Multisection of series

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1. *Introduction*

In a recent paper [1], the authors gave a combinatorial interpretation to sums of equally spaced binomial coefficients. Others have been interested in finding such sums, known as multisection of series. For example, Gould [2] derived interesting formulas but much of his work involved complicated manipulations of series. When the combinatorial approach can be implemented, it is neat and efficient. In this paper, we present another approach for finding equally spaced sums. We consider both infinite sums and partial finite sums based on generating functions and extracting coefficients.

While generating functions were first introduced by Abraham de Moivre at the end of seventeen century, its systematic use in combinatorial analysis was inspired by Leonhard Euler. Generating functions got a new birth in the twentieth century as a part of symbolic methods. As a central mathematical tool in discrete mathematics, generating functions are an essential part of the curriculum in the analysis of algorithms [3, 4]. They provide a bridge between discrete and continuous mathematics, as illustrated by the fact that the generating functions presented here appear as solutions to corresponding differential equations.

The multisection method that we describe goes back almost 250 years to Simpson [5], who evaluated equally spaced sums. Our goal is to acquaint the reader with the multisection method and to illustrate the beauty and power of generating functions in calculating sums of evenly spaced terms for various types of sums.

2. *Motivating examples*

An ordinary generating function is simply a formal power series centred at the origin; it is said to generate its sequence of coefficients. In general, computations with these series are purely *formal*. For example, consider the geometric series $F(x) = \sum_{k\geqslant 0}(rx)^k$. The familiar computation $F(x) - rxF(x) = 1$ leads to the well-known formula $\sum_{k\geqslant 0}(rx)^k = 1/(1-rx)$, $rx \neq 1$. Of course this geometric series converges when $|rx| < 1$ but as a rule, generating functions are not concerned with questions of convergence. However, most eminent results using generating functions are obtained when their sums are known. Reference [6] gives an excellent introduction to generating functions. Before describing how multisection works, we look at two examples.

Example 1: Our first example is concerned with the generating function for the sequence $\{1/k!\}$, which is the Maclaurin series for the exponential function, $e^x = \sum_{k\geqslant 0} x^k/k!$ (and is a solution to the differential equation

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$y' = y$). If we sum the even terms and sum the odd terms of this series, we obtain two familiar functions:

$$\sum_{k\geqslant 0}\frac{x^{2k}}{(2k)!}=\frac{e^x+e^{-x}}{2}=\cosh x,\qquad \sum_{k\geqslant 0}\frac{x^{2k+1}}{(2k+1)!}=\frac{e^x-e^{-x}}{2}=\sinh x.$$

Note that these two functions form a fundamental set of solutions to the differential equation $y'' = y$.

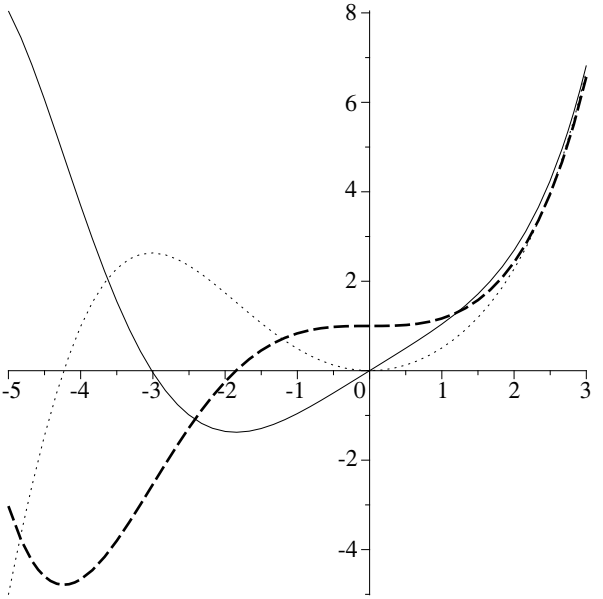


FIGURE 1: Graphs of the functions:
$\sum_{k\geqslant 0}x^{3k}/(3k)!$, dashed curve; $\sum_{k\geqslant 0}x^{3k+1}/(3k+1)!$, solid curve; and $\sum_{k\geqslant 0}x^{3k+2}/(3k+2)!$, dotted curve, plotted with *Maple*.

What sort of function does the sum of every third term in the e^x series produce? To this end, let ω be a primitive cube root of unity, that is, $\omega = e^{2\pi i/3}$, where i is the complex number represented by the unit vector from the origin in the vertical direction in the complex plane. Notice that ω. satisfies the equation $z^3 - 1 = (z-1)(z^2+z+1) = 0$. Thus

$$0 = 1+\omega+\omega^2 = 1+\omega^{-1}+\omega^{-2}.$$

Using this identity, we add the series expansions for e^x, $e^{\omega x}$, $e^{\omega^2 x}$, and divide by 3. The result is

$$\sum_{k\geqslant 0}\frac{x^{3k}}{3k!}=\frac{1}{3}\left(e^x+e^{\omega x}+e^{\omega^2 x}\right)=\frac{1}{3}\left(e^x+2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right).$$

In a similar way, we can compute the sums of every third term of $\sum_{k\geqslant 0}x^k/k!$ starting at $k = 1$ and then $k = 2$ as follows:

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$$\sum_{k\geqslant 0}\frac{x^{3k+1}}{(3k+1)!}=\frac{1}{3}\left(e^x+\omega^{-1}e^{\omega x}+\omega^{-2}e^{\omega^2 x}\right)=\frac{e^x}{3}+\frac{e^{-x/2}}{3}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}x\right)-\cos\left(\frac{\sqrt{3}}{2}x\right)\right);$$

$$\sum_{k\geqslant 0}\frac{x^{3k+2}}{(3k+2)!}=\frac{1}{3}\left(e^x+\omega^{-2}e^{\omega x}+\omega^{-4}e^{\omega^2 x}\right)=\frac{e^x}{3}-\frac{e^{-x/2}}{3}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}x\right)+\cos\left(\frac{\sqrt{3}}{2}x\right)\right).$$

These functions are displayed in Figure 1. Notice that for large x the graphs are very close and their sum approaches e^x, but when x moves to the left away from 1 the graphs diverge. These three functions form a fundamental set of solutions to the differential equation $y''' = y$.

Example 2: Consider the well known generating function for the binomial coefficients, namely

$$(1+x)^n=1+nx+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+\ \ldots+\binom{n}{n-1}x^{n-1}+x^n, \qquad (1)$$

where n is a positive integer. Note that $\binom{n}{k}=0$ if $k>n$. With $x=1$ in (1), we have the familiar expression for 2^n as a sum of binomial coefficients. If you replace x by $-x$ in (1), add the new equation to (1), and then multiply the sum by $\frac{1}{2}$, you wind up with the sum

$$y_1(x)=\sum_{k\geqslant 0}\binom{n}{2k}x^{2k}=\frac{1}{2}\left[(1+x)^n+(1-x)^n\right]$$

of every second term in (1). Notice that setting x equal to 1, you get

$$y_1(1)=\sum_{k\geqslant 0}\binom{n}{2k}=2^{n-1}$$

as expected. Similarly, the series of odd powers from (1) is

$$y_2(x)=\sum_{k\geqslant 0}\binom{n}{2k+1}=\frac{1}{2}\left[(1+x)^n-(1-x)^n\right].$$

The functions $y_1(x)$, $y_2(x)$ form a fundamental set of solutions to the differential equation

$$(1-x^2)y''+2x(n-1)y'-n(n-1)y=0.$$

We can access the sum of every third term using computations with $\omega = e^{2\pi i/3}$ similar to those in the previous example.

In (1), we replace x by ωx and then $\omega^2 x$ to obtain respectively

$$(1+\omega x)^n=1+n\omega x+\binom{n}{2}\omega^2x^2+\binom{n}{3}x^3+\binom{n}{4}\omega x^4+\binom{n}{5}\omega^2x^5+\binom{n}{6}x^6+\ldots+(\omega x)^n,$$

$$(1+\omega^2 x)^n=1+n\omega^2 x+\binom{n}{2}\omega x^2+\binom{n}{3}x^3+\binom{n}{4}\omega^2x^4+\binom{n}{5}\omega x^5+\binom{n}{6}x^6+\ldots+(\omega^2 x)^n.$$

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Adding these three binomial equations and dividing by 3, we obtain the finite sum

$$y_1(x) = \frac{(1+x)^n + (1+\omega x)^n + (1+\omega^2 x)^n}{3} = \sum_{k\geqslant 0}\binom{n}{3k}x^{3k}. \qquad (2)$$

The sum $y_2(x) = \sum_{k\geqslant 0}\binom{n}{3k+1}x^{3k+1}$ can be found by computing

$$y_2(x) = \frac{(1+x)^n + \omega^{-1}(1+\omega x)^n + \omega^{-2}(1+\omega^2 x)^n}{3} = \sum_{k\geqslant 0}\binom{n}{3k+1}x^{3k+1}.$$

Similarly, we find

$$y_3(x) = \frac{(1+x)^n + \omega^{-2}(1+\omega x)^n + \omega^{-4}(1+\omega^2 x)^n}{3} = \sum_{k\geqslant 0}\binom{n}{3k+2}x^{3k+2}.$$

These three functions form a fundamental set of solutions for the third order differential equation

$$(1+x^3)y''' - 3x^2(n-2)y'' + 3x(n-1)(n-2)y' - n(n-1)(n-2)y = 0.$$

They are displayed in Figure 2 for $n = 7$.

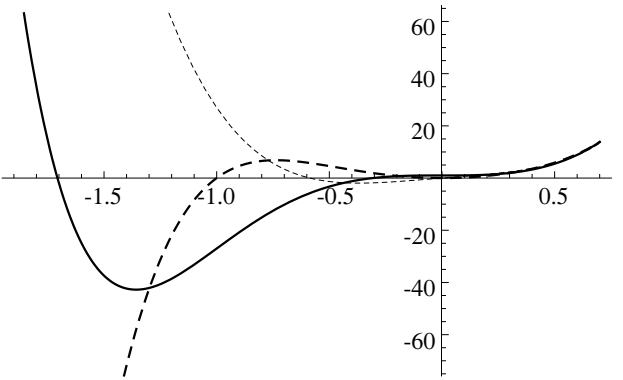


FIGURE 2: Graphs of the functions for $n = 7$: $\sum_{k\geqslant 0}\binom{n}{3k}x^{3k} = 1 + 35x^3 + 7x^6$, solid curve; $\sum_{k\geqslant 0}\binom{n}{3k+1}x^{3k+1} = 7x + 35x^4 + x^7$, dashed curve; and $\sum_{k\geqslant 0}\binom{n}{3k+2}x^{3k+2} = 21x^2 + 21x^5$, dotted curve, plotted with *Mathematica*.

Setting $x = 1$ and writing $\omega = \frac{-1+i\sqrt{3}}{2}$, $\omega^2 = \bar{\omega} = \frac{-1-i\sqrt{3}}{2}$, (2) yields

$$\sum_{k\geqslant 0}\binom{n}{3k} = \frac{1}{3}\left(2^n + \left(\frac{1+i\sqrt{3}}{2}\right)^n + \left(\frac{1-i\sqrt{3}}{2}\right)^n\right).$$

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Since

$$\left(\frac{1+i\sqrt{3}}{2}\right) = e^{\pi i/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \text{ and} \left(\frac{1-i\sqrt{3}}{2}\right) = e^{-\pi i/3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3},$$

the preceding displayed equation reduces to the nice formula

$$\sum_{k\geqslant 0}\binom{n}{3k} = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right).$$

3. *The general multisection method*

The previous examples illustrate a general procedure for finding the sum of equally spaced terms in a series $\sum_{k\geqslant 0} a_k z^k$. For integers r and l with $0 \leqslant l < r$, we shall find the sum $\sum_{j\geqslant 0} a_{rj+l} z^{rj+l}$ of every rth term starting at the lth term. In Example 1, we worked with ω, the cube root of unity. Now let ω_r be a primitive rth root of unity, that is, $\omega_r = e^{2\pi i/r}$. Since ω_r satisfies the equation

$$z^r - 1 = (z-1)\left(1 + z + z^2 + \ldots + z^{r-1}\right) = 0,$$

we have

$$1 + \omega_r + \omega_r^2 + \ldots + \omega_r^{r-1} = 0.$$

Since $\omega^{rj} = 1$, this primitive root satisfies the following 'circular' (orthogonal) relation

$$\sum_{j=0}^{r-1}\left(\omega_r^k\right)^j = \sum_{j=0}^{r-1}\omega_r^{kj} = \begin{cases} r, & \text{when } k \text{ is a multiple of } r, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Now let $A(z) = \sum_{k\geqslant 0} a_k z^k$ be a generating function for the sequence $\{a_k\}_{k\geqslant 0}$. The notation $[z^n]A(z)$ that we use below stands for the nth coefficient in the Maclaurin expansion of $A(z)$. Thus $[z^n]\sum_{k\geqslant 0} a_k z^k = a_n$. Using (3) we will find a generating function for equally spaced coefficients. For any non-negative integer s, we have

$$A\left(\omega_r^s\right) = \sum_{k\geqslant 0} a_k\left(\omega_r^s\right)^k = \sum_{k\geqslant 0} a_k\left(\omega_r^{ks}\right).$$

Multiplying the latter by ω_r^{-sl} and summing with respect to s, we get

$$\sum_{s=0}^{r-1} A\left(\omega_r^s\right)\omega_r^{-sl} = \sum_{s=0}^{r-1}\sum_{k\geqslant 0} a_k\omega_r^{ks-sl} = \sum_{k\geqslant 0} a_k\sum_{s=0}^{r-1}\omega_r^{ks-sl}. \tag{4}$$

Using (3), we see that

$$\sum_{s=0}^{r-1}\omega_r^{ks-ls} = \sum_{s=0}^{r-1}\omega_r^{(k-l)s} = \begin{cases} r, & \text{when } k-l \text{ is a multiple of } r, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if $k - l = jr$ for some integer j, we have

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$$\sum_{s=0}^{r-1} \omega_r^{ks-ls} = r \text{ for } k = jr + l.$$

If $k \neq jr + l$, all terms in the sum will be zero. Equation (4) may then be written as

$$\sum_{j \geqslant 0} a_{rj+l} = \frac{1}{r} \sum_{s=0}^{r-1} A(\omega_r^s)\, \omega_r^{-sl}. \tag{5}$$

This formula is obtained in [1] for the special case where $a_j = \binom{n}{j}$. When $l = 0$, we get

$$\sum_{j \geqslant 0} a_{rj} = \frac{1}{r} \sum_{s=0}^{r-1} A(\omega_r^s).$$

Formula (5) can be easily extended, yielding the following generating function for every r th term of the series $A(z) = \sum_{k \geqslant 0} a_k z^k$ starting with the l th term:

$$\sum_{j \geqslant 0} a_{rj+l} z^{rj+l} = \frac{1}{r} \sum_{s=0}^{r-1} A(\omega_r^s z)\, \omega_r^{-sl}. \tag{6}$$

This formula with $r = 3$ is illustrated with the examples in the preceding section.

4. *Finite Sums*

The examples in Section 2 reveal an interesting phenomenon: Example 1 involving infinite sums is simpler than Example 2 involving finite sums. As a rule, finding *finite* sums is much harder than infinite sums. Nevertheless, the multisection method can be used when an explicit (closed form) formula for a generating function is known for the entire sequence. Its application to finite sums is based on the fact that $A(z)/(1 - z)$ will be a generating function for the partial sums of the series $A(z)$. In other words,

$$\sum_{k=0}^{n} a_k = [z^n] \frac{A(z)}{1 - z}, \qquad \text{where } A(z) = \sum_{k \geqslant 0} a_k z^k. \tag{7}$$

Let us show that this formula is correct. The generating function for the sequence of partial sums $\sum_{k=0}^{n} a_k$ is

$$\sum_{n \geqslant 0} \left(\sum_{k=0}^{n} a_k \right) z^n = \sum_{k \geqslant 0} \sum_{n \geqslant k} a_k z^n = \sum_{k \geqslant 0} a_k \frac{z^k}{1 - z} = \frac{1}{1 - z} \sum_{k \geqslant 0} a_k z^k = \frac{A(z)}{1 - z}.$$

To illustrate this formula, let us derive the well-known formula for the sum of a finite geometric series $\sum_{k=0}^{n} q^k$. Since the generating function for the sequence $\{q^k\}$ is $\sum_{k \geqslant 0} (zq) = 1/(1 - zq)$, we apply (7), use partial fractions

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for $\dfrac{1}{(1-z)(1-zq)}$, and obtain

$$\begin{aligned}\sum_{k=0}^{n} q^k &= [z^n]\frac{1}{(1-z)}\frac{1}{(1-zq)}\\ &= [z^n]\frac{1}{(1-q)(1-z)} - [z^n]\frac{q}{(1-q)(1-zq)}\\ &= \frac{1}{1-q}[z^n]\sum_{k\geqslant 0} z^k - \frac{q}{1-q}[z^n]\sum_{k\geqslant 0}(qz)^k \qquad (8)\\ &= \frac{1-q^{n+1}}{1-q}.\end{aligned}$$

Notice that this *finite-sum* result follows by extracting nth coefficients using the corresponding formula for the sum of an infinite geometric series.

Now suppose we want to find a finite sum $\sum_{j=0}^{n} a_{rj}$ of equally spaced terms for which the generating function $\sum_{n\geqslant 0}\left(\sum_{j=0}^{n} a_{rj}\right)z^n$ is not immediately known, but one is known for the sequence $\{a_k\}$, namely, $A(z) = \sum_{k\geqslant 0} a_k z^k$. Then we can extend (7) to this case and obtain

$$\sum_{j=0}^{n} a_{rj} = \frac{1}{r}[z^{rn}]\frac{1}{1-z^r}\sum_{s=0}^{r-1} A\left(\omega_r^s z\right) = [z^{rn}]\frac{A_r(z)}{1-z^r}, \qquad (9)$$

where $A_r(z) = \sum_{j\geqslant 0} a_{rj}z^{rj} = \dfrac{1}{r}\sum_{s=0}^{r-1} A\left(\omega_r^s z\right)$ ((6) with $l = 0$). Rather than apply (9), we notice that $A_r(z)$ contains only powers of z^r. Thus

$$G_r(z) = A_r\left(z^{1/r}\right) = \sum_{j\geqslant 0} a_{rj}z^j$$

is also a generating function for the sequence $\{a_{rj}\}_{j\geqslant 0}$. Using (7) with $G_r(z)$ in place of $A(z)$, we obtain

$$\sum_{j=0}^{n} a_{rj} = [z^n]\frac{G_r(z)}{1-z}. \qquad (10)$$

As a practical example, suppose we plan to invest P dollars at the start of each month at an annual interest rate of $t\%$, compounded monthly. At the end of m months, and with the next deposit, we would have

$$P\sum_{k=0}^{m} q^k = \frac{P\left(1-q^{m+1}\right)}{1-q} \text{ dollars,}$$

where $q = 1 + \dfrac{t}{1200}$. But our times demand austerity and we can only

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afford to invest P dollars every other month. After $2m$ months the total would then be $P\sum_{k=0}^{m} q^{2k}$. Let us evaluate the sum using the multisection method. Since the generating function for the sequence $\{q^k\}$ is $A(z) = 1/(1 - qz)$, we compute $\frac{1}{2}\big(A(z) + A(-z)\big) = \sum_{k=0}^{\infty} q^{2k}z^{2k}$ and choose the generating function $G_2(z) = \sum_{j\geqslant 0} q^{2j}z^j$. Because of (10), we extract coefficients from the partial fraction decomposition of

$$\frac{G_2(z)}{1 - z} = \frac{1}{(1 - z)(1 - q^2z)}.$$

Of course, the result will be the same as that given by (8) but with n replaced by m and q replaced by q^2. In the following section we give less transparent examples involving finite sums.

5. *Fibonacci Sums*

Recall that the nth Fibonacci number F_n is defined as a solution of the second order recurrence $F_n = F_{n-1} + F_{n-2}$ subject to the initial conditions $F_0 = 0$ and $F_1 = 1$. It is well known [3, 6] that the sequence $\{F_n\}$ is generated by the function

$$F(x) = \sum_{k\geqslant 0} F_k x^k = \frac{x}{1 - x - x^2} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \ldots.$$

Now $1 - x - x^2 = (1 - \phi x)(1 - \varphi x)$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio and $\varphi = -1/\phi = \frac{1}{2}(1 - \sqrt{5})$. Thus

$$F(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}}\left[\frac{1}{1 - \phi x} + \frac{1}{1 - \varphi x}\right].$$

Using the geometric series formula $\sum_{n\geqslant 0} q^n = (1 - q)^{-1}$, we see that the nth Fibonacci number is expressed via a Binet formula as $F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \varphi^n\right]$. Computations in the following examples can be verified with pencil and paper using the identities $\phi\varphi = -1$, $\phi + \varphi = 1$ and $\phi - \varphi = \sqrt{5}$.

Example 3: Let us find a formula for the sum $\mathcal{A}(n) = \sum_{k=0}^{n} F_k$. According to (7), the generating function for the sequence $\{\mathcal{A}(n)\}_{n\geqslant 0}$ is

$$A(x) = \sum_{n\geqslant 0} A(n)x^n = \frac{F(x)}{1 - x}.$$

The partial fraction decomposition for $A(x)$ becomes

$$A(x) = \frac{x}{(1-x)(1-x-x^2)} = \frac{-1}{1-x} + \frac{1}{\sqrt{5}}\left[\frac{-\varphi^2}{1-\varphi x} + \frac{\phi^2}{1-\phi x}\right]. \qquad (11)$$

We extract coefficients and obtain

$$\mathcal{A}(n) = [x^n]\frac{x}{(1 - x)(1 - x - x^2)} = -1 + \frac{1}{\sqrt{5}}\left[-\varphi^{n+2} + \phi^{n+2}\right].$$

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Since $F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \varphi^n\right]$, we have

$$\mathcal{A}(n) = \sum_{k=0}^{n} F_k = F_{n+2} - 1. \tag{12}$$

Example 4: Let us separate the even and odd Fibonacci numbers and find the finite sums $\mathcal{B}_{\text{even}}(n) = \sum_{k=0}^{n} F_{2k}$ and $\mathcal{B}_{\text{odd}}(n) = \sum_{k=0}^{n} F_{2k+1}$. The corresponding generating functions are

$$\begin{aligned}
\mathcal{B}_{\text{even}}(x) &= \sum_{k=0}^{n} F_{2k}x^{2k} = \frac{F(x) + F(-x)}{2} \\
&= \frac{x}{2\left(1 - x - x^2\right)} - \frac{x}{2\left(1 + x - x^2\right)} = \frac{x^2}{1 - 3x^2 + x^4}, \\
\mathcal{B}_{\text{odd}}(x) &= \sum_{k=0}^{n} F_{2k+1}x^{2k+1} = \frac{F(x) - F(-x)}{2} \\
&= \frac{x}{2\left(1 - x - x^2\right)} + \frac{x}{2\left(1 + x - x^2\right)} = \frac{x\left(1 - x^2\right)}{1 - 3x^2 + x^4}.
\end{aligned}$$

Now we transition to the simpler generating functions:

$$\mathcal{G}_{\text{even}}(x) = \sum_{k\geqslant 0} F_{2k}x^k = \frac{x}{1 - 3x + x^2}, \qquad \mathcal{G}_{\text{odd}}(x) = \sum_{k\geqslant 0} F_{2k+1}x^k = \frac{1 - x}{1 - 3x + x^2}.$$

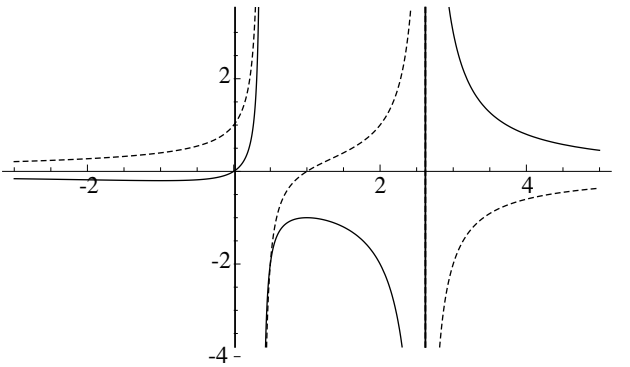


FIGURE 3: Graphs of the functions: $\mathcal{G}_{\text{even}}(x) = \sum_{k\geqslant 0} F_{2k}x^k = \frac{x}{1-3x+x^2}$, solid curve, and $\mathcal{G}_{\text{odd}}(x) = \sum_{k\geqslant 0} F_{2k+1}x^k = \frac{1-x}{1-3x+x^2}$, dashed curve, plotted with *Mathematica*.

These two functions, displayed in Figure 3, form a fundamental set of solutions to the second order differential equation:

$$\left(1 - 3x + x^2\right)y'' + 2(2x - 3)y' + 2y = 0.$$

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Following (10), we find the required finite sums by working with the functions

$$\mathcal{B}_{\text{even}}(x) = \sum_{k=0}^{n} F_{2k} = \frac{x}{(1 - 3x + x^2)(1 - x)},$$

$$\mathcal{B}_{\text{odd}}(x) = \sum_{k=0}^{n} F_{2k+1} = \frac{1 - x}{(1 - 3x + x^2)(1 - x)} = \frac{1}{1 - 3x + x^2}.$$

The zeroes of $1 - 3x + x^2$ are ϕ^2 and φ^2, and we note that $\phi^2 + \varphi^2 = 3$. Using partial fraction decomposition, we obtain

$$\mathcal{B}_{\text{even}}(x) = \frac{x}{(1-3x+x^2)(1-x)} = \frac{-1}{1-x} + \frac{1}{\sqrt{5}}\left[\frac{\phi}{1-\phi^2 x} - \frac{\varphi}{1-\varphi^2 x}\right]$$

$$\mathcal{B}_{\text{odd}}(x) = \frac{1}{1-3x+x^2} = \frac{1}{\sqrt{5}}\left[\frac{1}{\varphi^2 - x} - \frac{1}{\phi^2 - x}\right].$$

Extraction of coefficients leads to the formulas

$$\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1, \qquad \sum_{k=0}^{n} F_{2k+1} = F_{2n+2}. \tag{13}$$

Thus, the sum of the first n odd Fibonacci numbers is the next even one and the sum of the first n even Fibonacci numbers is the next odd Fibonacci number minus 1. The latter is evident comparing the generating functions $\mathcal{B}_{\text{odd}}(x)$ and $\mathcal{G}_{\text{even}}(x)$. Adding the equations (13) we find the sum of the first $2n + 1$ Fibonacci numbers to be $F_{2n+1} - 1 + F_{2n+2} = F_{2n+3} - 1$, consistent with formula (12).

The sum of every third Fibonacci number is given by $\sum_{k=0}^{n} F_{3k} = \frac{1}{2}F_{3n+2} - \frac{1}{2}$. Although this equation can be verified by mathematical induction, we close with an exercise for the reader to show its derivation using the methods we have described. Others have found a formula for the sum of every rth Fibonacci number using various methods (see [7], for example).

Exercise: Derive the formula for $\sum_{k=0}^{n} F_{3k}$ in the following four steps.

(a) Verify that

$$F(\omega x) + F(\omega^2 x) = \frac{\omega x}{1 - \omega x - (\omega x)^2} + \frac{\omega^2 x}{1 - \omega^2 x - (\omega^2 x)^2}$$

$$= \frac{x(x^2 - 2x - 1)}{1 + x + 2x^2 - x^3 + x^4},$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity, and where $F(x) = \dfrac{x}{1 - x - x^2}$ is a generating function for the sequence of Fibonacci numbers.

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(b) Observe that a generating function for the sequence $\{F_{3k}\}$ is

$$F_3(x) = \sum_{k \geqslant 0} F_{3k}x^{3k} = \frac{1}{3}\left[\frac{x}{1 - x - x^2} + \frac{x(x^2 - 2x - 1)}{1 + x + 2x^2 - x^3 + x^4}\right].$$

Verify that

$$G_3(x) = F_3\left(x^{1/3}\right) = \frac{2x}{1 - 4x - x^2},$$

which is also a generating function for the sequence $\{F_{3k}\}$.

(c) Find a partial fractions decomposition of $\dfrac{G_3(x)}{1 - x}$.

(d) Extracting coefficients from your result in part (c), derive the requested formula.

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