has uniform size of stable manifold at p_i for each i. Because the basins of distinct periodic sinks are disjoint, the previous discussion implies that the volume of this manifold is infinite, which is a contradiction. The proof is complete. □

5.1. *Proof of Theorem 5.1.* To prove Theorem 5.1 we need to use the scaled linear Poincaré flow.

Definition 5.2. The *scaled linear Poincaré flow* $\psi_t^* : \mathcal{N}_M \to \mathcal{N}_M$ is a flow such that, for any $v \in \mathcal{N}_x$, where $x \in M\backslash\mathrm{Sing}(Y)$,

$$\psi_t^*(v) = \frac{\|Y(x)\|}{\|Y(\phi_t(x))\|}\psi_t(v) = \frac{\psi_t(v)}{\|\Phi_t|_{\langle Y(x)\rangle}\|}.$$

LEMMA 5.3. *ψ_t^* is a bounded cocycle on $\mathcal{N}_M$ in the following sense: for any $\tau > 0$, there is $C_\tau > 0$ such that for any $t \in [-\tau, \tau]$,*

$$\|\psi_t^*\| \le C_\tau.$$

Moreover, the two cocycles ψ_t and ψ_t^* have the same Lyapunov exponent.

Proof. The uniform boundedness comes from the boundedness of Φ_t for $t \in [-\tau, \tau]$.

For ν-almost every x, we have $\lim \log\|\Phi_t(Y(x))\|/t = 0$. This implies that both cocycles ψ_t and ψ_t^* have the same Lyapunov exponent for the measure ν. □

We now show the strategy of the proof of Theorem 5.1. The proof is divided into two parts. The purpose of the first part (in §5.1.1) is to find pseudo-orbits such that the scaled linear Poincaré flows restricted on these pseudo-orbits have 'weak hyperbolicity' (see (1)). In the second part (§5.1.2) we explain the idea of the proof of Liao's shadowing lemma for the pseudo-orbits obtained above: we will take sections at the points where the 'hyperbolicity' can be seen, and analyze the holonomy maps between these sections; it turns out that these maps are uniformly continuous and have some 'hyperbolicity' (see (2)), and then one may apply Liao's shadowing lemma for discrete maps (see [**5**], for example).

5.1.1. *Quasi-contracting.*

Definition 5.4. An orbit arc $\phi^Y_{[0,T]}(x)$ is called $(\lambda, T_0)^*$*-quasi-contracting* with respect to the scaled linear Poincaré flow ψ_t^* if there exist $0 < \lambda < 1$ and a partition

$$0 = t_0 < t_1 < \cdots < t_l = T \quad \text{where } t_{i+1} - t_i \in [T_0, 2T_0],$$

such that, for $k = 0, 1, \ldots, l-1$,

$$\prod_{i=0}^{k-1} \|\psi^*_{t_{i+1}-t_i}|_{\mathcal{N}_{\phi_{t_i}(x)}}\| = \|\psi^*_{t_{i+1}}|_{\mathcal{N}_x}\| \le \lambda^k. \tag{1}$$

We now state a C^1 version of the Pesin block in the following sense.

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LEMMA 5.5. *There are L', η, $T_0 > 0$ and a positive ν measure subset $\Lambda \backslash \mathrm{Sing}(X)$ such that, for any x, $\phi_T^Y(x) \in \Lambda$ with $T > L'$, $\phi_{[0,T]}^Y(x)$ is $(\eta, T_0)^*$-quasi-contracting with respect to the scaled linear Poincaré flow ψ_t^*.*

Proof. Denote by $\lambda < 0$ the Lyapunov exponent of ν for ϕ^Y. Then, by Lemma 5.3, for ν-almost every x and $0 \neq v \in \mathcal{N}_x$,

$$\lim_{t\to\infty} \frac{1}{t} \log \|\psi_t^*(v)\| = \lambda.$$

For ν-almost every x, there is $t_x > 0$ such that, for any $t > t_x$,

$$\|\psi_t^*(v)\| \leq \exp(t\lambda/2).$$

Take L' sufficiently large such that $\nu(\{x;\, t_x \leq L'\}) > 0$. It is easy to check that $T_0 = 1$, $\eta = \exp(\lambda/2)$ and L' satisfy this lemma. □

5.1.2. *Liao's shadowing lemma for scaled linear Poincaré flow.* In this subsection we introduce Liao's shadowing lemma for scaled linear Poincaré flow [**14**].

THEOREM 5.6. *Given a compact set $\Lambda \cap \mathrm{Sing}(Y) = \emptyset$, and $\eta > 0$, $T_0 > 0$, for any $\varepsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that, for any $(\eta, T_0)^*$-quasi-contracting orbit arc $\phi_{[0,T]}(x)$ with respect to the scaled linear Poincaré flow ψ_t^* which satisfies x, $\phi_T^Y(x) \in \Lambda$ and $d(x, \phi_T(x)) \leq \delta$, there exist a point $p \in M$ and a C^1 strictly increasing function $\theta : [0, T] \to \mathbb{R}$ such that:*

(a) $\theta(0) = 0$ *and* $1 - \varepsilon < \theta'(t) < 1 + \varepsilon$;

(b) p *is a periodic sink with* $\tau(p) = \theta(T)$;

(c) $\mathrm{Orb}(p)$ *has uniform size of stable manifold at* p.

Proof. For simplicity, we assume that M is an open set in R^2, which implies that, for every regular point $x \in M$, $\mathcal{N}_x$ is a line. For $\beta > 0$, we denote by $\mathcal{N}_x(\beta)$ the segment contained in $\mathcal{N}_x$ with center x and radius β.

The first step is to translate the problem for flow to a sequence of one-dimensional maps: by [**6**, Lemmas 2.2, 2.3], there is β depending on T_0 such that the holonomy map induced by ϕ^Y is well defined between

$$\mathcal{P}_{x,\phi_t^Y(x)} : \mathcal{N}_x(\beta \|Y(x)\|) \to \mathcal{N}_{\phi_t^Y(x)} \quad \text{for any } t \in [T_0, 2T_0],$$

which is conjugate to the map

$$\mathcal{P}^*_{x,\phi_t^Y(x)}, \mathcal{N}_x(\beta) \to \mathcal{N}_{\phi_t^Y(x)} : \mathcal{P}^*_{x,\phi_t^Y(x)}(a) = \frac{\mathcal{P}_{x,\phi_t^Y(x)}(a\|Y(x)\|)}{\|Y(\phi_t^Y(x))\|}.$$

The tangent map of $\mathcal{P}^*_{x,\phi_t^Y(x)}$ is uniformly continuous.

Take the sequence of times $0 = t_0 < t_1 < \cdots < t_l = T$ in the definition of quasi-contracting orbit. We consider the local one-dimensional diffeomorphisms induced by the holonomy maps:

$$\mathcal{T}_i = \mathcal{P}^*_{\phi_{t_{i-1}}^Y(x),\phi_{t_i}^Y(x)} \quad \text{for } 1 \leq i \leq l-1;$$

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and $\mathcal{T}_l : \mathcal{N}_{\phi^Y_{t_{l-1}}(x)}(\beta) \to \mathcal{N}_x$ is defined by

$$\mathcal{T}_l = \mathcal{H} \circ \mathcal{P}^*_{\phi^Y_{t_{l-1}}(x), \phi^Y_{t_l}(x)},$$

where $\mathcal{H} : \mathcal{N}_{\phi^Y_T(x)} \to \mathcal{N}_x$ is the holonomy map induced by the flow. Because $d(x, \phi^Y_T(x))$ is sufficiently small, the last holonomy map $\mathcal{T}_l$ is well defined. For each $1 \le i \le l$,

$$T\mathcal{T}_i(0) = \psi^*_{t_i - t_{i-1}} | \mathcal{N}_{\phi^Y_{t_{i-1}}(x)}.$$

Then, by the definition of quasi-contracting, there is $0 < \lambda < 1$ such that, for $k = 1, \ldots, l$,

$$\prod_{i=1}^{k} \|\mathcal{T}_i\| \le \lambda^k. \tag{2}$$

We now apply the version of Liao's shadowing lemma on discrete quasi-hyperbolic maps [**5**, Theorem 1.1]; see also [**11**, **13**, **15**]. On the $d(x, \phi^Y_T(x))$-pseudo-orbit $\{0_x, 0_{\phi^Y_{t_1}(x)}, \ldots, 0_{\phi^Y_{t_{l-1}}(x)}\}$, there is $L > 0$ and a periodic point $p \in \mathcal{N}_x$ for the maps $\{\mathcal{T}_0, \ldots, \mathcal{T}_{l-1}\}$, whose orbit $Ld(x, \phi^Y_T(x))$-shadows the pseudo-orbit. Moreover, $\mathcal{T}_l \circ \cdots \circ \mathcal{T}_0$ has uniform size of basin at p. Using the flow tubular theorem [**6**, Lemmas 2.2], one concludes the proof of Theorem 5.6. □

5.1.3. *Proof of Theorem 5.1.* It is easy to see that Theorem 5.1 is a direct corollary of Lemma 5.5 and Theorem 5.6.

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A. *Appendix. Proof of Lemma 2.12*

For $s \in [0, t]$, denote $x_s = \phi^Y_s(x)$. We have

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s_0} (\ln \det \Phi_s|_{T_x M}) &= \lim_{\Delta s \to 0} \frac{1}{\Delta s} (\ln \det \Phi_{s_0+\Delta s}|_{T_x M} - \ln \det \Phi_{s_0}|_{T_x M}) \\
&= \lim_{\Delta s \to 0} \frac{1}{\Delta s} (\ln \det \Phi_{\Delta s}|_{T_{x_{s_0}} M}) \\
&= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \ln \det \begin{vmatrix} \left.\frac{\partial f_{\Delta s}}{\partial x}\right|_{x_{s_0}} & \left.\frac{\partial g_{\Delta s}}{\partial x}\right|_{x_{s_0}} \\ \left.\frac{\partial f_{\Delta s}}{\partial y}\right|_{x_{s_0}} & \left.\frac{\partial g_{\Delta s}}{\partial y}\right|_{x_{s_0}} \end{vmatrix} \\
&= \left.\frac{d}{d\Delta s}\right|_{\Delta s=0} \ln \left(\frac{\partial f_{\Delta s}}{\partial x} \frac{\partial g_{\Delta s}}{\partial y} - \frac{\partial g_{\Delta s}}{\partial x} \frac{\partial f_{\Delta s}}{\partial y} \right) \Bigg|_{x_{s_0}} \\
&= \frac{\frac{d}{d\Delta s}|_{\Delta s=0} \left(\frac{\partial f_{\Delta s}(x_{s_0})}{\partial x} \frac{\partial g_{\Delta s}(x_{s_0})}{\partial y} - \frac{\partial g_{\Delta s}(x_{s_0})}{\partial x} \frac{\partial f_{\Delta s}(x_{s_0})}{\partial y} \right)}{\det \Phi_0|_{T_{x_{s_0}} M}}.
\end{aligned}$$

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Note that

$$\frac{d}{d\Delta s}\bigg|_{\Delta s=0}\left(\frac{\partial f_{\Delta s}(x_{s_0})}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{d}{d\Delta x}\bigg|_{\Delta s=0}(f_{\Delta s}(x_{s_0}))\right)=\frac{\partial}{\partial x}(Y_1(x_{s_0})).$$

Similarity, we have that

$$\begin{aligned}\frac{d}{d\Delta s}\bigg|_{\Delta s=0}\left(\frac{\partial g_{\Delta s}(x_{s_0})}{\partial y}\right)&=\frac{\partial}{\partial x}(Y_2(x_{s_0})),\\ \frac{d}{d\Delta s}\bigg|_{\Delta s=0}\left(\frac{\partial g_{\Delta s}(x_{s_0})}{\partial x}\right)&=\frac{\partial}{\partial y}(Y_1(x_{s_0})),\\ \frac{d}{d\Delta s}\bigg|_{\Delta s=0}\left(\frac{\partial f_{\Delta s}(x_{s_0})}{\partial y}\right)&=\frac{\partial}{\partial y}(Y_2(x_{s_0})).\end{aligned}$$

Because $\Phi_0|_{T_{x_{s_0}}M}=\mathrm{ID}$, we have

$$\begin{pmatrix}\dfrac{\partial f_{\Delta s}(x_{s_0})}{\partial x} & \dfrac{\partial g_{\Delta s}(x_{s_0})}{\partial x}\\ \dfrac{\partial f_{\Delta s}(x_{s_0})}{\partial y} & \dfrac{\partial g_{\Delta s}(x_{s_0})}{\partial y}\end{pmatrix}_{\Delta s=0}=\mathrm{ID}.$$

By the above discussion, we have that

$$\frac{d}{ds}\bigg|_{s_0}(\ln\det\Phi_s|_{T_xM})=\frac{\partial Y_1}{\partial x}+\frac{\partial Y_2}{\partial y}=\mathrm{div}_Y(x_{s_0}).$$

Hence,

$$\ln\det\Phi_t|_{T_xM}=\int_0^t\frac{d}{ds}(\ln\det\Phi_s|_{T_xM})\,ds=\int_0^t\mathrm{div}_Y(x_s)\,ds.$$

The proof is complete.

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