B.A.J. **3**, II, 411–482 (1997)

MATURITY GUARANTEES REVISITED: ALLOWING FOR EXTREME STOCHASTIC FLUCTUATIONS USING STABLE DISTRIBUTIONS

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ABSTRACT

The paper examines the suitability of the stable family of distributions with the Maturity Guarantees Working Party's stochastic investment model (Ford *et al.*, 1980). It then examines the effect of replacing the Gaussian assumption made by the working party with a more general stable distribution. It also explains how the appropriate stable distribution can be fitted.

KEYWORDS

Stochastic Models; Stable Distributions; Guarantees; Reserves; Asset Liability Modelling

1. INTRODUCTION, AIMS AND MOTIVATION

1.1 In 1977 the Institute and Faculty of Actuaries established a Joint Working Party to investigate into suitable reserving bases for investment guarantees under investment-linked life assurance business. The Maturity Guarantees Working Party (MGWP) reported in 1980 (Ford *et al.*, 1980).

1.2 Ford *et al.* (1980) recommended the use of the following stochastic investment model to represent the movements of the unit price R_t, of a United Kingdom equity unit. Let:

P_t = the unit price at time t (excluding reinvestment of income);
D_t = the gross dividend per unit received during period t;
Yp_t = the prospective dividend yield at the end of period t; and
τ = the rate of tax on dividends (assumed constant).

Then:

$$R_t = R_{t-1} . (P_t + (1 - \tau)D_t) / P_{t-1} \qquad (1.1)$$

where:

$$P_t = D_{t+1} / Yp_t$$

$$\log D_t = \log D_{t-1} + \mu_D + Z_D(t)$$

$$\log Yp_t = A(\log Yp_{t-1} - \log \mu_Y) + \log \mu_Y + Z_Y(t)$$

and where:

$Z_D(t)$ is an error term assumed to be Normally distributed with mean 0 and variance $\sigma_D^2 = 0.13^2$; and

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$Z_Y(t)$ is an error term assumed to be Normally distributed with mean 0 and variance $\sigma_Y^2 = 0.20^2$, and is independent of $Z_D(t)$.

The complete parameterisation of the model is summarised below:

Dividend Yields	Dividend Growth	Tax
$A = 0.6$		
$\mu_Y = 5\%$	$\mu_D = 4\%$	
$\sigma_Y^2 = 0.20^2$	$\sigma_D^2 = 0.13^2$	$\tau = 0.375$

In addition, to run the model, initial values for R_0, P_0, and Yp_0 need to be chosen. R_0 and P_0 can arbitrarily be set to 1.0, while Yp_0 should be set to an initial value reflecting the circumstances actually prevailing at time $t = 0$.

The above parameter values are the original values that were put forward by the MGWP. They were derived from an analysis of experience up to and including 1979. Some of them, notably the tax rate, would need to be revised for use today.

1.3 The salient feature under examination in this article is the assumption that the error terms, $Z_D(t)$ and $Z_Y(t)$, are Normally distributed. This assumption is common with most of the stochastic models that are in widespread use within the actuarial profession. A well-known example is the so-called Wilkie model (Wilkie, 1984, 1995).

In order to examine the effect of changing the assumption of Normally distributed error terms in isolation, it will be useful to fix the parameters proposed by the MGWP. A comparison can then be made of the results obtained from using the original MGWP model and those that would have been obtained had an alternative probability distribution been used. For completeness, updated parameter estimates will also be provided, but this is very much of peripheral importance to the paper.

The paper is not concerned with alternative formulae to those given in Section 1.2 (except to the extent that the definitions of $Z_D(t)$ and $Z_Y(t)$ will change).

1.4 There is strong empirical evidence (see Section D2.13 of Ford *et al.* (1980) and Section 3.4 of this paper) that the Normal distribution is inappropriate. The main objections to the use of the Normal distribution stem from two main problems which often result when a stochastic model is fitted to financial data:

— the sample residuals tend to be much more skewed than those consistent with a Normal distribution; and
— the sample residuals tend to have a much higher kurtosis (tail thickness) than the Normal.

1.5 The problem of insufficient kurtosis with the Normal distribution could be significant. This is because the weight of the tails of the distribution will determine the extent of stochastic fluctuations in the model. Some authors, including Ford *et al.* (1980), Wilkie (1984, 1995) and Carter (1991), attempt to

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overcome this problem by somewhat arbitrarily increasing the standard deviation parameter of the distribution. This, however, is only a partial solution; since, while it may introduce larger stochastic fluctuations more frequently, it also results in an inappropriate frequency of smaller variations. This is illustrated in Figure 1, which shows a Cauchy distribution, a Normal distribution and an adjusted Normal distribution. In addition, the specific adjustment to the standard deviation parameter cannot be supported by the data.

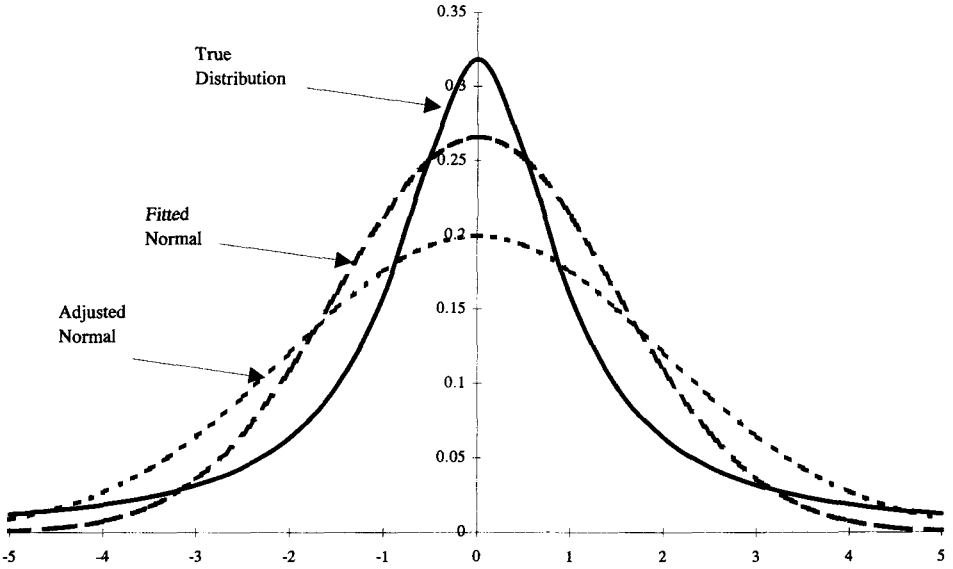


Figure 1.

1.6 The instability of sample standard deviations in financial data is discussed in Tenenbein (1988) and Becker (1991). The important point is that the volatility of a finite variance distribution, such as the Normal distribution, may not be sufficient to model the fluctuations of the actual investments.

1.7 A number of alternative (non-Normal) distributions have been proposed for use with financial data. These include the stable family (Mandelbrot, 1963; Fama & Roll, 1968, 1971; Tenenbein, 1988; Walter, 1991; Klein, 1993 and Wilkie, 1995), and Student's *t*-distribution (Praetz, 1972 and Blattberg & Gonedes, 1973). Geoghegan *et al.* (1992) also considered investigating heteroscedastic models. (These are models in which the standard deviation parameter at a point in time is itself obtained by sampling from a certain probability distribution.) Thomson (1994) used a number of different distributions in his model of various South African assets. In addition to the Normal distribution, Thomson uses a translated

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beta distribution for share dividend growth, and Student's *t*-distribution for long-term interest rates.

1.8 With the exception of Walter (1991) and Klein (1993), the stable distributions appear to have not yet found their way into widely used actuarial models (as confirmed in Geoghegan *et al.* (1992)). Investigation into the use of the stable distributions with actuarial models has been put forward as an area of further research by a number of authors including Ford *et al.* (1980), Wilkie (1984, 1995), Tenenbein (1988) and Geoghegan *et al.* (1992).

1.9 Previous attempts to model share prices with stable distributions appear to have concentrated on the simple univariate (log) price model. In addition, they concentrated on short-term (daily, weekly, monthly) price changes. The attempts include:

— Fama & Roll (1968) (U.S. data);
— Praetz (1972) (Australian data);
— Blattberg & Gonedes (1973) (U.S. data);
— Koutrouvelis (1980) (U.S. data); and
— Walter (1991) (French data).

Most of the attempts (all of the above except for Koutrouvelis) have restricted their attention to symmetric stable distributions. Fielitz & Smith (1972), and Leitch & Paulson (1975) investigated United States share log price changes, and concluded that the symmetry assumption is incorrect. Klein (1993) applied the symmetric stable distributions to modelling U.S. Treasury bond rates.

1.10 This paper will concentrate on the bivariate dividend yield/dividend growth model proposed by Ford *et al.* (1980), and the U.K. data on which it was based. No assumption about the symmetry of the probability distributions will be made.

1.11 The paper will have two aims:

(1) to determine how and whether a stable (non-Gaussian) distribution can be suitably fitted to the MGWP model; and
(2) to examine the effect of replacing the assumption of Normally distributed error terms by a more general stable distribution.

1.12 The remainder of the paper will be structured as follows:

— Section 2 : a review of the stable distributions;
— Section 3 : fitting the model;
— Section 4 : implications of using stable distributions; and
— Section 5 : conclusion.

1.13 A review of additional literature on the subject can be found in Finkelstein (1997). This review includes theoretical arguments why stable distributions may be more suitable than other distributions that provide a better fit to the data, as well as evidence against the use of stable distributions for univariate price models.

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2. A Review of the Stable Distributions

2.1 I shall begin with some examples of stable distributions.

First suppose that $Z_1, Z_2, \ldots, Z_n$ are independent Normal variables with mean δ and standard deviation $\gamma\sqrt{2}$. These variables have a probability density function given by:

$$f(z) = \frac{1}{2\gamma\sqrt{\pi}} \exp\left[-\frac{1}{4}\left(\frac{z-\delta}{\gamma}\right)^2\right]$$

and a characteristic function given by:

$$\Psi(t) = \mathrm{E}[e^{itz}] = \exp(i\delta t - \gamma^2 t^2)$$

where $i = \sqrt{-1}$. A very important property of the Normal distribution is that the sum $Z_1 + Z_2 + \ldots + Z_n$ also has the Normal distribution; in this case with mean $n\delta$ and standard deviation $\gamma\sqrt{2n}$.

As a second example, suppose instead that $Z_1, Z_2, \ldots, Z_n$ are independent Cauchy variables with location δ and dispersion γ. These variables have a probability density function given by:

$$f(z) = \frac{\gamma}{\pi[\gamma^2 + (z-\delta)^2]}$$

and a characteristic function given by:

$$\Psi(t) = \mathrm{E}[e^{itz}] = \exp(i\delta t - |\gamma t|).$$

Like the Normal distribution, the Cauchy distribution is closed under addition. Thus the sum $Z_1 + Z_2 + \ldots + Z_n$ also has a Cauchy distribution with location parameter $n\delta$ and dispersion parameter $n\gamma$.

As a final example, let us consider a Brownian motion, and define Z to be the first time that the Brownian motion moves above some level $\sqrt{\gamma}$. We say that Z has a Brownian exit distribution with parameter γ. (If $\gamma = 1$, then Z is also sometimes referred to as a Lévy distribution. It is also the reciprocal of a chi-squared variable with one degree of freedom, and is of type V in the Pearson system of frequency curves.) The probability density function of Z is:

$$f(z) = \sqrt{\frac{\gamma}{2\pi z^3}} \exp\left(\frac{-\gamma}{2z}\right)$$

and its characteristic function is:

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$$\Psi(t) = \mathrm{E}[e^{itz}] = \exp\left(-(1-i)\sqrt{|\gamma t|}\right).$$

Like the previous examples, the Brownian exit distribution is closed under addition. This time the sum $Z_1 + Z_2 + \ldots + Z_n$ will have a Brownian exit distribution with parameter $n^2\gamma$.

2.2 The examples in Section 2.1 are not the only instances of random variables that are closed under addition. They are, in fact, special cases of a class of distributions known as the *stable distributions*. Loosely speaking, a stable distribution is one which is closed under addition.

More formally, a random variable Z is said to be stable, or to have a stable distribution, if:

(1) for every positive integer n; and
(2) for every set of independent and identically distributed random variables $\{Z_1, Z_2, \ldots, Z_n\}$ with the same distribution as Z;

then there exist constants $a_n > 0$ and b_n such that the sum $Z_1 + Z_2 + \ldots + Z_n$ has the same distribution as $a_nZ + b_n$.

In the case of a Normal distribution we would choose $a_n = \sqrt{n}$ and $b_n = (n - \sqrt{n})\delta$.

In the case of the Cauchy distribution we would choose $a_n = n$ and $b_n = 0$.

In the case of the Brownian exit distribution, we would choose $a_n = n^2$ and $b_n = 0$.

In general, for any symmetric stable distribution centred at the origin, we ould choose $a_n = n^{\frac{1}{\alpha}}$ and $b_n = 0$.

2.3 Stability is the only assumption which we shall make for the error terms in the stochastic investment model. Clearly this is a much weaker assumption than Normality, which is, in fact, a special case of a stable distribution. Rejection of the stable model would, by definition, result in rejection of the Normal model too.

2.4 The definition given in Section 2.2 is very broad, and, in fact, there are infinitely many distributions which fall into the class. Except for three exceptions which were the examples in Section 2.1, the probability density function of the stable distributions cannot be written in closed form. Series expansions (Bergstrom, 1952) and approximations (Cramer, 1962) are, however, available. In addition, Zolotarev (1954, 1956, 1961) has derived representations of several stable densities in terms of higher transcendental functions (specifically the modified Bessel function and the Whittaker function). Holt & Crowe (1973) also produced numerical tabulations for a wide range of stable densities.

2.5 Although the density functions cannot generally be written in closed form, the characteristic function can be written in closed form. Paul Lévy (1924, 1925) showed that the stable distributions all have a characteristic function which takes the form:

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$$\Psi(t) = \mathrm{E}[e^{itZ}]$$

$$= \begin{cases} \exp(i\delta t) - |\gamma t|^{\alpha}(1 + i\beta\,\mathrm{sgn}(t)\tan(\pi\alpha/2)) & \text{if } \alpha \neq 1 \\ \exp(i\delta t) - |\gamma t|^{\alpha}(1 + i\beta\dfrac{2}{\pi}\mathrm{sgn}(t)\ln|t|) & \text{if } \alpha = 1 \end{cases} \tag{2.1}$$

where:

$\mathrm{sgn}(t)$ is +1 if t is positive, −1 if t is negative and 0 otherwise.

2.6 It follows that the entire class of stable distributions can be parameterised with the four parameters α, β, γ and δ.

The parameter δ is the location parameter. It simply shifts the distribution to the left or right. In the case of the symmetric distributions (e.g. the Gaussian), δ will be the mean of the distribution.

The parameter γ is the scale parameter. It compresses or expands the distribution about δ in proportion to γ. For the Normal distribution the standard deviation parameter σ and the scale parameter γ are related through $\sigma = \sqrt{2}\gamma$.

[Remark: One could argue that the degenerate distribution with $\gamma = 0$ is a fourth example of a stable distribution whose density function can be written down in closed form. Its distribution function will be $F(z) = 1$ if $z \geq \delta$ and 0 otherwise. Its characteristic function is $\Psi(t) = \exp(i\delta t)$ and, by definition, can be seen to be stable if one takes $a_n = n$ and $b_n = 0$.]

The parameter β determines the skewness of the distribution. This parameter is restricted to the range [−1, 1]. If $\beta = 0$, the distribution is symmetric about δ. If the mean is finite, it will coincide with the median. If $\beta > 0$, then the distribution is *negatively* skewed. This means that it has a long left hand tail and a heavy mass on the right hand side. In this case we have *mean* < *median* < *mode* (if the mean is finite). If $\beta < 0$, then the distribution is *positively* skewed.

The parameter α is known as the characteristic exponent. It lies in the range (0,2] and identifies a particular member of the family. If $\alpha = 2$, a Normal distribution results. (In this case the parameter β has no effect, since $\tan(\pi) = 0$.) If $\alpha = 1$ and $\beta = 0$, a Cauchy distribution results. If $\alpha = \frac{1}{2}$ and $\beta = -1$, the Brownian-Lévy distribution results. The smaller α is, the thicker the tail of the distribution. When $\alpha < 2$, moments of order α and greater do not exist. Thus, if $\alpha < 2$, then the variance of the distribution is infinite. This property is important to the paper, since it implies that a stable distribution with $\alpha < 2$ will introduce more extreme stochastic fluctuations in the investment model.

The first aim of this paper (see ¶1.11) involves determining the appropriate values for the parameters α, β, γ and δ.

In this paper, the abbreviation $S(\alpha, \beta, \gamma, \delta)$ will be used to denote a stable random variable with parameters α, β, γ and δ.

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2.7 If $Z_1, Z_2, \ldots Z_n$ are independent and identically distributed stable random variables with parameters α, β, γ and δ, then $S_n = \sum_{i=1}^{n} Z_i$ is stable (by definition), and with parameters α, β, $n^{1/\alpha}\gamma$ and $n\delta$. This property is needed for the results in Section 3.6.9 and ¶4.2.1.

2.8 The stable distribution can be standardised. If $X \sim S(\alpha, \beta, \gamma, \delta)$, then the variable $Z = \dfrac{X - \delta}{\gamma}$ is $S(\alpha, \beta, 1, 0)$. In this paper the abbreviation $S(\alpha, \beta)$ will be used to denote a standardised stable variable with parameters α and β. The standardisation property is important, since it reduces the problem of simulating from a general $S(\alpha, \beta, \gamma, \delta)$ to simulating from the less general $S(\alpha, \beta)$ distribution. If z is a sample point from an $S(\alpha, \beta)$ distribution, then $z' = \gamma z + \delta$ is a sample point from an $S(\alpha, \beta, \gamma, \delta)$ distribution.

2.9 A number of methods are available for simulating stable random variables. These include the methods of DuMouchel (1971, 121-24), Chambers *et al.* (1976) and Bartels (1978). Klein (1993) also describes a method that makes use of Bergstrom's series expansions for the symmetric stable distributions. The method of Chambers *et al.* appears to be the most widely used, and will be used in this paper. A brief description of the method is given in Appendix 4.

2.10 If $Z_1, Z_2, \ldots$ are independent random variables having a distribution function $F(z)$, and if there exist numbers A_n, B_n such that $B_n^{-1}\sum_{j=1}^{n} Z_i - A_n$ has a distribution which converges to the distribution function $G(z)$, then $F(z)$ is said to belong to the domain of attraction of $G(z)$. It is often believed that an important factor affecting the widespread use of the Normal distribution is the fact that this distribution has a domain of attraction stemming from the Central Limit Theorem. In fact all the stable distributions have a domain of attraction. There is a Generalised Central Limit Theorem (Feller, 1966, 1971), which states that, if a sum of independent and identically distributed random variables (from any distribution) has a limiting distribution, then the limiting distribution will be a member of the stable class.

If the summands have finite variance, then the central limit theorem applies, and the limiting distribution is the Normal distribution. If the summands have infinite variance, then the generalised limit theorem applies, and the limiting distribution will be stable (non-Normal). From this one can prove that a stable distribution cannot have finite variance if it is not the Normal distribution.

Necessary and sufficient conditions for a distribution to belong to the domain of attraction of the stable distributions are given in Gnedenko & Kolmogorov (1954).

2.11 Lévy (1925) also investigated the tail properties of the stable distributions. If $\alpha = 1$, $z\Pr(Z > z) = z\Pr(Z < -z) = 1/\pi$.

More generally, if $\alpha \neq 1$, there exist two constants C_1 and C_2, with $\beta = \dfrac{C_2 - C_1}{C_1 + C_2}$:

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$$z^{\alpha}\Pr(Z > z) \to C_1 \text{ as } z \to \infty$$

and

$$z^{\alpha}\Pr(Z \leq -z) \to C_2 \text{ as } z \to -\infty.$$

If $\alpha \neq 2$ then $C_1 = pC$ and $C_2 = qC = (1 - p)C$, where $p = \dfrac{1+\beta}{2}$ and

$$C = \begin{cases} \dfrac{\gamma^{\alpha}}{\Gamma(1-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha \neq 1 \\ \dfrac{2\gamma^{\alpha}}{\pi} & \text{if } \alpha = 1. \end{cases}$$

The constants C_1 and C_2 are non-zero if, and only if, $|\beta| \neq 1$. Thus *both* tails of *all* non-Gaussian stable laws (except those where $|\beta| = 1$) follow an asymptotic form of the law of Pareto. It is this property which explains how the parameters α and β affect the tail thickness of the distribution. It is also this property which explains why a number of authors (e.g. Mandelbrot, 1963; Blattberg & Sargent, 1971 and Wiener, 1975) refer to the distributions as 'Stable Paretian'.

2.12 The behaviour of the stable distribution when $\beta = \pm 1$ is covered in Skorokhod (1961). If $\beta = +1$, then the distribution is J-shaped. If, in addition, $\alpha < 1$, then there is no mass above δ. The corners of the J are smoothed slightly when $1 < \alpha < 2$. If $\beta = -1$, then the cumulative distribution function is the reflection about the vertical axis of the cumulative distribution function for the $\beta = +1$ case. If $\beta = -1$ and $\alpha < 1$, then there is no mass below δ.

2.13 Suppose X and Y are random variables with $X = \log Y$. If X is stable, then Y is said to be log-stable. When $\alpha < 2$, $|\beta| \neq 1$ and $\gamma \neq 0$, the log-stable distributions have infinite means as well as infinite variances. This applies even if the mean of the stable distribution is finite. This result ought not to be surprising, since if $Y = e^X$, one might intuitively reason that:

$$\mathrm{E}[Y] = \mathrm{E}[e^X] > \mathrm{E}[X^2] = \infty \text{ when } \alpha < 2.$$

To prove the result when $|\beta| \neq 1$, one can use the Paretian tail property of Section 2.11 to show that:

$$e^x f(x) \to e^x C_1 \alpha x^{-(\alpha+1)} \text{ as } x \to \infty.$$

Now since:

$$e^x C_1 \alpha x^{-(\alpha+1)} \to \infty \text{ as } x \to \infty$$

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it follows that the integral $\mathrm{E}\left[e^{X}\right] = \int_{-\infty}^{\infty} e^{x} f(x)dx$ diverges.

When $\beta = +1$, the Paretian tail property does not hold. In this case the distribution of X has a heavy left hand tail (i.e. significant probability mass when x is negative and e^{x} is small), and insufficient probability mass when x and e^{x} are large for the integral $\int_{-\infty}^{\infty} e^{x} f(x)dx$ to diverge. In this case $\mathrm{E}[e^{X}]$ is, therefore, finite.

If X is Normal (i.e. $\alpha = 2$), then it is well known that:

$$\mathrm{E}[Y] = \exp\left(\mathrm{E}[X] + \frac{\mathrm{Var}[X]}{2}\right)$$

and

$$\mathrm{Var}[Y] = \exp(2\mathrm{E}[X] + \mathrm{Var}[X])\exp(\mathrm{Var}[X] - 1).$$

3. Fitting the Model

3.1 *Restating the Model and Approach taken to Fit the Restated Model*

3.1.1 If the assumption of Normally distributed error terms in the MGWP model is changed to a more general stable distribution, then the model can be restated as:

$$P(t) = D(t+1) / Yp(t)$$

where:

$$\log D(t) = \log D(t-1) + \mu_D + Z_D(t)$$

$$\log Yp(t) = A(\log Yp(t-1) - \log \mu_Y) + \log \mu_Y + Z_Y(t)$$

and where:

$$Z_D(t) \sim S(\alpha_D, \beta_D, \gamma_D, 0)$$

and

$$Z_Y(t) \sim S(\alpha_Y, \beta_Y, \gamma_Y, 0).$$

3.1.2 Fitting the model with stable distributions to the data can then be split into two stages:

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(1) determining the parameters μ_D, μ_Y and A for the deterministic components of the series $D(t)$ and $Yp(t)$; and
(2) determining the parameters α, β and γ of the stable distributions for the error terms $Z_D(t)$ and $Z_Y(t)$.

3.1.3 These two stages have influenced the choice of structure for the remainder of the section, which is:
— 3.2: description of the data on which the model is based;
— 3.3: determination of the parameters μ_D, μ_Y and A;
— 3.4: non-normality of the residuals;
— 3.5: methods of estimating the parameters of a stable distribution;
— 3.6: determination of the appropriate stable distribution parameters for the error terms in the MGWP model; and
— 3.7: choice of the alternative model.

3.2 *Description of the Data*

3.2.1 The data used in fitting the model will be primarily the U.K. data used by the MGWP. These included :
— the DeZoete and Bevan Income Index (1919 to 1978), and a prospective dividend yield time series, which was derived from the DeZoete and Bevan Income and Price indices; and
— a dividend and prospective dividend yield index which was derived from price and historic yield indices published by the *Financial Times* and the Institute of Actuaries.

The data are described and tabulated in Appendix D of Ford *et al.* (1980).

3.2.2 The FT-Actuaries data commenced in 1928 with the Actuaries Investment Index (Ordinary Industrials). This was an unweighted geometric mean of share prices published by the Institute and Faculty of Actuaries. The data were superseded in 1962 by the FT-Actuaries Indices, published by the *Financial Times* in conjunction with the Institute and Faculty of Actuaries. The post-1962 data are based on the 500 Share Index (U.K. Ordinary Shares, Industrials and Oils), which was a weighted arithmetic index. The data were combined by separately chain-linking the price and dividend indices, and adjusting dividend yields pre-1962 accordingly.

3.2.3 One of the questions left open by the MGWP was the relative significance of basing the model on geometric and arithmetic time series. (Section D1.3 of Ford *et al.* (1980) refers.) This question is addressed in Section 3.6 of this paper. The analysis in Section 3.6 makes use of time series data which are a hybrid of the FT-Actuaries and DeZoete & Bevan data. The hybrid time series are based on the annual DeZoete & Bevan data until 1962, and the FT-Actuaries (December) data thereafter. The hybrid time series were constructed by separately chain-linking the dividend and price indices at December 1962. The dividend yield series was then constructed by using the relation $Y_{P_t} = D_{t+1}/P_t$ for all t.

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3.2.4 In addition to the above, consideration is given to data which have become available since Ford *et al.* (1980) published the MGWP report. For this purpose, the hybrid indices described above were extended to include the FT-Actuaries data ending December 1993.

3.2.5 Appendix 1 also shows the hybrid time series and the data needed for its construction. Thus the Appendix includes the DeZoete and Bevan data as used by the MGWP (Table 1.1), the FT-Actuaries data ending December 1993 (Table 1.2), and the hybrid indices (Table 1.3 for the period ending December 1978 and Table 1.4 for the period ending December 1993). In each table the actual data for the price, dividend and dividend yield series are shown in Columns (1), (2) and (8) respectively. Columns (3) and (9) show the transformed data $\log(D_t/D_{t-1})$ and $\log(Y_t)$ respectively. The other columns are explained in Section 3.3.

3.2.6 The following notation will be used:

Abbreviation	Description
Dividend yields	
DEZY	DeZoete and Bevan dividend yields Annual data 1.1.1919 to 1.1.1978 (Series D of Ford *et al.*, 1980)
FTAJUNY	FT-Actuaries dividend yields Annual data end June 1930 to end June 1977 (Series E of Ford *et al.*, 1980)
FTADECY	FT-Actuaries dividend yields Annual data end Dec 1930 to end Dec 1977
FTAQTRY	FT-Actuaries dividend yields Quarterly data 1930 to 1977
HYBRIDY	The Dividend Yield Index based on annual DEZY 1.1.1919 to 31.12.1961 December FT-Actuaries thereafter
Dividend growth	
DEZD	De-Zoete and Bevan Income Index Annual data 1919 to 1978 (Series A of Ford *et al.*, 1980)
FTAJUND	FT-Actuaries Derived Dividend Index Annual data end June 1930 to end June 1978 (Series B of Ford *et al.*, 1980)
FTADECD	FT-Actuaries Derived Dividend Index Annual data end Dec 1930 to End Dec 1978
FTAQTRD	FT-Actuaries Derived Dividend Index Quarterly data 1930 to 1978
HYBRIDD	Dividend Index based on Annual DEZD 1919 to 1961 December FTA thereafter

3.3. *Determination of the Parameters μ_D, μ_Y and A*

3.3.1 The parameters μ_D, μ_Y and A from equation (1.1) were estimated by the MGWP using the method of least squares. This is the method traditionally used when the error terms are assumed to be Normally distributed. This is because,

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when the Normal distribution applies, the least squares estimators are identical to the maximum likelihood estimators, and, by the Gauss-Markov theorem, are the most efficient of all unbiased estimators.

3.3.2 However, when a stable non-Normal distribution applies, it is not immediately obvious that the least squares estimators are consistent in probability. The problem was foreshadowed by Mandelbrot (1963) when he recounted Bienayme's argument in the discussion following Cauchy's 1853 paper (Bienayme, 1853). Bienayme argued that a method based on a minimisation of the sum of squares of sample deviations could not reasonably be used if the expected value of this sum was known to be infinite. In view of this, Mandelbrot (1963) applied the method of 'least P-power' of deviations, where $p < \alpha$. (This suggestion is taken up in ¶3.3.5.) The problem was also noted in Section 4.4 of Geoghegan *et al.* (1992), when they pointed out that it *may* be necessary to re-fit the parameters of the Wilkie model, if stable non-Normal residuals were assumed.

3.3.3 Method of moments estimators are also impractical, since moments of order greater than α do not exist, except when $\alpha = 2$. Davis & Resnick (1989) discuss a variation of the method of moments, whereby the parameters of the time series model are obtained by equating the sample and theoretical correlation functions.

3.3.4 Fortunately, the least squares estimates *are,* indeed, consistent — at least for the relatively simple MGWP model. The theory supporting this can be found in a series of papers including Kanter & Steiger (1974), Yohai & Maronna (1977), Hannan & Kanter (1977), Davis & Resnick (1989) and Cline (1989).

3.3.5 In addition, for the data at hand, the least squares estimates turn out to be the same (to 1 significant figure) as the estimates derived by minimising:

$$\sum_t \|z_t\|^p \quad (p = 1.5 \text{ for dividends and } 1.65 \text{ for dividend yields})$$

where z_t is a sample residual. (The value for p was deliberately chosen to be slightly less than a conservative (low) estimate for the relevant α, so that there is little doubt about the convergence of the summation.)

3.3.6 Paragraphs 3.3.4 and 3.3.5 are particularly fortunate, since it means that we can accept the values for μ_D, μ_Y and A given in Ford *et al.* (1980) as parameters for the model based on pre-1980 experience.

3.3.7 If experience since Ford *et al.* (1980) published their report is taken into account, then the parameters μ_D, μ_Y and A *do* need to be re-estimated, and the method of least squares can once again be used.

3.3.8 The least squares estimates derived from the hybrid data set over the period December 1918 to December 1993 are:

$$\begin{aligned} \mu_D &= 5.77\% \\ \mu_Y &= 5.11\% \\ A &= 0.60. \end{aligned}$$

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The estimates for μ_D and μ_Y are rounded downwards to 5.5% and and 5.0% respectively. Further accuracy seems unwarranted. The reasons for using the hybrid data set are given in Sections 3.6.6 and 3.6.7.

3.3.9 Strictly speaking, introducing new data (i.e. post-1980 experience) means that the appropriate shape of the model (i.e. a first order auto-regressive process for the log of dividend yields, and a once differenced process for the log of dividends) ought to be re-examined. This, however, involves a considerable amount of extra effort and is beyond the scope of this paper.

3.4 *Non-Normality of the Residuals*

3.4.1 In each of the tables in Appendix 1, columns (5) and (11) show the residuals that result when the model is fitted to the data. Column (5) shows the residuals for the dividend growth model, and is equal to column (3) – column (4). Column (11) shows the residuals for the dividend yield model, and is derived by subtracting column (10) from column (9).

3.4.2 If the error terms $Z_D(t)$ and $Z_Y(t)$ of the MGWP model are assumed to be Normal, then the above residuals should constitute a random sample from the relevant Normal distribution.

3.4.3 The comments in ¶¶1.4 and 1.5 suggest that, in assessing the suitability of the Normal distribution, the skewness and kurtosis of the data should be examined. Kendall & Stuart (1977) describe the following widely-used measures of skewness and kurtosis:

$$\text{Skewness } \sqrt{B_1} = \frac{\mu_3}{\mu_2^{3/2}}$$

$$\text{Kurtosis } B_2 = \frac{\mu_4}{\mu_2^2} - 3$$

where:

$$\mu_k = \mathrm{E}[(X - \mu)^k].$$

Under the assumption that the data (in our case residuals) constitute a random sample of size n from a Normal distribution, then $\sqrt{B_1}$ and B_2 are approximately Normal, with zero means and variances equal to $6/n$ and $24/n$ respectively.

3.4.4 The table below shows the skewness and kurtosis measures of the residuals that result when the MGWP model is fitted to the various data sets. The table also shows the sample standard deviations of the skewness and kurtosis measures, and the probability values of the observed measures. We test the null hypothesis that the error terms are Normally distributed, against the alternative that they are not.

It is noted that both the dividend growth and the dividend yield model can be

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rejected on the grounds of kurtosis. Regardless of which data set is used, the probability of Type-I error will be very low.

The dividend yield model can also be safely rejected on the grounds that the sample residuals are more skewed than is consistent with a Normal distribution, regardless of which data set is used. However, the dividend growth model can only be rejected on the grounds of skewness for the FT-Actuaries data, but not for the DeZoete & Bevan nor the Hybrid time series. This last anomaly is discussed further in Section 3.6.6.

	Skewness	Std	Prob value	Comment	Kurtosis	Std	Prob value	Comment
Dividend growth								
DEZD 1919 to 1978	−0.274	0.3189	0.19511	do not reject	1.633	0.6378	0.005228	reject
DEZD 1930 to 1978	−0.657	0.3536	0.03158	?	1.249	0.7071	0.03867	?
FTADECD	−1.985	0.3536	9.93E−09	reject	5.957	0.7071	0	reject
FTAJUND	−2.045	0.3536	3.67E−09	reject	5.463	0.7071	5.55E−15	reject
FTAQTRD	−1.774	0.1759	0	reject	6.519	0.3517	0	reject
HYBRIDD to 1978	−0.360	0.3189	0.12881	do not reject	2.159	0.6378	0.000355	reject
HYBRIDD to 1993	−0.318	0.2847	0.13201	do not reject	1.727	0.5695	0.001213	reject
Dividend yields								
DEZY 1919 to 1977	1.651	0.3162	8.9E−08	reject	7.961	0.6325	0	reject
DEZY 1930 to 1977	1.726	0.3536	5.28E−07	reject	7.336	0.7071	0	reject
FTADECY	2.097	0.3536	1.52E−09	reject	9.623	0.7071	0	reject
FTAJUNY	1.199	0.3536	0.000348	reject	3.362	0.7071	9.95E−07	reject
FTAQTRY	0.569	0.1759	0.000609	reject	4.821	0.3517	0	reject
HYBRIDY to 1977	1.665	0.3162	7.0E−08	reject	9.745	0.6325	0	reject
HYBRIDY to 1993	1.881	0.2828	1.5E−11	reject	9.177	0.5657	0	reject

3.4.5 Further justification for replacing the Normal distribution by a more general stable one can be found by examining the residuals directly. For each of the tables in Appendix 1, columns (6) and (12) show the probability values of the observed residuals of the dividend growth and dividend yield models respectively. The probability values are calculated as if the assumption of Normally distributed error terms were true. In columns (7) and (13), observations whose probability values are less than 0.05 and 0.005 are flagged as being unlikely and extremely unlikely respectively.

For example, the movements in the DeZoete & Bevan Dividend Index during 1921, and again in 1959, are considered to be unlikely with the least squares Normal model, and the movements during 1920 and 1930 are extremely unlikely. Three of the observations of the DeZoete & Bevan Dividend Yield series are considered to be unlikely and an additional one observation, 1974, extremely unlikely. With all the data sets, the frequency of unlikely and extremely unlikely observations is far higher than is consistent with the least squares Normal model.

3.5. *Methods for Estimating the Parameters of a Stable Distribution*

3.5.1 A number of methods are available for fitting a stable distribution to the sample residuals. The relative merits of these methods are discussed in Akgiray & Lamoureux (1989). The methods fall into three main classes:

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— fractile methods;
— maximum likelihood methods; and
— methods involving the sample characteristic function.

3.5.2 *The fractile methods*

Included in this class are the methods of Fama & Roll (1968, 1971) and McCulloch (1986). The idea is to base the parameter estimates on sample fractiles. For example, the location parameter δ could be based on the median, or on the truncated mean, as recommended by Fama & Roll. The scale parameter γ could be based on the inter-quartile range, or $\tilde{\gamma} = (z_{(0.72)} - z_{(0.28)})/1.654$, as used by Fama & Roll. Allowing for the other parameters of the stable distribution involves more complicated functions of the ordered statistics, the distributions of which have been found and tabulated. The methods then entail comparing the observed fractiles (or functions thereof) with the tabulated values, and hence deducing the parameters which are used to index into the tables.

Akgiray & Lamoureux (1989) report that, until recently, the most popular method has been that due to Fama & Roll (1968, 1971). The main advantage is that it is a relatively simple method, and easy to implement.

McCulloch's (1986) method is a significant improvement over Fama & Roll's. Without losing the relative simplicity of Fama & Roll's method, McCulloch relaxed the assumption that the distribution be symmetric (i.e. $\beta = 0$). McCulloch also relaxed the assumption that $\alpha \geq 1$. In addition, a problem relating to the discontinuity of the location parameter δ with asymmetrical cases as the index parameter α passes unity, was resolved. McCulloch's estimators for α and γ are similar to those of Fama & Roll, except that their asymptotic bias was eliminated.

3.5.3 *Maximum likelihood estimators*

Maximum likelihood estimates are theoretically attainable (see DuMouchel, 1971, 1973, 1975), but tend to be most impractical due to the lack of knowledge about the form of the density function in the general case. DuMouchel's strategy was to use the series expansions of Bergstrom (1952) for symmetric densities, and to use a multinomial approximation to the likelihood function in the general case.

It follows that the maximum likelihood estimators can only be approximated. Simulation experiments in Koutrouvelis (1980) suggest a downward bias in the maximum likelihood estimation of the index parameter α, and that this bias increases when α is close to 2. In addition, the method suffers from the disadvantage of requiring a relatively large amount of computational effort.

3.5.4 *Methods involving the sample characteristic function*

Included in this class are the methods of Press (1972), Paulson *et al.* (1975), Wiener (1975) and Koutrouvelis (1980, 1981). The method that has been recommended by Akgiray & Lamoureux (1989) is that due to Koutrouvelis.

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Koutrouvelis's method is recommended on the grounds of practicality (ease of computation), relative precision for the index parameter α and lack of bias. The method is also consistent in probability. It is noted that Koutrouvelis's method is in fact, an extension of Wiener's method to allow for both skewed and symmetric distributions.

Koutrouvelis's method is described in Appendix 2. A computer program that implements the method can be found in Finkelstein (1997).

3.6 *Determination of the Parameters of the Stable Distributions for the Error Terms*

3.6.1 The results of applying Koutrouvelis's method to the sample residuals derived from comparing the MGWP model and data, are given below:

Dividend yields	Alpha	Beta	Gamma
DEZY 1919-1978	1.77	(0.27)	0.099
DEZY 1930-1977	1.78	(0.46)	0.106
FTADECY	1.72	(0.85)	0.101
FTAJUNY	1.75	(0.72)	0.089
FTAQTRY	1.71	(0.47)	0.092
HYBRIDY 1919-1978	1.75	(0.39)	0.098
HYBRIDY 1919-1993	1.79	(0.71)	0.099
Dividend growth	**Alpha**	**Beta**	**Gamma**
DEZD 1919-1978	1.77	0.61	0.052
DEZD 1930-1978	1.88	1.00	0.049
FTADECD	1.60	0.92	0.045
FTAJUND	1.52	0.75	0.044
FTAQTRD	1.52	0.44	0.045
HYBRIDD 1919-1978	1.74	0.58	0.048
HYBRIDD 1919-1993	1.76	0.56	0.050

Note that the estimates for the scale parameters γ_D and γ_Y have been annualised using the formulae in Section 3.6.9.

3.6.2 In view of the choice of the least squares estimates for μ_D, μ_Y and A, the location parameters δ_D and δ_Y have been *a priori* set to zero.

3.6.3 With the dividend growth model, the natural choice for $\bar{\gamma}$ seems to be 0.05. There seems to be a significant difference between the estimates for α based on the DeZoete & Bevan data and the FT-Actuaries data. If $\bar{\alpha}$ is based on the DeZoete & Bevan data, then 1.77 seems to be an appropriate choice. However, if $\bar{\alpha}$ is based on the FT-Actuaries data, then the estimate would lower (between 1.5 and 1.6). This discrepancy needs to be explained (see Sections 3.6.5 to 3.6.7). In addition, there appears to be a wide range of possible estimates for β. This is examined further in Section 3.6.8. The choice of this parameter (for the purposes of the simulation experiments in Section 4), is deferred to Section 3.7.

3.6.4 With the dividend yield model, a natural choice for $\bar{\gamma}$ seems to be 0.10. There again seems to be a discrepancy between the estimates for α based on the DeZoete & Bevan data and the FT-Actuaries data, although this time the

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discrepancy is much smaller. The discrepancy between the α estimates is investigated in the following three sections. The wide range of possible values for β_Y is investigated in Sections 3.6.8 and 3.6.9; the final choice of the parameter again being deferred to Section 3.7.

3.6.5 Since the parameter α determines the tail thickness of the distribution, the estimate for α is very much influenced by the outliers in the data. An inspection of the data will reveal that the more extreme variations on the FT-Actuaries Indices occurred in 1931 and 1932, when the derived dividend index fell by 26.2% and 27.4% respectively. By contrast, the DeZoete and Bevan Income Index fell by 19.6% in 1931 and only by 10.0% the following year.

3.6.6 There are two possible reasons for the observations in the preceding paragraph. The first reason is that the two indices have different constituents. The second reason is that the two indices are constructed differently.

At this point it is noted that the DeZoete and Bevan Index was an arithmetically weighted index, while the Actuaries Investment Index (pre-1962) was a geometric index. (The effect of this difference was the question left open by the MGWP that was referred to in ¶3.2.3.) It is also noted that in the early 1930s the DeZoete and Bevan Index comprised 30 shares, while the Actuaries Investment Index comprised 123 shares. The precise constituents of the two indices are not known.

It is believed that the different construction of the two indices is the main reason for the discrepancies in ¶¶3.6.3 and 3.6.4, as well as the anomaly referred to in Section 3.4.4. This is supported by general reasoning when one notes that a geometric index is more sensitive to large falls in prices (or dividends) of its constituents. In an extreme case, if any one of the constituents falls in value by 100%, the whole geometric index collapses. By contrast, the impact on an arithmetic index depends on the weight given to the relevant constituent in the index, and the impact is normally much less severe. In addition, at least some of the 30 constituents in the DeZoete and Bevan Index would have also been in the Actuaries Investment Index.

In order to be able to quantify accurately the relative effects of the different constitution and construction of the two indices, details of the actual constituents and their weightings in the early 1930s are needed. The amount of work involved does not seem justified, and is outside the scope of this paper.

3.6.7 Instead, a new hybrid index was constructed. The hybrid index is based on the annual DeZoete & Bevan data until 1962, and the FT-Actuaries data thereafter. The construction of the Hybrid time series is described in ¶3.2.3, and is tabulated in Appendix 1.

The results of fitting the stable distribution to the residuals that result from comparing the MGWP model to the hybrid data are also shown in ¶3.6.1. Using the hybrid data set leads to parameter estimates much more like those derived from the DeZoete & Bevan data than those from the FT-Actuaries data. This is reasonable, considering that the Hybrid consists of 40 years of DeZoete & Bevan data and approximately twenty years of FT-Actuaries data. However, it also

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confirms, to some extent, that the main reason for the large difference between the DeZoete & Bevan results and the FT-Actuaries results is the data before 1962, and in particular the outliers in 1930 and 1931.

3.6.8 Another important point to be made is that the parameters of the stable distribution cannot be reliably estimated with only 60 data points. This applies especially to the skewness parameter β. This fact can be illustrated with the following simulation experiment, which makes use of Koutrouvelis's method.

SIMULATION EXPERIMENT

(1) Simulate a random sample of size n (e.g. 60) from a population with known α, β, γ and δ.
(2) Use Koutrouvelis's Method to estimate the parameters from this sample.
(3) Repeat the simulation and estimation procedures a number of times.
Hence derive estimates for the standard errors of the Koutrouvelis estimators.

The standard errors depend on the true values of n, α, β, γ and δ. The results of this experiment for a particular set of values for n, α, β, γ and δ are shown below.

Actual values					Average of estimates				STD errors			
α	β	γ	δ	n	α	β	γ	δ	α	β	γ	δ
1.75	–0.5	0.10	0	2,000	1.750	–0.499	0.100	0.0003	0.005	0.024	0.001	0.002
1.75	–0.5	0.10	0	240	1.753	–0.505	0.100	0.0005	0.008	0.031	0.001	0.003
1.75	–0.5	0.10	0	60	1.745	–0.441	0.097	0.0015	0.065	0.284	0.008	0.016

More detailed sampling statistics and simulation experiments of this kind are given in Koutrouvelis (1980). The experiment provides a useful means of checking the simulation and parameter estimation software against each other.

3.6.9 One might try reducing the problem of unreliable estimators associated with short time series by using more frequent observations. The idea is to use, say, quarterly (non-overlapping) data, and then use the definition of stability to derive an equivalent annual model. This approach was used in Section D2.12 of Ford *et al.* (1980), and uses the additivity property (see ¶2.7) to establish the following facts:

(1) If:

$$\log Y_t = \log \mu_Y + A.(\log Y_{t-1} - \log \mu_Y) + Z_t \quad \text{with } Z_t \sim S(\alpha, \beta, \gamma, 0)$$

then:

$$\log Y_{t+4} = \log \mu_Y + A^4(\log Y_t - \log \mu_Y) + Z_t'$$

with :

$$Z_t' \sim S(\alpha, \beta, (1 + A^\alpha + A^{2\alpha} + A^{3\alpha})^{1/\alpha}\gamma, 0).$$

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(2) If:

$$\log D_t = \log D_{t-1} + \mu_D + Z_t \text{ with } Z_t \sim S(\alpha, \beta, \gamma, 0)$$

then:

$$\log D_{t+4} = \log D_t + 4\mu_D + Z_t' \text{ with } Z_t' \sim S(\alpha, \beta, 4^{1/\alpha}\gamma, 0).$$

In the case of the Normal distribution, the idea of reducing uncertainty by sampling more frequently only works for the scale parameter. The only way of reducing the uncertainty of the location parameter δ is to sample over a longer time period. While the idea seems intuitively reasonable, the author remains uncertain about the behaviour (in terms of consistency and reliability) of the Koutrouvelis estimators for α, β and γ with frequent sampling.

The equivalent annual model, which results when the FT-Actuaries quarterly time-series is used, is shown in the table in ¶3.6.1. The reasonably close agreement between the annualised quarterly estimates and the estimates derived from annual data can be taken as evidence that the data come from a stable distribution (or at least are well within its domain of attraction).

In the final analysis, it must be noted that the DeZoete & Bevan time series is available only on an annual basis. This is restrictive, since we noted (in Section 3.6.6) that the FT-Actuaries data are based on a geometric index until 1962, and this makes them less suitable in many situations.

3.7 *A Possible Alternative Model*

3.7.1 *Model for pre-1980 experience*

Based on the preceding evidence, it is suggested that a possible refinement to the MGWP model is one which uses the following stable distributions instead of the relevant Normal distributions:

For dividend yields	$\alpha_Y = 1.75$	$\beta_Y = -0.5$	$\gamma_Y = 0.10$	$\delta_Y = 0$
For dividend growth	$\alpha_D = 1.75$	$\beta_D = 0.5$	$\gamma_D = 0.05$	$\delta_D = 0.$

The final choice of parameters is ultimately a matter of judgement. I have based my choice primarily on the DeZoete & Bevan data set, which was the longest arithmetic index available at the time the model was published. I have also applied some rounding to the parameter estimates. This avoids giving a misleading impression of accuracy, given that the parameter estimates cannot be regarded as reliable due to the limited data available. The rounded parameter estimates match closely the parameter estimates based on the hybrid time series, especially for pre-1980 experience.

It is noted that downward rounding adjustments to the index (and location) parameters are cautious, in the sense that the likelihood of the model under-

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estimating the maturity guarantee reserve required for a given probability of ruin is reduced. Equivalently, the likelihood of the model under-estimating the 'true' probability of ruin with a given level of reserves is reduced by using lower index and location parameters.

The choice of skewness parameters deserves special mention, since the estimates for these were found in ¶3.6.8 to be exceptionally unreliable. Referring to the results in ¶3.6.8, it could be argued that symmetric distributions can be used, on the grounds that the estimates for β_D and β_Y are not significantly different from zero. However, the estimates for β_D and β_Y are also not significantly different from the values chosen. In addition, a positive skewness assumption (negative β) for dividend yields and a negative skewness assumption (positive β) for dividend growth is supported by the empirical evidence presented in Section 3.4.4.

A negative skewness for the dividend growth model is consistent with the observation that companies tend to increase dividends more often than they decrease them (the distribution has a positive mass), but that reductions in dividends tend to be greater than increases (the distribution has a long left hand tail). (When dividends are cut, they are often passed altogether.)

While dividends are fixed, a positive skewness for the dividend yield model is consistent with a negative skewness assumption for the distribution of price increases. This is consistent with the observation that prices tend to increase more often than they decrease, but that the falls in share prices tend to be more extreme.

The alternative model deliberately makes explicit assumptions about skewness. This allows the impact of the skewness parameters (mainly in the context of reserving for investment guarantees) to be investigated (Sections 4.3.3 and 4.4). The possibility of simplifying the alternative model by removing either or both of the skewness parameters could be kept in mind, and the implications of doing this will be apparent from the investigations in Section 4.

3.7.2 *Updated model to allow for post-1980 experience*

If the experience since the MGWP published its report is taken into account, then the parameters need to be updated. The following values were derived from the least squares estimates applicable for the hybrid time series over the period December 1918 to December 1993:

$$\begin{aligned} \mu_D &= 5.5\% \\ \mu_Y &= 5.0\% \\ A &= 0.60. \end{aligned}$$

The least squares estimates have again been rounded downwards to avoid introducing spurious accuracy. Downward adjustments again result in a conservative basis. The hybrid time series were used, as they are based on the longest arithmetic indices available.

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The same stable distributions can be used for the error terms in the model for post-1980 experience as were used in the model for pre-1980 experience. There is a case for increasing the value chosen for α_Y to 1.80, but this seems unnecessary in view of the relatively large uncertainties about parameter estimates derived from small data sets.

3.7.3 *Goodness of fit*

The Kolmogorov-Smirnov test has been used to test the null hypothesis that the sample residuals, derived from comparing the alternative model and the hybrid data series, come from the stable distributions specified in Section 3.7.1. In order to apply the tests, it was first necessary to generate the cumulative distribution functions of the relevant stable distributions by simulation. The tests failed to reject the null hypothesis for both the dividend growth and the dividend yield models at the 5% significance level. Further details of goodness of fit testing can be found in Finkelstein (1997).

3.7.4 *An improvement to the existing model*

Despite the relatively large uncertainties surrounding the appropriate stable law parameters, the proposed refinement should still improve the MGWP model. This is because the Normal distribution is, in fact, a special case of a stable distribution, and rejection of a stable distribution by definition implies rejection of the Normal distribution too.

3.7.5 *Sensitivity to choice of parameters and rounding adjustments*

The sensitivity of the model to the choice of parameters is investigated in Section 4.3. The effect of the particular rounding adjustments that have been made is examined in Section 4.4.

4. Impact of using the Stable Distributions

This section is structured by introducing the notation, definitions and liability models that were used in the MGWP report, and which have been extended in this paper.

The next sub-section gives some theoretical implications for replacing the Normal distributions by stable non-Normal distributions. Probability distributions for forecasts of the dividend and dividend yield variables will be specified. In addition, it will be proved that the annual investment return has infinite mean (as well as infinite variance). This leads to the failure of the sample means and variances as measures of risk and reward and to the inapplicability of a large part of financial economics based on mean-variance optimisation and power law utility functions.

The impact of changing the parameters of the investment model (including the parameters of the stable distributions) are investigated with sensitivity tests in Section 4.3. Separate sensitivity tests are used for:

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— the location, scale and auto-correlation parameters (Section 4.3.1);
— the index (or tail thickness) parameters of the stable distributions (Section 4.3.2);
— the skewness parameters (Section 4.3.3); and
— initial yields (Section 4.3.4).

The possible alternative models (including the original MGWP model) that could be derived from the various data sets are compared in Section 4.4. In this section, the differences between the MGWP model and the proposed alternative are quantified in detail.

Finally, the behaviour of the proposed alternative investment model with various different liability portfolios is investigated in Section 4.5.

4.1 *Notation and Definitions*

4.1.1 Let $I(t)$ denote the rate of return in year t of the projection; and let $G(t)$ denote the geometric return over the t year period.

Thus, if $R(t)$ is the share price or equity unit price (allowing for reinvestment of dividend income) at time t, then:

$$1 + I(t) = \frac{R(t)}{R(t-1)}$$

and

$$1 + G(t) = \left(\frac{R(t)}{R(0)}\right)^{\frac{1}{t}} = \left(\prod_{s=1}^{t}(1 + I(s))\right)^{\frac{1}{t}}.$$

The model for $R(t)$ is given in Section 1.2, and re-stated in ¶3.1.1.

4.1.2 In view of the fact (see Sections 4.2.2 and 4.2.3) that the annual investment returns have infinite means and variances, the effect of using a stable distribution will be investigated by examining the sample fractiles.

If X is a random variable, define $Q_X(p)$ such that:

$$\Pr(X \geq Q_X(p)) = p.$$

Equivalently:

$$Q_X(p) = F_X^{-1}(1-p)$$

where $F_X^{-1}(t)$ is the inverse of the cumulative distribution function of X.

The median of X is Med[X] = $Q_X(0.5)$. Exactly half the population values are greater than the median, and exactly half the observations are less than it.

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The inter-quartile range of X is IQR[X] = $Q_X(0.25) - Q_X(0.75)$. The inter-quartile range is the central interval in which 50% of the population values lie.

The effects of using the stable distributions on investment performance will be investigated by examining:

— Med[$I(t)$]: the median of the annual investment return in the tth year of projection;
— Med[$G(t)$]: the median of the geometric investment return over a t-year period;
— IQR[$I(t)$]: the inter-quartile range of the annual investment return; and
— IQR[$G(t)$]: the inter-quartile range of the geometric investment return.

The inter-quartile range is a widely used measure of the spread of values that the variable X can take. However, it ignores observations in the tail of the distribution. Since these observations are critical to our investigation on extreme stochastic fluctuations, it will also be necessary to examine some of the fractiles in more detail. The values $Q_{I(t)}(0.005)$ and $Q_{G(t)}(0.005)$ illustrate how large some (0.5%) of the observations are. The values $Q_{I(t)}(0.995)$ and $Q_{G(t)}(0.995)$ illustrate how small some of the observations are.

In all cases, the sample fractiles, derived from $n = 5000$ simulations, are used as the estimates for the population fractiles.

4.1.3 The MGWP investment model was developed primarily for the purpose of managing maturity guarantees under equity-linked insurance policies. It, therefore, seems sensible to investigate the effect of using stable non-Gaussian distributions with a model for equity-linked insurance policies with maturity guarantees. The same liability model that was used by the MGWP will be used here. A summary of this model is given below. Appendix E of Ford *et al.*(1980) provides more details.

4.1.4 Consider an insurance policy with a premium of 1 p.a. invested for a term of n years. At maturity, the policyholder is promised that his benefit will be the accumulated amount of his investments, subject to a guaranteed minimum being a return of the premiums paid.

Let S_n = the accumulated amount of the investments of 1 p.a. after n years. The maturity guarantee is then n and the maturity benefit is $B_n = \max(n, S_n)$.

Define the guarantee claim amount $G_n = \max(0, n - S_n)$.

It is noted that B_n, S_n and G_n are related through $B_n = G_n + S_n$.

4.1.5 Now consider a cohort of policies written simultaneously at time $t = 0$. Ignore mortality, so that all policies reach maturity.

Let P_t denote the premium payable for policies maturing in year t ($t = m_1, \ldots, m_2$). The quantities m_1 and m_2 can be defined to be the earliest and latest maturity dates of policies in the cohort. The total sum assured is given by:

$$T = \sum_{t=m_1}^{m_2} tP_t.$$

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The total guarantee claim for a particular realisation is:

$$C = \sum_{t=m_1}^{m_2} P_t G_t.$$

The total discounted claim for a particular realisation is:

$$DC = \sum_{t=m_1}^{m_2} P_t G_t v^t.$$

where $v = \dfrac{1}{(1+i)}$ and i is the discount rate of interest. Ford *et al.*(1980) fixed i at 4%.

The claim ratio, as a percentage of the total sum assured, is defined as:

$$CR = 100 \times \frac{C}{T}$$

and the discounted claim ratio, as a percentage of total sum assured, is defined as:

$$DCR = 100 \times \frac{DC}{T}.$$

4.1.6 The statistics to be examined for each set of n (typically 5000) simulations are:

— *NZ*, the number of simulations in which *CR* and *DCR* are non-zero;
— *MNZ*, the mean of the *NZ* non-zero observations of *CR* and *DCR*; and
— specific quantiles $Q_{CR}(p)$ and $Q_{DCR}(p)$ of the simulated distributions of *CR* and *DCR* respectively. $Q_{CR}(p)$ and $Q_{DCR}(p)$ are defined analogously to the variables $Q_{I(t)}(p)$ and $Q_{G(t)}(p)$ in Section 4.1.2. The specific quantiles to be examined correspond to p = 0.001, 0.005, 1% and 5%.

The quantity *MNZ* provides a measure of the expected severity of a claim. The quantile $Q_{CR}(k/n)$ provides an estimate for the contingency reserve needed, as a percentage of the total sum assured, to ensure that the probability of ruin is limited to k/n. The quantile $Q_{DCR}(k/n)$ gives the reserve needed, as a percentage of total sum assured, if discounting is allowed. The ruin probability will be limited to k/n if it is assumed that the reserves can be invested in a risk-free investment and earn 4% p.a.

4.1.7 It can be argued that the MGWP's approach of calculating the reserves by discounting the claims at a certain rate of interest is questionable. The objection arises in a stochastic environment where there is no such thing as a 4%

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risk-free investment return. Furthermore, if one chooses to discount at a fixed rate, then what should the value of this discount rate be? Noting the similarity between the cost of a maturity guarantee and the cost of a put option, one might be tempted to price the guarantee using an option pricing model. However, under most standard pricing models (e.g. CAPM or the Black-Scholes limit), the implied discount rate in put option prices is often substantially below gilt rates, and is sometimes even negative.

Possible alternative approaches include:

— discounting at 0% (i.e. the reserves are kept under the mattress);
— discounting at the rate earned on the underlying unitised fund; or
— discounting at money market r_t, which are *stochastic* risk-free interest rates, and which are consistent with the remainder of the stochastic model.

In order to compare like with like, I shall adopt the approach used by the MGWP, but concentrate on the undiscounted claim ratios.

4.1.8 The MGWP investigated the reserving requirements for various different insurance portfolios. The portfolios investigated include (amongst others):

— a cohort of identical policies all written for a term of 10 years;
— a cohort of identical policies all written for a term of 20 years; and
— a standard portfolio being chosen as representative of a plausible typical mix of new business. The standard portfolio is reproduced in Appendix 3. Except where otherwise indicated, all comparisons will be made by reference to this standard portfolio.

4.1.9 The MGWP assumed a rate of tax τ, of 37.5% on dividend income. U.K. tax rates have subsequently changed a number of times. Except where otherwise indicated, a gross investor will be assumed.

4.1.10 In order to be able to run the model, it is necessary to assign a value to the initial yield Y_0. In practice, the dividend yields prevailing at the time of application would be used. Except where otherwise indicated, Y_0 will be set to equal μ_Y. The sensitivity of the model to Y_0 will be investigated in Section 4.3.4.

4.2 *Some Theoretical Implications of Using Stable Distributions*

4.2.1 *Forecasting more than one step ahead*

The distribution of $\log D_{t+s}$, given the value of dividends D_t at time t, is:

$$\log D_{t+s} \Big| D_t \sim S(\alpha_D, \beta_D, s^{\frac{1}{\alpha}}\gamma_D, \log D_t + s\mu_D)$$

and the distribution of $\log Yp_{t+s}$, given the value of dividend yields Yp_t at time t, is:

$$\log Yp_{t+s} \Big| Yp_t \sim S\left(\alpha_Y, \beta_Y, \left[\frac{A^{\alpha s}-1}{A^{\alpha}-1}\right]^{\frac{1}{\alpha}}\gamma_Y, A^s \log Yp_t + (1-A^s)\log \mu_Y\right).$$

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These facts can be established by re-writing the model (as re-stated in ¶3.1.1) in the above form (with $s = 1$) to establish an inductive basis. One can then use the techniques outlined in Section 3.6.9 to show that the result holds true for $s = 2$, 3, 4, ... and that, if the result holds true for a general s, then it holds true for $s+1$, $s+2$, ... and so on. The principle of induction establishes the general result.

4.2.2 *Moments of the annual investment return*

In this section I shall prove that:

(a) if either α_D or α_Y is less than 2, then the variance of the annual investment return of the model is infinite; and

(b) if $\alpha_D < 2$ and $|\beta_D| \neq 1$, then the mean of the annual investment return is also infinite.

One might intuitively expect the variance of the annual investment return to be infinite when the error terms $Z_D(t)$ and/or $Z_Y(t)$ have infinite variance (i.e. case (a)). Less intuitive is the result that the annual investment return has infinite mean, even when α_D and α_Y are greater than 1, so that the error terms have finite means (i.e. case (b)).

To prove the result, first introduce some notation and let:

$$\left(\frac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$$

denote the variable $\dfrac{P_t}{P_{t-1}}$ given that $Yp_{t-1} = y_{t-1}$ and $Yp_t = y_t$. Next assume that condition (b) holds and note that:

$$P_t = \frac{D_{t+1}}{Yp_t} \Rightarrow \frac{P_t}{P_{t-1}} = \frac{D_{t+1}}{D_t}\frac{Yp_{t-1}}{Yp_t}$$

$$\Rightarrow \log\left(\frac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right) = \log\left(\frac{D_{t+1}}{D_t}\right) + \log\left(\frac{y_{t-1}}{y_t}\right).$$

Thus, $\log\left(\dfrac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$ is a stable (non-Gaussian) random variable. Consequently, $\left(\dfrac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_{t-1};\ Yp_t = y_t\right)$ is *log*-stable (non-Gaussian) and therefore, by the result in Section 2.13, it has infinite mean and variance. Now since:

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$$\mathrm{E}\left[\frac{P_t}{P_{t-1}}\right] = \mathrm{E}\left[\mathrm{E}\left[\frac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_1;\ Yp_t = y_2\right]\right]$$

$$= \iint \mathrm{E}\left[\frac{P_t}{P_{t-1}}\middle| Yp_{t-1} = y_1;\ Yp_t = y_2\right] f_{Yp_{t-1},Yp_t}(y_1;\ y_2)dy_1dy_2$$

it follows that the unconditional mean of $\frac{P_t}{P_{t-1}}$ is infinite. (The integrands in the above integration are either zero, when the joint probability density $f_{Yp_{t-1},Yp_t}(y_1;y_2) = 0$, or infinite, when the joint probability density $f_{Yp_{t-1},Yp_t}(y_1;y_2) > 0$.)

Using:

$$\mathrm{Var}\left[\frac{P_t}{P_{t-1}}\right] = \mathrm{E}\left[\mathrm{Var}\left[\frac{P_t}{P_{t-1}}\middle| Yp_{t-1}, Yp_t\right]\right] + \mathrm{Var}\left[\mathrm{E}\left[\frac{P_t}{P_{t-1}}\middle| Yp_{t-1}, Yp_t\right]\right]$$

establishes the fact that the unconditional variance of $\frac{P_t}{P_{t-1}}$ is infinite when α_D is less than 2.

Similar reasoning can be used for the case when α_Y is less than 2, to show that the total annual investment return $\frac{R_t}{R_{t-1}} = \frac{P_t}{P_{t-1}} + \frac{D_t}{P_{t-1}}$ has infinite variance when condition (a) holds and also infinite mean when condition (b) holds.

4.2.3 *Failure of the sample mean and standard deviation as measures of risk and reward*

In the previous section it was shown that if a stable non-Gaussian distribution is used, then the population means and standard deviations for the investment returns will be infinite. However, for any finite number of simulations, the *sample* means and standard deviation *will* be finite. One might be tempted, therefore, to compute the sample means and standard deviations and use them as measures of the amount of risk and reward inherent in the model under consideration.

The idea of using the sample moments as measures of risk and reward is flawed. An explanation follows. Suppose that E_n equals the average annual rate of return in year t after n simulations, and S_n equals the sample standard deviation of $I(t)$. If the sample moments were useful measures of risk or reward, then the variables E_n and S_n should be able to fluctuate within some band for large enough n. This is necessary in order that some useful information can be inferred from a given simulation (e.g. to form comparisons with alternative models). The following proof shows that the sample moments do not fluctuate within any band,

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regardless of the number of simulations, and both E_n and S_n tend to infinity for large n.

Let I_n denote the value of $I(t)$ on the nth simulation. The variables $I_1, I_2, \ldots, I_n$ are independent and identically distributed with the same distribution as $I(t)$. Note that $I_n \geq -1$, since asset prices cannot go below zero. Let $M > 0$ be an arbitrarily large constant, and consider the variable $\min\{I_n, k\}$ for large k. This variable is bounded in $[-1, k]$, and therefore has finite mean. By the monotone convergence theorem, and the result in Section 4.2.2, we have:

$$\lim_{k\to\infty} \mathrm{E}[\min\{I(t), k\}] = \lim_{k\to\infty} \mathrm{E}[\min\{I_n, k\}] = \mathrm{E}[I_n] = \infty.$$

In particular, we can find K such that $\mathrm{E}[\min\{I_n, K\}] > M$. Now define:

$$E_n = \frac{I_1 + I_2 + \cdots + I_n}{n}$$

and

$$E_n^* = \frac{\min\{I_1, K\} + \min\{I_2, K\} + \cdots + \min\{I_n, K\}}{n}.$$

By the strong law of large numbers (and by hypothesis on K) we have, with probability 1:

$$\lim_{n\to\infty} E_n^* = \mathrm{E}[\min\{I(t), K\}] > M.$$

However, $E_n \geq E_n^*$, and therefore $\liminf_{n\to\infty} E_n \geq \liminf_{n\to\infty} E_n^* > M$.

Now, since M was arbitrary, it follows that $\liminf_{n\to\infty} E_n = \infty$ with probability 1. This concludes the proof for E_n. A similar (but slightly longer) proof is possible for S_n.

The implication of the above developments is that a large part of financial economics, based on mean-variance optimisation or power-law utility functions, will be inapplicable under a stable (or rather log-stable non-Gaussian) probability framework.

4.3 *Sensitivity Tests*

4.3.1 *The location, scale and auto-correlation parameters*

In Section E2.5 of their paper, the MGWP investigated the sensitivity of changing the basis. It did this by individually changing each of the parameters μ_D, μ_Y, σ_D, σ_Y and A both upwards and downwards by a small and large amount.

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This experiment was repeated with the error terms assumed to follow the proposed alternative model, and with the tax rate set to zero.

The results are shown in Tables 5.1(a), 5.1(b), and 5.1(c) respectively. Note that the scale parameters σ_D and σ_Y of the Normal distribution are re-written using the general stable distribution notation γ_D and γ_Y, where $\gamma_D = \sigma_D/\sqrt{2}$ and $\gamma_\gamma = \sigma_\gamma/\sqrt{2}$.

The results show that the same general conclusions that were made by the MGWP can be made with the proposed alternative model. These are summarised as follows:

(1) The effect of a change in basis is, in general, far from linear.

(2) The results tend to be most sensitive to the dividend location, followed by the yield location and then the auto-regressive parameter A.

(3) The parameters affect the different distributions in different ways, and the ranking order just quoted depends on which statistic one is considering. For example, the volatility of the investment returns as measured with sample standard deviations (Normal case) or interquartile ranges (non-Normal case) seem to be most sensitive to the scale parameter γ_γ, while the claim frequency rate is most sensitive to changes in μ_D.

(4) Care is needed when choosing the basis. A too optimistic view of the dividend growth location parameter (the mean growth rate when Normal distributions are involved), may have a large effect in reducing the required contingency reserves, but may be unjustified in the light of experience. Given the difficulties and uncertainties surrounding the choice of parameter estimates, some implicit margins for caution could be built into the reserving basis. This could come in the form of conservative adustments to the parameters of the investment model, and/or reserving at lower probabilities of ruin.

(5) If adjustments to the parameter values are to be incorporated in the basis (e.g. for rounding or to cater for possible model mis-specification error), then:
   — for the location parameters, downwards adjustments are conservative; and
   — for the scale parameters, upwards adjustments will be conservative.

   Low values for the location parameters mean that a low performing asset which is less capable of supporting the maturity guarantees would be modelled. High values for the scale parameters result in more volatile stochastic fluctuations, and increased frequency and severity of guarantee claims.

If the more general stable (non-Gaussian) distributions are used, then two additional types of parameter are introduced. These are the index and skewness parameters, which are investigated in the following two sections.

4.3.2 *Tail thickness and the index parameters α_D and α_Y*

In this experiment we shall fix the skewness, scale and location parameters at

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the values of the proposed alternative model, and investigate the effect of choosing different index parameters.

Tables 5.2(a) to (i) (in Appendix 5) show the values of Med[$I(30)$], IQR[$I(30)$], $Q_{I(30)}(0.995)$, $Q_{I(30)}(0.005)$ *NZ*, *MNZ*, $Q_{CR}(0.001)$, $Q_{CR}(0.01)$ and $Q_{CR}(0.05)$, with the skewness, scale and location parameters fixed at:

$$\begin{array}{lll} \beta_D = +0.5 & \gamma_D = 0.05 & \delta_D = 0 \quad (\mu_D = 0.04) \\ \beta_Y = -0.5 & \gamma_Y = 0.10 & \delta_Y = 0 \quad (\mu_Y = 0.05) \end{array}$$

and the index parameters varying between 1 and 2 in 0.1 increments.

The results illustrate how the fluctuations within the model increase when either or both of the index parameters are reduced. For example, IQR[$I(30)$] increases from 25.5% when both index parameters are 2 (Normally distributed error terms) to 40.7% when both index parameters are 1. The results are even more noticeable when tail fractiles are considered. For example, $Q_{I(30)}(0.995)$ reduces from –29.6% when both index parameters are 2 to –98.6% when the yield index parameter is 1, and to –84% when the dividend growth index parameter is 1. At the same time $Q_{I(30)}(0.005)$ increases from 70% when both index parameters are 2 to 5.6×10^9% when they are both 1. In the same sets of simulations, the reserving requirement (for a 1:1,000 chance of ruin) increases from 3% of total sum assured to 86% of total sum assured (a 2,800% increase in relative terms).

By reducing the index parameters, the kurtosis and tail thickness of the distributions are increased, according to the law of Pareto (Section 2.11). Very fat tails result in a greater proportion of random sample points having extreme values, and the stochastic fluctuations therefore become more extreme. This contributes to more frequent and more severe guarantee claims, with a consequential increase in the reserving requirement.

The reserving requirements at the less extreme ruin probabilities (e.g. 1% or 5%) are less sensitive to changes in the index parameters. For example, reducing the index parameters from 2.0 to 1.9 has no material impact on the reserving requirement corresponding to a 5% ruin probability. By contrast, the same reduction in index parameters causes the reserving requirement for a 1:1,000 ruin probability to increase five times. A *very* small reduction in either index parameter (e.g. from 2 to 1.999) does not have a significant effect on the reserving requirement, except perhaps at extremely low ruin probabilities.

It is also interesting to note that such small changes in index parameters do not materially impact on the number or severity of claims. For example, when both index parameters are 1.999, the number and size of claims are *NZ* = 82.5 and *MNZ* = 0.50 respectively. These observations should be contrasted with the fact that, even with a *very* small reduction in index parameters, the variance of the investment returns becomes infinite. This illustrates the point that volatility, as measured in terms of variance or standard deviation, is not a satisfactory measure of risk under a stably distributed framework.

Table 5.2(a) shows that, while the median of the annual investment return

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increases slightly with reductions in α_D, the same is not necessarily true with reductions in α_Y. This suggests that it would be difficult to replace the mean-variance optimisation theory in financial economics with optimisations involving, say, the median and the inter-quartile range.

4.3.3 *Skewness*

In this section we shall fix the index, scale and location parameters at the values of the proposed alternative model, and investigate the effect of choosing different skewness parameters.

Tables 5.3(a) to (i) show the values of Med[$I(30)$], IQR[$I(30)$], $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, *NZ*, *MNZ*, $Q_{CR}(0.001)$, $Q_{CR}(0.01)$ and $Q_{CR}(0.05)$ with the index, scale and location parameters fixed at:

$$\begin{array}{lll} \alpha_D = 1.75 & \gamma_D = 0.05 & \delta_D = 0 \ (\mu_D = 0.04) \\ \alpha_Y = 1.75 & \gamma_Y = 0.10 & \delta_Y = 0 \ (\mu_Y = 0.05) \end{array}$$

and the skewness parameters β_D and β_Y varying between -1 and $+1$ in increments of 0.5.

It is important to state, at the outset, that the relative significance of the skewness parameter β depends also on the parameters α, γ and δ of the stable distribution concerned. For example, as $\alpha \to 2$ the value of β becomes irrelevant. (This is because the term $\tan(\pi\alpha/2)$ in equation (2.1) approaches 0 as $\alpha \to 2$.)

The following observations were made with the particular parameter values associated with the proposed alternative model:

(1) For a given β_Y, the median of the annual investment return Med[$I(30)$], increases as β_D increases. By contrast, for a given β_D, Med[$I(30)$] decreases as β_Y increases.

(2) The inter-quartile range of the annual investment return IQR[$I(30)$] is not materially affected by changes in the skewness parameters.

(3) The above two observations again illustrate the difficulty of attempting to optimise any trade-offs between the median and the inter-quartile range of the investment return. The inter-quartile range can be relatively insensitive to changes in a particular parameter if the quantiles $Q_{I(30)}(0.25)$ and $Q_{I(30)}(0.75)$ both change by the same amount and in the same direction. The more extreme fractiles can also convey useful information, and should be examined in conjunction with the claim statistics. Tables 5.3(c) and (d) show that the large returns $Q_{I(30)}(0.005)$ decrease as β_D increases as do the small (negative) returns $Q_{I(30)}(0.995)$. By contrast, $Q_{I(30)}(0.005)$ and $Q_{I(30)}(0.995)$ increase as β_Y increases.

(4) The claim frequency statistic *NZ* increases with increasing β_D. The behaviour of *NZ* with changes in β_Y depends on β_D. If $\beta_D > 0$, then *NZ* decreases with increasing β_Y.

(5) The mean claim severity *MNZ* reacts similarly. It increases with increasing

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β_D, and its behaviour with changes in β_Y depends on β_D. If $\beta_D > 0$, then *MNZ* increases with increasing β_Y.

Overall, the reserving requirement $Q_{CR}(0.001)$, which depends on both the number and severity of claims, appears to be more sensitive to β_D than to β_Y. The reserving requirement increases with increasing β_D, and its behaviour with changes in β_Y also depends on the value of β_D. If $\beta_D > 0$, then the reserving requirement tends to increase with increasing β_Y.

(6) When considering the contingency reserves required for maturity guarantees, we can say that positive β assumptions (negative skewnesses) are conservative. The data supported a positive assumption for β_D and a negative assumption for β_Y. It was noted, however (see Section 3.6.8), that the limited data availability meant that the β parameters could not be reliably estimated.

(7) Overall, the skewness parameters appear to have much less impact on the simulation results than do the index, scale and location parameters. The observation that the model is relatively insensitive to the choice of skewness parameters (especially β_Y when β_D is fixed) is an important factor to consider when settling on the final basis.

Recall that positive βs imply negatively skewed distributions for the error terms. This means that the distributions have positive masses and long left hand tails. (The opposite applies with negative βs.) The tail follows an asymptotic form of the law of Pareto (Section 2.11). A long left hand tail for the dividend growth model means that the probability of large *falls* in dividends remains relatively large. This explains why a positive β_D assumption is conservative.

The above is consistent with the work of Leitch & Paulson (1975), who concluded that the skewness parameter can be used as a measure of 'downside risk' in a univariate price model. Leitch & Paulson reached this conclusion by observing that the location parameter lies to the left of the median of distributions which are negatively skewed (i.e. have a positive β). Since δ is interpreted as the rate of return (in a univariate price model), or the rate of dividend growth (in the MGWP dividend growth model), it follows that negative skewness is undesireable for reserving purposes. Put another way, a negative skewness (or positive β) assumption is conservative.

To understand the behaviour of the model to changes in β_Y, one must realise that the dividend yield model is a 'reversion to trend' one. Thus, if, at a point in time, the prevailing level of yields is above/(below) the long-term trend μ_Y, then the model (ignoring the error terms) will project downward/(upward) moving yields respectively. This corresponds to upward/(downward) moving prices respectively.

If the model is projecting upward moving yields (and downward moving prices), then large positive error terms would increase the fall in prices. In this case, a distribution for which the probability of a large positive error term is relatively large (i.e. a positively skewed distribution with a negative β_Y) is conservative. The converse applies if the model is projecting downward moving yields.

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4.3.4 *The initial yield parameter* Y_0

In order to be able to run the model, it is necessary to assign a value to the initial yield Y_0. This is not really a basis assumption, since, in a practical application, the dividend yields prevailing at the time of application would be used. Despite this, the sensitivity of the model to this parameter needs to be examined, in order to understand the behaviour of the model in different conditions.

The results of running the proposed alternative model with values of $Y_0 = 2.5\%$, 5% and 7.5% are shown in Appendix 5, Table 5.1(d). The results are similar to those observed by the MGWP. The main features are:

— Low values for Y_0 tend to result in lower investment returns in the early years of projection, and have less impact in the later years. Overall, the median geometric rate of return over the 30-year period is increased/(decreased) by approximately the same amount by which Y_0 exceeds/(is less than) μ_Y.

— The inter-quartile ranges of the investment returns are not very sensitive to the initial yield Y_0.

— Since low initial yields tend to depress the long-term geometric investment returns, the ability of the assets to support the maturity guarantees reduces as the prevailing yields reduce. Consequently, the claim frequencies and reserving requirements are higher if the initial yields are lower than average.

The behaviour of the model to changes in initial conditions is very much dependent on the fact that the model uses an auto-regressive process to project successive dividend yields. If the parameter A were zero, then the successive yields in the early years of the projection would be much less dependent on the initial yield, and a very different pattern of investment returns would result.

4.4 *Simulation Investigations into the Differences between the Alternative Models*

4.4.1 *Initial comparison of the alternative models*

Let model A denote the proposed alternative model (Section 3.7.1). This model is based primarily on the DeZoete & Bevan time series, but with rounding of the parameter values. The adjusted parameter values could also be derived by fitting the model to the Hybrid data series, which was based on the DeZoete & Bevan index until 1962 and the FT-Actuaries thereafter.

Let model Z be a model whose stable distribution parameters would be based on the DeZoete & Bevan time series, without adjustment. Referring to ¶3.5.1, the parameters for this model might be:

$$\begin{array}{llll} \alpha_D = 1.77 & \beta_D = 0.6 & \gamma_D = 0.05 & \delta_D = 0 \\ \alpha_Y = 1.77 & \beta_Y = -0.3 & \gamma_Y = 0.10 & \delta_Y = 0. \end{array}$$

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A comparison of the simulation results for models A and Z reveal the extent of any margins that have been incorporated as a result of adjusting the parameter values.

Let model F be a model whose stable distribution parameters would be based on the FT-Actuaries (quarterly) data. The annualised parameters for this model might be:

$$\begin{array}{llll} \alpha_D = 1.52 & \beta_D = 0.4 & \gamma_D = 0.045 & \delta_D = 0 \\ \alpha_Y = 1.70 & \beta_Y = -0.5 & \gamma_Y = 0.095 & \delta_Y = 0. \end{array}$$

Let M denote the model proposed by the MGWP (see Section 1.2).

The models are compared in Table 5.4(a). The standard portfolio with no tax was used. The following observations are made:

(1) Model F appears to have the most volatile stochastic fluctuations. Although the inter-quartile (and also inter-decile) ranges of the annual and geometric investment returns are lower for model F than with model M, the tail quantiles $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, $Q_{G(30)}(0.005)$ and $Q_{G(30)}(0.995)$, are more extreme under model F than any of the other models. In addition, the claim frequencies, claim severities and reserving requirements are highest under model F. These observations are a result of model F having the lowest index parameters.

(2) In all cases, the geometric investment statistics tend to be smaller than the corresponding year-on-year statistics. The observation that $\text{Med}[G(t)] < \text{Med}[I(t)]$ follows from the fact that the geometric mean of a set of non-equal positive numbers $\{(1+I(1)), (1+I(2)), \ldots, (1+I(t))\}$ is less than the arithmetic mean.

(3) Models A and Z are very similar, with model A being slightly more volatile. Model A was derived from model Z after rounding the parameters. The differences shown in Table 5.4(a) reflect the effects of these adjustments. Overall, the effect of rounding the parameters on the reserving requirements was almost negligable. The slightly higher volatility in Model A will be due to the fact that it has slightly lower index parameters, although this is confused by the slightly different skewnesses.

(4) When comparing model M (the MGWP model) with model A (the stable non-Gaussian alternative), we observe that under model A:
   - The inter-quartile range of the investment returns are smaller, but the tail quantiles of the investment returns are more extreme.
   - The expected severity of a guarantee claim is greater.
   - The guarantee claim frequency rate is lower. (The number *NZ* of non-zero claims, per 5,000 simulations, is smaller.)
   - The reserving requirements at the extreme tail probabilities are larger, but at the less extreme ruin probabilities (e.g. 5%), the reserving requirements are similar.

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	Model M	Model A
α_D	2.00	1.75
α_Y	2.00	1.75
β_D	0.00	+0.50
β_Y	0.00	–0.50
γ_D	$0.0919 = 0.13/\sqrt{2}$	0.05
γ_Y	$0.1414 = 0.20/\sqrt{2}$	0.10

The different simulation results for models A and M are entirely attributable to their different parameterisations. The fact that there are as many as six different parameters makes accounting for the different models a little complicated.

On the one hand, the fact that model A has lower index parameters suggests that this model should be more volatile. The population variances of the investment returns are infinite, and the associated extreme stochastic fluctuations should result in higher claim frequencies, claim severities and reserving requirements.

On the other hand, the fact that model A has smaller scale parameters will tend to make it less volatile. (In the extreme case of the scale parameters approaching zero, the model would degenerate into a deterministic one, and there would be no stochastic fluctuations at all.) Note that a reduction in the scale parameter is a natural consequence of fitting a model with more parameters to a given data set.

The different simulation results will also be partly due to the different skewnesses in the two models. The relative impact of changing the individual parameters on the difference between models A and M is explored in more detail in the next sub-section.

4.4.2 *Quantifying the effect of individual parameter changes on the differences between model A and model M*

Let $Q(\alpha_D, \beta_D, \gamma_D, \delta_D, \alpha_Y, \beta_Y, \gamma_Y, \delta_Y) = Q_{CR}(0.001)$ using a stochastic model whose parameters are α_D, β_D, γ_D, δ_D, α_Y, β_Y, γ_Y and δ_Y. Let superscripts A and M be used to denote models A and M respectively.

We know from Section 4.4.1 (and Table 5.4(a)) that:

$$Q(\text{model A}) = Q(1.75, 0.5, 0.05, 0, 1.75, -0.5, 0.10, 0) = 28\%$$
$$Q(\text{model M}) = Q(2.0, 0.0, 0.0919, 0, 2.0, 0.0, 0.1414, 0) = 12\%.$$

Let $\Delta = Q(\text{model A}) - Q(\text{Model M})$. Then (using an analysis of surplus approach to quantify the difference between the models):

$$\begin{aligned}\Delta &= Q(\alpha_D^A, \beta_D^A, \gamma_D^A, \delta_D^A, \alpha_Y^A, \beta_Y^A, \gamma_Y^A, \delta_Y^A) - Q(\alpha_D^M, \beta_D^M, \gamma_D^M, \delta_D^M, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, \delta_Y^M) \\ &= Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^A, \gamma_Y^A, 0) \quad - Q(\alpha_D^M, \beta_D^M, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0)\end{aligned}$$

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$$
\begin{aligned}
&= Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^A, \gamma_Y^A, 0) && - Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^A, \gamma_Y^M, 0)\\
&+ Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^A, \gamma_Y^M, 0) && - Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^M, \gamma_Y^M, 0)\\
&+ Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^A, \beta_Y^M, \gamma_Y^M, 0) && - Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0)\\
&+ Q(\alpha_D^A, \beta_D^A, \gamma_D^A, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0) && - Q(\alpha_D^A, \beta_D^A, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0)\\
&+ Q(\alpha_D^A, \beta_D^A, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0) && - Q(\alpha_D^A, \beta_D^M, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0)\\
&+ Q(\alpha_D^A, \beta_D^M, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0) && - Q(\alpha_D^M, \beta_D^M, \gamma_D^M, 0, \alpha_Y^M, \beta_Y^M, \gamma_Y^M, 0).
\end{aligned}
$$

A similar analysis can be done to quantify the source of change in variables other than $Q_{CR}(0.001)$. Table 5.4(b) shows the results of 5,000 computer simulations of the variables *NZ*, *MNZ*, $Q_{CR}(0.001)$, $Q_{CR}(1\%)$, $Q_{CR}(5\%)$, Med[$I(30)$], IQR[$I(30)$], $Q_{I(30)}(0.005)$, $Q_{I(30)}(0.995)$, Med[$G(30)$], IQR[$G(30)$], $Q_{G(30)}(0.005)$ and $Q_{G(30)}(0.995)$ for the 7 relevant values for α_D, β_D, γ_D, α_Y, β_Y and γ_Y. From these results we deduce the effect of changing each individual stable law parameter from the MGWP value to the new alternative value.

The effects of the individual parameter changes are shown in the table below. Positive entries in this table record increases in the relevant variable as a result of the change; negative values record decreases.

Parameter change	Impact on variable				
	$Q(0.001)$	$Q(0.01)$	$Q(0.05)$	*NZ*	*MNZ*
γ_Y	–3	–2	–1	–427	0.3
β_Y	–2	+1	0	+147	0.0
α_Y	+8	+2	+1	+466	–0.3
γ_D	–19	–12	–3	–631	–1.0
β_D	+7	+3	+1	–43	+0.6
α_D	+25	+12	+2	+368	+1.9
Total	+16	+4	0	–120	+1.5

Parameter change	Med[$I(30)$]	IQR[$I(30)$]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
γ_Y	–0.3	–10.1	+12.3	–83.4
β_Y	+0.8	+0.1	–2.4	–43.2
α_Y	+0.5	+2.7	–22.0	+166.0
γ_D	–0.6	–4.0	+6.7	–17.1
β_D	+0.7	+0.1	–1.9	–16.9
α_D	+0.2	+1.3	–6.6	+28.6
Total	+1.3	–9.9	–13.9	+34.0

Parameter change	Med[$G(30)$]	IQR[$G(30)$]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
γ_Y	–0.1	–0.3	–0.1	–6.5
β_Y	0.0	0.0	–0.1	+4.3
α_Y	+0.1	+0.3	–0.5	+7.7
γ_D	–0.2	–1.8	+3.5	–4.4
β_D	+0.2	0.0	–0.9	–5.7
α_D	+0.1	0.9	–3.7	+9.3
Total	+0.1	–0.9	–1.8	+4.7

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The above tables illustrate the fact that changes in the parameters affect the distributions of the different variables in different ways, and the relative significance of each variable depends on which statistic one is considering. Similar observations were made by the MGWP in its experiments on sensitivity to changes in basis.

For example, we find that the expected claim severities (*MNZ*) are most affected by the reduction in the index parameter α_D from 2 to 1.75. The second most influential parameter change on *MNZ* is the reduction in the dividend scale parameter γ_D from $0.13/\sqrt{2}$ to 0.05, and this acted in the opposite direction to the change in the index parameter. By contrast, the claim frequency variable *NZ* is most affected by the change in γ_D, followed by the reduction in the yield index parameter α_Y.

Overall, the reserve requirement for a 1:1,000 chance of ruin is higher for the alternative model, and this is mainly due to the reduction in α_D. The second most influential factor was the reduction in γ_D, which only partially offsets the change in α_D.

It is also relevant to note that the reserving requirements for less extreme ruin probabilities (e.g. $Q_{CR}(1\%)$, or $Q_{CR}(5\%)$), are much less sensitive to the index parameters. For these statistics, the reductions in the index and corresponding scale parameters were about the same. This is because of the fact that the index parameter controls the stochastic fluctuations from the extreme tail of a distribution (see Section 2.11), and the scale parameter affects the fluctuations elsewhere.

The consequences of Section 2.11 are also evident in the investment statistics. The inter-quartile range, which is a measure of less extreme stochastic fluctuations, is more sensitive to a change in the scale parameter of either the dividend yield or dividend growth model than it is to a change in the corresponding index parameter. By contrast, the statistics $Q_{I(30)}(0.995)$, $Q_{I(30)}(0.005)$, $Q_{G(30)}(0.995)$ and $Q_{G(30)}(0.005)$, which measure more extreme stochastic fluctuations, are more sensitive to changes in the index parameters than scale parameters.

Introducing a skewness parameter into either the dividend yield or the dividend growth model generally has the smallest impact on the claim and reserving statistics. This point could be taken into account when deciding on a final basis (i.e. whether or not to include any allowance for skewness).

4.5 *Different Liability Portfolios*

4.5.1 The same set of simulations was used to record the usual statistics for a variety of 'simple' portfolios, other than the standard portfolio. The simulation results are shown in Tables 5.5(a) and (b) for models M and A respectively.

4.5.2 The portfolios chosen included:

— a single policy of 1 p.a. for a term of 5, 10, 20 and 30 years (Columns 1 to 4 respectively);

— a group of five policies, each one with a premium of 1p.a., and maturing in each of years 10, 15, 20, 25, 30 respectively (Column 5);

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— a group of twenty-one policies, each one with a premium of 1p.a., and maturing in each of years 10, 11, 12, ..., 30 respectively (Column 6);
— a group of ten policies, each one with a premium of 1p.a., and maturing in each of years 1 to 10 (Column 7); and
— a group of twenty policies, each one with a premium of 1p.a., and maturing in each of years 1 to 20 (Column 8).

4.5.3 The simulations are done on the basis of zero tax. A comparison of the figures in Table 5.5(a) with those in Table E2.6 of Ford *et al.* (1980) shows the effect of changing the tax basis from 37.5% to zero. It can be seen that the main impact of operating in a gross tax environment is to reduce the expected claim frequencies, claim severities and consequential reserving requirements. Eliminating the imposition of tax improves the performance of the asset and increases its ability to support the maturity guarantees. The benefit of a gross tax environment increases with increasing term.

4.5.4 Consider now the single policy portfolios (Columns 1 to 4 of Tables 5.5(a) and (b)). The MGWP found that the number *NZ* of non-zero claims reduces substantially as the term increases. The MGWP also observed that the mean of the non-zero claims *MNZ*, was relatively stable over time. They found that the most extreme claims were large and similar for all policy durations, although the inner quantiles reduced with term. Overall, the rate of decline of *MNZ* with policy term was very slow.

4.5.5 By comparison, we find that the number of non-zero claims with the alternative model also reduces substantially with term, although the rate of reduction is noticeably less. The mean of the non-zero claims is less stable with the alternative model, and, in fact, increases with policy term. The overall effect is that the reserving requirement for very low ruin probabilities decreases substantially more slowly with the alternative model. The reserving requirements for less extreme ruin probabilities behave similarly to those given by model M. These results reflect the fact that the main impact of using the alternative model is to generate more extreme stochastic fluctuations, due to the smaller index parameters, at the expense of fewer less extreme stochastic fluctuations.

4.5.6 As pointed out by the MGWP, the discounted claim ratio for a single policy is simply the claim ratio for that policy, discounted by its term. No further information is derived from it.

4.5.7 The MGWP also investigated the effect of a spread of maturity dates, using the portfolios in Columns 5 and 6 of Tables 5.5(a) and (b). The portfolio in Column 5 has only five policies, while that in Column 6 has twenty-one. Both portfolios have approximately the same mean term — twenty years. It is, therefore, interesting to compare the results of these two portfolios with each other, and also with the results of the single policy (term 20) portfolio in Column 3.

4.5.8 We find that the claim frequency rates are higher for the larger portfolios, but the mean claim severities (as a percentage of total sum assured)

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decrease as the spread in the portfolios increase. Overall the reserving requirements (for low ruin probabilities) decrease as the size of the portfolio increases. These results apply to both model A and model M. These results can be deduced by general reasoning. The greater the number of policies in the portfolio, the greater the probability that at least some of them will become claims. Also, the greater the spread in the portfolio, the larger the total sum assured relative to the size of any of the claims that do occur.

4.5.9 The portfolios in Columns 7 and 8 suffer relatively high claim frequency rates. This illustrates the fact that most of the claims in such a portfolio would occur in the early years (in respect of the short-term policies in the portfolio). The expected claim severities are also relatively high, decreasing as the number of policies and size of total sum assured is increased. The discounted claim ratios increase significantly as the term of the portfolio decreases. These results apply to the alternative model in the same way as they do to the MGWP model.

4.5.10 It is unlikely that any office would, in practice, have a liability portfolio like any of the ones which have been considered. The portfolios do, however, provide some insight as to the implications of providing maturity guarantees with portfolios of different terms and spreads. In addition, one gains some insight as to the position for an on-going portfolio in respect of its future premiums, although the accumulated past premiums and asset shares also need to be taken into account. Investigating the on-going experience of an existing portfolio, and allowing for its past experience, is beyond the scope of this paper.

5. Summary and Conclusion

5.1 *An Alternative Model*

5.1.1 The MGWP model can be rejected on the grounds that the residuals which result from fitting both the dividend yield and the dividend growth models to the data have more kurtosis than can be supported by the assumption of a Normal distribution.

The dividend yield model can also be safely rejected on the grounds of skewness in the residuals, with all the data sets considered. However, the dividend growth model can only be rejected on the grounds of skewness with the FT-Actuaries data, but not with the DeZoete & Bevan nor the Hybrid time series.

This last anomaly is due to the fact that the Actuaries Investment Index was a geometric index until 1962, while the DeZoete and Bevan and the Hybrid Indices were arithmetically weighted over the whole period of investigation.

5.1.2 The model can, however, be refined by replacing the Normality assumptions by more general stable distributions, with index parameters less than two.

5.1.3 The results of Kanter & Steiger (1974), Yohai & Maronna (1977), Hannan & Kanter (1977) and Davis & Resnick (1989) establish that the least

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squares estimates for the parameters μ_D, μ_Y and A are consistent in probability when the error terms have a stable distribution. We can thus accept the values for these parameters that were recommended by the MGWP for the pre-1980 experience.

5.1.4 If experience since the MGWP published its report is taken into account, then the parameters μ_D, μ_Y and A need to be updated, and the method of least squares can again be used. The following values were derived from the Hybrid time series over the period December 1918 to December 1993, using the method of least squares (and some rounding):

$$\begin{aligned} \mu_D &= 5.5\% \\ \mu_Y &= 5.0\% \\ A &= 0.60. \end{aligned}$$

5.1.5 The method that has been recommended for fitting the appropriate stable distributions is due to Koutrouvelis (1980, 1981). Koutrouvelis's method is recommended on the grounds of ease of computation, lack of bias, and also relative precision of the index parameter. The results of applying Koutrouvelis's method to fit the model to the data are given in ¶3.6.1.

5.1.6 The discrepancy in the estimates that result from fitting the model to the FT-Actuaries and the DeZoete and Bevan time series are primarily due to the fact that, until 1962, the Actuaries' Investment Index was a geometric index. By contrast, the DeZoete and Bevan was an arithmetically weighted one. Geometric indices are more sensitive to sharp downward falls of the individual assets. This could result in larger downward fluctuations, lower index parameters, and negative skewness.

5.1.7 The model with stable distribution parameters:

$$\begin{array}{llll} \alpha_D = & 1.75 & \alpha_Y = & 1.75 \\ \beta_D = & +0.5 & \beta_Y = & -0.5 \\ \gamma_D = & 0.05 & \gamma_Y = & 0.10 \end{array}$$

has been proposed for consideration as an alternative to the MGWP model. These parameters can be used for the error terms in the model for pre-1980 experience, as well as in the updated model.

5.1.8 The stable distribution is a generalisation of the Normal distribution. Rejection of the stable model implies rejection of the Normal model too.

5.1.9 A detailed consideration of the potential problem areas with the stable model can be found in Chapter 5 of Finkelstein (1997), together with suggestions for further research.

5.2 *Impact of Using Stable (Non-Normal) Distributions*

5.2.1 If the Normal distributions are replaced by more general stable (non-Normal) distributions, then the population variances of the variables $\log D(t)/D(t-1)$ and $\log Y_p(t)$ become infinite. This leads to the annual investment return having both infinite mean and infinite variance.

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5.2.2 The lower the index parameters, the thicker are the tails of the distributions and the more extreme are the stochastic fluctuations. This will tend to increase guarantee claim frequencies, mean claim severities and the reserving requirements for low probabilities of ruin.

5.2.3 An important implication of the above is that a large part of financial economics that is based on mean-variance optimisation or power law utility functions becomes inapplicable under a stably distributed framework.

5.2.4 Another consequence of replacing the Normal distribution by a more general stable distribution is the fact that lower scale parameters result. This means that a better fit to the data is achieved. This is not surprising, in view of the fact that more parameters are introduced into the model. Lower scale parameters tend to reduce the stochastic fluctuations in the model. (In the limit, as the scale parameters approach zero, the model degenerates into a deterministic one, and there are no stochastic fluctuations at all.)

5.2.5 The relative impact of changing the different stable distribution parameters is investigated using simulation techniques. The overall impact of replacing the MGWP model by the proposed alternative is to:

— decrease the inter-quartile range of both the annual and the geometric investment returns (lower scale parameters dominating);
— decrease the values of $Q_{I(30)}(0.995)$ and $Q_{G(30)}(0.995)$ and increase the values of $Q_{I(30)}(0.005)$ and $Q_{G(30)}(0.005)$ (i.e. lower index parameters causing tail-quantiles to become more extreme);
— decrease the frequency of guarantee claims (lower scale parameters dominating);
— increase the mean severity of guarantee claims (lower index parameters dominating); and
— increase the reserving requirements for low ruin probabilities (lower index parameters dominating).

It was also noted that the reserving requirements for the less extreme ruin probabilities were similar with the two models. (Here the reductions in scale and index parameters offset each other almost exactly.)

5.2.6 The dividend yield skewness parameter was generally the least significant parameter on the claim and reserving statistics of the dividend yield model. Similarly, the dividend growth skewness parameter was generally the least significant parameter on the claim and reserving statistics of the dividend growth model. However, it was not necessarily the case that the skewness parameter of the dividend growth model was less significant (on the claim and reserving statistics) than either the scale or the index parameters of the dividend yield model. Also, for some of the annual investment statistics, the dividend yield skewness parameter was more significant than any of the dividend growth parameters.

These points, together with the purpose of the investigation and the lack of certainty about the correct values for the skewness parameters, should be taken

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into account if consideration is being given to simplifying the models by eliminating the skewness parameters.

5.2.7 When liability portfolios of different terms are used, we find that the claim frequency rates and reserving requirements (at low ruin probabilities) decrease with increasing term, although the rates of reduction are less with the alternative model than with the MGWP model. The behaviour of the reserving requirements at the less extreme ruin probabilities is once again similar in the two models. These results reflect the fact that the main impact of using the alternative model is to generate more extreme stochastic fluctuations, at the expense of fewer less extreme fluctuations.

ACKNOWLEDGEMENTS

The author wishes to thank Professor Rob Dorrington and Dr Iain Macdonald for their supervision of his research efforts at the University of Cape Town. The author also acknowledges with thanks the advice of Dr David Bowie, Associate Professor Darren Cline, Dr Andrew Gene, Andrew Smith, Professor David Wilkie and the *Journal* referees.

Acknowledgement is also due to the Centre for Science and Development of the Human Sciences Research Council (HSRC) of South Africa, and also to the Berkowitz and Noonan families for their financial support during the research.

Notwithstanding the above acknowledgements, the author is responsible for the information presented in the paper. Any errors or omissions are due to him.

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https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

APPENDIX 1

DATA

Table 1.1(a). The De Zoete & Bevan dividend data

	Price (1)	D (2)	$\ln(D/D)$ (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1918	100.00						
1919	135.50	6.50					
1920	93.73	8.91	0.31537	0.04	0.27537	0.00097	OUTLIER
1921	84.74	7.48	–0.17494	0.04	–0.21494	0.00775	UNLIKELY
1922	119.80	7.22	–0.03538	0.04	–0.07538	0.19801	
1923	125.87	8.13	0.11871	0.04	0.07871	0.18774	
1924	145.61	7.93	–0.02491	0.04	–0.06491	0.23243	
1925	182.88	8.78	0.10182	0.04	0.06182	0.24317	
1926	179.20	9.69	0.09862	0.04	0.05862	0.25461	
1927	200.60	9.82	0.01333	0.04	–0.02667	0.38196	
1928	231.70	10.60	0.07643	0.04	0.03643	0.34082	
1929	191.68	11.71	0.09959	0.04	0.05959	0.25112	
1930	158.90	10.60	–0.09959	0.04	–0.13959	0.05800	
1931	125.70	8.52	–0.21844	0.04	–0.25844	0.00181	OUTLIER
1932	162.27	7.67	–0.10510	0.04	–0.14510	0.05115	
1933	204.90	8.00	0.04212	0.04	0.00212	0.49046	
1934	245.59	9.11	0.12993	0.04	0.08993	0.15562	
1935	269.90	10.02	0.09521	0.04	0.05521	0.26708	
1936	310.85	10.60	0.05627	0.04	0.01627	0.42732	
1937	259.10	12.03	0.12655	0.04	0.08655	0.16489	
1938	220.57	12.10	0.00580	0.04	–0.03420	0.35009	
1939	213.68	11.58	–0.04393	0.04	–0.08393	0.17233	
1940	191.97	12.03	0.03812	0.04	–0.00188	0.49157	
1941	224.25	11.71	–0.02696	0.04	–0.06696	0.22543	
1942	252.64	11.19	–0.04542	0.04	–0.08542	0.16806	
1943	270.94	10.99	–0.01803	0.04	–0.05803	0.25672	
1944	293.30	11.19	0.01803	0.04	–0.02197	0.40233	
1945	299.25	11.38	0.01684	0.04	–0.02316	0.39712	
1946	340.45	12.03	0.05555	0.04	0.01555	0.43052	
1947	319.49	13.72	0.13145	0.04	0.09145	0.15156	
1948	294.26	12.62	–0.08357	0.04	–0.12357	0.08205	
1949	264.64	13.33	0.05473	0.04	0.01473	0.43411	
1950	279.25	13.92	0.04331	0.04	0.00331	0.48514	
1951	287.30	15.61	0.11459	0.04	0.07459	0.20050	
1952	270.64	16.52	0.05666	0.04	0.01666	0.42560	
1953	318.91	17.24	0.04266	0.04	0.00266	0.48805	
1954	453.63	19.90	0.14349	0.04	0.10349	0.12195	
1955	480.57	23.09	0.14868	0.04	0.10868	0.11052	
1956	413.61	23.74	0.02776	0.04	–0.01224	0.44520	
1957	384.60	24.32	0.02414	0.04	–0.01586	0.42912	
1958	542.60	25.82	0.05985	0.04	0.01985	0.41157	
1959	810.83	29.20	0.12302	0.04	0.08302	0.17494	
1960	789.70	35.19	0.18659	0.04	0.14659	0.04940	UNLIKELY

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Table 1.1(a) (continued). The De Zoete & Bevan dividend data

	Price (1)	D (2)	ln(D/D) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1961	766.40	36.49	0.03628	0.04	–0.00372	0.48328	
1962	732.50	36.81	0.00873	0.04	–0.03127	0.36239	
1963	927.80	38.11	0.03471	0.04	–0.00529	0.47624	
1964	846.20	44.03	0.14439	0.04	0.10439	0.11990	
1965	863.30	49.63	0.11972	0.04	0.07972	0.18467	
1966	759.40	50.34	0.01420	0.04	–0.02580	0.38573	
1967	979.67	48.45	–0.03827	0.04	–0.07827	0.18908	
1968	1310.97	47.22	–0.02571	0.04	–0.06571	0.22966	
1969	1071.70	50.21	0.06140	0.04	0.02140	0.40481	
1970	931.79	51.12	0.01796	0.04	–0.02204	0.40201	
1971	1281.50	46.31	–0.09882	0.04	–0.13882	0.05901	
1972	1317.66	47.80	0.03167	0.04	–0.00833	0.46263	
1973	935.04	50.73	0.05949	0.04	0.01949	0.41314	
1974	437.19	54.70	0.07535	0.04	0.03535	0.34531	
1975	1065.89	58.54	0.06785	0.04	0.02785	0.37693	
1976	1035.28	63.74	0.08510	0.04	0.04510	0.30578	
1977	1407.99	72.78	0.13263	0.04	0.09263	0.14847	
1978	1451.70	84.62	0.15073	0.04	0.11073	0.10623	

Table 1.1(b). The De Zoete & Bevan dividend yield data

	Yp (8)	ln(Yp) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1918	6.50	–2.73337	–2.73337	0.00000	0.50000	
1919	6.58	–2.72114	–2.83831	0.11718	0.26015	
1920	7.98	–2.52823	–2.83097	0.30274	0.04836	UNLIKELY
1921	8.52	–2.46275	–2.71523	0.25248	0.08300	
1922	6.79	–2.68972	–2.67595	–0.01377	0.46988	
1923	6.30	–2.76462	–2.81212	0.04750	0.39719	
1924	6.03	–2.80842	–2.85707	0.04864	0.39478	
1925	5.30	–2.93746	–2.88335	–0.05412	0.38327	
1926	5.48	–2.90407	–2.96077	0.05671	0.37786	
1927	5.28	–2.94124	–2.94073	–0.00051	0.49888	
1928	5.05	–2.98578	–2.96304	–0.02274	0.45035	
1929	5.53	–2.89498	–2.98976	0.09478	0.30153	
1930	5.36	–2.92621	–2.93528	0.00908	0.48014	
1931	6.11	–2.79524	–2.95402	0.15877	0.19185	
1932	4.93	–3.00983	–2.87544	–0.13439	0.23046	
1933	4.44	–3.11452	–3.00419	–0.11032	0.27249	
1934	4.08	–3.19907	–3.06700	–0.13207	0.23435	
1935	3.93	–3.23653	–3.11774	–0.11879	0.25728	
1936	3.87	–3.25192	–3.14021	–0.11170	0.26999	
1937	4.67	–3.06401	–3.14944	0.08543	0.31964	
1938	5.25	–2.94694	–3.03670	0.08976	0.31120	
1939	5.63	–2.87706	–2.96646	0.08940	0.31190	
1940	6.10	–2.79688	–2.92453	0.12765	0.24186	
1941	4.99	–2.99773	–2.87642	–0.12131	0.25284	

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Table 1.1(b) (continued). The De Zoete & Bevan dividend yield data

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1942	4.35	−3.13499	−2.99693	−0.13806	0.22439	
1943	4.13	−3.18689	−3.07929	−0.10760	0.27748	
1944	3.88	−3.24934	−3.11043	−0.13891	0.22300	
1945	4.02	−3.21389	−3.14789	−0.06599	0.35865	
1946	4.03	−3.21140	−3.12663	−0.08478	0.32092	
1947	3.95	−3.23145	−3.12514	−0.10632	0.27984	
1948	4.53	−3.09445	−3.13717	0.04272	0.40735	
1949	5.26	−2.94504	−3.05496	0.10992	0.27323	
1950	5.59	−2.88419	−2.96532	0.08113	0.32813	
1951	5.75	−2.85597	−2.92881	0.07284	0.34472	
1952	6.37	−2.75357	−2.91188	0.15830	0.19255	
1953	6.24	−2.77419	−2.85044	0.07625	0.33786	
1954	5.09	−2.97789	−2.86281	−0.11509	0.26389	
1955	4.94	−3.00780	−2.98503	−0.02278	0.45028	
1956	5.88	−2.83361	−3.00298	0.16936	0.17639	
1957	6.71	−2.70157	−2.89846	0.19689	0.14002	
1958	5.38	−2.92248	−2.81924	−0.10325	0.28554	
1959	4.34	−3.13730	−2.95178	−0.18551	0.15438	
1960	4.62	−3.07478	−3.08067	0.00589	0.48710	
1961	4.80	−3.03655	−3.04316	0.00660	0.48555	
1962	5.20	−2.95651	−3.02023	0.06371	0.36333	
1963	4.75	−3.04703	−2.97220	−0.07483	0.34071	
1964	5.86	−2.83702	−3.02651	0.18949	0.14926	
1965	5.83	−2.84215	−2.90051	0.05835	0.37443	
1966	6.38	−2.75200	−2.90358	0.15158	0.20280	
1967	4.82	−3.03240	−2.84949	−0.18290	0.15782	
1968	3.83	−3.26231	−3.01773	−0.24457	0.08982	
1969	4.77	−3.04282	−3.15568	0.11285	0.26791	
1970	4.97	−3.00175	−3.02399	0.02224	0.45145	
1971	3.73	−3.28876	−2.99934	−0.28942	0.05616	
1972	3.85	−3.25710	−3.17155	−0.08555	0.31941	
1973	5.85	−2.83873	−3.15255	0.31382	0.04256	UNLIKELY
1974	13.39	−2.01066	−2.90153	0.89087	0.00000	OUTLIER
1975	5.98	−2.81675	−2.40469	−0.41206	0.01189	UNLIKELY
1976	7.03	−2.65498	−2.88834	0.23336	0.10022	
1977	6.01	−2.81175	−2.79128	−0.02046	0.45531	

Table 1.2(a). The FT–Actuaries (December) dividend data

	Price (1)	*D* (2)	ln(*D*/*D*) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1930	64.30	4.205					
1931	49.29	3.103	−0.30390	0.04	−0.34390	0.0002	OUTLIER
1932	56.69	2.252	−0.32055	0.04	−0.36055	0.0001	OUTLIER
1933	69.61	2.190	−0.02792	0.04	−0.06792	0.2418	
1934	73.91	2.460	0.11626	0.04	0.07626	0.2158	
1935	80.81	2.697	0.09198	0.04	0.05198	0.2960	

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Table 1.2(a) (continued). The FT–Actuaries (December) dividend data

	Price (1)	D (2)	ln(D/D) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1936	90.83	3.031	0.11675	0.04	0.07675	0.2143	
1937	73.80	3.287	0.08108	0.04	0.04108	0.3359	
1938	61.49	3.419	0.03937	0.04	–0.00063	0.4974	
1939	56.41	3.151	–0.08163	0.04	–0.12163	0.1049	
1940	49.51	2.905	–0.08129	0.04	–0.12129	0.1055	
1941	60.70	2.753	–0.05374	0.04	–0.09374	0.1669	
1942	71.99	2.919	0.05855	0.04	0.01855	0.4242	
1943	77.80	3.126	0.06851	0.04	0.02851	0.3844	
1944	86.10	3.249	0.03859	0.04	–0.00141	0.4942	
1945	85.60	3.346	0.02942	0.04	–0.01058	0.4566	
1946	101.09	3.805	0.12855	0.04	0.08855	0.1806	
1947	98.33	4.173	0.09232	0.04	0.05232	0.2948	
1948	94.38	4.058	–0.02794	0.04	–0.06794	0.2418	
1949	81.30	4.114	0.01371	0.04	–0.02629	0.3931	
1950	86.44	4.203	0.02140	0.04	–0.01860	0.4240	
1951	88.53	4.738	0.11982	0.04	0.07982	0.2052	
1952	84.85	4.925	0.03871	0.04	–0.00129	0.4947	
1953	98.45	5.143	0.04331	0.04	0.00331	0.4864	
1954	132.39	6.292	0.20164	0.04	0.16164	0.0478	UNLIKELY
1955	134.50	6.879	0.08919	0.04	0.04919	0.3060	
1956	122.39	7.193	0.04464	0.04	0.00464	0.4809	
1957	118.31	7.532	0.04605	0.04	0.00605	0.4751	
1958	157.57	7.503	–0.00386	0.04	–0.04386	0.3255	
1959	225.92	8.156	0.08345	0.04	0.04345	0.3271	
1960	215.29	9.744	0.17790	0.04	0.13790	0.0775	
1961	209.88	10.715	0.09499	0.04	0.05499	0.2853	
1962	218.84	10.242	–0.04515	0.04	–0.08515	0.1900	
1963	259.19	10.835	0.05628	0.04	0.01628	0.4333	
1964	233.53	12.423	0.13677	0.04	0.09677	0.1592	
1965	245.35	13.323	0.06994	0.04	0.02994	0.3788	
1966	227.58	13.288	–0.00263	0.04	–0.04263	0.3301	
1967	286.03	13.186	–0.00771	0.04	–0.04771	0.3114	
1968	412.18	13.395	0.01573	0.04	–0.02427	0.4012	
1969	348.63	13.701	0.02259	0.04	–0.01741	0.4288	
1970	314.00	14.381	0.04844	0.04	0.00844	0.4653	
1971	434.94	14.918	0.03666	0.04	–0.00334	0.4863	
1972	480.49	16.193	0.08201	0.04	0.04201	0.3324	
1973	332.31	16.947	0.04551	0.04	0.00551	0.4773	
1974	151.55	18.443	0.08459	0.04	0.04459	0.3228	
1975	365.74	20.592	0.11022	0.04	0.07022	0.2345	
1976	362.04	23.316	0.12424	0.04	0.08424	0.1925	
1977	512.14	27.193	0.15382	0.04	0.11382	0.1203	
1978	532.18	30.334	0.10931	0.04	0.06931	0.2374	
1979	545.70	38.962					
1980	680.51	42.599					
1981	733.33	42.825					
1982	934.59	46.354					
1983	1114.65	50.046					
1984	1441.16	62.256					

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Table 1.2(a) (continued). The FT–Actuaries (December) dividend data

	Price (1)	D (2)	ln(D/D) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1985	1660.28	70.726					
1986	2028.60	80.533					
1987	2121.76	89.960					
1988	2235.22	104.605					
1989	2903.77	121.374					
1990	2520.76	133.092					
1991	2956.74	143.693					
1992	3361.86	143.883					
1993	3950.40	136.680					

Table 1.2(b). The FT–Actuaries (December) dividend yield data

	Yp (8)	ln(Yp) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1930	4.826	−3.03115	−3.03115	0.00000	0.5000	
1931	4.569	−3.08588	−3.01698	−0.06889	0.3648	
1932	3.863	−3.25373	−3.04982	−0.20391	0.1531	
1933	3.534	−3.34274	−3.15053	−0.19221	0.1674	
1934	3.649	−3.31072	−3.20394	−0.10678	0.2960	
1935	3.751	−3.28315	−3.18472	−0.09842	0.3107	
1936	3.619	−3.31897	−3.16818	−0.15079	0.2246	
1937	4.633	−3.07197	−3.18968	0.11771	0.2774	
1938	5.124	−2.97123	−3.04147	0.07024	0.3622	
1939	5.150	−2.96617	−2.98103	0.01486	0.4703	
1940	5.561	−2.88939	−2.97800	0.08860	0.3283	
1941	4.809	−3.03468	−2.93193	−0.10275	0.3031	
1942	4.342	−3.13684	−3.01910	−0.11773	0.2773	
1943	4.176	−3.17582	−3.08039	−0.09542	0.3160	
1944	3.886	−3.24779	−3.10378	−0.14401	0.2349	
1945	4.445	−3.11339	−3.14697	0.03358	0.4331	
1946	4.128	−3.18738	−3.06633	−0.12105	0.2718	
1947	4.127	−3.18762	−3.11072	−0.07690	0.3498	
1948	4.359	−3.13293	−3.11086	−0.02206	0.4559	
1949	5.170	−2.96230	−3.07805	0.11575	0.2807	
1950	5.481	−2.90388	−2.97567	0.07179	0.3593	
1951	5.563	−2.88903	−2.94062	0.05159	0.3979	
1952	6.061	−2.80330	−2.93171	0.12842	0.2596	
1953	6.391	−2.75028	−2.88027	0.12999	0.2571	
1954	5.196	−2.95728	−2.84846	−0.10882	0.2925	
1955	5.348	−2.92845	−2.97266	0.04421	0.4122	
1956	6.154	−2.78807	−2.95536	0.16729	0.2006	
1957	6.342	−2.75798	−2.87113	0.11316	0.2851	
1958	5.176	−2.96114	−2.85308	−0.10806	0.2938	
1959	4.313	−3.14354	−2.97498	−0.16856	0.1988	
1960	4.977	−3.00034	−3.08441	0.08407	0.3366	
1961	4.880	−3.02002	−2.99850	−0.02153	0.4570	
1962	4.951	−3.00558	−3.01031	0.00473	0.4905	

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Table 1.2(b) (continued). The FT–Actuaries (December) dividend yield data

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1963	4.793	−3.03801	−3.00164	−0.03637	0.4276	
1964	5.705	−2.86383	−3.02110	0.15727	0.2150	
1965	5.416	−2.91581	−2.91659	0.00078	0.4985	
1966	5.794	−2.84835	−2.94778	0.09943	0.3089	
1967	4.683	−3.06123	−2.90730	−0.15393	0.2199	
1968	3.324	−3.40400	−3.03503	−0.36897	0.0320	UNLIKELY
1969	4.125	−3.18810	−3.24069	0.05259	0.3959	
1970	4.751	−3.04682	−3.11116	0.06434	0.3734	
1971	3.723	−3.29064	−3.02638	−0.26426	0.0924	
1972	3.527	−3.34472	−3.17268	−0.17205	0.1940	
1973	5.550	−2.89137	−3.20513	0.31375	0.0577	
1974	13.588	−1.99598	−2.93312	0.93713	0.0000	OUTLIER
1975	6.375	−2.75279	−2.39588	−0.35690	0.0366	UNLIKELY
1976	7.511	−2.58880	−2.84996	0.26116	0.0950	
1977	5.923	−2.82633	−2.75157	−0.07475	0.3538	
1978	7.321					
1979	7.806					
1980	6.293					
1981	6.321					
1982	5.355					
1983	5.585					
1984	4.908					
1985	4.851					
1986	4.435					
1987	4.930					
1988	5.430					
1989	4.583					
1990	5.700					
1991	4.866					
1992	4.066					

Table 1.3(a). The Hybrid dividend series (period ending 1978)

	Price (1)	*D* (2)	ln(*D*/*D*) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1918	100.00						
1919	135.50	6.500					
1920	93.73	8.910	0.31537	0.04	0.27537	0.0007	OUTLIER
1921	84.74	7.480	−0.17494	0.04	−0.21494	0.0063	UNLIKELY
1922	119.80	7.220	−0.03538	0.04	−0.07538	0.1906	
1923	125.87	8.130	0.11871	0.04	0.07871	0.1803	
1924	145.61	7.930	−0.02491	0.04	−0.06491	0.2254	
1925	182.88	8.780	0.10182	0.04	0.06182	0.2363	
1926	179.20	9.690	0.09862	0.04	0.05862	0.2480	
1927	200.60	9.820	0.01333	0.04	−0.02667	0.3783	
1928	231.70	10.600	0.07643	0.04	0.03643	0.3361	
1929	191.68	11.710	0.09959	0.04	0.05959	0.2444	

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Table 1.3(a) (continued). The Hybrid dividend series (period ending 1978)

	Price (1)	D (2)	$\ln(D/D)$ (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1930	158.90	10.600	–0.09959	0.04	–0.13959	0.0524	
1931	125.70	8.520	–0.21844	0.04	–0.25844	0.0013	OUTLIER
1932	162.27	7.670	–0.10510	0.04	–0.14510	0.0459	UNLIKELY
1933	204.90	8.000	0.04212	0.04	0.00212	0.4902	
1934	245.59	9.110	0.12993	0.04	0.08993	0.1481	
1935	269.90	10.020	0.09521	0.04	0.05521	0.2606	
1936	310.85	10.600	0.05627	0.04	0.01627	0.4250	
1937	259.10	12.030	0.12655	0.04	0.08655	0.1573	
1938	220.57	12.100	0.00580	0.04	–0.03420	0.3456	
1939	213.68	11.580	–0.04393	0.04	–0.08393	0.1648	
1940	191.97	12.030	0.03812	0.04	–0.00188	0.4913	
1941	224.25	11.710	–0.02696	0.04	–0.06696	0.2183	
1942	252.64	11.190	–0.04542	0.04	–0.08542	0.1605	
1943	270.94	10.990	–0.01803	0.04	–0.05803	0.2501	
1944	293.30	11.190	0.01803	0.04	–0.02197	0.3993	
1945	299.25	11.380	0.01684	0.04	–0.02316	0.3939	
1946	340.45	12.030	0.05555	0.04	0.01555	0.4283	
1947	319.49	13.720	0.13145	0.04	0.09145	0.1440	
1948	294.26	12.620	–0.08357	0.04	–0.12357	0.0756	
1949	264.64	13.330	0.05473	0.04	0.01473	0.4320	
1950	279.25	13.920	0.04331	0.04	0.00331	0.4847	
1951	287.30	15.610	0.11459	0.04	0.07459	0.1931	
1952	270.64	16.520	0.05666	0.04	0.01666	0.4233	
1953	318.91	17.240	0.04266	0.04	0.00266	0.4877	
1954	453.63	19.900	0.14349	0.04	0.10349	0.1146	
1955	480.57	23.090	0.14868	0.04	0.10868	0.1034	
1956	413.61	23.740	0.02776	0.04	–0.01224	0.4435	
1957	384.60	24.320	0.02414	0.04	–0.01586	0.4269	
1958	542.60	25.820	0.05985	0.04	0.01985	0.4088	
1959	810.83	29.200	0.12302	0.04	0.08302	0.1674	
1960	789.70	35.190	0.18659	0.04	0.14659	0.0443	UNLIKELY
1961	766.40	36.490	0.03628	0.04	–0.00372	0.4827	
1962	732.50	36.810	0.00873	0.04	–0.03127	0.3582	
1963	867.54	38.941	0.05628	0.04	0.01628	0.4250	
1964	781.66	44.648	0.13676	0.04	0.09676	0.1305	
1965	821.21	47.883	0.06995	0.04	0.02995	0.3639	
1966	761.74	47.757	–0.00263	0.04	–0.04263	0.3102	
1967	957.39	47.390	–0.00771	0.04	–0.04771	0.2897	
1968	1379.63	48.141	0.01572	0.04	–0.02428	0.3890	
1969	1166.91	49.241	0.02259	0.04	–0.01741	0.4199	
1970	1050.99	51.685	0.04844	0.04	0.00844	0.4609	
1971	1455.82	53.615	0.03666	0.04	–0.00334	0.4845	
1972	1608.27	58.197	0.08200	0.04	0.04200	0.3128	
1973	1112.27	60.907	0.04551	0.04	0.00551	0.4745	
1974	507.24	66.284	0.08460	0.04	0.04460	0.3022	
1975	1224.18	74.007	0.11021	0.04	0.07021	0.2074	
1976	1211.80	83.797	0.12424	0.04	0.08424	0.1639	
1977	1714.20	97.731	0.15382	0.04	0.11382	0.0930	
1978	1781.28	109.020	0.10931	0.04	0.06931	0.2104	

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Table 1.3(b). The Hybrid dividend yield data (period ending 1978)

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1918	6.50	−2.73337	−2.73337	0.00000	0.5000	
1919	6.50	−2.73337	−2.83831	0.10495	0.2872	
1920	6.58	−2.72114	−2.83831	0.11718	0.2653	
1921	7.98	−2.52823	−2.83097	0.30274	0.0526	
1922	8.52	−2.46275	−2.71523	0.25248	0.0883	
1923	6.79	−2.68972	−2.67595	−0.01377	0.4706	
1924	6.30	−2.76462	−2.81212	0.04750	0.3997	
1925	6.03	−2.80842	−2.85707	0.04864	0.3973	
1926	5.30	−2.93746	−2.88335	−0.05412	0.3860	
1927	5.48	−2.90407	−2.96077	0.05671	0.3808	
1928	5.28	−2.94124	−2.94073	−0.00051	0.4989	
1929	5.05	−2.98578	−2.96304	−0.02274	0.4516	
1930	5.53	−2.89498	−2.98976	0.09478	0.3060	
1931	5.36	−2.92621	−2.93528	0.00908	0.4806	
1932	6.11	−2.79524	−2.95402	0.15877	0.1977	
1933	4.93	−3.00983	−2.87544	−0.13439	0.2360	
1934	4.44	−3.11452	−3.00419	−0.11032	0.2774	
1935	4.08	−3.19907	−3.06700	−0.13207	0.2398	
1936	3.93	−3.23653	−3.11774	−0.11879	0.2625	
1937	3.87	−3.25192	−3.14021	−0.11170	0.2750	
1938	4.67	−3.06401	−3.14944	0.08543	0.3237	
1939	5.25	−2.94694	−3.03670	0.08976	0.3155	
1940	5.63	−2.87706	−2.96646	0.08940	0.3162	
1941	6.10	−2.79688	−2.92453	0.12765	0.2472	
1942	4.99	−2.99773	−2.87642	−0.12131	0.2581	
1943	4.35	−3.13499	−2.99693	−0.13806	0.2300	
1944	4.13	−3.18689	−3.07929	−0.10760	0.2823	
1945	3.88	−3.24934	−3.11043	−0.13891	0.2286	
1946	4.02	−3.21389	−3.14789	−0.06599	0.3620	
1947	4.03	−3.21140	−3.12663	−0.08478	0.3250	
1948	3.95	−3.23145	−3.12514	−0.10632	0.2847	
1949	4.53	−3.09445	−3.13717	0.04272	0.4096	
1950	5.26	−2.94504	−3.05496	0.10992	0.2782	
1951	5.59	−2.88419	−2.96532	0.08113	0.3321	
1952	5.75	−2.85597	−2.92881	0.07284	0.3483	
1953	6.37	−2.75357	−2.91188	0.15830	0.1984	
1954	6.24	−2.77419	−2.85044	0.07625	0.3416	
1955	5.09	−2.97789	−2.86281	−0.11509	0.2690	
1956	4.94	−3.00780	−2.98503	−0.02278	0.4515	
1957	5.88	−2.83361	−3.00298	0.16936	0.1823	
1958	6.71	−2.70157	−2.89846	0.19689	0.1460	
1959	5.38	−2.92248	−2.81924	−0.10325	0.2903	
1960	4.34	−3.13730	−2.95178	−0.18551	0.1604	
1961	4.62	−3.07478	−3.08067	0.00589	0.4874	
1962	4.80	−3.03655	−3.04316	0.00660	0.4859	
1963	5.32	−2.93445	−3.02023	0.08578	0.3231	
1964	5.15	−2.96695	−2.95896	−0.00799	0.4829	
1965	6.13	−2.79263	−2.97846	0.18584	0.1600	
1966	5.82	−2.84473	−2.87387	0.02914	0.4380	

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Table 1.3(b) (continued). The Hybrid dividend yield data (period ending 1978)

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1967	6.22	−2.77724	−2.90513	0.12789	0.2468	
1968	5.03	−2.99015	−2.86464	−0.12551	0.2509	
1969	3.57	−3.33288	−2.99238	−0.34050	0.0342	UNLIKELY
1970	4.43	−3.11700	−3.19802	0.08103	0.3323	
1971	5.10	−2.97573	−3.06849	0.09276	0.3098	
1972	4.00	−3.21938	−2.98373	−0.23564	0.1036	
1973	3.79	−3.27360	−3.12992	−0.14368	0.2210	
1974	5.96	−2.82027	−3.16245	0.34218	0.0335	UNLIKELY
1975	14.59	−1.92483	−2.89045	0.96562	0.0000	EXTREME OUTLIER
1976	6.85	−2.68165	−2.35319	−0.32846	0.0394	UNLIKELY
1977	8.07	−2.51764	−2.80728	0.28965	0.0605	
1978	6.36	−2.75514	−2.70887	−0.04627	0.4022	

Table 1.4(a). The Hybrid dividend data (updated to 1993)

	Price (1)	*D* (2)	ln(*D*/*D*) (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1918	100.000						
1919	135.500	6.500					
1920	93.730	8.910	0.31537	0.055	0.26037	0.0013	EXTREME
1921	84.740	7.480	−0.17494	0.055	−0.22994	0.0040	EXTREME
1922	119.800	7.220	−0.03538	0.055	−0.09038	0.1488	
1923	125.870	8.130	0.11871	0.055	0.06371	0.2314	
1924	145.610	7.930	−0.02491	0.055	−0.07991	0.1785	
1925	182.880	8.780	0.10182	0.055	0.04682	0.2947	
1926	179.200	9.690	0.09862	0.055	0.04362	0.3076	
1927	200.600	9.820	0.01333	0.055	−0.04167	0.3155	
1928	231.700	10.600	0.07643	0.055	0.02143	0.4024	
1929	191.680	11.710	0.09959	0.055	0.04459	0.3036	
1930	158.900	10.600	−0.09959	0.055	−0.15459	0.0374	UNLIKELY
1931	125.700	8.520	−0.21844	0.055	−0.27344	0.0008	EXTREME
1932	162.270	7.670	−0.10510	0.055	−0.16010	0.0325	UNLIKELY
1933	204.900	8.000	0.04212	0.055	−0.01288	0.4410	
1934	245.590	9.110	0.12993	0.055	0.07493	0.1939	
1935	269.900	10.020	0.09521	0.055	0.04021	0.3215	
1936	310.850	10.600	0.05627	0.055	0.00127	0.4942	
1937	259.100	12.030	0.12655	0.055	0.07155	0.2048	
1938	220.570	12.100	0.00580	0.055	−0.04920	0.2853	
1939	213.680	11.580	−0.04393	0.055	−0.09893	0.1271	
1940	191.970	12.030	0.03812	0.055	−0.01688	0.4229	
1941	224.250	11.710	−0.02696	0.055	−0.08196	0.1724	
1942	252.640	11.190	−0.04542	0.055	−0.10042	0.1235	
1943	270.940	10.990	−0.01803	0.055	−0.07303	0.1999	
1944	293.300	11.190	0.01803	0.055	−0.03697	0.3350	
1945	299.250	11.380	0.01684	0.055	−0.03816	0.3300	
1946	340.450	12.030	0.05555	0.055	0.00055	0.4975	
1947	319.490	13.720	0.13145	0.055	0.07645	0.1891	

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Table 1.4(a) (continued). The Hybrid dividend data (updated to 1993)

	Price (1)	D (2)	$\ln(D/D)$ (3)	Model (4)	Residual (5)	Prob value (6)	Remark (7)
1948	294.260	12.620	–0.08357	0.055	–0.13857	0.0551	
1949	264.640	13.330	0.05473	0.055	–0.00027	0.4988	
1950	279.250	13.920	0.04331	0.055	–0.01169	0.4464	
1951	287.300	15.610	0.11459	0.055	0.05959	0.2461	
1952	270.640	16.520	0.05666	0.055	0.00166	0.4924	
1953	318.910	17.240	0.04266	0.055	–0.01234	0.4434	
1954	453.630	19.900	0.14349	0.055	0.08849	0.1539	
1955	480.570	23.090	0.14868	0.055	0.09368	0.1401	
1956	413.610	23.740	0.02776	0.055	–0.02724	0.3768	
1957	384.600	24.320	0.02414	0.055	–0.03086	0.3610	
1958	542.600	25.820	0.05985	0.055	0.00485	0.4777	
1959	810.830	29.200	0.12302	0.055	0.06802	0.2165	
1960	789.700	35.190	0.18659	0.055	0.13159	0.0647	
1961	766.400	36.490	0.03628	0.055	–0.01872	0.4146	
1962	732.500	36.810	0.00873	0.055	–0.04627	0.2969	
1963	867.542	38.941	0.05628	0.055	0.00128	0.4941	
1964	781.660	44.648	0.13676	0.055	0.08176	0.1730	
1965	821.207	47.883	0.06995	0.055	0.01495	0.4316	
1966	761.739	47.757	–0.00263	0.055	–0.05763	0.2532	
1967	957.393	47.390	–0.00771	0.055	–0.06271	0.2349	
1968	1379.631	48.141	0.01572	0.055	–0.03928	0.3254	
1969	1166.909	49.241	0.02259	0.055	–0.03241	0.3544	
1970	1050.987	51.685	0.04844	0.055	–0.00656	0.4699	
1971	1455.815	53.615	0.03666	0.055	–0.01834	0.4163	
1972	1608.271	58.197	0.08200	0.055	0.02700	0.3778	
1973	1112.271	60.907	0.04551	0.055	–0.00949	0.4565	
1974	507.242	66.284	0.08460	0.055	0.02960	0.3665	
1975	1224.182	74.007	0.11021	0.055	0.05521	0.2623	
1976	1211.802	83.797	0.12424	0.055	0.06924	0.2124	
1977	1714.195	97.731	0.15382	0.055	0.09882	0.1273	
1978	1781.281	109.020	0.10931	0.055	0.05431	0.2657	
1979	1826.518	140.030	0.25033	0.055	0.19533	0.0122	UNLIKELY
1980	2277.771	153.103	0.08925	0.055	0.03425	0.3465	
1981	2454.564	153.917	0.00530	0.055	–0.04970	0.2834	
1982	3128.218	166.601	0.07919	0.055	0.02419	0.3902	
1983	3730.902	179.870	0.07663	0.055	0.02163	0.4015	
1984	4823.772	223.753	0.21831	0.055	0.16331	0.0299	UNLIKELY
1985	5557.197	254.193	0.12755	0.055	0.07255	0.2015	
1986	6790.004	289.440	0.12985	0.055	0.07485	0.1941	
1987	7101.839	323.322	0.11070	0.055	0.05570	0.2604	
1988	7481.603	375.958	0.15083	0.055	0.09583	0.1347	
1989	9719.328	436.226	0.14868	0.055	0.09368	0.1401	
1990	8437.354	478.343	0.09217	0.055	0.03717	0.3342	
1991	9896.640	516.444	0.07664	0.055	0.02164	0.4015	
1992	11252.623	517.126	0.00132	0.055	–0.05368	0.2680	
1993	13222.562	491.236	–0.05136	0.055	–0.10636	0.1101	

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Table 1.4(b). The Hybrid dividend yield data (updated to 1993)

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1918	6.500	–2.73337	–2.73337	0.00000	0.5000	
1919	6.580	–2.72114	–2.83831	0.11718	0.2560	
1920	7.980	–2.52823	–2.83097	0.30274	0.0452	UNLIKELY
1921	8.520	–2.46275	–2.71523	0.25248	0.0789	
1922	6.790	–2.68972	–2.67595	–0.01377	0.4693	
1923	6.300	–2.76462	–2.81212	0.04750	0.3952	
1924	6.030	–2.80842	–2.85707	0.04864	0.3928	
1925	5.300	–2.93746	–2.88335	–0.05412	0.3810	
1926	5.480	–2.90407	–2.96077	0.05671	0.3755	
1927	5.280	–2.94124	–2.94073	–0.00051	0.4989	
1928	5.050	–2.98578	–2.96304	–0.02274	0.4494	
1929	5.530	–2.89498	–2.98976	0.09478	0.2980	
1930	5.360	–2.92621	–2.93528	0.00908	0.4798	
1931	6.110	–2.79524	–2.95402	0.15877	0.1872	
1932	4.930	–3.00983	–2.87544	–0.13439	0.2261	
1933	4.440	–3.11452	–3.00419	–0.11032	0.2685	
1934	4.080	–3.19907	–3.06700	–0.13207	0.2300	
1935	3.930	–3.23653	–3.11774	–0.11879	0.2531	
1936	3.870	–3.25192	–3.14021	–0.11170	0.2660	
1937	4.670	–3.06401	–3.14944	0.08543	0.3163	
1938	5.250	–2.94694	–3.03670	0.08976	0.3078	
1939	5.630	–2.87706	–2.96646	0.08940	0.3085	
1940	6.100	–2.79688	–2.92453	0.12765	0.2376	
1941	4.990	–2.99773	–2.87642	–0.12131	0.2487	
1942	4.350	–3.13499	–2.99693	–0.13806	0.2199	
1943	4.130	–3.18689	–3.07929	–0.10760	0.2736	
1944	3.880	–3.24934	–3.11043	–0.13891	0.2185	
1945	4.020	–3.21389	–3.14789	–0.06599	0.3560	
1946	4.030	–3.21140	–3.12663	–0.08478	0.3176	
1947	3.950	–3.23145	–3.12514	–0.10632	0.2760	
1948	4.530	–3.09445	–3.13717	0.04272	0.4056	
1949	5.260	–2.94504	–3.05496	0.10992	0.2693	
1950	5.590	–2.88419	–2.96532	0.08113	0.3250	
1951	5.750	–2.85597	–2.92881	0.07284	0.3418	
1952	6.370	–2.75357	–2.91188	0.15830	0.1879	
1953	6.240	–2.77419	–2.85044	0.07625	0.3348	
1954	5.090	–2.97789	–2.86281	–0.11509	0.2598	
1955	4.940	–3.00780	–2.98503	–0.02278	0.4493	
1956	5.880	–2.83361	–3.00298	0.16936	0.1717	
1957	6.710	–2.70157	–2.89846	0.19689	0.1353	
1958	5.380	–2.92248	–2.81924	–0.10325	0.2818	
1959	4.340	–3.13730	–2.95178	–0.18551	0.1497	
1960	4.620	–3.07478	–3.08067	0.00589	0.4868	
1961	4.800	–3.03655	–3.04316	0.00660	0.4853	
1962	5.316	–2.93445	–3.02023	0.08578	0.3157	
1963	5.146	–2.96695	–2.95896	–0.00799	0.4822	
1964	6.126	–2.79263	–2.97846	0.18584	0.1492	
1965	5.815	–2.84473	–2.87387	0.02914	0.4352	
1966	6.221	–2.77724	–2.90513	0.12789	0.2371	

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Table 1.4(b) (continued). The Hybrid dividend yield data (updated to 1993)

	Yp (8)	ln(*Yp*) (9)	Model (10)	Residual (11)	Prob value (12)	Remark (13)
1967	5.028	–2.99015	–2.86464	–0.12551	0.2413	
1968	3.569	–3.33288	–2.99238	–0.34050	0.0284	UNLIKELY
1969	4.429	–3.11700	–3.19802	0.08103	0.3252	
1970	5.101	–2.97573	–3.06849	0.09276	0.3019	
1971	3.998	–3.21938	–2.98373	–0.23564	0.0937	
1972	3.787	–3.27360	–3.12992	–0.14368	0.2107	
1973	5.959	–2.82027	–3.16245	0.34218	0.0278	UNLIKELY
1974	14.590	–1.92483	–2.89045	0.96562	0.0000	EXTREME OUTLIER
1975	6.845	–2.68165	–2.35319	–0.32846	0.0331	UNLIKELY
1976	8.065	–2.51764	–2.80728	0.28965	0.0526	
1977	6.360	–2.75514	–2.70887	–0.04627	0.3979	
1978	7.861	–2.54326	–2.85138	0.30812	0.0424	UNLIKELY
1979	8.382	–2.47908	–2.72425	0.24516	0.0851	
1980	6.757	–2.69459	–2.68574	–0.00885	0.4803	
1981	6.787	–2.69016	–2.81505	0.12489	0.2424	
1982	5.750	–2.85597	–2.81239	–0.04358	0.4037	
1983	5.997	–2.81391	–2.91188	0.09796	0.2918	
1984	5.270	–2.94314	–2.88664	–0.05650	0.3760	
1985	5.208	–2.95497	–2.96418	0.00920	0.4795	
1986	4.762	–3.04450	–2.97128	–0.07322	0.3410	
1987	5.294	–2.93860	–3.02499	0.08640	0.3144	
1988	5.831	–2.84198	–2.96145	0.11947	0.2519	
1989	4.922	–3.01146	–2.90348	–0.10797	0.2729	
1990	6.121	–2.79344	–3.00517	0.21172	0.1181	
1991	5.225	–2.95172	–2.87436	–0.07736	0.3326	
1992	4.366	–3.13132	–2.96932	–0.16200	0.1824	

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APPENDIX 2

KOUTROVELIS'S METHOD OF ESTIMATING STABLE LAW PARAMETERS

Koutrouvelis's method involves evaluating the modulus of the sample characteristic function at a number of points close to zero. The following describes how the method works. Computer code which implements the method is provided in Finkelstein (1997).

Given a random sample z_1, z_2, ..., z_n from a random variable Z, the sample characteristic function is defined to be:

$$\psi_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{iz_t}$$

where $i = \sqrt{-1}$.
The modulus at a particular point t is evaluated using:

$$|\psi_n(t)| = \tfrac{1}{n}\sqrt{\left(\sum_j \cos(z_j t)\right)^2 + \left(\sum_j \sin(z_j t)\right)^2}.$$

In addition, Koutrouvelis's method makes use of the following relations which are derived from the log characteristic function, equation (2.1):

$$\log(-\log|\psi(t)|^2) = \log(2\gamma^\alpha) + \alpha\log|t|$$

$$\frac{\mathrm{Im}(\psi(t))}{\mathrm{Re}(\psi(t))} = \tan\left(\delta t - \beta\gamma^\alpha \tan\left(\frac{\pi\alpha}{2}\right)\mathrm{sgn}(t)|t|^\alpha\right) \qquad \alpha \neq 1$$

where $\mathrm{Re}(\psi(t))$ and $\mathrm{Im}(\psi(t))$ are the real and imaginary parts of $\psi(t)$ respectively.

Having evaluated $|\psi_n(t_k)|$ at a number of points ($k = 1, 2, ..., K$), the method involves estimating α and γ by regressing $y = \log(-\log|\psi_n(t)|^2)$ on $w = \log(|t|)$ in the model:

$$y_k = c + \alpha w_k + \varepsilon_k$$

where $c = \log(2\gamma^\alpha)$ and ε_k is an error term.

The slope of the line of best fit gives the estimate for α. This and the y-intercept is then used to estimate γ using $\gamma = (e^c/2)^{1/\alpha}$.

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A second regression is used, together with the estimates for α and γ, to estimate β and δ. The second regression model is:

$$z_l = \delta u_l - \beta\gamma^{\alpha}\tan\frac{\pi\alpha}{2}\operatorname{sgn}(u)|u_l|^{\alpha} + \eta_l$$

where:

u_l are the points at which $|\psi_n(u)|$ is evaluated; and
η_l are error terms.

This last regression simplifies if δ is assumed *a priori* to be zero.

Koutrouvelis's method shows that it is best to evaluate $|\psi_n(t)|$ at points close to zero, and recommends using $t_k = \pi k/25$ ($k = 1, 2, ..., K$) for the first regression, and $u_l = \pi l/50$ ($l = 1, 2, ..., L$) for the second regression. The optimal numbers K and L of points for various sample sizes and index parameters were found by simulation and tabulated in his 1980 paper.

Koutrouvelis also points out that some of the inter-dependencies of the estimators can be reduced by using standardised data. This is effected by transforming each residual z_t to $z_t' = (z_t - \delta)/\gamma_0 = z_t/\gamma_0$, where γ_0 is an initial estimate for γ and when $\delta = 0$. If γ_1 is the estimate for the scale parameter after the data has been standardised by dividing by γ_0, then $\tilde{\gamma} = \gamma_1\gamma_0$ will be the estimate for the scale parameter for the unstandardised data. The initial estimate can be found using one of the fractile methods.

The above results in an iterative process whereby the ith iteration will use the results of the $(i-1)$th iteration to standardise the data before performing the regressions to update the parameter estimates. The process halts when the differences between the results of the successive iterations are satisfactorily small.

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APPENDIX 3

THE STANDARD LIABILITY PORTFOLIO

Term t	Premium P_t	Sum assured tP_t
10	30	300
11	6	66
12	7	84
13	8	104
14	9	126
15	50	750
16	10	160
17	10	170
18	10	180
19	10	190
20	50	1000
21	9	189
22	8	176
23	7	161
24	6	144
25	30	750
26	5	130
27	5	135
28	5	140
29	5	145
30	10	300

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

APPENDIX 4

THE SIMULATION METHOD OF CHAMBERS, MALLOWS AND STUCK

Chambers *et al.* (1976) deduced the following representation for the standardised random variable $S(\alpha, \beta)$:

$$S(\alpha,\ \beta) = \begin{cases} \dfrac{\sin\{\alpha(\phi-\phi_0)\}}{\{\cos\ \phi\}^{1/\alpha}} \left[\dfrac{\cos(\phi-\alpha(\phi-\phi_0))}{W}\right]^{(1-\alpha)/\alpha} & \text{if } \alpha \neq 1 \\ \dfrac{2}{\pi}\left(\left(\dfrac{\pi}{2}+\phi\beta\right)\tan\phi - \beta\ln\left(\dfrac{(\pi/2)W\cos\phi}{(\pi/2)+\phi\beta}\right)\right) & \text{if } \alpha = 1 \end{cases} \tag{A4.1}$$

where:

$\phi \sim U\left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$;

$W \sim \text{expo}(1)$;

ϕ and W are mutually independent;

$\phi_0 = -\dfrac{k(\alpha)}{2\alpha}\pi\beta$; and

$k(\alpha) = 1 - |1-\alpha|$.

It follows that, to generate a sample point from an $S(\alpha, \beta)$ distribution, one needs to:

(1) Generate two random numbers u_1, u_2 which are independent sample points from a uniform (0,1) random variable. A linear congruential random number generator can be used for this. (See Ripley (1987) for more details.)
(2) Let $\phi_1 = \pi(u_1 - 0.5)$. Now ϕ_1 is a sample point from a distribution which is uniform over the interval $(-\pi/2, \pi/2)$.
(3) Let $w_1 = -\ln(u_2)$. Now w_1 is a sample point from the standard exponential distribution, and is independent of ϕ_1.
(4) The sample point s from the $S(\alpha, \beta)$ distribution can be obtained by using equation (A4.1), replacing ϕ by ϕ_1 and w by w_1 respectively.
(5) Finally, a sample point s' from an unstandardised $S(\alpha, \beta, \gamma, \delta)$ distribution can be obtained using $s' = \delta + \gamma s$.

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

APPENDIX 5

TABLES OF SIMULATION RESULTS

5.1 *Sensitivity of the Proposed Alternative Model to Changes in the Location, Scale, Auto-Correlation and Initial Yield Parameters*

Table 5.1(a). Sensitivity of the model to changes in the location parameters

	Dividend means					Yield means				
μ_D	8%	6%	4%	2%	0%	4%	4%	4%	4%	4%
γ_D	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	5.5%	5.25%	5%	4.75%	4.5%
γ_Y	10%	10%	10%	10%	10%	10%	10%	10%	10%	10%
Y_o	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
A	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
Med[$I(30)$]	14.2	12.0	9.9	7.8	5.8	10.4	10.1	9.9	9.7	9.4
Med[$G(30)$]	13.7	11.5	9.4	7.3	5.3	9.5	9.5	9.4	9.4	9.3
IQR[$I(30)$]	29.0	28.5	27.9	27.4	26.9	28.0	28.0	27.9	27.9	27.8
IQR[$G(30)$]	2.7	2.7	2.6	2.6	2.5	2.6	2.6	2.8	2.7	2.8
NZ	264	437	679	1,197	2,133	600	634	679	732	786
MNZ (per mille)	29	26	26	24	25	26	26	26	26	27
$Q_{CR}(0.001)$	15	22	28	36	45	27	27	28	28	29
$Q_{CR}(0.005)$	10	12	15	19	27	14	14	15	16	17
$Q_{CR}(0.01)$	6	7	9	12	18	9	9	9	10	10
$Q_{CR}(0.05)$	1	2	2	4	6	2	2	2	3	3
$Q_{DCR}(0.001)$	9	12	14	18	21	14	14	14	15	15
$Q_{DCR}(0.005)$	5	7	8	11	14	8	8	8	9	9
$Q_{DCR}(0.01)$	4	4	5	7	9	5	5	5	5	6
$Q_{DCR}(0.05)$	1	1	2	2	4	2	2	2	2	2

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.1(b). Sensitivity of the model to changes in the scale parameters

	Dividend scale					Yield scale				
μ_D	4%	4%	4%	4%	4%	4%	4%	4%	4%	4%
γ_D	1%	3%	5%	7%	9%	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
γ_Y	10%	10%	10%	10%	10%	5%	7.5%	10%	12.5%	15%
Y_o	5%	5%	5%	5%	5%	5%	5%	5%	5%	5%
A	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
Med[$I(30)$]	9.6	9.7	9.9	10.2	10.4	9.8	9.9	9.9	10.0	10.2
Med[$G(30)$]	9.2	9.3	9.4	9.5	9.6	9.3	9.3	9.4	9.5	9.6
IQR[$I(30)$]	24.9	26.5	27.9	30.1	32.5	16.8	22.3	27.9	33.9	40.1
IQR[$G(30)$]	1.2	1.8	2.6	3.5	4.3	2.4	2.5	2.6	2.8	3.0
NZ	434	542	679	920	1,198	308	463	679	679	1196
MNZ (per mille)	15	19	26	30	35	36	30	26	26	23
$Q_{CR}(0.001)$	13	17	28	42	47	23	26	28	30	32
$Q_{CR}(0.005)$	8	10	15	23	27	15	15	15	15	16
$Q_{CR}(0.01)$	4	6	9	14	19	7	9	9	10	12
$Q_{CR}(0.05)$	1	2	2	3	5	1	2	2	3	4
$Q_{DCR}(0.001)$	7	8	14	21	23	12	14	14	15	16
$Q_{DCR}(0.005)$	4	5	8	11	14	8	8	8	9	9
$Q_{DCR}(0.01)$	2	4	5	7	9	4	5	5	6	6
$Q_{DCR}(0.05)$	1	1	2	2	3	1	1	2	2	2

Table 5.1(c). Sensitivity to changes in auto–correlation

μ_D	4%	4%	4%	4%	4%	4%
γ_D	5%	5%	5%	5%	5%	5%
μ_Y	5%	5%	5%	5%	5%	5%
γ_Y	10%	10%	10%	10%	10%	10%
Y_o	5%	5%	5%	5%	5%	5%
A	0.0	0.2	0.4	0.6	0.8	1.0
Med[$I(30)$]	9.5	9.8	9.8	9.9	10.1	12.6
Med[$G(30)$]	9.4	9.4	9.4	9.4	9.5	10.4
IQR[$I(30)$]	34.1	23.1	30.1	27.9	26.7	26.1
IQR[$G(30)$]	2.5	2.5	2.5	2.6	2.8	3.7
NZ	669	659	672	679	745	760
MNZ (per mille)	23	24	25	26	27	30
$Q_{CR}(0.001)$	24	25	27	28	26	34
$Q_{CR}(0.005)$	14	14	14	15	16	17
$Q_{CR}(0.01)$	9	9	9	9	10	11
$Q_{CR}(0.05)$	2	2	2	2	3	3
$Q_{DCR}(0.001)$	13	13	14	14	14	18
$Q_{DCR}(0.005)$	8	8	8	8	9	9
$Q_{DCR}(0.01)$	5	5	5	5	6	6
$Q_{DCR}(0.05)$	2	2	2	2	2	2

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.1(d). Sensitivity to changes in initial yields

μ_D	4%	4%	4%
γ_D	5%	5%	5%
μ_Y	5%	5%	5%
γ_Y	10%	10%	10%
Y_o	2.5%	5%	7.5%
A	0.6	0.6	0.6
Med[I(30)]	9.9	9.9	9.9
Med[G(30)]	6.8	9.4	11.0
IQR[I(30)]	27.9	27.9	27.9
IQR[G(30)]	2.5	2.6	2.6
NZ	1,291	679	489
MNZ (per mille)	20	26	29
$Q_{CR}(0.001)$	31	28	26
$Q_{CR}(0.005)$	17	15	13
$Q_{CR}(0.01)$	11	9	8
$Q_{CR}(0.05)$	3	2	2
$Q_{DCR}(0.001)$	16	14	13
$Q_{DCR}(0.005)$	10	8	8
$Q_{DCR}(0.01)$	6	5	4
$Q_{DCR}(0.05)$	2	2	2

5.2 *Sensitivity of the Model to Changes in the Index Parameters*

Table 5.2(a). Effect of changes in index parameters on Med[I(30)]

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	11.81	11.01	10.40	9.99	9.55	9.31	9.00	8.88	8.60	8.44	8.25
1.1	12.92	12.03	11.29	10.84	10.40	10.14	9.85	9.59	9.49	9.24	9.09
1.2	13.55	12.74	12.15	11.56	11.01	10.66	10.28	10.24	10.10	9.93	9.82
1.3	13.94	13.11	12.34	11.80	11.33	10.94	10.66	10.48	10.24	10.08	10.07
1.4	14.20	13.16	12.46	11.95	11.45	11.06	10.82	10.65	10.41	10.25	10.11
1.5	14.10	13.15	12.46	11.94	11.40	10.98	10.64	10.38	10.27	10.16	10.10
1.6	14.01	13.19	12.41	11.76	11.39	10.99	10.59	10.37	10.24	10.07	9.99
1.7	13.70	12.79	12.10	11.51	11.08	10.80	10.37	10.15	10.01	9.82	9.82
1.8	13.29	12.47	11.78	11.22	10.82	10.51	10.13	9.88	9.67	9.62	9.47
1.9	12.67	12.07	11.38	10.82	10.43	10.14	9.84	9.55	9.35	9.17	9.14
2.0	12.20	11.64	11.04	10.55	10.12	9.73	9.47	9.18	8.99	8.89	8.79

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Table 5.2(b). Effect of changes in index parameters on IQR[$I(30)$]

α_D / α_γ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	40.70	40.10	39.29	38.42	37.30	36.69	36.40	36.11	35.80	35.23	34.93
1.1	38.80	38.34	37.95	37.07	36.34	36.16	35.61	35.06	34.66	34.18	34.06
1.2	37.78	37.04	36.36	35.69	34.77	34.29	33.92	33.55	33.18	32.78	32.57
1.3	36.11	35.67	34.94	34.40	33.78	33.15	32.68	32.31	31.85	31.40	31.00
1.4	34.81	34.52	34.03	33.37	32.69	32.39	31.89	31.44	30.62	30.37	30.09
1.5	33.50	33.22	32.36	31.94	31.45	31.34	31.04	30.48	29.89	29.62	29.26
1.6	32.40	31.93	31.26	30.92	30.63	30.09	29.89	29.33	28.98	28.55	28.35
1.7	31.62	31.12	30.49	30.18	29.77	29.36	28.94	28.35	28.02	27.74	27.45
1.8	31.10	30.52	29.91	29.51	29.15	28.73	28.21	27.95	27.39	26.97	26.85
1.9	30.20	29.93	29.45	28.73	28.35	27.82	27.43	27.06	26.77	26.47	26.26
2.0	29.69	28.99	28.55	28.02	27.42	26.99	26.63	26.14	25.88	25.66	25.51

Table 5.2(c). Effect of changes in index parameters on $Q_{I(30)}(0.995)$

α_D / α_γ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55	–98.55
1.1	–96.25	–96.25	–96.21	–96.21	–96.14	–96.14	–96.14	–96.14	–96.14	–96.13	–96.10
1.2	–95.48	–95.48	–95.35	–95.14	–95.14	–95.12	–95.06	–95.06	–95.06	–95.06	–94.93
1.3	–94.90	–94.67	–94.29	–94.09	–93.66	–93.64	–92.89	–92.77	–92.78	–92.78	–92.14
1.4	–93.80	–93.02	–92.99	–91.77	–90.42	–89.05	–89.07	–88.11	–88.14	–88.16	–87.29
1.5	–92.40	–91.44	–89.53	–88.37	–86.51	–82.40	–80.01	–80.05	–79.58	–79.63	–79.20
1.6	–91.78	–89.58	–86.31	–83.84	–81.55	–78.65	–73.81	–70.91	–71.03	–69.97	–69.27
1.7	–90.22	–88.99	–83.79	–77.79	–73.76	–72.32	–67.82	–61.98	–61.16	–60.22	–59.05
1.8	–86.40	–84.60	–81.97	–75.93	–68.25	–62.53	–60.00	–57.06	–51.13	–49.68	–46.95
1.9	–85.33	–81.42	–76.55	–70.01	–64.37	–55.96	–50.50	–48.36	–41.53	–38.78	–36.99
2.0	–83.95	–80.35	–73.46	–65.20	–56.28	–49.33	–41.83	–35.85	–33.94	–31.37	–29.56

Table 5.2(d). Effect of changes in index parameters on $Q_{I(30)}(0.005)$

α_D / α_γ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	5.6E+9	3.9E+6	3.5E+6	3.4E+6	3.4E+6	3.4E+6	3.4E+6	3.4E+6	3.3E+6	3.3E+6	3.0E+6
1.1	2.5E+6	2.3E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6	1.2E+6
1.2	2.9E+5	7.7E+4	4.5E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4	2.8E+4
1.3	7.6E+4	6.1E+3	4.5E+3	4.0E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3	3.6E+3
1.4	2.3E+4	3.0E+3	1.4E+3	1.3E+3	1.2E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3	1.1E+3
1.5	7.2E+3	1.7E+3	8.6E+2	6.6E+2	5.1E+2	5.0E+2	4.9E+2	4.8E+2	4.8E+2	4.9E+2	5.0E+2
1.6	2.9E+3	9.6E+2	5.3E+2	4.2E+2	3.3E+2	2.7E+2	2.7E+2	2.6E+2	2.6E+2	2.7E+2	2.7E+2
1.7	2.2E+3	5.7E+2	4.0E+2	2.8E+2	2.3E+2	2.0E+2	1.8E+2	1.7E+2	1.7E+2	1.7E+2	1.7E+2
1.8	1.5E+3	4.8E+2	2.6E+2	2.0E+2	1.7E+2	1.4E+2	1.4E+2	1.2E+2	1.1E+2	1.1E+2	1.1E+2
1.9	8.8E+2	4.3E+2	2.2E+2	1.5E+2	1.1E+2	1.0E+2	9.2E+1	8.9E+1	8.7E+1	8.4E+1	8.3E+1
2.0	8.7E+2	4.3E+2	2.0E+2	1.2E+2	9.9E+1	8.6E+1	7.9E+1	7.4E+1	7.1E+1	7.0E+1	7.0E+1

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.2(e). Effect of changes in index parameters on *NZ*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	1980	2109	2175	2198	2199	2185	2147	2100	2079	2049	1038
1.1	1854	2006	2062	2081	2064	2075	2036	1975	1932	1902	1885
1.2	1755	1897	1959	1961	1929	1893	1853	1797	1779	1746	1709
1.3	1660	1774	1814	1820	1783	1713	1655	1588	1555	1521	1475
1.4	1545	1680	1699	1668	1626	1551	1471	1396	1362	1316	1267
1.5	1421	1530	1558	1529	1451	1386	1321	1240	1186	1125	1076
1.6	1317	1392	1389	1346	1278	1185	1110	1003	941	885	846
1.7	1208	1267	1263	1206	1129	1007	915	812	740	667	623
1.8	1111	1179	1156	1082	980	847	745	630	552	493	422
1.9	1057	1088	1052	977	862	728	617	496	392	314	254
2.0	1030	1010	975	878	748	598	464	345	242	165	82

Table 5.2(f). Effect of changes in index parameters on *MNZ*

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	7.90	7.43	6.78	6.12	5.53	5.05	4.69	4.46	4.23	4.07	4.10
1.1	7.34	6.80	6.18	5.47	4.87	4.27	3.86	3.63	3.42	3.24	3.02
1.2	7.14	6.57	5.87	5.15	4.52	3.95	3.49	3.22	2.94	2.75	2.54
1.3	7.27	6.72	6.00	5.15	4.43	3.85	3.36	3.07	2.79	2.56	2.34
1.4	7.54	6.81	6.08	5.27	4.45	3.76	3.25	2.94	2.60	2.36	2.11
1.5	7.93	7.17	6.29	5.38	4.58	3.74	3.09	2.72	2.37	2.12	1.83
1.6	8.33	7.61	6.73	5.76	4.80	3.94	3.15	2.75	2.33	2.01	1.63
1.7	8.86	8.12	7.12	6.09	5.06	4.20	3.33	2.79	2.29	1.89	1.45
1.8	9.44	8.50	7.51	6.49	5.49	4.59	3.63	3.01	2.37	1.79	1.26
1.9	9.75	9.03	8.01	6.89	5.90	4.95	3.93	3.22	2.56	1.84	1.00
2.0	9.80	9.52	8.41	7.39	6.45	5.62	4.74	3.98	3.20	2.16	0.47

Table 5.2(g). Effect of changes in index parameters on Q_{CR} (0.001)

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	86	86	86	86	86	86	86	86	86	86	86
1.1	80	79	79	79	80	80	80	80	80	80	80
1.2	57	57	57	57	56	53	49	43	33	23	18
1.3	56	56	55	54	53	51	47	41	31	21	17
1.4	53	53	52	52	50	50	47	39	29	18	17
1.5	53	53	52	52	49	49	45	38	28	18	16
1.6	52	52	52	49	49	47	43	35	26	18	14
1.7	51	51	50	49	47	44	42	34	25	17	13
1.8	50	50	51	48	47	44	40	32	23	17	12
1.9	50	50	51	48	46	42	37	30	21	15	8
2.0	49	49	50	48	46	42	36	27	19	10	3

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.2(h). Impact of changes in index parameters on $Q_{CR}(1\%)$

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	44	43	41	37	33	32	27	22	19	18	16
1.1	39	38	35	32	27	23	20	18	17	15	14
1.2	36	36	34	31	26	22	18	16	14	14	13
1.3	36	35	33	30	26	21	17	15	14	13	12
1.4	35	34	33	30	26	21	17	14	13	12	11
1.5	35	35	33	31	26	21	16	13	12	10	9
1.6	35	35	32	30	26	20	15	12	11	9	6
1.7	35	35	33	30	26	20	15	11	9	6	5
1.8	35	34	33	30	27	21	14	11	7	5	4
1.9	34	34	33	30	27	21	15	10	6	4	2
2.0	34	34	33	30	27	21	15	10	5	2	1

Table 5.2(i). Impact of changes in index parameters on $Q_{CR}(5\%)$

α_D / α_Y	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.0	20	19	17	15	14	13	11	10	10	9	8
1.1	18	18	16	14	13	11	9	8	7	6	6
1.2	18	17	15	13	11	9	8	7	6	6	5
1.3	18	17	14	12	10	8	7	6	5	5	4
1.4	18	17	14	12	10	7	6	5	5	4	4
1.5	17	16	14	11	9	7	5	4	4	4	3
1.6	17	17	13	11	8	6	4	4	3	3	3
1.7	17	16	13	11	8	5	4	3	3	3	2
1.8	17	16	13	10	7	5	3	2	2	2	1
1.9	17	16	13	10	7	4	3	2	1	1	1
2.0	16	16	13	10	6	3	2	1	1	1	1

5.3 *Sensitivity of the Model to Changes in the Skewness Parameters*

Table 5.3(a). Values of Med[$I(30)$] with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	9.59	9.81	10.18	10.59	10.83
−0.5	8.95	9.21	9.58	9.92	10.30
0.0	8.31	8.60	9.01	9.37	9.76
+0.5	7.80	8.13	8.45	8.73	9.10
+1.0	7.11	7.45	7.79	8.11	8.43

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.3(b). Values of IQR[$I(30)$] with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	27.00	27.28	27.47	27.41	27.17
−0.5	27.82	27.91	27.92	27.93	27.72
0.0	27.84	28.10	28.12	28.22	27.82
+0.5	27.23	27.64	27.71	27.73	27.52
+1.0	26.51	26.61	26.86	26.64	26.35

Table 5.3(c). Values of $Q_{I(30)}$ (.005) with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	162.36	158.75	145.68	112.99	98.76
−0.5	171.41	164.40	151.93	143.37	131.26
0.0	190.90	180.16	165.56	160.06	160.00
+0.5	185.46	177.15	166.03	159.74	160.97
+1.0	199.21	194.30	177.90	155.26	156.34

Table 5.3(d). Values of $Q_{I(30)}$ (.995) with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	−55.47	−58.95	−59.17	−60.16	−60.70
−0.5	−53.85	−55.58	−56.01	−55.95	−58.74
0.0	−51.24	−53.41	−53.60	−56.15	−58.18
+0.5	−48.24	−48.76	−50.43	−53.25	−55.29
+1.0	−42.65	−44.96	−46.85	−47.87	−51.94

Table 5.3(e). Claim frequency rates with different skewness parameters (number NZ of claims per 5,000 simulations)

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	691	737	747	753	751
−0.5	613	660	672	679	680
0.0	481	540	572	580	584
+0.5	357	415	448	468	491
+1.0	161	243	283	311	337

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.3(f). Mean claim severities *MNZ* with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	1.41	1.84	2.18	2.50	2.78
−0.5	1.29	1.79	2.23	2.59	2.89
0.0	1.22	1.85	2.32	2.74	3.08
+0.5	0.98	1.85	2.46	2.95	3.23
+1.0	0.51	2.12	3.06	3.70	4.06

Table 5.3(g). Reserving requirements $Q_{CR}(0.001)$ with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	13	17	19	23	29
−0.5	13	17	19	28	36
0.0	10	17	22	30	38
+0.5	9	15	24	32	38
+1.0	4	11	25	33	39

Table 5.3(h). Reserving requirements $Q_{CR}(1\%)$ with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	5	7	9	10	11
−0.5	4	6	8	9	11
0.0	4	5	7	9	11
+0.5	3	5	6	8	10
+1.0	3	3	6	7	10

Table 5.3(i). Reserving requirements $Q_{CR}(5\%)$ with different skewness parameters

β_D / β_Y	−1.0	−0.5	0.0	+0.5	+1.0
−1.0	2	2	3	3	3
−0.5	2	2	2	2	3
0.0	2	2	2	2	2
+0.5	1	1	2	2	2
+1.0	1	1	1	1	1

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.4(a). Comparison of the alternative models

	M	A	F	Z
Med[$I(30)$]	8.6	9.9	10.1	9.6
IQR[$I(30)$]	37.8	27.9	30.2	27.8
$Q_{I(30)}(0.995)$	−42.1	−56.0	−65.7	−54.3
$Q_{I(30)}(0.005)$	109.4	143.4	202.0	140.3
Med[$G(30)$]	9.3	9.4	9.5	9.4
IQR[$G(30)$]	3.5	2.6	2.5	2.6
$Q_{G(30)}(0.995)$	2.9	1.1	−0.2	1.3
$Q_{G(30)}(0.005)$	16.2	20.9	30.7	18.2
NZ	799	679	895	601
MNZ (per mille)	12	26	29	26
$Q_{CR}(.001)$	12	28	30	28
$Q_{CR}(.005)$	7	15	18	14
$Q_{CR}(0.01)$	5	9	12	8
$Q_{CR}(0.05)$	2	2	3	2
$Q_{DCR}(.001)$	7	14	16	15
$Q_{DCR}(.005)$	4	8	11	8
$Q_{DCR}(0.01)$	3	5	7	5
$Q_{DCR}(0.05)$	2	2	2	2

Table 5.4(b). Simulation investigation into the relative impact of individual parameter changes

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	*NZ*	*MNZ*	$Q_{CR}(.001)$	$Q_{CR}(1\%)$	$Q_{CR}(5\%)$
1.75	0.5	0.05	1.75	(0.5)	0.10	679	0.026	28	9	2
1.75	0.5	0.05	1.75	(0.5)	0.1414	1,106	0.023	31	11	3
1.75	0.5	0.05	1.75	0	0.1414	959	0.023	33	10	3
1.75	0.5	0.05	2.00	0	0.1414	493	0.026	25	8	2
1.75	0.5	0.0919	2.00	0	0.1414	1,124	0.036	44	20	5
1.75	0	0.0919	2.00	0	0.1414	1,167	0.030	37	17	4
2.00	0	0.0919	2.00	0	0.1414	799	0.011	12	5	2

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	Med[$I(30)$]	IQR[$I(30)$]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
1.75	0.5	0.05	1.75	(0.5)	0.10	9.9	27.9	−56.0	143.4
1.75	0.5	0.05	1.75	(0.5)	0.1414	10.2	38.0	−68.3	226.8
1.75	0.5	0.05	1.75	0	0.1414	9.4	37.9	−65.9	270.0
1.75	0.5	0.05	2.00	0	0.1414	8.9	35.2	−43.9	104.0
1.75	0.5	0.0919	2.00	0	0.1414	9.5	39.2	−50.6	121.1
1.75	0	0.0919	2.00	0	0.1414	8.8	39.1	−48.7	138.0
2.00	0	0.0919	2.00	0	0.1414	8.6	37.8	−42.1	109.4

α_D	β_D	γ_D	α_Y	β_Y	γ_Y	Med[$G(30)$]	IQR[$G(30)$]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
1.75	0.5	0.05	1.75	(0.5)	0.10	9.4	2.6	1.1	20.9
1.75	0.5	0.05	1.75	(0.5)	0.1414	9.5	2.9	1.2	27.4
1.75	0.5	0.05	1.75	0	0.1414	9.5	2.9	1.3	23.1
1.75	0.5	0.05	2.00	0	0.1414	9.4	2.6	1.8	15.4
1.75	0.5	0.0919	2.00	0	0.1414	9.6	4.4	−1.7	19.8
1.75	0	0.0919	2.00	0	0.1414	9.4	4.4	−0.8	25.5
2.00	0	0.0919	2.00	0	0.1414	9.3	3.5	2.9	16.2

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

Table 5.4(c). Relative impact of changing individual parameters from model M to model A values

Parameter change	$Q_{CR}(.001)$	$Q_{CR}(0.01)$	$Q_{CR}(0.05)$	*NZ*	*MNZ*
γ_Y	−3	−2	−1	−427	+0.3
β_Y	−2	+1	0	+147	0.0
α_Y	+8	+2	+1	+466	−0.3
γ_D	−19	−12	−3	−631	−1.0
β_D	+7	+3	+1	−43	+0.6
α_D	+25	+12	+2	+368	+1.9
Total	+16	+4	0	−120	+1.5

Parameter change	Med[*I*(30)]	IQR[*I*(30)]	$Q_{I(30)}(.995)$	$Q_{I(30)}(.005)$
γ_Y	−0.3	−10.1	+12.3	−83.4
β_Y	+0.8	+0.1	−2.4	−43.2
α_Y	+0.5	+2.7	−22.0	+166.0
γ_D	−0.6	−4.0	+6.7	−17.1
β_D	+0.7	+0.1	−1.9	−16.9
α_D	+0.2	+1.3	−6.6	+28.6
Total	+1.3	−9.9	−13.9	+34.0

Parameter change	Med[*G*(30)]	IQR[*G*(30)]	$Q_{G(30)}(.995)$	$Q_{G(30)}(.005)$
γ_Y	−0.1	−0.3	−0.1	−6.5
β_Y	0.0	0.0	−0.1	+4.3
α_Y	+0.1	+0.3	−0.5	+7.7
γ_D	−0.2	−1.8	+3.5	−4.4
β_D	+0.2	0.0	−0.9	−5.7
α_D	+0.1	+0.9	−3.7	+9.3
Total	+0.1	−0.9	−1.8	+4.7

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press

5.5 *Simulation Investigation into Model Behaviour with Different Liability Portfolios*

Table 5.5(a). Simulation results for the MGWP model with different liability portfolios (gross investor)

Portfolio	All at term 5 (1)	All at term 10 (2)	All at term 20 (3)	All at term 30 (4)	1 each at 10,15,...30 (5)	1 each at 10,11,...30 (6)	1 each at 1...10 (7)	1 each at 1...20 (8)
MNZ	13.4	13.1	11.5	7.5	1.8	0.9	2.4	0.9
NZ	965	307	39	4	419	799	3,807	3,938
$Q_{CR}(.001)$	47	48	27	1	10	11	25	15
$Q_{CR}(.005)$	38	32	5	1	6	5	18	10
$Q_{CR}(0.01)$	33	23	1	1	5	4	16	8
$Q_{CR}(0.05)$	18	5	1	1	2	2	9	4
$Q_{DCR}(.001)$	39	33	13	1	6	6	20	10
$Q_{DCR}(.005)$	31	22	3	1	4	3	14	7
$Q_{DCR}(0.01)$	28	16	1	1	3	3	12	6
$Q_{DCR}(0.05)$	15	4	1	1	2	1	7	3

Table 5.5(b). Simulation results for the alternative model with different liability portfolios (gross investor)

Portfolio	All at term 5 (1)	All at term 10 (2)	All at term 20 (3)	All at term 30 (4)	1 each at 10,15,...30 (5)	1 each at 10,11,...30 (6)	1 each at 1...10 (7)	1 each at 1...20 (8)
NZ	1,620	164	47	27	306	679	3,122	3,332
MNZ	15.1	19.8	37.9	36.7	5.1	2.3	2.0	1.0
$Q_{CR}(.001)$	82	76	80	68	26	23	38	29
$Q_{CR}(.005)$	49	40	37	6	16	13	21	18
$Q_{CR}(0.01)$	38	23	1	1	11	7	17	9
$Q_{CR}(0.05)$	11	1	1	1	2	3	7	4
$Q_{DCR}(.001)$	68	52	37	22	13	11	30	17
$Q_{DCR}(.005)$	41	27	17	2	8	7	16	11
$Q_{DCR}(0.01)$	31	16	1	1	6	4	13	6
$Q_{DCR}(0.05)$	9	1	1	1	1	2	6	3

https://doi.org/10.1017/S1357321700004992 Published online by Cambridge University Press