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Automorphisms of automatic shifts

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Abstract. In this paper we continue the study of automorphism groups of constant-length substitution shifts and also their topological factors. We show that, up to conjugacy, all roots of the identity map are letter-exchanging maps, and all other non-trivial automorphisms arise from *twisted* compressions of another constant-length substitution. We characterize the group of roots of the identity in both the measurable and topological setting. Finally, we show that any topological factor of a constant-length substitution shift is topologically conjugate to a constant-length substitution shift via a letter-to-letter code.

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1. *Introduction*

In this paper we continue the study of the automorphism groups of constant-length substitution shifts and complete some of our work in [**6**]. In particular, there we gave an algorithm for the computation of the automorphism group $\mathrm{Aut}(X_\theta, \sigma)$ of a constant-length substitution shift. Here, we ask what are the possible elements of this group, and also, what kinds of factors these shifts can have. In both cases, the answer is a form of rigidity.

Our strategy in [**6**] was to attach a rational number $\kappa(\Phi)$ to each automorphism Φ, which signifies by how much points are being shifted while being acted on by Φ (Theorem 6 below). In particular, a simple example of an automorphism whose κ-value equals zero is that coming from a permutation on letters which plays well with the substitution θ, by which we mean that it commutes with some power of the substitution. These automorphisms are natural and indicate that there is a symmetry in the substitution

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rule, up to a shuffling of letters. Now one can construct examples of substitutions with automorphisms whose κ-value is zero, but which are not of this nature. However we show, in Theorem 32 that, up to conjugacy, only the natural case manifests.

What remains to consider are the automorphisms with a non-zero rational κ-value $1/k$. It is not difficult to create such situations. Namely, one starts with a substitution η, and one takes its *k-compression*. This is the process whereby we rewrite a sequence as a concatenation of words of length k, and where the shift map moves from one word to the next. Cobham [**4**] used this terminology, and he showed that constant-length substitutions are robust under this operation: a k-compression of a substitution shift (X_η, σ) of length r gives us another substitution shift (X_θ, σ) of length r. By construction, $\mathrm{Aut}(X_\theta, \sigma)$ will contain a shadow of the shift on X_η, and this manifests as a kth root of the shift map on X_θ. Can automorphisms Φ with $\kappa(\Phi) = 1/k$ arise in any other way? We discover in Theorem 46 that, up to conjugacy, the only other way that they can arise is if we *twist* after we compress η (Definition 45). At some level, our results say that the automorphisms of our substitution shift X_θ are only constrained by fixed points either of θ or of another substitution.

Can one hope that the letter-exchanging maps of θ are related to those of η? In our world, this is generally not the case; we see this in Examples 51 and 52. There is a fundamental obstruction discussed in Remark 53. Hence we fall short of giving a general structure theorem. We could formulate one, but it would not be succinct.

We also characterize, in Theorem 37, the letter-exchanging maps that define automorphisms of X_θ, generalizing results due to Lemańczyk and Mentzen in the bijective case [**16**]. In particular, they characterized the measurable essential centralizer of a bijective substitution as being the centralizer of the group generated by the columns of θ; the topological statement is identical. To characterize the letter-exchanging maps of an arbitrary constant-length substitution, we work with *minimal sets*, a notion that was implicit in the algorithm of [**6**], but which was defined and used extensively by the first author in [**20**, §2]. We obtain characterizations in both the topological setting, in Theorem 37, and the measurable setting, in Theorem 44.

These are our principal results, but along the way to discovering them, we obtained some other results of interest. The first, which follows from Lemma 3 and Theorem 22, tells us that any topological shift factor of a constant-length substitution shift X_θ is a letter-to-letter coding of a constant-length substitution shift. This answers a question of Durand and Leroy in [**9**, §8]. It also suggests that the question that Coven, Dekking and Keane ask in [**5**], namely to list all *constant-length* substitutions whose shifts are conjugate to a given constant-length substitution shift, is not constraining.

Finally, we investigate the relationship between $\mathrm{Aut}(X, \sigma)$ and the existence of a uniform-to-one factor for X. This has been studied by Herning [**12**] in the case where $X = X_\theta$ and the factor is metrically a rotation. In Theorem 24, we show that $\mathrm{Aut}(X, \sigma)$ contains a square root of the identity if and only if X has a two-to-one factor. In Example 25, we show that there is no possibility of extending this result.

2. *Preliminaries*

Let $\mathcal{A}$ be a finite alphabet, with the discrete topology. We endow $\mathcal{A}^{\mathbb{Z}}$ with the product topology, and let $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ denote the (left) shift map. We consider only infinite

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minimal shifts (X, σ), that is, those such that the σ-orbit of any point in X is dense in X. The *language* $\mathcal{L}_X$ of a shift (X, σ) is the set of all finite words $w = w_1 \dots w_n$ that appear as a subword of some x in X, that is, $w = x_j \dots x_{j+n-1}$ for some j.

2.0.1. *Automorphism groups.* A *factor map* from (X, σ) to (Y, σ) is a map $\Phi : X \to Y$ which is continuous and commutes with σ. If $X = Y$, Φ is called an endomorphism. If Φ is bijective, it is called a *conjugacy*, and if Φ is a self-conjugacy then it is called an *automorphism*. We say $\Phi : X \to X$ has (finite) *order* m if m is the least positive integer such that Φ^m is the identity map Id, and we say that Φ is a *kth root of the shift* if $\Phi^k = \sigma$. Let $\mathrm{Aut}(X, \sigma)$ denote the automorphism group of (X, σ); it contains the (normal) subgroup generated by the shift, denoted by $\langle\sigma\rangle$.

Let $X \subset \mathcal{A}^{\mathbb{Z}}$ and let $Y \subset \mathcal{B}^{\mathbb{Z}}$. By the Curtis–Hedlund–Lyndon theorem, for any factor map $\Phi : X \to Y$, there exist ℓ, r, and a map $f : \mathcal{A}^{\ell+r+1} \to \mathcal{B}$ with the property that $(\Phi(x))_n = f(x_{n-\ell}, \dots, x_n, \dots, x_{n+r})$ for all x and n. Let ℓ, r be the smallest possible integers so that such an f exists. We call ℓ and r the left and right radius of Φ, respectively. We say Φ has radius R if its left radius and right radius are both at most R.

2.0.2. *Constant-length substitutions.* A *substitution* is a map from $\mathcal{A}$ to the set of non-empty finite words on $\mathcal{A}$. We use concatenation to extend θ to a map on finite and infinite words from $\mathcal{A}$. We say that θ is *primitive* if there is some $k \in \mathbb{N}$ such that for any $a, a' \in \mathcal{A}$, the word $\theta^k(a)$ contains at least one occurrence of a'. By iterating θ on any fixed letter in $\mathcal{A}$, we obtain one-sided right-infinite points $u = u_0 \dots$ such that $\theta^j(u) = u$ for some natural j. Similarly, we can obtain one-sided left-infinite points $v = \dots v_{-1}$ such that $\theta^j(v) = v$. A bi-infinite periodic point for θ is a concatenation of a left-infinite periodic point $v = \dots v_{-1}$ and a right-infinite periodic point $u = u_0 \dots$ provided that $v_{-1}u_0$ appears in some word $\theta^k(a)$. The pigeonhole principle implies that θ-periodic points always exist, and, for primitive substitutions, we define X_θ to be the shift orbit closure of any one of these bi-infinite θ-periodic points and call (X_θ, σ) a *substitution shift*. The substitution θ has *(constant) length* r if for each $a \in \mathcal{A}$, $\theta(a)$ is a word of length r.

Let θ be a primitive length-r substitution such that X_θ is infinite. We say that θ is *recognizable* if any $x \in X_\theta$ can be written in a unique way as $x = \sigma^k(\theta(y))$, where $y \in X_\theta$ and $0 \le k < r$. Our substitutions are recognizable; see [**13**] for a proof in the injective case, which is all we will need.

Let θ be a length-r substitution. We write

$$\theta(a) = \theta_0(a) \dots \theta_{r-1}(a);$$

with this notation we see that for each $0 \le i \le r-1$, we have a map $\theta_i : \mathcal{A} \to \mathcal{A}$ where $\theta_i(a)$ is the $(i+1)$th letter of $\theta(a)$. Similarly, for any k we write $(\theta^k)_i$ to denote the map which sends $a \in \mathcal{A}$ to the $(i+1)$th letter of $\theta^k(a)$.

Let θ be a primitive length-r substitution with fixed point u, and with (X_θ, σ) infinite. The *height* $h = h(\theta)$ of θ is defined as

$$h(\theta) := \max\{n \ge 1 : \gcd(n, r) = 1,\ n \mid \gcd\{a : u_a = u_0\}\}.$$

If $h > 1$, this means that $\mathcal{A}$ decomposes into h disjoint subsets $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_h$, where a symbol from $\mathcal{A}_i$ is always followed by a symbol from $\mathcal{A}_{i+1}$.

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If θ has non-trivial height $h \geq 2$, then the shift (X_θ, σ) is topologically conjugate to a constant-height suspension of a height-1 substitution shift $(X_{\theta'}, \sigma)$, where θ' is called the *pure base* of θ. In this case $(X_\theta, \sigma) \cong (X_{\theta'} \times \{0, \ldots, h-1\}, T)$, where

$$T(x, i) := \begin{cases} (x, i+1) & \text{if } 0 \leq i < h-1, \\ (\sigma'(x), 0) & \text{if } i = h-1. \end{cases}$$

We refer the reader to [**7**, Remark 9, Lemmas 17 and 19] for details.

For every $\Psi \in \mathrm{Aut}(X_{\theta'}, \sigma')$ and $0 \leq i \leq h-1$ we can define $\Psi_i \in \mathrm{Aut}(X_{\theta'} \times \{0, \ldots, h-1\}, T)$ as

$$\Psi_i(x, j) := \begin{cases} (\Psi(x), j+i) & \text{if } j+i < h, \\ (\Psi(\sigma'(x)), j+i \bmod h) & \text{if } j+i \geq h. \end{cases}$$

The following is proved in [**6**, Proposition 3.5].

PROPOSITION 1. *Let θ be a primitive length-r substitution, of height h, and such that (X_θ, σ) is infinite. Let θ' be a pure base of θ. Then* $\mathrm{Aut}(X_\theta, \sigma) = \{\Psi_j : \Psi \in \mathrm{Aut}(X_{\theta'}, \sigma')$ *and* $0 \leq j < h\}$.

Proposition 1 tells us that we need only study automorphism groups of shifts (X_θ, σ) where θ has height 1, and henceforth this will almost always be a standing assumption.

2.0.3. *The maximal equicontinuous factor of a constant-length substitution shift.* The *maximal equicontinuous factor* of a dynamical system (X, σ) encodes its continuous eigenvalues. For a precise definition, see [**6**]. The following is shown by Dekking [**7**], with partial results by Kamae [**15**] and Martin [**18**].

THEOREM 2. *Let θ be a primitive length-r substitution, of height h, and such that (X_θ, σ) is infinite. Then the maximal equicontinuous factor of (X_θ, σ) is $(\mathbb{Z}_r \times \mathbb{Z}/h\mathbb{Z}, +(1, 1))$.*

2.0.4. *Automatic shifts and sliding block code representations.* Let $\theta : \mathcal{A} \to \mathcal{A}^r$ be a primitive constant-length substitution. If $\tau : \mathcal{A} \to \mathcal{B}$ and $Y := \tau(X_\theta)$, then Y is closed and σ-invariant, and we call (Y, σ) an *r-automatic shift*, specifying that it is generated by (θ, τ) when necessary. We use this terminology since for u one-sided and θ-fixed, $\tau(u)$ is known as an *r-automatic* sequence. Note that (Y, σ) is a factor of (X_θ, σ) via a radius-0 factor map.

It can happen that X_θ is infinite, but $\tau(X_\theta)$ is finite. We are not interested in this case and will always require that the automatic shifts we consider are infinite. If (X_θ, σ) is minimal and $Y = \tau(X_\theta)$, then (Y, σ) is minimal.

Given a constant-length substitution $\theta : \mathcal{A} \to \mathcal{A}^r$, $\ell \in \mathbb{N}$ and $0 \leq k < r$, we define $\theta^{(\ell,k)} : \mathcal{A}^\ell \to (\mathcal{A}^{(\ell)})^r$, the *$k$-shifted ℓ-sliding block representation of θ*, which appears in [**21**, §5.4]. Namely, if $(\alpha_1 \ldots \alpha_\ell) \in \mathcal{A}^\ell$ and $\theta(\alpha_1 \ldots \alpha_\ell) = a_1 \ldots a_{\ell r}$, let

$$\theta^{(\ell,k)}(\alpha_1 \ldots \alpha_\ell) := (a_{k+1} \ldots a_{k+\ell})(a_{k+2} \ldots a_{k+\ell+1}) \ldots (a_{k+r} \ldots a_{k+r+\ell-1}).$$

If $k = 0$ we will write $\theta^{(\ell)}$ for $\theta^{(\ell,0)}$. It is straightforward to show that for each $\ell \geq 1$ and each $0 \leq k < r$, $(X_{\theta^{(\ell,k)}}, \sigma)$ is topologically conjugate to (X_θ, σ). The ℓ-sliding block representation is particularly useful as it allows us to change the radius of a factor map.

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LEMMA 3. *Let (Y, σ) be a shift and let $\Phi : X_\theta \to Y$ be a shift-commuting continuous map with left radius ℓ and right radius r. Then there is a shift-commuting map $\Phi^{(\ell+r+1)} : X_{\theta^{(\ell+r+1)}} \to Y$, with left and right radius 0.*

2.1. *Maximal equicontinuous factors of automatic shifts.* Let (Y, σ) be an automatic shift generated by (θ, τ). As (Y, σ) is a topological factor of (X_θ, σ), its maximal equicontinuous factor must be a factor of that of (X_θ, σ), which is $(\mathbb{Z}_r \times \mathbb{Z}/h\mathbb{Z}, +(1, 1))$. In principle, $(\mathbb{Z}_r, +1)$ may have smaller non-finite factors, in particular when r is not prime. In the next lemmas we show that as long as (Y, σ) is not finite it always has $(\mathbb{Z}_r, +1)$ as an equicontinuous factor. We briefly describe the required set-up, referring the reader to [**7**] and [**15**] for more detail. Recall that a partition $\{P_1, \ldots, P_k\}$ of a shift (X, σ) is *cyclic* if $\sigma(P_i) = P_{i+1 \bmod k}$. A partition is *$\sigma^n$-minimal* if each of its elements is σ^n-minimal, that is, each (P_i, σ^n) is a minimal shift. For each n, any minimal shift has a cyclic σ^n-minimal cyclic partition, which by minimality is unique up to cyclic permutation of its members. Let $\gamma(n)$ be the cardinality of the cyclic σ^n-minimal partition. In the case where we have a substitution shift (X_θ, σ), Dekking [**7**, Lemma II.7] shows that $\gamma(r^n) = r^n$, and that if $P_0 := \theta^n(X_\theta)$, then P_0 generates a σ^{r^n}-cyclic partition.

The proof of the following lemma is very similar to that of [**7**, Lemma 7].

LEMMA 4. *If (Y, σ) is an infinite primitive r-automatic shift, then $\gamma(r^n) = r^n$.*

Proof. First we show that there are at most $C + C^2(\gamma(r^n) - 1)$ words of length r^n in the language $\mathcal{L}_Y$ defined by Y. If $Y = \tau(X_\theta)$, with θ defined on $\mathcal{A}$ of cardinality C, let $v = \tau(u)$ for any u bi-infinite and θ-fixed. Now $v \in \tau(\theta^n(X_\theta))$ for each n. Therefore $\sigma^{k\gamma(r^n)}(v) \in \tau(\theta^n(X_\theta))$, so that v is composed of overlapping blocks of the form $\theta^n(a)$ spaced at intervals $\gamma(r^n)$. As there are at most C words $\theta^n(a)$, and at most C^2 words $\theta^n(a)\theta^n(b)$, this implies that there are at most $C + C^2(\gamma(r^n) - 1)$ words of length r^n in $\mathcal{L}_Y$. Now if $\gamma(r^n) < r^n$, then, by [**7**, Lemma 3(vi)], we have $\gamma(r^n)/r^n \to 0$. This implies that for n large, $\mathcal{L}_Y$ contains fewer than r^n words of length r^n, contradicting the assumption that Y is infinite. □

We note that, for each k, $\tau(\theta^k(X_\theta))$ is a closed, minimal σ^{r^k}-invariant set. This follows from the fact $\theta^k(X_\theta)$ satisfies these properties. Also, $\tau(\theta^k(X_\theta))$ must generate a σ^{r^k}-invariant partition.

We fix the maximal equicontinuous factor map that we consider henceforth. First suppose we consider a substitution shift (X_θ, σ). We write $\Lambda_{r^n}(x) = i$ if $x \in \sigma^i\theta^n(X_\theta)$. Note that if $\Lambda_n(x) = i$, then $\Lambda_{n+1}(x) \equiv i \pmod{r^n}$. Using this, we can define $\pi(x) := \ldots x_2\, x_1\, x_0$ where, for each $n \in \mathbb{N}$, $\Lambda_n(x) = \sum_{i=0}^{n-1} r^i x_i$.

If θ has trivial height, we have defined a maximal equicontinuous factor map. If θ has positive height h, then X_θ also has a σ^h-cyclic partition. Here we choose an arbitrary but fixed base, and use it to extend π to a maximal equicontinuous factor map.

Finally, if we consider an automatic shift (Y, σ) generated by θ, the convention we use is that we take the image of these partitions by τ.

Now, by Lemma [**7**, Lemma 11], (X_θ, σ) has no continuous irrational eigenvalues, and so neither does (Y, σ). Thus the maximal equicontinuous factor of (Y, σ) is generated by

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the σ^n-minimal cyclic partitions [**10**]. On the one hand, the maximal equicontinuous factor of (Y, σ) must be a factor of $(\mathbb{Z}_r \times \mathbb{Z}/h\mathbb{Z}, +(1, 1))$. On the other hand, Lemma 4 tells us that each $(\mathbb{Z}/r^n\mathbb{Z}, +1)$ is a factor of (Y, σ). We have proved the following theorem.

THEOREM 5. *If (Y, σ) is an infinite r-automatic shift generated by (θ, τ), where θ is primitive and of height h, then $(\mathbb{Z}_r \times \mathbb{Z}/\bar{h}\mathbb{Z}, +(1, 1))$ is the maximal equicontinuous factor of (Y, σ) for some $\bar{h}$ dividing h. Furthermore, we can take the maximal equicontinuous factor maps π_X and π_Y to satisfy $\pi_X = \pi_Y \circ \tau$.*

2.2. *Automorphisms of constant-length substitution shifts.* We recall key ingredients that will be essential in our study of automorphism groups of automatic shifts.

Let $\mathrm{End}(X, \sigma)$ be the set of endomorphisms of (X, σ). Given two shifts (X, σ) and (Y, σ), and let $\mathrm{Fac}(X, Y)$ denote the set of factor maps from (X, σ) to (Y, σ). The following theorem is proved in [**6**, Theorem 3.3].

THEOREM 6. *Let (X, σ) and (Y, σ) be infinite minimal shifts. Suppose that the group rotation (G, R) is the maximal equicontinuous factor of both (X, σ) and (Y, σ) and let π_X and π_Y be the respective factor maps. Then there is a map $\kappa : \mathrm{Fac}(X, Y) \to G$ such that*

$$\pi_Y(\Phi(x)) = \kappa(\Phi) + \pi_X(x)$$

for all $x \in X$ and $\Phi \in \mathrm{Fac}(X, Y)$. Also

(1) *if (Z, σ) is another shift which satisfies the assumptions on (X, σ), then $\kappa(\Psi \circ \Phi) = \kappa(\Psi) + \kappa(\Phi)$ for $\Phi \in \mathrm{Fac}(X, Y)$, $\Psi \in \mathrm{Fac}(Y, Z)$; and*

(2) *if $\min_{g\in G} |\pi_X^{-1}(g)| = \min_{g\in G} |\pi_Y^{-1}(g)| = c < \infty$, then κ is at most c-to-one. In particular, if π_X and π_Y are somewhere one-to-one, then κ is an injection.*

A constant-length substitution is called *injective* if $\theta(i) \neq \theta(j)$ for any distinct i and j in $\mathcal{A}$. It will be convenient to deal with injective substitutions in what follows. The following theorem of Blanchard, Durand and Maass allows us to restrict our attention to that situation.

THEOREM 7. [**3**] *Let θ be a constant-length substitution such that X_θ is infinite. Then there exists an injective constant-length substitution θ' such that (X_θ, σ) is topologically conjugate to $(X_{\theta'}, \sigma)$.*

We say that θ *has column number* c if

$$c = c(\theta) := \min_{0\le j<r^k, k\in\mathbb{N}} |\theta^k(\mathcal{A})_j|. \tag{1}$$

In other words, for some $k \in \mathbb{N}$, and some $(i_1, \ldots, i_k)$, $|\theta_{i_1} \circ \ldots \circ \theta_{i_k}(\mathcal{A})| = c$ and c is the least such number.

The following proposition tells us that, up to a shift, every automorphism has a small radius.

PROPOSITION 8. [**6**, Proposition 3.10] *Let θ be an injective primitive length-r substitution of height 1, and such that (X_θ, σ) is infinite. If $\Phi \in \mathrm{Aut}(X_\theta, \sigma)$ has the*

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property that $\kappa(\Phi) \in \mathbb{Z}_r \setminus \mathbb{Z}$ *is periodic, then* Φ *has right radius* 0 *and left radius at most* 1.

Further, if $\kappa(\Phi) = k/(1 - r^p)$ *for* $0 < k < r^p - 1$, *then*

$$\theta^{-c!p} \circ \sigma^{-k(1+r^p+r^{2p}+\ldots+r^{(c!-1)p})} \circ \Phi \circ \theta^{c!p} = \Phi, \tag{2}$$

where c *is the column number.*

If $\kappa(\Phi) = 0$, *then* Φ *has left and right radius at most* 1 *and* Φ *satisfies*

$$\theta^{-c!} \circ \Phi \circ \theta^{c!} = \Phi. \tag{3}$$

Example 9. In this example we show that for automatic shifts, it is no longer the case that automorphisms with a zero κ-value have radius at most 1, as in Proposition 8. Consider the substitution θ:

$$\begin{aligned} \theta(a) &= ac, \\ \theta(b) &= bd, \\ \theta(c) &= ab, \\ \theta(d) &= ba. \end{aligned}$$

Now X_θ has an automorphism Φ whose κ-value equals zero, given by the local rule f defined by the permutation $(ab)(cd)$. Next we consider the coding $\tau : \{a, b, c, d\} \to \{x, y, z\}$ defined by

$$\tau(a) = x, \quad \tau(b) = y, \quad \tau(c) = x \quad \text{and} \quad \tau(d) = z,$$

and let Y be the automatic shift defined by (θ, τ). One can verify that τ is injective, so there exists an automorphism of Y that corresponds to Φ, whose local rule is given by g, partially given by

$$\begin{aligned} g(y) &= x, \\ g(z) &= x. \end{aligned}$$

However, the local rule is a bit more complicated when considering the image of x under g, as we need to consider the preceding and following two elements, and the rule is given in Table 1. This table contains all appearing blocks of length 5 such that the middle element is x. To be able to define this local rule, it is crucial that every appearance of such a block uniquely defines the preimage of the middle element.

The next result, [**6**, Proposition 3.21] gives us conditions to check whether a local rule defines an element of $\mathrm{Aut}(X_\theta, \sigma)$. If Φ has local rule f, left radius ℓ and right radius r, we write $\Phi = \Phi^f_{\ell,r}$.

PROPOSITION 10. *Let* θ *be an injective primitive length-*r *substitution on* $\mathcal{A}$, *with column number* c, *of height* 1, *and such that* (X_θ, σ) *is infinite. Let* $p \geq 1$ *and* $0 < k < r^p - 1$.

Let $f : \mathcal{A}^2 \to \mathcal{A}$. *Then* $\Phi := \Phi^f_{1,0}$ *satisfies* $\Phi \in \mathrm{Aut}(X_\theta, \sigma)$ *and* $\kappa(\Phi) = k/(1 - r^p)$ *if and only if:*

(1) *if* $x_0x_1x_2 \in \mathcal{L}_{X_\theta}$, *then* $f(x_0, x_1) f(x_1, x_2) \in \mathcal{L}_{X_\theta}$; *and*

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TABLE 1. The local rule for the factor map in Example 9.

	x	y	z
$xxxx$	y	z	
$xxxy$	y	y	
$xyxx$	y	y	
$yxxx$		z	
$yxxy$			z
$yzxx$	y	y	
$zxxx$	z	z	
$zxxy$			z
$zyxx$	y		
$zyxy$			y

(2) *for each* $x_{-1}x_0x_1 \in \mathcal{L}(X_\theta)$ *and* $0 \le i < r^{c!p}$,

$$f(\theta^{c!p}(x_{-1} \cdot x_0)_{i-1}, \theta^{c!p}(x_{-1} \cdot x_0)_i) = \theta^{c!p}(f(x_{-1}, x_0)f(x_0, x_1))_{N+i},$$

where $N = k(1 + r^p + \ldots + r^{(c!-1)p})$ *and* $x_{-1} \cdot x_0$ *denotes a finite segment of a bi-infinite sequence with* x_{-1} *in the* -1*th coordinate and* x_0 *in the* 0*th coordinate.*

Similarly, given $g : \mathcal{A}^3 \to \mathcal{A}$, $\Phi^g_{1,1}$ *satisfies* $\Phi^g_{1,1} \in \mathrm{Aut}(X_\theta, \sigma)$ *and* $\kappa(\Phi^g_{1,1}) = 0$ *if and only if*

(3) $g(x_0, x_1, x_2)g(x_1, x_2, x_3) \in \mathcal{L}_{X_\theta}$ *whenever* $x_0x_1x_2x_3 \in \mathcal{L}_{X_\theta}$*; and*

(4) *for any* $x_{-1}x_0x_1 \in \mathcal{L}_{X_\theta}$ *and* $0 \le i < r^{c!}$,

$$g(u_{i-1}, u_i, u_{i+1}) = (\theta^{c!}g(x_{-1}, x_0, x_1))_i,$$

where $u = u_{-r^{c!}}u_{-r^{c!}+1} \ldots u_{-1} \cdot u_0 \ldots u_{2r^{c!}-1}$ *is the word obtained by applying* $\theta^{c!}$ *to the word* $x_{-1} \cdot x_0x_1$.

Remark 11. Proposition 8 implies that if Φ is a (topological) automorphism, then it is also a measure-preserving (measurable) automorphism of (X_θ, σ, μ), where μ is the unique σ-invariant measure on X_θ. Conversely, Host and Parreau [**14**] show that if Φ is a measure-preserving (measurable) automorphism of (X_θ, σ, μ), with the additional conditions that θ has column number at least 2, and that θ is *reduced*, then Φ is almost everywhere defined by a sliding block code of radius 1, and that it satisfies a version of (2) or (3). Now Proposition 10 tells us that Φ is therefore (almost everywhere equal to) a topological automorphism. We note that reduced substitutions (defined in [**14**]) are always strongly injective, a notion we define and study in §4.

3. *Topological factors of substitution shifts*

In this section we study automorphism groups of factors of constant-length substitution dynamical systems. We show, in particular, that we can both produce new automorphisms, and lose automorphisms, when we pass to a factor. Moreover, we show that any shift factor is indeed topologically conjugate to a constant-length substitution dynamical system, via a code (a letter-to letter map), that is, it is an automatic shift, so that we can explicitly describe its automorphism group. In the last part we consider the special case when the factor map is uniform-to-one.

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3.1. *Automorphism groups and factors.* Our motivation for the main results in §3 arose as we noticed that in general there is no relationship between the automorphism group of a substitution shift (X_θ, σ) and that of one of its topological factors $(Y, \sigma) = (\tau(X_\theta), \sigma)$. We summarize this with the following two examples.

Example 12. An example where $\mathrm{Aut}(\mathbf{X}_\theta, \sigma) \setminus \mathrm{Aut}(\mathbf{Y}, \sigma)$ *is non-empty.* Define θ as

$$\begin{aligned}\theta(a) &= ad,\\ \theta(b) &= bc,\\ \theta(c) &= ab,\\ \theta(d) &= ba.\end{aligned}$$

This substitution has column number 2 and the fixed points, namely $b.a$, $c.a$, $a.b$, $d.b$, comprise the only π-fibre with more than two elements, where $\pi : X_\theta \to \mathbb{Z}_2$ is the maximal equicontinuous factor map. Now X_θ has a non-trivial automorphism $\Phi \in \mathrm{Aut}(X_\theta, \sigma)$ corresponding to the local rule given by the letter-exchanging map $(ab)(cd)$. Note that Φ has κ-value zero, and it acts by permuting elements within the fibres of X_θ.

Now consider the projection $\tau : \{a, b, c, d\} \to \{x, y\}$ given by $\tau(a) = \tau(b) = x$, $\tau(c) = \tau(d) = y$. We find that $Y = X_{\bar\theta}$, where

$$\begin{aligned}\bar\theta(x) &= xy,\\ \bar\theta(y) &= xx,\end{aligned}$$

where θ has two fixed points $x.x$ and $y.x$. Since θ has a coincidence, that is, $c(\theta) = 1$, it does not have any non-trivial automorphisms with a zero κ-value. In other words, the map τ is everywhere two-to-one, and it collapses the fibres for X_θ. Note that if Φ is sent to an automorphism via τ, it must be an automorphism which also permutes within fibres. There is no non-trivial $\Phi' \in \mathrm{Aut}(Y, \sigma)$ such that $\kappa(\Phi') = 0$ as $\bar\theta$ has a coincidence. In other words, here both the identity map and Φ are collapsed by τ to the identity.

It is appropriate to put the next example here, even though our claims use ideas from §4.

Example 13. An example where $\mathrm{Aut}(\mathbf{Y}, \sigma) \setminus \mathrm{Aut}(\mathbf{X}_\theta, \sigma)$ *is non-empty.* We consider the following primitive substitution θ:

$$\begin{aligned}\theta(a) &= aac,\\ \theta(b) &= bca,\\ \theta(c) &= bba.\end{aligned}$$

Note that this substitution is *strongly injective* (Definition 30), and so if there exists $\Phi \in \mathrm{Aut}(X_\theta)$ with $\kappa(\Phi) = 0$ then, by Theorem 37, it must have radius 0, that is, it must exchange letters. Now the fixed points of θ are $a.a$, $c.a$, $a.b$, $c.b$ and so if there is a non-trivial automorphism it must send a to b and a to c, a contradiction. Thus $\Phi = \mathrm{Id}$. However, now we consider the coding $\tau(a) = x$, $\tau(b) = \tau(c) = y$, and we find that $Y = X_{\bar\theta}$, where

$$\begin{aligned}\bar\theta(x) &= xxy,\\ \bar\theta(y) &= yyx.\end{aligned}$$

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One easily finds that there exists a non-trivial $\Phi \in \mathrm{Aut}(Y, \sigma)$ with κ-value zero, namely (xy).

We remark here that $\Phi \in \mathrm{Aut}(Y, \sigma)$ does define a measurable automorphism F of (X_θ, σ). In particular, the only points where F is not defined are on the orbits of the fixed points, and also the orbit of the points in $\pi^{-1}(\ldots 111) = \pi^{-1}(-\frac{1}{2})$, with $\pi : X_\theta \to \mathbb{Z}_3$ the maximal equicontinuous factor map. The map F is not defined almost everywhere by a radius-1 sliding block code (or by any finite sliding block code, for that matter), which highlights that the result of Host and Parreau [**14**] is not true for non-reduced substitutions.

3.2. *Automatic shifts as substitution shifts.* In this section we prove that any shift factor of a primitive constant-length substitution shift is topologically conjugate, via a radius-0 code, to a constant-length substitution shift. Note that, by Lemma 3, we can assume that the factor is automatic to begin with.

Definition 14. We call $a \neq b \in \mathcal{A}$ a *periodic pair* if there exist $p = p(a, b)$ and $j < r^p$ such that $\theta^p(a)_j = a$ and $\theta^p(b)_j = b$. Note that this is equivalent to the requirement that for all $k > 0$ there exists $j < r^{kp}$ such that $\theta^{kp}(a)_j = a$ and $\theta^{kp}(b)_j = b$. We define

$$p(\theta) = lcm\{p(a, b) : a, b \text{ are a periodic pair}\},$$

and call a substitution θ with $p(\theta) = 1$ *pair aperiodic*.

Readers who are familiar with techniques used in substitutions will be aware that one often replaces a substitution θ by an appropriate power so that, without loss of generality, one can assume that all θ-periodic points are θ-fixed. In a similar spirit, we show in the following lemma that without loss of generality we can work with pair-aperiodic substitutions.

LEMMA 15. *Let θ be a substitution of constant length. Then $\theta^{p(\theta)}$ is pair aperiodic.*

Proof. A simple computation shows that if a, b are a periodic pair for θ with period p_1, then they are also a periodic pair for θ^n with period $p_2 = p_1/\gcd(p_1, n)$. Furthermore, one finds directly that any periodic pair for θ^n is also a periodic pair for θ. The result follows. □

Definition 16. We call $a, b \in \mathcal{A}$ an *asymptotic disjoint pair if for all $k > 0$ there exists $j_k < r^k$ such that $\theta^k(a)_{j_k} \neq \theta^k(b)_{j_k}$.*

Definition 17. We call a pair $(c, d) \in \mathcal{A}^2$ k-reachable from $(a, b) \in \mathcal{A}^2$ if there exists $j < r^k$ such that $\theta^k(a)_j = c$ and $\theta^k(b)_j = d$.

It is straightforward to check that reachability is transitive, that is, if (b_1, b_2) is k_1-reachable from (a_1, a_2) and (c_1, c_2) is k_2-reachable from (b_1, b_2), then (c_1, c_2) is $(k_1 + k_2)$-reachable from (a_1, a_2).

LEMMA 18. *Let θ be a pair-aperiodic substitution of constant length and (a, b) an asymptotic disjoint pair. Then there exists a periodic pair (a', b') such that for any $k \geq 2|\mathcal{A}|^2$, and any (c, d) that is reachable from (a', b'), it is also k-reachable from (a, b).*

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Proof. As (a, b) is an asymptotic disjoint pair, there exists for any $k \in \mathbb{N}$ some j_k such that $\theta^k(a)_{j_k} \neq \theta^k(b)_{j_k}$. We consider the finite sequence $(\theta^k(a)_{j_k}, \theta^k(b)_{j_k})_{k=0,\ldots,|\mathcal{A}|^2}$. We note that we can choose the j_k such that $j_k = \lfloor j_{k+1}/r \rfloor$. (For example, one first chooses $j_{|\mathcal{A}|^2}$.)

As every element in this sequence belongs to $\mathcal{A}^2$ and the sequence has $|\mathcal{A}|^2 + 1$ elements, there exist (a', b') and $0 \leq k_1 < k_2 \leq |\mathcal{A}|^2$ such that $\theta^{k_i}(a)_{j_{k_i}} = a'$ and $\theta^{k_i}(b)_{j_{k_i}} = b'$ for $i = 1, 2$. We claim that (a', b') are a periodic pair. For we know that $j_{k_2} = j_{k_1} r^{k_2 - k_1} + j'$, which gives

$$\begin{aligned} a' &= \theta^{k_2}(a)_{j_{k_2}} = \theta^{k_1 + (k_2 - k_1)}(a)_{j_{k_1} r^{k_2 - k_1} + j'} = \theta^{k_2 - k_1}(\theta^{k_1}(a)_{j_{k_1}})_{j'} \\ &= \theta^{k_2 - k_1}(a')_{j'}, \end{aligned}$$

and the analogous statement for b, b', and the claim follows. Since θ is pair aperiodic, (a', b') is 1-reachable from (a', b'). By the transitivity of reachability it follows that (a', b') is reachable from (a, b) by k steps for any $k \geq |\mathcal{A}|^2$. Finally, if (c, d) is reachable from (a', b'), then, by the pigeonhole principle, it must be k'-reachable for some $k' \leq |\mathcal{A}|^2$. The result follows by transitivity. □

In this last proof, we used pair aperiodicity strongly. We use it in a similar fashion in the next lemma.

LEMMA 19. *Let θ be a primitive, pair-aperiodic substitution. If (a, b) is not an asymptotic disjoint pair, then $\theta^{|\mathcal{A}|^2}(a) = \theta^{|\mathcal{A}|^2}(b)$.*

Proof. Since (a, b) is not asymptotic disjoint, there exists a minimal K such that $\theta^K(a) = \theta^K(b)$. Suppose that $K > |\mathcal{A}|^2$. Then for each $0 \leq i \leq K - 1$ there is a j_i such that $\alpha_i := (\theta^i(a))_{j_i} \neq \beta_i := (\theta^i(a))_{j_i}$ and where if $i < \bar{i}$ then $(\alpha_{\bar{i}}, \beta_{\bar{i}})$ is reachable from (α_i, β_i). Since $K - 1 \geq |\mathcal{A}|^2$, we have $(\alpha_i, \beta_i) = (\alpha_{\bar{i}}, \beta_{\bar{i}})$ for some $i < \bar{i}$. Thus (α_i, β_i) is a periodic pair. But now since θ is pair aperiodic, (α_i, β_i) appears, for each n, as $(\theta^n(a))_{k_n} = (\theta^n(b))_{k_n}$ for some k_n. This contradicts the fact that $\theta^K(a) = \theta^K(b)$. □

Definition 20. Let (θ, τ) be an r-automatic pair. We call two letters $a \neq b$ *indistinguishable* (by (θ, τ)) if for any $k \geq 0$ and $0 \leq j < r^k$ we have $\tau(\theta^k(a)_j) = \tau(\theta^k(b)_j)$. We call the pair (θ, τ) *minimal* if every pair of letters is distinguishable by (θ, τ).

Note that the letters a, b are indistinguishable if and only if for any (c, d) reachable from (a, b) (which includes (a, b) by 0 steps) we have $\tau(c) = \tau(d)$.

In the next lemma, a fundamental result from automata theory enables us to replace an r-automatic pair by a minimal r-automatic pair.

LEMMA 21. *Let (Y, σ) be an infinite r-automatic shift generated by $(\theta', \tau')n$ where θ' is primitive and pair aperiodic. Then there exists a minimal r-automatic pair (θ, τ) which generates (Y, σ), where θ is primitive and pair aperiodic.*

Proof. We define an equivalence relation $a \sim b$ if a, b are indistinguishable. This equivalence relation is obviously compatible with θ', so we can define

$$\theta([a])_j := [\theta'(a)_j],$$

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and it is also compatible with τ, so we set

$$\tau([a]) := \tau(a).$$

We need to check that θ is primitive and pair aperiodic. As θ' is primitive, we know that there exists some n such that for all $a, b \in \mathcal{A}'$ we have that $(\theta')^n(a)$ contains b as a subword. Let us take $[c], [d] \in \mathcal{A}$. We know that $(\theta')^n(c)$ contains d as a subword and therefore $\theta^n([c])$ contains $[d]$ as a subword. In other words, θ is primitive.

Let us now assume that $\theta^k([a])_j = [a]$ and $\theta^k([b])_j = [b]$, that is, $([a], [b])$ is a periodic pair. This implies that $(\theta')^k_j$ is a map from $[a]$ to itself and also from $[b]$ to itself. Thus a power must have a fixed point:

$$((\theta')^k_j)^n(a_0) = a_0,$$
$$((\theta')^k_j)^m(b_0) = b_0,$$

for some $a_0 \in [a]$, $b_0 \in [b]$. Therefore we have

$$((\theta')^k_j)^{nm}(a_0) = a_0,$$
$$((\theta')^k_j)^{nm}(b_0) = b_0,$$

which means that (a_0, b_0) is a periodic pair for θ'. As θ' is pair aperiodic we know that there exists j_0 such that

$$\theta'(a_0)_{j_0} = a_0,$$
$$\theta'(b_0)_{j_0} = b_0$$

and, therefore,

$$\theta([a])_{j_0} = [a],$$
$$\theta([b])_{j_0} = [b].$$

This shows that θ is pair aperiodic.

Furthermore, by definition,

$$\tau'((\theta')^k(a)) = \tau(\theta^k([a])),$$

which shows that the language of $(Y\sigma)$ is the same as the language of $(\tau(X_\theta), \sigma)$. Since shifts are uniquely determined by their language, the result follows. □

We can now prove the main result of this section.

THEOREM 22. *Let (Y, σ) be an infinite r-automatic shift generated by (θ', τ'), where θ' is primitive. Then there exists an r-automatic pair (θ, τ) that generates (Y, σ) and such that $\tau : X_\theta \to Y$ is a bijection.*

Proof. By Lemmas 15 and 21, we can assume that (Y, σ) is generated by the minimal r-automatic pair (θ, τ), where θ is pair aperiodic.

First we suppose that there exist $x \neq \tilde{x} \in X_\theta$ such that $\tau(x) = \tau(\tilde{x})$. As X_θ is recognizable, [**19**] we have x', $\tilde{x}'$ such that $x = \sigma^\ell \theta^{2|\mathcal{A}|^2}(x')$ and $\tilde{x} = \sigma^{\tilde{\ell}} \theta^{2|\mathcal{A}|^2}(\tilde{x}')$. Now Theorem 5 tells us that $\ell = \tilde{\ell}$. Thus, we have x', $\tilde{x}'$ such that $\theta^{2|\mathcal{A}|^2}(x') \neq \theta^{2|\mathcal{A}|^2}(\tilde{x}')$ and $\tau(\theta^{2|\mathcal{A}|^2}(x')) = \tau(\theta^{2|\mathcal{A}|^2}(\tilde{x}'))$.

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We consider the pair $(x'_n, \tilde{x}'_n) \in \mathcal{A}^2$. Suppose that for some $n \in \mathbb{Z}$, $(x'_n, \tilde{x}'_n)$ is an asymptotic disjoint pair. By Lemma 18 we know that there exists a periodic pair (a', b') such that for any (c, d) that is reachable from (a', b'), we have that (c, d) is $(2|\mathcal{A}|^2)$-reachable from (a, b). As $\tau(\theta^{2|\mathcal{A}|^2}(x'_n)) = \tau(\theta^{2|\mathcal{A}|^2}(\tilde{x}'_n))$, this gives that $\tau(c) = \tau(d)$ and therefore that a' and b' are indistinguishable. This contradicts our assumption that (θ, τ) is minimal.

Thus $(x'_n, \tilde{x}'_n)$ is not an asymptotic disjoint pair for any $n \in \mathbb{Z}$. By Lemma 19 we know that $\theta^{2|\mathcal{A}|^2}(x'_n) = \theta^{2|\mathcal{A}|^2}(\tilde{x}'_n)$ for each $n \in \mathbb{Z}$, that is, $x = \tilde{x}$, another contradiction. The result follows. □

The following example tells us that starting with a pair-aperiodic substitution before we minimize is necessary in Theorem 22.

Example 23. We consider the substitution $\theta' : \{a, b, c, d\} \to \{a, b, c, d\}^3$ and $\tau' : \{a, b, c, d\} \to \{x, y, z\}$, where

$$\begin{aligned} \theta'(a) &= abd, \\ \theta'(b) &= aad, \\ \theta'(c) &= add, \\ \theta'(d) &= acd \end{aligned}$$

and

$$\tau'(a) = \tau'(c) = x, \quad \tau'(b) = y, \quad \tau'(d) = z.$$

We find that (θ', τ') is minimal, as the only possible indistinguishable pair is (a, c), but $\tau'(\theta'(a)_1) = y \neq \tau'(\theta'(c)_1) = z$.

However, $\tau' : X_{\theta'} \to Y$ is not injective. Let $x = \lim_{n\to\infty} \sigma^{(3^{2n}-1)/2}\theta'^{2n}(a)$ and let $\tilde{x} = \lim_{n\to\infty} \sigma^{(3^{2n}-1)/2}\theta'^{2n}(c)$. One can verify that $x_n = \tilde{x}_n$ for all $n \neq 0$ and $x_0 = a$, $\tilde{x}_0 = c$. Thus $\tau'(x) = \tau'(\tilde{x})$.

The 'problem' here is that (a, c) is a periodic pair of period 2 and we need to actually consider θ'^2 instead of θ':

$$\begin{aligned} \theta'^2(a) &= abdaadacd, \\ \theta'^2(b) &= abdabdacd, \\ \theta'^2(c) &= abdacdacd, \\ \theta'^2(d) &= abdaddacd. \end{aligned}$$

Here we see that (a, c) is indeed indistinguishable, so that (θ'^2, τ) is not minimal. In this case we minimize and we find that the new substitution

$$\begin{aligned} \theta(x) &= xyzxxzxxz, \\ \theta(y) &= xyzxyzxxz, \\ \theta(z) &= xyzxzzxxz \end{aligned}$$

along with $\tau(a) = x$, $\tau(b) = y$, $\tau(d) = z$ gives us the correct representation.

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3.3. *Automorphism groups and uniform-to-one extensions.* In this subsection we investigate topological factors of minimal dynamical systems (X, T), where X is compact and metric, and where the factor map is uniform-to-one.

Let (X, T) be a such a dynamical system. If there exists an automorphism Φ of (X, T) with $\Phi^k = \mathrm{Id}$ for some minimal k, then we claim that we can define an everywhere k-to-one factor of (X, T). First note that if k is minimal, then for no $\ell < k$ do we have $\Phi^\ell(x) = x$ for some $x \in X$. For if that were the case, then Φ^ℓ equals the identity on a dense subset of X. By minimality, $\Phi^\ell = \mathrm{Id}$, contradicting minimality of k. Now we can identify points $\{x, \Phi(x), \ldots, \Phi^{k-1}(x)\}$ that are in the same orbit, and this is a well defined T-commuting map on X.

This assertion extends to the measurable setting. Namely, if there exists a measurable ergodic bijection Φ of (X, T, μ), then we can use ergodicity to assert that there is a minimal n such that for μ-almost all x, $\Phi^n(x) = x$ and n is minimal for that x. Now we can proceed with that n as above.

In particular, if θ has column number c and $\Phi^c = \mathrm{Id}$ with c minimal, then the factor defined above is an almost everywhere one-to-one extension of its maximal equicontinuous factor, and this is a convenient topological factor for a dynamical system to have; see, for example, [**1**], [**17**], or [**22**] for uses. It is not always the case that such a factor exists [**12**]. It is natural to investigate how closely related automorphisms and everywhere k-to-one factors are. The following result shows that for the case $k = 2$ we have a one-to-one correspondence. We are grateful to Downarowicz and Lemańczyk who pointed this general result out to us via private communication [**8**].

THEOREM 24. *Let (X, T) and (Y, S) be minimal topological dynamical systems on compact metric spaces X and Y. If $\pi : (X, T) \to (Y, S)$ is a uniformly two-to-one topological factor map, then there exists $\Phi \in \mathrm{Aut}(X, T)$ such that $\Phi^2 = \mathrm{Id}$.*

Proof. Several times in this proof, compactness allows us to drop to a subsequence; we will do this without reindexing.

For $y \in Y$, write $\{x, \tilde{x}\} = \pi^{-1}(y)$; define $\Phi(x) = \tilde{x}$ and $\Phi(\tilde{x}) = x$. It is clear that Φ is bijective, and the fact that $\pi \circ T = S \circ \pi$ implies that $\Phi \circ T = T \circ \Phi$. It remains to show that Φ is continuous.

Define $D : Y \to [0, \infty)$ by $D(y) = d(x, \tilde{x})$. If $y_n \to y$ then it can be verified that $D(y) \geq \limsup D(y_n)$, that is, D is upper semicontinuous. Suppose that D is bounded away from 0. Then Φ is continuous. Let $x_n \to x$ and $\{x_n, \tilde{x}_n\} = \pi^{-1}(y_n)$. By compactness, $\tilde{x}_n \to \tilde{x}$ along one subsequence, and let us assume that $\tilde{x}_n \to x'$ along another subsequence, where $\tilde{x} \neq x'$. Since D is bounded away from 0, we also have $x' \neq x$. But since π is continuous, we have $\pi(x) = \pi(\tilde{x}) = \pi(x') = y$, a contradiction.

It remains to show that D is bounded away from 0. Suppose that $D(y_n) \to 0$. The continuity of T implies that for each j, $D(S^j y_n)) \to 0$ as $n \to \infty$. If $y_n \to y$, the minimality of S implies that in any open subset of Y we can find points whose D value is arbitrarily small. By upper semicontinuity $\{y : D(y) < r\}$ is open, and by the above it is also dense. Hence $D = 0$ on a dense G_δ set, which is a contradiction. □

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TABLE 2. The local rule for the factor map in Example 25.

	a	b	c
a	z	y	x
b	y	x	z
c	x	z	y

Unfortunately Theorem 24 is no longer true for a uniform k-to-one factor map, where $k > 2$.

Example 25. Let $\theta(a) = abb$, $\theta(b) = bac$, $\theta(c) = cca$, and let $\eta(x) = yxz$, $\eta(y) = yxx$, $\eta(z) = yxy$. Then η has column number 1, and (X_θ, σ) is an almost everywhere three-to-one extension of $(\mathbb{Z}_3, +1)$. One verifies that the left radius-1, right radius-0 rule in Table 2 defines an exactly three-to-one shift-commuting map $\pi : X_\theta \to X_\eta$. However, (X_θ, σ) has a trivial automorphism group.

4. *Strongly injective substitutions and language automorphisms*

Definition 26. Let $\mathcal{L} \subset \mathcal{A}^*$ be a language. We say that the bijection $\Phi : \mathcal{L} \to \mathcal{L}$ is an $\mathcal{L}$-*automorphism* if $\Phi(w_1 w_2) = \Phi(w_1)\Phi(w_2)$ whenever w_1, w_2 and $w_1 w_2$ belong to $\mathcal{L}$.

We have the following proposition which links language automorphisms to automorphisms of the corresponding dynamical system. Recall that any shift space X is defined by its language, which is closed under the taking of subwords, and where every word is left and right extendable to a word in $\mathcal{L}$. Conversely, any language $\mathcal{L}$ satisfying these two properties defines a shift $X_\mathcal{L}$.

PROPOSITION 27. *Let $\mathcal{L}$ be the language of a shift (X, σ). Then Φ is an* $\mathcal{L}$-automorphism *if and only if $\Phi \in \mathrm{Aut}(X, \sigma)$ has radius 0.*

Proof. Suppose that Φ is an $\mathcal{L}$-automorphism. Since Φ is bijective, it must map letters to letters, that is, $\Phi|_\mathcal{A}$ is a bijection $\phi : \mathcal{A} \to \mathcal{A}$. Since it is a language morphism, ϕ defines Φ. We can extend Φ, via concatenation, to a map on bi-infinite sequences u in X, namely, $\Phi((u_n)_{n\in\mathbb{Z}} = (\phi(u_n))_{n\in\mathbb{Z}}$. Let $u \in X$ and let $v := \Phi(u)$. Since Φ maps $\mathcal{L}$ to $\mathcal{L}$, $v \in X$ and so the image of $\Phi(X)$ is contained in X. Since Φ is defined by a local rule it is shift-commuting, and it is injective on X since it is injective on $\mathcal{L}$. By minimality, Φ is surjective.

The converse is straightforward. □

Remark 28. In the substitutional case, we note that if Φ has radius 0 then $\kappa(\Phi) = 0$. For if Φ has radius 0, then for some n, $\Phi^n = \mathrm{Id}$. But then $n\kappa(\Phi) = 0$, and since $\mathbb{Z}_r$ is torsion-free, our assertion follows.

In an ideal world any automorphism Φ with $\kappa(\Phi) = 0$ would have radius 0. Unfortunately this is not the case.

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Example 29. Let $\theta(a) = ab$, $\theta(b) = ca$ and $\theta(c) = ba$ and define the local rule ϕ with left radius 1, as $\phi(b) = a$, $\phi(c) = a$, $\phi(aa) = c$, $\phi(ba) = b$ and $\phi(ca) = b$. Then ϕ is the local rule of an involution in $\mathrm{Aut}(X_\theta, \sigma)$.

Definition 30. We say that θ is *strongly right-injective (left-injective)* if θ is injective and does not have any right-infinite (left-infinite) fixed points which differ only in their 0th entry, and we say that θ is *strongly injective* if it is both strongly right- and left-injective.

Definition 31. Let $\theta : \mathcal{A} \to \mathcal{A}^r$ be a length-r substitution with column number c. A *minimal set* (for θ) is a set of letters $\{\alpha_1, \ldots, \alpha_c\} \subset \mathcal{A}$ of cardinality c such that this set appears as $(\theta^n(\mathcal{A}))_i$ for some $n \in \mathbb{N}$ and some $0 \le i < r^n$. Let $\mathcal{X}$ denote the collection of all minimal sets:

$$\mathcal{X} := \{M \in \mathcal{A}^{c(\theta)} : \exists k,\ j < r^k \text{ s.t. } \theta^k(\mathcal{A})_j = M\}.$$

THEOREM 32. *Let θ be a primitive length-r substitution such that X_θ is infinite. Then (X_θ, σ) is topologically conjugate to (X_η, σ) where η is a strongly injective length-r substitution.*

Proof. Note that if a substitution θ has column number c, then it has at least c right-infinite and c left-infinite fixed points. We claim that if θ has exactly c right-infinite (left-infinite) fixed points, then it must be right-strongly injective (left-strongly injective). For in this case there is a minimal set $\mathcal{A}_r$ ($\mathcal{A}_l$) of c letters that generates the c right-infinite (left-infinite) fixed points. The fact that θ has column number c implies that these fixed points are pairwise distinct at each index $n \ge 0$, and our claim follows.

Now suppose that θ is not strongly injective. By replacing θ by a power if necessary, we can assume that there are three consecutive indices $k-1$, k, $k+1$, with $k \in \{1, \ldots r-1\}$, where θ's column number is achieved jointly for these indices, that is, $|\{\theta(a)[k-1, k+1]\colon\ a \in \mathcal{A}\}| = c$. Now let $\bar{\eta} := \theta^{(2,k)}$ be the k-shifted 2-sliding block representation of θ. By construction, $\bar{\eta}$ has exactly c right-infinite fixed points and c left-infinite fixed points. If $\bar{\eta}$ is injective on letters, then by our previous claim, it is strongly injective and in this case let $\eta = \bar{\eta}$. If $\bar{\eta}$ is not injective on letters, define an equivalence relation on its alphabet where $\alpha \sim \beta$ if $\bar{\eta}(\alpha) = \bar{\eta}(\beta)$. Let $\mathcal{B}$ be the set of equivalence classes for $\sim$ and define $\eta([\beta]) := \bar{\eta}(\beta)$. Then (X_η, σ) is topologically conjugate to $(X_{\bar{\eta}}, \sigma)$ via the map $\Phi((x_n)_{n\in\mathbb{Z}}) = ([x_n])_{n\in\mathbb{Z}}$. (See, for example, the proof of Theorem 7 in [**3**].) By construction, η is injective on letters. Since $\kappa(\phi) = 0$, the fixed points of $\bar{\eta}$ are in one-to-one correspondence with those of η. The result follows by our claim. □

THEOREM 33. *Let θ be a strongly injective primitive length-r substitution such that X_θ is infinite. If $\kappa(\Phi) = 0$ then Φ has radius 0 and so, in particular, it is a language automorphism.*

Proof. We suppose that Φ is non-trivial; then it must permute fixed points, and it cannot fix either the right ray or the left ray of a fixed point. We assume that Φ sends the fixed point u, whose right-infinite ray is $u_{[0,\infty)} = \lim_k \theta^k(a)$, to the fixed point $v \ne u$, whose right-infinite ray is $v_{[0,\infty]} = \lim_k \theta^k(b)$.

If Φ does not have radius 0, then there exist letters c, d and e, with $d \ne e$, and indices $i \ne j$ such that $u_i = c = u_j$, $v_i = d$ and $v_j = e$. This implies that for all n, $\theta^n(c)$ appears

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starting at the indices $r^n i$ and $r^n j$ in u, $\theta^n(d)$ appears starting at index $r^n i$ in v, and $\theta^n(e)$ appears starting at index $r^n j$ in v. Suppose that the left radius of Φ is $\ell > 0$. Then for all large n, $\theta^n(d)_{[\ell, r^n-1]} = \theta^n(e)_{[\ell, r^n-1]}$ and thus we have two fixed points $u' = \ldots \cdot d' D' x$ and $v' = \ldots \cdot e' E' x$ which disagree on their initial entries: $d' \neq e'$, where the length of D' equals that of E', and where the suffix of D' does not equal that of E', and finally where u' and v' agree on their right rays x starting at index at most ℓ. If $D' \neq \emptyset$, so that also $E' \neq \emptyset$, then the image of their suffixes under θ must agree, and this contradicts our assumption that θ is injective. (Here we are using the fact that r must be at least 2, of course.) Thus $D' = \emptyset = E'$ and thus we have two fixed points $u' = \ldots \cdot d' x$ and $v' = \ldots \cdot e' x$ which disagree on their initial entry, $d' \neq e'$, and which agree on their right rays starting at index 1. This contradicts our assumption that θ is strongly right-injective. If Φ has right radius $r > 0$, we would obtain a contradiction to our assumption that θ is strongly left-injective. □

Our next goal is to characterize the automorphisms of (X_θ, σ) which have radius 0. This is particularly interesting as, by Theorem 32, we can assume without loss of generality that θ is a strongly injective substitution. Furthermore, by Theorem 33, we know that in this situation any Φ with $\kappa(\Phi) = 0$ has radius 0. In particular, the local rule is just a map from $\mathcal{A}$ to itself.

The characterization that we will give is inspired by the special case of bijective substitutions, and it is instructive to recall this special case first. We note that bijective substitutions are strongly injective, so any element of $\ker \kappa$ is a letter-exchanging map. Lemańczyk and Mentzen [**16**, Theorem 5] gave a closed form for the automorphism group of bijective substitutions in the measurable setting. In particular, they consider the group G of permutations generated by the bijections $\{\theta_i : 1 \leq i \leq r\}$ of $\mathcal{A}$, and they show that a permutation τ defines a shift-commuting measurable bijection of (X_θ, σ) if and only if it belongs to the centralizer of G in the symmetric group $S_{|\mathcal{A}|}$. This statement is still true if we pass to the topological setting. The fact that $\ldots 00$ is the only singular fibre of the equicontinuous factor map tells us that the automorphisms of X_θ must have κ-value zero. Our claim can now be seen using Proposition 10, which is effectively telling us that τ must commute with G. In fact, we only need a simpler version of Proposition 10, namely, one that takes into account the fact that τ must have radius 0.

We recall the definition of the column number in (1), which denotes the minimal number of elements of $\mathcal{A}$ one can see in one column when writing the words $\theta^k(a)$ for $a \in \mathcal{A}$ underneath each other. Recall that we use $\mathcal{X} \subseteq \mathcal{P}(\mathcal{A}^{c(\theta)})$ to denote the collection of all possible minimal sets for θ. The minimality of $M \in \mathcal{X}$ implies some useful properties: for any $a, b \in M$ we have that $\theta^k(a)_j \neq \theta^k(b)_j$ for all $k \geq 1$ and $j < r^k$. Otherwise, we would have a contradiction to the minimality of the column number c. This implies that θ is well defined on $\mathcal{X}$ and we denote it by $\tilde{\theta}$. It follows quite easily that $\tilde{\theta}$ is primitive (see [**17**, Proposition 6.3]).

LEMMA 34. *$\mathcal{X}$ covers $\mathcal{A}$. In other words, every letter of $\mathcal{A}$ belongs to a set in $\mathcal{X}$.*

Proof. This is [**17**, Equation (37)], but we include the verification here for completeness. We need to show that for any $b \in \mathcal{A}$ there exists $M' \in \mathcal{X}$ with $b \in M'$. Obviously $\mathcal{X} \neq \emptyset$.

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Thus, there exist some M and a such that $a \in M \in \mathcal{X}$. As θ is primitive, there exist $k,\ j < r^k$ with $\theta^k(a)_j = b$. Now obviously $b \in \tilde{\theta}^k(M)_j =: M'$. □

This allows us to find our first restrictions for automorphisms with radius 0.

LEMMA 35. *Suppose that τ is an automorphism of (X_θ, σ) with radius 0. Then, for any $M \in \mathcal{X}$, we have that $\tau|_M$ is a bijection.*

Proof. We know by a modified version of Proposition 10 (see, for example, [**2**, Lemma 2]) that

$$\tau \circ \theta^{c!} = \theta^{c!} \circ \tau.$$

Thus by changing to $\theta^{c!}$ we can assume without loss of generality that

$$\tau \circ \theta = \theta \circ \tau. \tag{4}$$

By the surjectivity of τ, it follows directly that τ is a bijection on $\mathcal{A}$. We know that for any $M \in \mathcal{X}$ there exist $k \in \mathbb{N},\ j < r^k$ with $\theta^k(\mathcal{A})_j = M$. Thus, for any $a \in M$ there exists $b \in \mathcal{A}$ such that $\theta^k(b)_j = a$. This gives, by (4),

$$\tau(a) = \tau \circ \theta_j^k(b) = \theta_j^k \circ \tau(b) \in M.$$

So τ maps M to itself and as τ is injective it is bijective on M. □

So we know that τ is a bijection on every minimal set, and, by (4), we would expect these bijections to be compatible. Next we aim to relate elements of minimal sets to each other. Therefore we fix some $M_0 \in \mathcal{X}$ and k_0 with

$$\theta^{k_0}(\mathcal{A})_{j_0} = M_0, \quad \text{for some } j_0 < r^{k_0}. \tag{5}$$

Furthermore, we can assume without loss of generality that $\theta^{k_0}(a)_{j_0} = a$ for every $a \in M_0$. This is due to the fact that $\theta_{j_0}^{k_0}$ defines a bijection on M_0 and taking a power yields the result. Now we fix an arbitrary bijection $f_0 : M_0 \to \{1, \ldots, c\}$ which allows us to define $f : \mathcal{A} \to \{1, \ldots, c\}$ by

$$f(a) = f_0(\theta^{k_0}(a)_{j_0}). \tag{6}$$

In particular, we have $f|_{M_0} = f_0$. Note also that the minimality of c implies that for any $M \in \mathcal{X}$ we have that $f|_M$ is a bijection between M and $\{1, \ldots, c\}$.

LEMMA 36. *There exists a bijection τ' on $\{1, \ldots, c\}$ such that $f \circ \tau = \tau' \circ f$.*

Proof. Let us take $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. We need to show that $f(\tau(a_1)) = f(\tau(a_2))$. By the definition of f, we have

$$\theta^{k_0}(a_1)_{j_0} = \theta^{k_0}(a_2)_{j_0}.$$

By (4), this gives

$$\begin{aligned} f(\tau(a_1)) &= f_0(\theta_{j_0}^{k_0} \circ \tau(a_1)) = f_0(\tau \circ \theta_{j_0}^{k_0}(a_1)) \\ &= f_0(\tau \circ \theta_{j_0}^{k_0}(a_2)) = f_0(\theta_{j_0}^{k_0} \circ \tau(a_2)) = f(\tau(a_2)). \end{aligned}$$

□

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However, there is still one more condition that τ has to satisfy. For this remaining condition we use a generalization of Lemańczyk and Mentzen's idea for bijective substitutions to assign to a general θ a group of permutations called G. The following argument was presented by the first author and Lemańczyk in [**17**], and we follow their notation. Let us fix an arbitrary $M \in \mathcal{X}$. Then θ_j is a bijection from M to $\tilde{\theta}(M)_j$. Thus it corresponds via f to a permutation of $\{1, \ldots, c\}$ which we denote by $\sigma_{M,j}$. More precisely, we have $\sigma_{M,j}(m) = n$ if there exist $a \in M$, $b \in \tilde{\theta}(M)_j$ with $f(a) = n$, $f(b) = m$ such that $\theta(a)_j = b$. Another way to define it is via the relation

$$\sigma_{M,j}^{-1}(f(a)) = f(\theta(a)_j). \tag{7}$$

Furthermore, we denote by G the group generated by all the $\sigma_{M,j}$:

$$G(\theta) := \langle \sigma_{M,j} : M \in \mathcal{X}, 0 \le j \le r-1 \rangle.$$

We have gathered all the information needed for the following theorem. In what follows the function f is that defined in Equation (6).

THEOREM 37. *Let θ be a primitive and injective length r substitution of height* 1 *such that X_θ is infinite. Then τ is an automorphism with radius* 0 *if and only if:*

(i) *τ is a bijection on every minimal set $M \in \mathcal{X}(\theta)$;*

(ii) *τ is well defined on $\{1, \ldots, c\}$ via f, which we denote by τ', that is, $f \circ \tau = \tau' \circ f$;*

(iii) *τ' belongs to the centralizer of $G(\theta)$.*

Proof. First, we still need to show that any automorphism τ with radius 0 fulfils (iii): Obviously it is sufficient to show that τ' commutes with every $\sigma_{M,j}^{-1}$. We find for any $a \in M$,

$$\begin{aligned}\tau' \circ \sigma_{M,j}^{-1}(f(a)) = \tau' \circ f(\theta(a)_j) = f \circ \tau(\theta(a)_j) = f(\theta(\tau(a))_j) \\ = \sigma_{M,j}^{-1} \circ f \circ \tau(a) = \sigma_{M,j}^{-1} \circ \tau'(f(a)).\end{aligned}$$

Here the first and fourth equations are due to (7), the second and fifth equations follow from (ii) and the third equation from (4).

It remains to show that any τ that satisfies (i)–(iii) indeed gives an automorphism. It is sufficient to check (4) as this implies $\tau(\theta^k(a)[j-1, j+1]) = \theta^k(\tau(a))[j-1, j+1] \in \mathcal{L}_\theta$. Let $a \in \mathcal{A}$ be an arbitrary element and $j < r$. By Lemma 34, we know that $a \in M'$ for some $M' \in \mathcal{X}$. By (i), we have that $\tau(a) \in M'$. It follows by the definition of $\tilde{\theta}$ that $\theta(a)_j, \theta(\tau(a))_j \in \tilde{\theta}(M')_j =: M$. This gives

$$\begin{aligned}\theta(\tau(a))_j &= f|_M^{-1} \circ f(\theta(\tau(a))_j) \\ &= f|_M^{-1} \circ \sigma_{M,j}^{-1} \circ f \circ \tau(a) \\ &= f|_M^{-1} \circ \sigma_{M,j}^{-1} \circ \tau' \circ f(a) \\ &= f|_M^{-1} \circ \tau' \circ \sigma_{M,j}^{-1} \circ f(a) \\ &= f|_M^{-1} \circ \tau' \circ f(\theta(a)_j) \\ &= \tau \circ f|_M^{-1} \circ f(\theta(a)_j) \\ &= \tau(\theta(a))_j.\end{aligned}$$

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Here the second and fifth equations hold by (7), the third and sixth equations by (ii) and the fourth equation by (iii). □

Remark 38. In the case of bijective substitution we have that for any element in the centralizer of G there exists an automorphism. This is no longer the case in the general situation, as the following example shows.

Example 39. We consider the strongly injective substitution

$$\theta(a) = aac,$$
$$\theta(b) = bba,$$
$$\theta(c) = bca.$$

We find directly that $\mathcal{X} = \{\{a, b\}, \{a, c\}\}$. We have $\theta(a)_0 = a$, $\theta(b)_0 = b$, $\theta(c)_0 = b$. Thus we define $f(a) = 1$, $f(b) = f(c) = 2$. This gives

$$\sigma_{\{a,b\},0} = \mathrm{id},$$
$$\sigma_{\{a,b\},1} = \mathrm{id},$$
$$\sigma_{\{a,b\},2} = (12),$$
$$\sigma_{\{a,c\},0} = \mathrm{id},$$
$$\sigma_{\{a,c\},1} = \mathrm{id},$$
$$\sigma_{\{a,c\},2} = (12).$$

Therefore, we have $G(\theta) = \{\mathrm{id}, (12)\}$ and the centralizer of G is G itself. However, as $a \in \{a, b\}$ and $a \in \{a, c\}$ we know by (i) that $\tau(a) = a$ and therefore $\tau(b) = b$, $\tau(c) = c$. This gives an automorphism with $\tau = \mathrm{id}$ which corresponds to $\tau' = \mathrm{id}$, but we have no automorphism corresponding to $\tau' = (12)$.

Because of the situation in Example 39, Theorem 37 is not entirely satisfactory. We recall that Lemańczyk and Mentzen were working in a measurable context, and this raises the question of whether we can hope for a better version of Theorem 37 if we move to the measurable setting. It turns out that this is indeed the case, as we shall next show in Theorem 44. To do this we recall the notion of a *reduced* substitution, due to Host and Parreau [**14**].

We denote by $d^* : (\mathcal{A}^2)^* \to \mathbb{R}$,

$$d^*(a_1 \dots a_n, b_1 \dots b_n) = \frac{|\{1 \le i \le n : a_i \ne b_i\}|}{n}.$$

Definition 40. We call a primitive length-r substitution *reduced* if for all $a, b \in \mathcal{A}$ we have

$$\lim_{k\to\infty} d^*(\theta^k(a), \theta^k(b)) > 0.$$

One can reduce a substitution by identifying all $a, b \in \mathcal{A}$ for which

$$\lim_{k\to\infty} d^*(\theta^k(a), \theta^k(b)) = 0.$$

When reducing a substitution, the related dynamical system does not change measure-theoretically.

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LEMMA 41. *Let θ be a primitive length-r substitution. Then we have, for any $a, b \in \mathcal{A}$,*

$$\lim_{k\to\infty} d^*(\theta^k(a), \theta^k(b)) = 0 \tag{8}$$

if and only if, for each k, we have

$$f(\theta^k(a)_j) = f(\theta^k(b)_j), \quad \text{for all } j < r^k. \tag{9}$$

Proof. Let $a \in M \in \mathcal{X}(\theta)$. Then $\theta^k(a)_j$ is uniquely determined by $f(\theta^k(a)_j)$ and $\tilde{\theta}^k(M)_j$. Furthermore, as $c(\tilde{\theta}) = 1$ we know that

$$\lim_{k\to\infty} d^*(\tilde{\theta}^k(M_1), \tilde{\theta}^k(M_2)) = 0,$$

for all $M_1, M_2 \in \mathcal{X}(\theta)$. This shows that (9) implies (8).

Let us now assume that there exist k' and $j' < r^{k'}$ such that

$$f(\theta^{k'}(a)_{j'}) \neq f(\theta^{k'}(b)_{j'}).$$

Recall the definition of M_0, k_0, j_0 and f via (5) and (6). We know that

$$f_0(\theta^{k_0+k'}(a)_{j'r^{k_0}+j_0}) = f_0(\theta^{k_0}(\theta^{k'}(a)_{j'})_{j_0}) \neq f_0(\theta^{k_0}(\theta^{k'}(b)_{j'})_{j_0}) = f_0(\theta^{k_0+k'}(b)_{j'r^{k_0}+j_0}).$$

As f_0 is a bijection between $\{1, \ldots, c\}$ and M_0 we know that

$$a' := \theta^{k_0+k'}(a)_{j'r^{k_0}+j_0} \neq \theta^{k_0+k'}(b)_{j'r^{k_0}+j_0} =: b'$$

and also $a', b' \in M_0$. We know by the definition of minimal sets that, for all k and $j < r^k$,

$$\theta^k(a')_j \neq \theta^k(b')_j.$$

Thus we find that

$$\lim_{k\to\infty} d^*(\theta^k(a), \theta^k(b)) \geq \frac{1}{r^{k'+k_0}}. \qquad \square$$

COROLLARY 42. *Let θ be a reduced and primitive substitution of constant length r. Then every τ' in the centralizer of $G(\theta)$ defines an automorphism $\tau \in \mathrm{Aut}(X_\theta, \sigma)$.*

Proof. As θ is reduced this implies that θ is strongly injective, so we know that any automorphism with κ-value 0 has radius 0. We need to define $\tau : \mathcal{A} \to \mathcal{A}$ that satisfies the conditions of Theorem 37. Fix any f defined as in (6). Let $a \in M \in \mathcal{X}$. Then we define

$$\tau(a) = f|_M^{-1}(\tau'(f(a))), \tag{10}$$

and we need to make sure that this definition is independent of the choice of M.

Let $a \in M_1 \cap M_2$. Then we set

$$b_i = f|_{M_i}^{-1}(\tau'(f(a))).$$

So we have $b_i \in M_i$ and

$$f(b_i) = \tau'(f(a)).$$

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We compute

$$\begin{aligned}\tau'(f(\theta^k(a)_j)) &= \tau' \circ (\sigma^{(k)}_{M_i,j})^{-1}(f(a)) \\ &= (\sigma^{(k)}_{M_i,j})^{-1}(\tau'(f(a))) \\ &= (\sigma^{(k)}_{M_i,j})^{-1}(f(b_i)) \\ &= f(\theta^k(b_i)_j),\end{aligned}$$

for $i = 1, 2$. Thus we know that $f(\theta^k(b_1)_j) = f(\theta^k(b_2)_j)$ for all $k,\ j < r^k$. By Lemma 41, we have

$$\lim_{k\to\infty} d^*(\theta^k(b_1), \theta^k(b_2)) = 0.$$

As θ is reduced we conclude that $b_1 = b_2$, so that (10) is well defined and one checks easily that τ fulfils the conditions of Theorem 37. □

LEMMA 43. *Let θ be a primitive length-r substitution and let θ' be the reduced substitution of θ. Then $G(\theta) = G(\theta')$.*

Proof. Let $\mathcal{A}'$ be the alphabet on which θ' is defined. By Lemma 41, we can define f on $\mathcal{A}'$ as in particular it says that if $a \sim b$ then $f(a) = f(b)$, so we can simply define $f([a]) := f(a)$. This shows that $|\theta'^k([M])_j| = c(\theta)$ for all k and $j < r^k$, which means that $c(\theta') \geq c(\theta)$. As θ' is constructed by identifying letters of $\mathcal{A}$, we have $c(\theta) \geq c(\theta')$ and so $c(\theta) = c(\theta')$. It suffices to show that

$$\{\sigma_{M,j} : M \in \mathcal{X}(\theta),\ j < r\} = \{\sigma'_{M,j} : M \in \mathcal{X}(\theta'),\ j < r\}. \tag{11}$$

First we find directly that for all $M \in \mathcal{X}(\theta)$ we have $[M] \in \mathcal{X}(\theta')$, and as $f([a]) = f(a)$ we know that

$$\{\sigma_{M,j} : M \in \mathcal{X}(\theta),\ j < r\} \subseteq \{\sigma'_{M,j} : M \in \mathcal{X}(\theta'),\ j < r\}.$$

Furthermore, for any $M' \in \mathcal{X}(\theta')$ there exist k' and $j' < r^{k'}$ such that $\theta'^{k'}(\mathcal{A}')_{j'} = M'$. We recall that as in (5), we have $M_0 = \theta^{k_0}(\mathcal{A})_{j_0}$ and, therefore, we find

$$\mathcal{X}(\theta) \ni \theta^{k_0+k'}(\mathcal{A})_{j_0 r^{k'}+j'} = \theta^{k'}(M_0)_{j'} \subseteq M'.$$

Thus we can write $M' = [M]$ for some $M \in \mathcal{X}(\theta)$, which shows that

$$\{\sigma_{M,j} : M \in \mathcal{X}(\theta),\ j < r\} \supseteq \{\sigma'_{M,j} : M \in \mathcal{X}(\theta'),\ j < r\}$$

and in total (11), which finishes the proof. □

Recall that if θ is primitive, then there is a unique probability measure μ such that (X_θ, σ, μ) is a measure-preserving dynamical system. The *essential centralizer* of (X_θ, σ, μ) is the set of all measure-preserving maps of X_θ which commute with σ, quotiented out by $\{\sigma^n : n \in \mathbb{Z}\}$. Combining Corollary 42 and Lemma 43, we deduce the following result.

THEOREM 44. *Let θ be a primitive length-r substitution. Then $G(\theta)$ corresponds to those elements of the essential centralizer of (X_θ, σ, μ) that are a root of the identity.*

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5. *Rational orders in an automorphism group, twisted substitutions and k-compressions*
Theorem 32 tells us that we can assume that our substitution is strongly injective, and henceforth we do this. Theorem 33 tells us that all automorphisms whose κ-value is 0 arise from alphabet permutations. In this section we investigate the structure of substitutions θ which possess other automorphisms, namely, automorphisms with a non-integer κ-value. We show in Theorem 46 that these automorphisms indicate that θ has a structure coming from some other constant-length substitution.

Given a sequence $(x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, we can consider what Cobham [**4**] calls its *k-compression*, which is the sequence $(y_n)_{n\in\mathbb{Z}} \in (\mathcal{A}^k)^{\mathbb{Z}}$ where $y_n := x_{nk} \dots x_{(n+1)k-1}$. The operation of k-compression also applies to words whose length is a multiple of k. If $\theta : \mathcal{A} \to \mathcal{A}^r$, and $\alpha_1 \dots \alpha_k \in \mathcal{A}^k$, then the k-compression of the word $\theta(\alpha_1 \dots \alpha_k)$ is a word in $\mathcal{A}^k$ of length r; this defines a substitution $\theta^{(k)} : \mathcal{B} \to \mathcal{B}^r$, where $\mathcal{B} \subset \mathcal{A}^k$ is the set of words $\mathcal{B} = \{u[mk, (m+1)k-1] : m \geq 0\}$ where $(u_n)_{n\in\mathbb{N}}$ is a fixed point of θ. Cobham studied the notion of k-compression and showed that if x is r-automatic and primitive, then, for any k, its k-compression is also r-automatic and primitive, generated by the substitution $\theta^{(k)}$. We call $\theta^{(k)}$ (respectively, $(X_{\theta^{(k)}}, \sigma)$) the *k-compression of* θ (respectively, (X_θ, σ)). It is straightforward to see that (X_θ, σ^k) is topologically conjugate to $(X_{\theta^{(k)}}, \sigma)$.

Definition 45. Let $\theta : \mathcal{A} \to \mathcal{A}^r$ be a substitution and let $\tau \in \mathrm{Aut}(X_\theta, \sigma)$ with radius 0 satisfying $\kappa(\tau) = 0$ and $\tau^r = \tau$. Define the substitution $\theta_\tau : \mathcal{A} \to \mathcal{A}^r$ by

$$\theta_\tau(\alpha)_j = \tau^j(\theta(\alpha)_j).$$

We call $(X_{\theta_\tau}, \sigma)$ the *τ-twist of* (X_θ, σ).

Recall $\kappa_\theta : \mathrm{Aut}(X_\theta, \sigma) \to \mathbb{Z}_r$, the group homomorphism defined in Theorem 6. By [**6**, Theorem 3.16 and Lemma 3.2], $\kappa(\mathrm{Aut}(X_\theta, \sigma))$ is always cyclic and generated by $1/k$ for some k.

THEOREM 46. *Let θ be a strongly injective primitive length-r substitution of height* 1 *such that X_θ is infinite. If $\kappa_\theta(\mathrm{Aut}(X_\theta, \sigma)) = \langle 1/k \rangle$ then (X_θ, σ) is topologically conjugate to a twist of a k-compression of (X_η, σ), with η a primitive length-r substitution.*

It is often the case that $\tau = \mathrm{Id}$, and in this case one can conclude that (X_θ, σ) is topologically conjugate to a k-compression. However, situations where τ is non-trivial do occur; see Example 50.

To prove Theorem 46 we prove some preliminary lemmas.

LEMMA 47. *Let θ be a primitive length-r substitution of height* 1 *such that X_θ is infinite. If $1/k \in \kappa(\mathrm{Aut}(X_\theta, \sigma))$ then, by taking a power of θ if necessary, there are an automorphism Φ with $\kappa(\Phi) = 1/k$ satisfying*

$$\Phi^r \circ \theta = \theta \circ \Phi, \tag{12}$$

and an automorphism $\tau \in \mathrm{Aut}(X_\theta, \sigma)$ such that $\tau^r = \tau$ and

$$\Phi^k = \sigma \circ \tau. \tag{13}$$

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Proof. Let Φ be any automorphism with $\kappa(\Phi) = 1/k$. Then we have

$$\pi(\Phi^r \circ \theta(x)) = \frac{r}{k} + \pi(\theta(x)) = r\left(\frac{1}{k} + \pi(x)\right),$$

and since $1/k \in \mathbb{Z}_r$, this implies that $\Phi^{-1} \circ \theta^{-1} \circ \Phi^r \circ \theta : X_\theta \to X_\theta$ is a well-defined bijection which also commutes with σ, that is, Φ is an automorphism. We also have $\kappa(\Phi^{-1} \circ \theta^{-1} \circ \Phi^r \circ \theta) = 0$. We claim, by taking a power of θ if necessary, that there exists an automorphism Φ such that $\kappa(\Phi) = 1/k$ and Equation (12) holds.

If κ is injective, then $\kappa(\Phi^{-1} \circ \theta^{-1} \circ \Phi^r \circ \theta) = 0$ implies that $\Phi^{-1} \circ \theta^{-1} \circ \Phi^r \circ \theta = \mathrm{Id}$ and we are done. If κ is not injective, consider the map $f : \kappa^{-1}(1/k) \to \kappa^{-1}(1/k)$ given by $\Phi \mapsto \theta^{-1}\Phi^r\theta$. For some $n \geq 1$, and for some Φ, we have $f^n(\Phi) = \Phi$. One verifies that in this case Φ satisfies $\theta^n\Phi = \Phi^{r^n}\theta^n$. As $X_{\theta^n} = X_\theta$, we can replace θ by θ^n if necessary, to assume that we started with a θ for which there exists an automorphism Φ satisfying (12).

For this Φ satisfying (12), we have that $\Phi^k = \sigma \circ \tau$ for some $\tau \in \ker \kappa$. It remains to show that $\tau^r = \tau$. Considering the power $\theta^{c!p}$ instead of θ, we have, by Equation (2), that $\sigma^N \circ \Phi \circ \theta = \theta \circ \Phi$ for some integer N. Using Equation (12), we have $\sigma^N \circ \Phi \circ \theta = \Phi^r \circ \theta$, so that $\sigma^N\Phi = \Phi^r$. Using Equation (13), we have

$$\sigma^r\tau^r = \Phi^{kr} = \sigma^{Nk}\Phi^k = \sigma^{Nk+1}\tau.$$

Both sides being automorphisms with necessarily equal κ-values, this implies that $Nk + 1 = r$, and the result follows. □

LEMMA 48. *Let θ be a strongly injective primitive length-r substitution of height* 1 *such that X_θ is infinite. If $\Phi \in (X_\theta, \sigma)$ satisfies*

$$\kappa(\Phi) = \frac{1}{k}, \quad \Phi^r \circ \theta = \theta \circ \Phi \text{ and } \Phi^k = \sigma \circ \tau,$$

for some automorphism $\tau \in \mathrm{Aut}(X_\theta, \sigma)$ such that $\tau^r = \tau$, then there exists a substitution η such that (X_θ, Φ) is topologically conjugate to (X_η, σ).

Proof. By Theorem 33, τ has radius 0. If θ is defined on the alphabet $\mathcal{A}$, define a map $\pi : X_\theta \to \mathcal{A}^{\mathbb{Z}}$ by $\pi(x)_n = (\Phi^n x)_0$. We assume that $k > 1$, so, by an analogue of Proposition 8, Φ has left radius 0 and right radius at most 1. We notice that

$$\pi \circ \Phi = \sigma \circ \pi, \tag{14}$$

where we use σ to denote the shift on either of our spaces. Let $\bar{X} := \pi(X_\theta)$; then $\bar{X}$ is closed and shift invariant. We have

$$\pi(x)_{nk} = (\Phi^{nk}(x))_0 = (\sigma^n\tau^n(x))_0 = (\tau^n(x))_n = \tau^n(x_n),$$

and since τ must be injective on letters, this implies that π is injective.

Since π conjugates (X_θ, Φ) to $(\bar{X}, \sigma)$, and $\Phi^k = \sigma \circ \tau$, it follows that π conjugates $(X_\theta, \sigma \circ \tau)$ to $(\bar{X}, \sigma^k)$. Since (X_θ, σ) is minimal and τ is an automorphism, $(X_\theta, \sigma \circ \tau)$ is minimal and so $(\bar{X}, \sigma^k)$ is minimal; hence $(\bar{X}, \sigma)$ is also minimal.

We define a map on $\bar{X}$ by $\eta = \pi \circ \theta \circ \pi^{-1}$. We have

$$\eta \circ \sigma = \pi \circ \theta \circ \pi^{-1} \circ \sigma = \pi \circ \theta \circ \Phi \circ \pi^{-1} = \pi \circ \Phi^r \circ \theta \circ \pi^{-1} = \sigma^r \circ \eta, \tag{15}$$

where the second, third and fourth equalities are by (14), (12) and (14), respectively.

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We claim that η is a length-r substitution. By (15), it suffices to check that if $z, z' \in \bar{X}$ satisfy $z_0 = z'_0$, then $\eta(z)_i = \eta(z')_i$ for $i = 0, \ldots, r-1$. Suppose then that z and $z' \in \bar{X}$ satisfy $z_0 = z'_0$. Let $x = \pi^{-1}z$ and $x' = \pi^{-1}z'$. Notice that if $z = \pi(x)$, then $z_0 = x_0$, so that $x_0 = x'_0$. Hence $\theta(x)_i = \theta(x')_i$ for $0 \le i < r$. As Φ has right radius at most 1 and left radius 0, we have that $\eta(z)_i = (\pi \circ \theta(x))_i = (\pi \circ \theta(x'))_i = \eta(z')_i$ for $0 \le i \le r-1$. It follows that η is a length-r substitution map, as required.

If $x \in X_\theta$ is a fixed point of θ, then $\pi(x) \in \bar{X}$ is a fixed point of η. Furthermore, by minimality, its orbit is dense in $\bar{X}$, so that $\bar{X}$ is the substitution space of η. By (14), π conjugates $\Phi\colon X_\theta \to X_\theta$ to $\sigma : X_\eta \to X_\eta$. The result follows. □

Proof of Theorem 46. We continue with the notation of Lemmas 47 and 48. By Lemma 47, there exist automorphisms Φ and τ, with κ values $1/k$ and 0 respectively, and satisfying Equations (12) and (13). Also $\tau^r = \tau$. By Lemma 48, $(X_\theta, \Phi^k) = (X_\theta, \sigma \circ \tau)$ is conjugate to the k-compression $(X_{\eta^{(k)}}, \sigma)$ of the substitution shift (X_η, σ). Let $\bar{\pi} : (X_\theta, \sigma \circ \tau) \to (X_{\eta^{(k)}}, \sigma)$ be a conjugacy. Since τ^{-1} is an automorphism of $(X_\theta, \sigma \circ \tau)$, this implies that (X_θ, σ) is conjugate to $(X_{\eta^{(k)}}, \sigma \circ \bar{\tau})$ for some automorphism $\bar{\tau}$ of $(X_{\eta^{(k)}}, \sigma)$ with $\bar{\tau} \in \ker \kappa_{\eta^{(k)}}$.

We claim that $\bar{\tau}$ has radius 0. To see this it is sufficient to show that η is strongly injective, as this will imply that $\eta^{(k)}$ is also strongly injective and we are done by Theorem 33.

Note that the column number of η is at most the column number of θ. If θ has column number c, we can assume, by considering the rotation in the proof of Theorem 32 if necessary, where we replace θ by the appropriate k-shifted 2-sliding block representation, that θ has exactly c right-infinite fixed points and c left-infinite fixed points. The map Φ has left radius 0 and right radius 1, and each right-infinite θ-fixed point is mapped to a right-infinite η-fixed point. Furthermore, any right-infinite η-fixed point is the image under π of a right-infinite θ-fixed point. Thus η has c right-fixed points and hence its column number must also be c. Now since $\kappa(\Phi^{-1}) = -\kappa(\Phi)$, Φ^{-1} has left radius 1 and right radius 0. Working with Φ^{-1} (i.e. moving up this spacetime diagram, instead of down it), we see as above that η has exactly c left fixed points. By the discussion in the proof of Theorem 32, η is strongly injective.

Thus $\bar{\tau}$ has radius 0. Suppose that $\eta^{(k)}$ is defined on an alphabet $\mathcal{B}$. Define a map $p : X_{\eta^{(k)}} \to \mathcal{B}^{\mathbb{Z}}$ by $p(x)_n = ((\sigma \circ \bar{\tau})^n(x))_0$. One verifies that if $x \in X_{\eta^{(k)}}$ is a fixed point for $\eta^{(k)}$, then $p(x)_n = \bar{\tau}^n(x_n)$, that is, $p(x)$ is a $(\eta^{(k)})_{\bar{\tau}}$-fixed point. Now p is an injection, since $\bar{\tau}$ is an automorphism with radius 0. Furthermore, $p \circ \sigma \circ \bar{\tau} = \sigma \circ p$, that is, p is a conjugacy between $(X_{\eta^{(k)}}, \sigma \circ \bar{\tau})$ and $(X_{(\eta^{(k)})_{\bar{\tau}}}, \sigma)$. We have shown that (X_θ, σ) is conjugate to $(X_{\eta^{(k)}}, \sigma \circ \bar{\tau})$, which is conjugate to $(X_{(\eta^{(k)})_{\bar{\tau}}}, \sigma)$, that is, that (X_θ, σ) is conjugate to the twist of a k-compression. □

We specialize to the following case, which includes the case where $\mathrm{Aut}(X_\theta, \sigma)$ is cyclic, generated by an automorphism whose κ-value is $1/k$. This happens, for instance, if θ has a coincidence, and more generally if κ is injective and θ has height coprime to k [**6**, Corollary 3.7]. Then we can restate our result as follows.

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TABLE 3. Local rule table for ϕ, Example 50. The entry in the row labelled α and in the column labelled β is $\phi(\alpha\beta)$. The map ϕ need only be defined on the language of θ.

	a	b	c	d	e	f	g	h
a						b		b
b	f		f	f	h	h		
c		a						
d		g				e		
e		f		f				
f	d	d			b		b	b
g						e		
h		a				c		

COROLLARY 49. *Let θ be a primitive length-r substitution such that X_θ is infinite. If $\Phi \in \mathrm{Aut}(X_\theta, \sigma)$ and $\Phi^k = \sigma$ then (X_θ, σ) is topologically conjugate to the k-compression of a substitution shift (X_η, σ).*

The following example shows that the twist in Theorem 46 is indeed necessary.

Example 50. Consider the substitution

$$\begin{aligned}
\theta(a) &= afbcb,\\
\theta(b) &= ahfgf,\\
\theta(c) &= afbdf,\\
\theta(d) &= ahfhb,\\
\theta(e) &= ebfgf,\\
\theta(f) &= edbcb,\\
\theta(g) &= ebfhb,\\
\theta(h) &= edbdf.
\end{aligned}$$

It has an automorphism Φ with κ-value $1/3$, and whose local rule ϕ is given in Table 3.

Let

$$\begin{aligned}
\eta(a) &= abfbh,\\
\eta(b) &= abfdg,\\
\eta(c) &= abfbh,\\
\eta(d) &= abfdg,\\
\eta(e) &= efbfd,\\
\eta(f) &= efbhc,\\
\eta(g) &= efbfd,\\
\eta(h) &= efbhc.
\end{aligned}$$

Now η has an order-2 automorphism τ defined by the permutation $(ae)(bf)(cg)(dh)$, (X_θ, Φ) is conjugate to (X_η, σ), and (X_θ, σ) is conjugate to a twist (by τ) of the 3-compression of (X_η, σ).

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In general, what can one say about the κ-kernel of the substitution θ as compared to that of the substitution η in the statement of Theorem 46? The following examples tell us that, in general, there is no elegant statement. In this next example, we were still able to cling to a sense that we could set things up so that the κ-kernels of the relevant maps are isomorphic.

Example 51. Consider the substitution θ on the alphabet $\mathcal{A}$:

$$\begin{aligned}
\theta(A) &= ABC,\\
\theta(B) &= AAD,\\
\theta(C) &= EFG,\\
\theta(D) &= HIB,\\
\theta(E) &= EJJ,\\
\theta(F) &= FCB,\\
\theta(G) &= KKC,\\
\theta(H) &= HLL,\\
\theta(I) &= IDG,\\
\theta(J) &= FEL,\\
\theta(K) &= KGD,\\
\theta(L) &= IHJ.
\end{aligned}$$

Then θ has one non-trivial automorphism τ belonging to $\ker \kappa_\theta$, defined by the permutation $(AK)(BG)(CD)(EH)(FI)(JL)$. It also has two square roots of the shift, one of which has local rule in Table 4 and which we call Φ, the other of which is $\tau \circ \Phi$. Now using Φ, and working through the proof of Lemma 48, we can show that (X_θ, Φ) is topologically conjugate to (X_η, σ), where

$$\begin{aligned}
\theta(a) &= acb,\\
\theta(a) &= bda,\\
\theta(c) &= cef,\\
\theta(d) &= dfe,\\
\theta(e) &= ead,\\
\theta(f) &= fbc.
\end{aligned}$$

So θ is the twist of a compression of η, but $\ker \kappa_\eta$ is isomorphic to S_3, so here it is not the case that the κ-kernels of the two maps θ and η are the same. However, note that if we use $\tau \circ \Phi$ in Lemma 48, we obtain that $(X_\theta, \tau \circ \Phi)$ is topologically conjugate to (X_{η_τ}, σ), and that $\ker \kappa_{\eta_\tau}$ *is* isomorphic to $\ker \kappa_\theta$.

However, whatever hope we clung to with Example 51 was dashed with this next example.

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TABLE 4. The local rule for the square root Φ of the shift in Example 51.

	A	B	C	D	E	F	G	H	I	J	K	L
A	J	J		F								
B			I		I						L	
C	H	H						D				
D					C		E				E	
E						B				B		A
F			C		C		E					
G	J			F				F				
H									G	K		G
I		H		D				D				
J						K			G	K		
K			I				L				L	
L						B			A			A

Example 52. Consider

$$\begin{aligned}\theta(A) &= ABC,\\ \theta(B) &= DEC,\\ \theta(C) &= AFG,\\ \theta(D) &= DGB,\\ \theta(E) &= HAC,\\ \theta(F) &= DHG,\\ \theta(G) &= HIB,\\ \theta(H) &= HCG,\\ \theta(I) &= ADB.\end{aligned}$$

This is a two-compression of (X_{η_1}, σ) where

$$\begin{aligned}\eta_1(a) &= abb,\\ \eta_1(b) &= bac,\\ \eta_1(c) &= cca.\end{aligned}$$

Note that θ has a non-trivial kernel automorphism τ given by the permutation $(ADH)(BGC)(EIF)$, and so, since it is a 2-compression, it also has three automorphisms with κ value $1/2$, Φ_1: $\Phi_2 = \tau \circ \Phi_1$ and $\Phi_3 = \tau^2 \circ \Phi_1$. Note also that τ and Φ_1 do not commute. Finally, $\Phi_1^2 = \Phi_2^2 = \Phi_3^2 = \sigma$, and, using Φ_i in Lemma 48, we get that (X_θ, Φ_i) is conjugate to a 2-compression of (X_{η_i}, σ) ,where

$$\begin{aligned}\eta_1(a) &= abb \quad \eta_2(a) = acb \quad \eta_3(a) = aab,\\ \eta_1(b) &= bac \quad \eta_2(b) = bbc \quad \eta_3(b) = bcc,\\ \eta_1(c) &= cca \quad \eta_2(c) = cba \quad \eta_3(c) = cba,\end{aligned}$$

all three of whose automorphism groups are trivial.

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Remark 53. What causes this breakdown? We have seen in Theorem 37 that the automorphisms of η with κ-value 0 can be characterized in terms of the minimal sets and the group

$$G(\eta) := \langle \sigma_{M,j} : M \in \mathcal{X},\ 0 \le j \le r-1 \rangle.$$

Let us now consider the k-compression $\eta^{(k)}$ of η for some $k \mid r-1$. One can then show that

$$G(\eta^{(k)}) = \langle \sigma_{M,j} : M \in \mathcal{X},\ 0 \le j \le r-1,\ k \mid j \rangle.$$

Thus, we might be in the situation where $G(\eta^{(k)})$ is strictly smaller than $G(\eta)$, or one where the centralizer of $G(\eta^{(k)})$ is strictly larger then the centralizer of $G(\eta)$.

This is precisely what happens in this last example, where $G(\eta_i) = S_3$, so that η_i has a trivial automorphism group, but $G(\eta_i^{(2)}) = A_3 = \{\mathrm{Id}, (123), (132)\} = C(G(\eta_i^{(2)}))$, so that we have non-trivial automorphisms with κ value 0.

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