BENDING ANALYSIS OF THIN PLATES WITH VARIABLE THICKNESS RESTING ON ELASTIC FOUNDATION BY ELEMENT FREE GALERKIN METHOD

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ABSTRACT

Element Free Galerkin method was used to analyze bending of thin plates with variable thickness resting on one parameter elastic foundation. Thickness of plate is considered as linearly varying in one direction. Formulation could be applied to plates of any shape with general boundary conditions and loadings. Convergence of solution was examined for different number of nodes, thickness variation and foundation parameters. It was found that for deflection good results were achieved even with small number of nodes regardless of boundary condition, thickness variation and foundation parameters. Accuracy of method is checked against available results and good agreements were found. Applicability of method is demonstrated by solving numerical examples with different boundary conditions, thickness and foundation parameters, and loadings.

Keywords: Element free galerkin method, Thin plate, Variable thickness, Elastic foundation.

1. INTRODUCTION

Plates of variable thickness are widely used in many branches of engineering technology and, in particular, in civil, mechanical, architectural, aeronautical, and marine engineering. The use of variable thickness helps to reduce the weight of structural elements and improve the utilization of material. Consideration of bending of such plates is essential for an efficient and reliable design which was done by using various approaches.

Bending of plates resting on elastic foundation has attracted attention of many researchers. Svec [1] has studied bending of thick plates on elastic foundation by finite element method (FEM). Ng and Chan [2] have used the collocation least square method for bending analysis of clamped isotropic plates on elastic foundations. Ng and Das [3] have applied Galerkin method for bending analysis of skew sandwich plates on elastic foundation. Kobayashi and Sonoda [4] have used Levy-type method for bending of Mindlin plates on elastic foundation. Qin [5,6] has used Hybrid-Trefftz finite element approach for bending of Kirchhoff and Reissner plates on elastic foundation. Han and Liew [7] have used differential quadrature method for Reisnner-Mindlin plates on two parameters foundations. Boundary Element method (BEM) has been used by other researchers for bending of plate on elastic foundation [8-13].

Meshless method is an approximate solution to many boundary value problems which approximations are entirely based on nodal parameters. Varity of Meshless methods have been introduced, namely the smoothed particle hydrodynamics (SPH) [14], the Meshless local Petrov Galerkin method (MLPG) [15], the Reproducing Kernel Particle method (RKPM) [16], the Element Free Galerkin method (EFG) [17-20] and the radial basis function approach [21-24].

Sladek *et al.* [25] have used meshless local boundary integral method for bending analysis of simply supported and clamped plates on elastic foundation. Chinnaboon *et al.* [26] have used a BEM based meshless for plates on bi-parametric elastic foundation with internal supports. Zhuojia *et al.* [27] have applied boundary particle method and Ferreira *et al.* [28] have applied the radial basis function for analysis of plates on elastic foundation. Katsikadelis and Yiotis [29] have used BEM to study bending of plates with variable thickness on two parameter elastic foundation. Their study is limited to circular and annular plates with linearly varying thickness along the radius and subjected to axisymmetric loading and rectangular plates with unidirectional thickness variation compliant to Levy-type solutions.

EFG method is regarded as one of the meshless method which is employing moving least squares (MLS) approximations and has been successfully applied to elasticity, crack growth, and other discontinuity problems. In this method, the shape functions constructed by MLS approximation do not have the property of

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delta functions. Hence the essential boundary conditions cannot be imposed as conveniently as FEM. Some techniques associated with imposing essential boundary conditions in the EFG method have also been developed, such as Lagrange multipliers [30], penalty method [31], coupling with FEM [32], and coupling with boundary element method [33].

Krysl *et al.* [34] were the first who used EFG method for analyzing bending of thin plate based on classical plate theory. It is the main objective of this paper to use EFG method for bending analysis of plates with variable thickness resting on one-parameter elastic foundation. The plate is discretized by a set of scattered nodes. Based on the classical plate theory (CPT) basic equations has been formulated. MLS approximation procedure is employed to produce displacement shape functions and Penalty method is used for imposing essential boundary condition. Accuracy and convergence of method is examined and some examples are solved.

2. APPROXIMATION OF DISPLACEMENT BY MLS METHOD

In this method the general displacement of a point **x**, say $u(x)$, are approximated with a displacement approximation function $u^h(\mathbf{x})$ in the following form:

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})\, a_j(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\, a(\mathbf{x}) \tag{1}$$

where $\mathbf{P}^{\mathrm{T}}(\mathbf{x})$ is basis function of spatial coordinates, $\mathbf{x}^T = [x, y]$ for two dimensional problem, and m is number of basis. The commonly used bases are linear basis as follows,

$$\mathbf{P}^{\mathrm{T}} = [1,\ x] \ \text{ in } \ 1D,\ m = 2 \tag{2}$$

$$\mathbf{P}^{\mathrm{T}} = [1,\ x,\ y] \ \text{ in } \ 2D,\ \ m = 3 \tag{3}$$

and quadratic basis as follows,

$$\mathbf{P}^{\mathrm{T}} = [1,\ x,\ x^2] \ \text{ in } \ 1D,\ \ m = 3 \tag{4}$$

$$\mathbf{P}^{\mathrm{T}} = [1,\ x,\ y,\ x^2,\ xy,\ y^2] \ \text{ in } \ 2D,\ \ m = 6 \tag{5}$$

In this study, quadratic basis is adopted. The unknown coefficients $a_j(x)$ can be determined by minimizing a function of weighted residual as follows,

$$J = \sum_{I=1}^{n} w(\mathbf{x} - \mathbf{x}_I)\left[\mathbf{P}^{\mathrm{T}}(\mathbf{x}_I)\, a(\mathbf{x}) - u_I\right]^2 \tag{6}$$

where n is the number of nodes in the neighborhood of x which is also called influence domain of **x**, $w(\mathbf{x} - \mathbf{x}_I)$ is weight function, and u is nodal parameter at node I. At an arbitrary point **x**, **a**(**x**) is chosen from following condition,

$$\frac{\partial J}{\partial a} = 0 \tag{7}$$

which yields following equation system:

$$\mathbf{A}(\mathbf{x})\,\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\, u \tag{8}$$

where symmetrical matrix **A**(**x**) and non-symmetrical matrix **B**(**x**) are defined as follows,

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^{n} w(\mathbf{x} - \mathbf{x}_I)\,\mathbf{p}(\mathbf{x}_I)\,\mathbf{p}^{\mathrm{T}}(\mathbf{x}_I) \tag{9}$$

$$\mathbf{B}(\mathbf{x}) = \left[w(\mathbf{x} - \mathbf{x}_I)\,\mathbf{p}(\mathbf{x}_I),\ ...,\ w(\mathbf{x} - \mathbf{x}_n)\,\mathbf{p}(\mathbf{x}_n)\right] \tag{10}$$

Substituting Eq. (8) into Eq. (1) yields,

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \mathbf{p}^{\mathrm{T}}(\mathbf{x})\,\mathbf{A}^{-1}(\mathbf{x})\ \mathbf{B}_I(\mathbf{x})\, u_I = \sum_{I=1}^{n} \phi_I(\mathbf{x})\ u_I \tag{11}$$

where $\mathbf{B}_I$ is the I-th element of matrix **B** and ϕ_I is the **MLS** shape function given as follows,

$$\phi_I(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\,\mathbf{A}^{-1}(\mathbf{x})\,\mathbf{B}_I(\mathbf{x}) \tag{12}$$

For simplicity, the following definitions are made,

$$\Phi(\mathbf{x}) = \left[\phi_1(\mathbf{x}),\ ...,\ \phi_n(\mathbf{x})\right]$$

$$\gamma(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\,\mathbf{p}(\mathbf{x})$$

Therefore shape function can be written in the following form,

$$\Phi(\mathbf{x}) = \gamma^{\mathrm{T}}(\mathbf{x})\,\mathbf{B}(\mathbf{x}) \tag{13}$$

Following relationship between partial derivatives are exist,

$$\mathbf{A}\gamma_{,x} = \mathbf{p}_{,x} - \mathbf{A}_{,x}\gamma \tag{14}$$

$$\mathbf{A}\gamma_{,y} = \mathbf{p}_{,y} - \mathbf{A}_{,y}\gamma \tag{15}$$

$$\mathbf{A}\gamma_{,xx} = \mathbf{p}_{xx} - (\mathbf{A}_{,xx}\,\gamma + 2\mathbf{A}_{,x}\,\gamma_{,x}) \tag{16}$$

$$\mathbf{A}\gamma_{,xy} = \mathbf{p}_{,xy} - (\mathbf{A}_{,xy}\,\gamma + \mathbf{A}_{,x}\,\gamma_{,y} + \mathbf{A}_{,y}\,\gamma_{,x}) \tag{17}$$

$$\mathbf{A}\gamma_{,yy} = \mathbf{p}_{,yy} - (\mathbf{A}_{,yy}\,\gamma + 2\mathbf{A}_{,y}\,\gamma_{,y}) \tag{18}$$

By using Eqs. (14) to (18), the partial derivatives of shape function **Φ**(**x**) can be written as follows,

$$\mathbf{\Phi}_{I,x} = \gamma^{\mathrm{T}}_{,x}\,\mathbf{B}_I + \gamma^{\mathrm{T}}\mathbf{B}_{I,x} \tag{19}$$

$$\mathbf{\Phi}_{I,y} = \gamma^{\mathrm{T}}_{,y}\,\mathbf{B}_I + \gamma^{\mathrm{T}}\mathbf{B}_{I,y} \tag{20}$$

$$\mathbf{\Phi}_{I,xx} = \gamma^{\mathrm{T}}_{,xx}\,\mathbf{B}_I + 2\gamma^{\mathrm{T}}_{,x}\,\mathbf{B}_{I,x} + \gamma^{\mathrm{T}}\mathbf{B}_{I,xx} \tag{21}$$

$$\mathbf{\Phi}_{I,xy} = \gamma^{\mathrm{T}}_{,xy}\,\mathbf{B}_I + \gamma^{\mathrm{T}}_{,x}\,\mathbf{B}_{I,y} + \gamma^{\mathrm{T}}_{,y}\,\mathbf{B}_{I,x} + \gamma^{\mathrm{T}}\mathbf{B}_{I,xy} \tag{22}$$

$$\mathbf{\Phi}_{I,yy} = \gamma^{\mathrm{T}}_{,yy}\,\mathbf{B}_I + 2\gamma^{\mathrm{T}}_{,y}\,\mathbf{B}_{I,y} + \gamma^{\mathrm{T}}\mathbf{B}_{I,yy} \tag{23}$$

The domain of influence of a point can be controlled through weight function. Variety of weight functions

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have been used in EFG method. For classical thin plate, the weight function needs to be non-negative and its first and second partial derivatives must be non-singular. To satisfy these requirements, quartic spline weight function is chosen as follows,

$$w(\mathbf{x}-\mathbf{x}_I) \equiv w(r) = \begin{cases} 1-6r^2+8r^3-3r^4, & r \le 1 \\ 0, & r > 1 \end{cases} \tag{24}$$

$$r = \frac{\| \mathbf{x}-\mathbf{x}_I \|}{d_I} \tag{25}$$

where d_I is radius of support of the Ith node. For rectangular influence domain in 2D problems, weight function is as follows,

$$w(\mathbf{x}-\mathbf{x}_I) = w(r_x).w(r_y) \tag{26}$$

where r_x and r_y are given by

$$r_x = \frac{\| x-x_I \|}{d_{Ix}} \tag{27}$$

$$r_y = \frac{\| y-y_I \|}{d_{Iy}} \tag{28}$$

where

$$d_{Ix} = d_{\max}.\Delta x \tag{29}$$

$$d_{Iy} = d_{\max}.\Delta y \tag{30}$$

where $d_{\max}$ is scaling parameter, Δx and Δy are mesh size in x and y directions, respectively. Size of the support can be any arbitrary value, but it must have sufficient number of nodes to ensure that matrix **A** is non-singular. In this paper, $d_{\max}$ is chosen as 3.9 [30].

3. GOVERNING EQUATION

3.1 Classical Plate Theory (CPT)

A thin plate with thickness h and Cartesian coordinate system with origin at center is considered (Fig. 1). Displacements in x, y and z directions are denoted by u, v and w, respectively. Based on classical assumption of thin plate, displacement field can be expressed as follows [35]:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} -z\dfrac{\partial}{\partial x} & -z\dfrac{\partial}{\partial y} & 1 \end{Bmatrix}^{\mathrm{T}} w = \mathbf{L}_u\, w \tag{31}$$

The pseudo-strains of the plate are obtained by

$$\boldsymbol{\varepsilon}_p = \begin{Bmatrix} -\dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial y^2} & -2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} \tag{32}$$

The pseudo-stresses of the plate are given by

$$\boldsymbol{\sigma}_p = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{33}$$

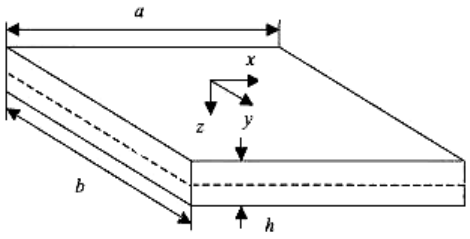


Fig. 1 Geometry and coordinate system of plate

The pseudo-strains and the pseudo-stress are related as

$$\boldsymbol{\sigma}_p = D\boldsymbol{\varepsilon}_p \tag{34}$$

where **D** is the stiffness matrix and is defined as follows,

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{35}$$

For isotropic plates, **D** would be as follows,

$$\mathbf{D} = \frac{\mathrm{Eh}^3}{12(1-v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} \tag{36}$$

where E is Young's module and v is Poisson's ratio.

3.2 Variational form of System Equations

Boundary conditions of a plate can be written as follows,

$$\sigma.\mathbf{n} = \bar{\mathbf{t}} \text{ on } \Gamma_\sigma,\ \tilde{\mathbf{u}} = \bar{\mathbf{u}} \text{ on } \Gamma_u \tag{37}$$

where $\bar{\mathbf{t}}$ and $\bar{\mathrm{u}}$ are the prescribed traction and displacement, respectively, and **n** denote outward normal direction on boundary of plate. Vector $\tilde{\mathbf{u}}$ is given by

$$\tilde{\mathbf{u}} = \mathbf{L}_b\, w$$

For clamped boundary conditions, $\mathbf{L}_b$ is defined as,

$$\mathbf{L}_b = \begin{Bmatrix} 1 & \dfrac{\partial}{\partial n} \end{Bmatrix}^{\mathrm{T}} \tag{38}$$

For simply supported boundary conditions, $\mathbf{L}_b$ is defined as,

$$\mathbf{L}_b = \{1 \quad 0\}^{\mathrm{T}} \tag{39}$$

Static equations of a plate can be derived by principle of minimum total potential energy,

$$\delta\Pi = \delta(U-W) = 0 \tag{40}$$

where U and W are strain energy and work done by external applied force, respectively. Since the MLS shape function does not satisfy the Kronecker delta

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condition, *i.e.* $\phi_I(x_J) \neq \delta_{IJ}$ at each node, the essential boundary conditions have to be imposed in a proper manner. In this paper, a penalty method is employed to enforced essential boundary conditions. Hence, after expanding Eq. (40), additional term appears in the variational forms of the equilibrium equation as follows,

$$\begin{aligned}&\int_S \delta\boldsymbol{\varepsilon}_p{}^{\mathrm{T}}\,\boldsymbol{\sigma}_p\,dS+\int_S \delta\mathbf{u}^{\mathrm{T}}\,\mathbf{p}\,dS-\int_V \delta\mathbf{u}^{\mathrm{T}}\,\mathbf{b}\,dV\\&-\int_{\Gamma\sigma}\delta\mathbf{u}^{\mathrm{T}}\,\bar{\mathbf{t}}\,d\Gamma\\&-\frac{1}{2}\int_{\Gamma u}\delta(\tilde{\mathbf{u}}-\bar{\mathbf{u}})^{\mathrm{T}}\,\boldsymbol{\alpha}\,(\tilde{\mathbf{u}}-\bar{\mathbf{u}})d\Gamma=0\,,\end{aligned}\tag{41}$$

where **b** is body force vector, **p** is the surface pressure vector of foundation, and α is a diagonal matrix of penalty factors which are usually very large numbers. For a Winkler-type elastic foundation, **p** is defined as follows,

$$\mathbf{p}=\begin{Bmatrix}0\\0\\k_f\,w\end{Bmatrix}\tag{42}$$

where k_f denote the foundation stiffness. Substituting the known variables into Eq. (41), it can be rewritten as

$$\begin{aligned}&\int_S \delta(\mathbf{L}\,w)^{\mathrm{T}}\,\mathbf{D}\,(\mathbf{L}\,w)\,dS+\int_S \delta(\mathbf{L}_u\,w)^{\mathrm{T}}\,\mathbf{P}\,dS\\&-\int_V \delta(\mathbf{L}_u\,w)^{\mathrm{T}}\,\mathbf{b}\,dV-\int_{\Gamma\sigma}\delta(\mathbf{L}_u\,w)^{\mathrm{T}}\,\bar{\mathbf{t}}\,d\Gamma\\&-\frac{1}{2}\int_{\Gamma u}\delta(\mathbf{L}_b\,w-\bar{u})^{\mathrm{T}}\;\alpha(\mathbf{L}_b\,w-\bar{u})\,d\Gamma=0\,,\end{aligned}\tag{43}$$

3.3 Approximation of Field Variables

Based on CPT only the deflection w is independent and needs to be approximated.

$$w^h(\mathbf{x})=\sum_{I=1}^{n}\phi_I(\mathbf{x})\,w_I\tag{44}$$

3.4 Discrete System Equations

By substituting Eq. (44) into variational form (Eq. 43), final discrete system equations can be written as follows,

$$(\mathbf{K}^{\mathrm{P}}+\mathbf{K}^{\mathrm{F}}+\tilde{\mathbf{K}})\,\mathbf{U}=\mathbf{F}\tag{45}$$

where $\mathbf{K}^{\mathrm{P}}$ is the stiffness matrix of plate and defined as follows:

$$\mathbf{K}^{\mathrm{P}}=\int_S \mathbf{B}_I^T\,\mathbf{D}\,\mathbf{B}_J\,dS\tag{46}$$

where

$$\mathbf{B}_I=\mathbf{L}\phi_I=\begin{Bmatrix}-\phi_{I,xx}\\-\phi_{I,yy}\\-2\phi_{I,xy}\end{Bmatrix}\tag{47}$$

and $\mathbf{K}^{\mathrm{F}}$ is the corresponding elastic foundation stiffness matrix and defined as follows,

$$\mathbf{k}^{\mathrm{F}}=\int_S \phi_I\;k_f\;\phi_J\,dS\tag{48}$$

matrix $\tilde{\mathbf{K}}$ is obtained using following relation,

$$\tilde{\mathbf{K}}_{IJ}=\int_{\Gamma u}\boldsymbol{\Psi}_I^{\mathrm{T}}\,\alpha\,\boldsymbol{\Psi}_J\,d\Gamma\tag{49}$$

where $\boldsymbol{\Psi}_I$ depends on boundary condition.

For clamped boundary condition, $\boldsymbol{\Psi}_I$ is defined as follows,

$$\boldsymbol{\Psi}_I=\begin{Bmatrix}\phi_I\\\phi_{I,n}\end{Bmatrix}\tag{50}$$

and for simply support boundary condition is defined as follows,

$$\boldsymbol{\Psi}_I=\begin{Bmatrix}\phi_I\\0\end{Bmatrix}\tag{51}$$

The force vector, **F** is assembled by using the nodal force given by

$$\begin{aligned}f_I&=\int_V(\mathbf{L}_u\phi_I)^{\mathrm{T}}\,\mathbf{b}\,dV+\int_{\Gamma\sigma}(\mathbf{L}_u\,\phi_I)^{\mathrm{T}}\;\bar{\mathbf{t}}\,d\Gamma\\&=\int_V\mathbf{B}_u{}^{\mathrm{T}}\,\mathbf{b}\,dV+\int_{\Gamma\sigma}\mathbf{B}_u{}^{\mathrm{T}}\;\bar{\mathbf{t}}\,d\Gamma\end{aligned}\tag{52}$$

where

$$\mathbf{B}_u=\begin{Bmatrix}-z\phi_{I,x}\\-z\phi_{I,y}\\\phi_I\end{Bmatrix}\tag{53}$$

4. NUMERICAL RESULTS AND DISCUSSION

For comparison purpose, a dimensionless parameter for foundation is defined as follows:

$$\mathrm{K}=\left(\frac{k_f\,a^4}{D_0}\right)^{1/4}\tag{54}$$

where a is characteristic length of plate and D_0 is its flexural stiffness at the origin of coordinate system. Linear thickness variation in x direction is considered as follows (Fig. 2),

$$h=h_0\left(1+\frac{\beta x}{a}\right)\tag{55}$$

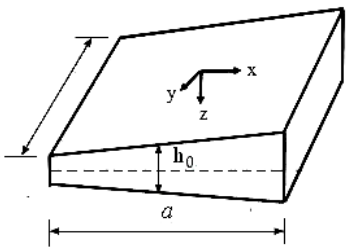


Fig. 2 Geometry and coordinate system of plate with variable thickness

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where h_0 is thickness at origin and β denote thickness variation parameter. To show accuracy and convergence of present method, a computer code in Matlab is developed and several numerical examples are solved for different boundary conditions, thickness and foundation parameters. A 4 × 4 gauss integration scheme is used to perform the integration in computing the system matrixes.

For convenience, following dimensionless parameters for deflection and bending moment are defined,

$$\bar{w} = \frac{w}{\left(\frac{qa^4}{D_0}\right)\times 10^{-3}}$$

$$\bar{M}_x = \frac{M_x}{qa^2 \times 10^{-2}}$$

$$\bar{M}_y = \frac{M_y}{qa^2 \times 10^{-2}}$$

4.1 Square Plate with Variable Thickness (Without Elastic Foundation)

First rectangular plate with varying thickness without elastic foundation is analysed. Poisson's ratio is taken as 0.25, thickness variation parameter is chosen as β = 0.2 and uniform distributed load, $\boldsymbol{q}$ is considered. Regularly distributed field nodes 26 × 26 are used for field variable approximation. In Tables 1 and 2, non-dimensional displacements and bending moments of a square plate with SSSS and CSCS boundary conditions on centerline are depicted, respectively and are compared with Petrian and Conway [36].

As can be seen for SSSS boundary condition very good agreement for deflection and resultant moments are achieved. For CSCS boundary condition small deviations for resultant moments at some points can be noticed, however deflection has good agreement.

4.2 Square Plate with Constant Thickness on an Elastic Foundation

In Figs. 3 ~ 11 accuracy and convergence of present method using different densities of nodes, foundation parameter, and three types of boundary conditions are compared with reference [4]. A square plate with constant thickness resting on Winkler foundation subjected to uniform distributed loading is considered. Poisson's ratio of 0.3 and thickness to span ratio, h/a = 0.01 are taken for all cases. In these figures error is calculated from following equation,

$$\text{Error} = \frac{|\xi_{\text{Ref}[4]} - \xi_{EFG}|}{|\xi_{\text{Ref}[4]}|}\times 100 \qquad (56)$$

where $\xi_{\text{Ref}[4]}$ is deflection/ moment taken from reference [4] and ξ_{EFG} is the calculated deflection/moment by present method. One way of computing the error is L2 error norm which use the root mean square of the

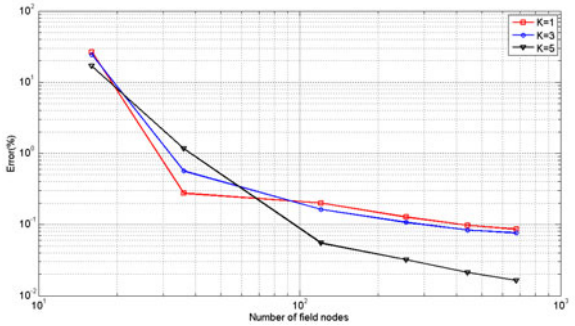


Fig. 3 Convergence curve for deflection of a SSSS square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

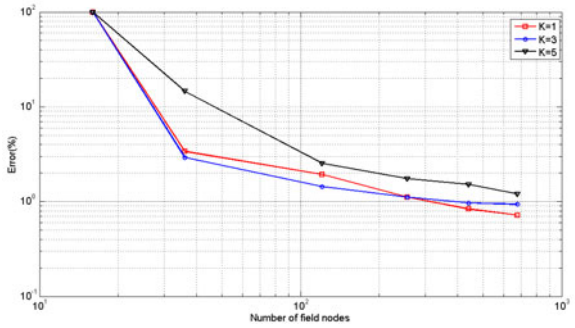


Fig. 4 Convergence curve for M_x of a SSSS square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

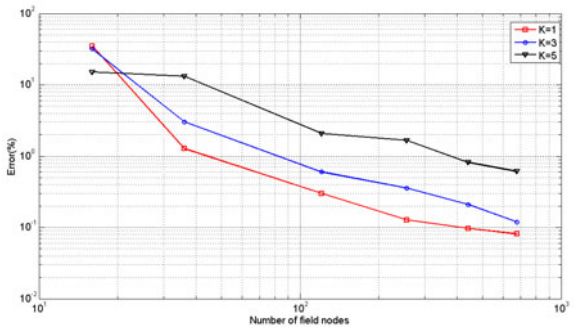


Fig. 5 Convergence curve for M_y of a SSSS square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

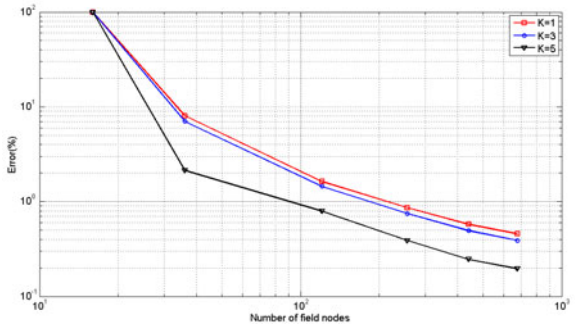


Fig. 6 Convergence curve for deflection of a SCSC square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

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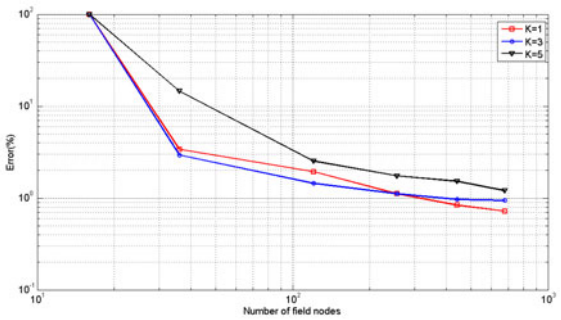


Fig. 7 Convergence curve for M_x of a SCSC square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

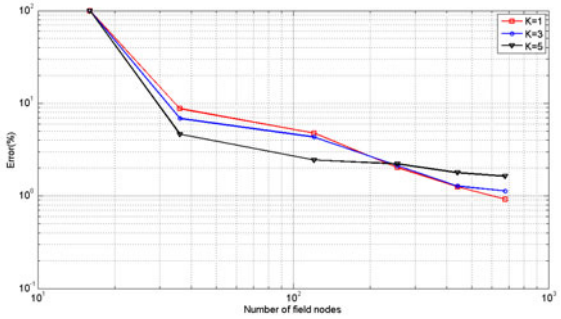


Fig. 8 Convergence curve for M_y of a SCSC square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

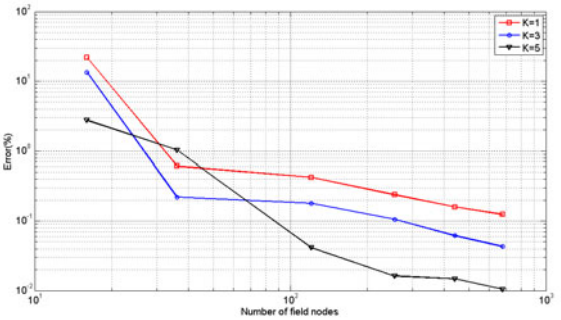


Fig. 9 Convergence curve for deflection of a SFSF square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

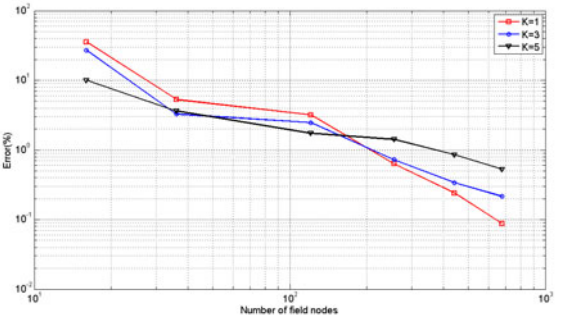


Fig. 10 Convergence curve for M_x of a SFSF square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

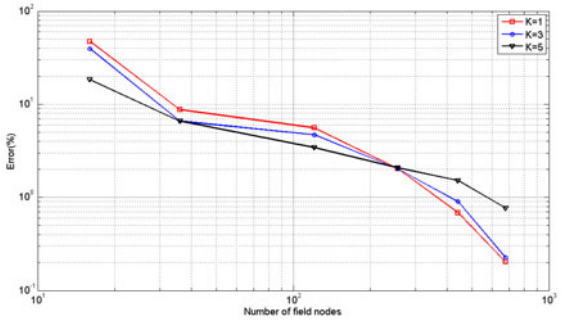


Fig. 11 Convergence curve for M_y of a SFSF square plate, constant thickness resting on Winkler type of elastic foundation and under uniform loading

Table 1 Deflections and bending moments on center-line of a square SSSS plate under uniform distributed load and $\beta = 0.2$

x/a	$\bar{w}$		$\bar{M}_x$		$\bar{M}_y$	
	Present work	[36]	Present work	[36]	Present work	[36]
–0.5	0.000	0.000	0.000	0.000	0.000	0.000
–0.375	0.020	0.020	0.026	0.026	0.018	0.018
–0.25	0.035	0.036	0.039	0.039	0.032	0.032
–0.125	0.044	0.045	0.045	0.045	0.042	0.042
0	0.046	0.046	0.046	0.046	0.046	0.047
0.125	0.041	0.042	0.042	0.043	0.044	0.044
0.25	0.031	0.031	0.035	0.035	0.036	0.036
0.375	0.017	0.017	0.023	0.022	0.021	0.021
0.5	0.000	0.000	0.000	0.000	0.000	0.000

Table 2 Deflections and bending moments on center-line of a square CSCS plate under uniform distributed load and $\beta = 0.2$

x/a	$\bar{w}$		$\bar{M}_x$		$\bar{M}_y$	
	Present work	[36]	Present work	[36]	Present work	[36]
–0.5	0.000	0.000	-0.062	-0.062	-0.016	-0.016
–0.375	0.005	0.005	-0.013	-0.013	-0.001	-0.001
–0.25	0.014	0.014	0.014	0.015	0.011	0.012
–0.125	0.020	0.020	0.029	0.029	0.020	0.020
0	0.022	0.022	0.032	0.032	0.023	0.023
0.125	0.018	0.019	0.025	0.025	0.020	0.020
0.25	0.011	0.012	0.006	0.008	0.011	0.011
0.375	0.004	0.004	-0.024	-0.024	-0.003	-0.003
0.5	0.000	0.000	-0.078	-0.079	-0.020	-0.020

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components of the error and convert a set of errors distributed over the meshed area. Since in literatures only results at some specific points were quoted, so this error norm cannot be used.

As can be seen from these figures, by increasing number of nodes accuracy of results are increases. For deflection (Figs. 3, 6 and 9) results are converged very fast and one can conclude that number of field nodes equal to 11 × 11 yields good results for all boundary conditions and foundation parameter. For bending moment in some cases of boundary conditions, the results converge very fast (SSSS, and SCSC) however in some instances the results are converge very slow (SFSF). In this cases, number of field nodes should be at least 26 × 26.

In Table 3 results of present method for a square plate with three different boundary conditions and constant thickness resting on Winkler type of elastic foundation with 26 × 26 number of nodes are compared with Kobayashi and Sonoda [4] and Ferreira *et al.* [28]. Again for simply supported boundary conditions, very good agreements are achieved for both deflection and bending moment. For combination of free and simply supported boundary conditions and clamped and simply supported boundary conditions, less agreement can be seen. The differences between present works and [28] can be reached as high as 22 percent for bending moment and 3 percent for deflection. The differences between present works and [4] is about 0.5 percent for deflection and 2 percent for bending moment.

4.3 Square Plate with Linearly Varying Thickness on Elastic Foundation

A square plate with linearly varying thickness resting on Winkler type of elastic foundation has been studied. The plate is subjected to uniform load and Poisson's ratio of 0.3 is assumed. In Tables 4 ~ 6 convergence study for different foundation and thickness parameters with SSSS boundary condition are given.

As can be seen deflection converge very fast for all values of thickness and foundation parameters. The differences between 11 × 11 number of nodes and 26 × 26 number of nodes is about 0.1 percent. Convergence of bending moment depends on foundation parameter. When foundation parameter is one, the difference between 11 × 11 number of nodes and 26 × 26 number of nodes is about 0.9 percent, while for foundation parameter of five this difference reaches to 2.9 percent. Therefore for high value of foundation parameters, number of nodes should be increase. Thickness parameter has very little effect of results.

Table 3 Deflections and moments at center of a uniformly loaded square plate with constant thickness resting on Winkler type of elastic foundation

Boundary condition	Foundation parameter, K	$\bar{w}$			$\bar{M}_x$			$\bar{M}_y$		
		Present work	[28]	[4]	Present work	[28]	[4]	Present work	[28]	[4]
SSSS	1	4.051	4.050	4.054	4.762	4.776	4.775	4.771	4.784	4.775
	3	3.347	3.345	3.349	3.863	3.877	3.875	3.870	3.885	3.875
	5	1.506	1.505	1.506	1.531	1.545	1.540	1.531	1.552	1.540
SCSC	1	1.927	1.868	1.918	2.454	2.629	2.437	3.350	2.876	3.320
	3	1.751	1.701	1.744	2.205	2.371	2.184	3.022	2.578	2.989
	5	1.071	1.047	1.069	1.227	1.369	1.212	1.736	1.423	1.709
SFSF	1	12.944		12.96	12.109		12.12	2.662		2.667
	3	6.985		6.988	6.381		6.395	1.446		1.449
	5	1.680		1.680	1.313		1.320	0.337		0.34

Table 4 Deflections and bending moments at center of a square SSSS plate with linearly varying thickness, β = 0.01 resting on Winkler type of elastic foundation

Foundation parameter K	Number of field nodes	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$
1	11 × 11	4.046	4.717	4.717
	21 × 21	4.050	4.791	4.791
	26 × 26	4.051	4.762	4.762
3	11 × 11	3.344	3.844	3.844
	21 × 21	3.346	3.894	3.894
	26 × 26	3.346	3.862	3.862
5	11 × 11	1.506	1.572	1.572
	21 × 21	1.506	1.566	1.566
	26 × 26	1.506	1.527	1.527

Table 5 Deflections and bending moments at center of a square SSSS plate with linearly varying thickness, β = 0.1 resting on Winkler type of elastic foundation

Foundation parameter K	Number of field nodes	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$
1	11 × 11	4.048	4.709	4.716
	21 × 21	4.052	4.781	4.790
	26 × 26	4.053	4.752	4.761
3	11 × 11	3.345	3.837	3.843
	21 × 21	3.347	3.886	3.893
	26 × 26	3.348	3.853	3.861
5	11 × 11	1.506	1.569	1.571
	21 × 21	1.506	1.562	1.564
	26 × 26	1.506	1.524	1.526

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Table 6 Deflections and bending moments at center of a square SSSS plate with linearly varying thickness, β = 0.15 resting on Winkler type of elastic foundation

Foundation parameter K	Number of field nodes	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$
1	11 × 11	4.051	4.699	4.716
	21 × 21	4.055	4.770	4.789
	26 × 26	4.056	4.739	4.760
3	11 × 11	3.346	3.828	3.841
	21 × 21	3.349	3.876	3.891
	26 × 26	3.349	3.842	3.858
5	11 × 11	1.506	1.565	1.569
	21 × 21	1.506	1.558	1.563
	26 × 26	1.505	1.519	1.524

Table 7 Deflections and Moments for square plate with linearly varying thickness and β = 0.001 resting on Winkler type of elastic foundation

Loading	Boundary condition	Foundation Parameter	$\bar{w}$		$\bar{M}_x$	
			Present work	[29]	Present work	[29]
Hydro-static load	CCCC	K = 0.134	0.624×10^{-3}	0.614×10^{-3}	0.116×10^{-1}	0.113×10^{-1}
		K = 3.76	0.537×10^{-3}		0.980×10^{-2}	
	SSSS	K = 0.134	0.203×10^{-2}		0.240×10^{-1}	
		K = 3.76	0.133×10^{-2}		0.150×10^{-1}	
Uniform loading	CCCC	K = 0.134	0.126×10^{-2}		0.233×10^{-1}	
		K = 3.76	0.110×10^{-2}		0.196×10^{-1}	
	SSSS	K = 0.134	0.406×10^{-2}		0.482×10^{-1}	
		K = 3.76	0.265×10^{-2}	0.265×10^{-2}	0.0300	0.0299

Also when thickness parameter is low, say β = 0.01, bending moment in x and y directions are almost the same, which is an indication of symmetry plate.

In Table 7 results of present method for square plate with linearly varying thickness on elastic foundation for two different boundary conditions and loadings are compared with Katsikadelis and Yiotis [29]. Again for deflection very good result is achieved for SSSS plate, and for CCCC plate the difference is about 2 percent. For bending moment differences in SSSS is less than one percent but for CCCC is about 3 precent.

To show applicability of this method, deflection and bending moment at center of a square plate with any boundary conditions and foundation parameters are calculated and shown in Table 8.

Table 8 Deflections and Moments for square plate with linearly varying thickness, β = 0.1 resting on Winkler type of elastic foundation

Boundary condition	Foundation parameter	$\bar{w}$	$\bar{M}_x$	$\bar{M}_y$
CCCC	1	1.261	2.236	2.242
	3	1.184	2.080	2.085
	5	0.834	1.370	1.373
SSCC	1	2.088	2.948	2.978
	3	1.876	2.619	2.644
	5	1.103	1.418	1.427
SCSC	1	1.910	2.403	3.278
	3	1.738	2.153	2.950
	5	1.067	1.190	1.680
FFCC	1	8.398	−0.258	0.195
	3	3.807	0.857	1.017
	5	1.165	0.793	0.801
FFSS	1	51.443	6.064	7.000
	3	8.165	2.341	2.427
	5	1.611	0.783	0.770
FFSC	1	17.569	2.136	−0.169
	3	5.551	1.632	1.250
	5	1.368	0.718	0.857
FSFS	1	12.948	2.630	12.085
	3	6.982	1.434	1.434
	5	1.680	0.338	1.308
FCFC	1	2.546	1.063	4.015
	3	2.190	0.915	3.415
	5	1.120	0.457	1.615
FSFC	1	5.116	1.449	6.008
	3	3.807	1.089	4.413
	5	1.391	0.399	1.466
SFSS	1	7.881	7.902	3.860
	3	5.126	5.084	2.681
	5	1.596	1.412	0.914
CFCC	1	1.881	3.083	1.640
	3	1.677	2.725	1.487
	5	0.979	1.491	0.904
SFSC	1	5.633	5.561	2.747
	3	3.909	3.838	2.140
	5	1.375	1.242	0.983
CFCS	1	2.235	3.647	1.726
	3	1.970	3.182	1.525
	5	1.094	1.644	0.820
FFFS	1	751.670	0.550	3.054
	3	10.493	0.428	2.185
	5	1.637	0.165	0.655
FFFC	1	42.322	−2.177	−11.16
	3	6.799	0.085	0.231
	5	1.637	0.160	0.638
CFFF	1	46.958	−11.082	−2.090
	3	6.996	0.384	0.125
	5	1.385	0.733	0.200
SFFF	1	751.670	3.0515	0.548
	3	10.513	2.165	0.422
	5	1.638	0.613	0.153
FSFF	1	751.670	0.550	3.054
	3	10.493	0.428	2.185
	5	1.637	0.160	0.638
FCFF	1	42.322	−2.177	−11.16
	3	6.799	0.085	0.231
	5	1.367	0.200	0.733
FFCF	1	38.393	−11.291	−2.223
	3	6.608	0.073	0.053
	5	1.349	0.731	0.201

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5. CONCLUSIONS

In this paper Element Free Galerkin method is used for bending analysis of thin plates with linearly varying thickness, under general loading and boundary condition. Convergence of solution was examined for different foundation and thickness parameters. It was found that, the convergence of method is very good for deflection regardless of boundary condition, foundation and thickness parameters. Convergence of solution for bending moment depends on foundation and thickness parameters, but for later case to smaller extent. Accuracy of method is demonstrated by comparing results with available results in literature and good agreements were observed for all types of boundary condition, loading and foundation and thickness parameters. Applicability of method is shown for very general boundary conditions.

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