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Some problems of integral geometry on Anosov manifolds

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Abstract. In this paper we prove that on an Anosov manifold the space of symmetric m-tensor fields of vanishing energy is finite dimensional modulo the space of potential tensor fields for an arbitrary m and coincides with the latter for $m = 0$ and $m = 1$. For $m = 2$ this question relates to the spectral rigidity problem.

1. *Posing the problem and formulating the results*

The authors are grateful to B. Kleiner for the following remark that has been used as a starting point for the present work: there is an analogy between the modified horizontal derivative constructed in [**Sh1, Sh3**] and the asymptotic Jacobi fields first introduced by Hopf [**H**] and then developed by many authors [**G, Es, Eb**]. Indeed, the absence of conjugate points plays a crucial role in both cases, and the same Riccati equation is used.

Let (M, g) be a closed (i.e. compact without boundary) Riemannian manifold. We are interested in questions like the following: to what extent is a smooth function $f \in C^\infty(M)$ determined by its integrals over all closed geodesics?

Of course, the question is sensible only in the case when (M, g) has sufficiently many closed geodesics. Therefore, the question was first investigated for symmetric Riemannian manifolds. The simplest of such manifolds is the sphere with the standard metric. Funk [**F**] proved that the even part of a function on the two-dimensional sphere is determined by integrals over great circles. This work is traditionally considered the start of integral geometry. Later the problem was investigated for some other symmetric spaces [**Hl**].

Anosov manifolds constitute another wide class of manifolds with sufficiently many closed geodesics. A closed Riemannian manifold is said to be an *Anosov manifold* if its geodesic flow is of Anosov type [**An**]. For such a manifold, the set of closed geodesics is dense in the set of all geodesics. The class of Anosov manifolds is wider than the class of closed negatively curved manifolds. On the other hand, the class of Anosov manifolds is opposite in some sense to the class of homogeneous Riemannian manifolds because the isometry group of an Anosov manifold is discrete [**Es**].

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THEOREM 1.1. *Let (M, g) be an Anosov manifold. If a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic then f must itself be zero.*

A closed Riemannian manifold is said to have a simple length spectrum if there do not exist two different closed geodesics such that the ratio of their lengths is a rational number. This is a generic condition.

COROLLARY 1.2. *Let (M, g) be an Anosov manifold with simple length spectrum, and let $\Delta : C^\infty(M) \to C^\infty(M)$ be the corresponding Laplace–Beltrami operator. If real functions $q_1, q_2 \in C^\infty(M)$ are such that the operators $\Delta + q_1$ and $\Delta + q_2$ have coincident spectra, then $q_1 \equiv q_2$.*

Theorem 1.1 implies Corollary 1.2 by the same arguments as in [**GK**].

A similar result is valid for 1-forms.

THEOREM 1.3. *Let (M, g) be an Anosov manifold, and let f be a smooth 1-form on M. If f integrates to zero around every closed geodesic, then f is an exact form.*

We now consider the corresponding question for symmetric tensor fields of arbitrary degree. Let τ'_M be the cotangent bundle of a closed Riemannian manifold M, and $C^\infty(S^m\tau'_M)$ be the space of smooth symmetric covariant tensor fields of degree m. For such a field $f \in C^\infty(S^m\tau'_M)$ and a closed geodesic $\gamma : [a, b] \to M$, we may consider the integral

$$If(\gamma) = \oint_\gamma \langle f, \dot\gamma^m \rangle \, dt = \int_a^b f_{i_1 \dots i_m}(\gamma(t))\dot\gamma^{i_1}(t) \dots \dot\gamma^{i_m}(t) \, dt. \tag{1.1}$$

The integrand of (1.1) is written with use made of local coordinates. Nevertheless, it is evidently invariant, i.e. independent of the choice of coordinates. Let $Z(S^m\tau'_M)$ denote the subspace of $C^\infty(S^m\tau'_M)$ that consists of all fields such that $If(\gamma) = 0$ for every closed geodesic γ. For $m > 0$ this subspace is not zero as is seen from the following argument. Introduce the first-order differential operator

$$d = \sigma\nabla : C^\infty(S^{m-1}\tau'_M) \to C^\infty(S^m\tau'_M), \tag{1.2}$$

where ∇ is the covariant derivative and σ is the symmetrization. This operator is called the *inner differentiation*. A tensor field f is said to be a *potential field* if it can be represented in the form $f = dv$ for some $v \in C^\infty(S^{m-1}\tau'_M)$. Let $P(S^m\tau'_M)$ denote the space of all potential fields. If $f = dv$, then the integrand of (1.1) equals $d(v_{i_1 \dots i_{m-1}}(\gamma(t))\dot\gamma^{i_1}(t) \dots \dot\gamma^{i_{m-1}}(t))/dt$. Therefore we have the inclusion

$$P(S^m\tau'_M) \subset Z(S^m\tau'_M). \tag{1.3}$$

The principal question of the integral geometry of tensor fields is formulated as follows: For what classes of closed Riemannian manifolds and for what values of m is the inclusion (1.3) in fact equality? For instance, Theorems 1.1 and 1.3 say that (1.3) is equality for an Anosov manifold in the cases when $m = 0$ and $m = 1$. Equality is also proved in the case of an arbitrary m for some symmetric spaces [**M1, M2**].

The following result is proved in [**CSh**].

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THEOREM 1.4. *For an Anosov manifold of non-positive sectional curvature, the inclusion (1.3) is an equality for all m.*

In [**CSh**] this theorem is formulated for negatively curved manifolds. Nevertheless, it is only the non-positivity of curvature and the Anosov type of the geodesic flow that are used in the proof.

Without constraining the curvature, we have the following.

THEOREM 1.5. *For an Anosov manifold, the inclusion (1.3) has finite codimension for every m.*

In the case of $m = 2$, the question under consideration relates to the spectral rigidity problem. Let us recall the corresponding definitions. A smooth one-parameter family $g^\tau (-\varepsilon < \tau < \varepsilon)$ of metrics on a manifold M is said to be a *deformation* of a metric g if $g^0 = g$. Such a family is called an *isospectral deformation* if the spectrum of the Laplace–Beltrami operator Δ^τ of the metric g^τ is independent of τ. A deformation g^τ is said to be *trivial* if there exists a family φ^τ of diffeomorphisms of M such that $g^\tau = (\varphi^\tau)^* g$. A manifold (M, g) is said to be *spectrally rigid* if it does not admit a non-trivial isospectral deformation.

The following result is proved in [**GK**].

THEOREM 1.6. *An Anosov manifold is spectrally rigid if the inclusion (1.3) is an equality for $m = 2$.*

The theorem is formulated in [**GK**] for negatively curved manifolds. Nevertheless, the same proof applies to Anosov manifolds. Thus, Theorem 1.5 asserts that the space of isospectral deformations of an Anosov manifold has a finite dimension modulo trivial deformations.

Our method of proving Theorem 1.5 is a combination of the methods used in [**Sh2**] and [**CSh**]. The principal idea of [**Sh2**] consists in constructing a semibasic tensor field $a = (a^{ij})$ such that the curvature tensor of the modified horizontal derivative $\overset{a}{\nabla}$ meets the curve $\overset{a}{R}_{ijkl}\xi^i\xi^k = 0$. In this case the Pestov identity does not involve the term dependent on curvature. The stable and unstable distributions enable us to construct a similar modifying tensor field $a = (a^{ij})$ in the case of an Anosov manifold. Although this tensor field is not smooth, we can approximate it by a smooth field without losing control over the corresponding curvature tensor in the process of approximation. Theorems 1.1 and 1.3 are proved likewise.

We conclude the introduction by posing the following problem.

Problem 1.7. Does there exist an Anosov manifold such that the inclusion (1.3) is not an equality for some value of m?

2. *Reduction of Theorem 1.5 to an estimate for the kinetic equation*

Given a Riemannian manifold (M, g), denote its tangent space by TM. The points of TM are viewed as the pairs (x, ξ), where $x \in M$ and $\xi \in T_x M$. Let ΩM be the manifold of unit tangent vectors. Denote the geodesic flow by $G^t : TM \to TM$ and

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denote the corresponding vector field on TM by H. At the points of ΩM, the field H is tangent to ΩM. Therefore, H can be considered as a first-order differential operator $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$.

The operator

$$-\delta : C^\infty(S^m\tau'_M) \to C^\infty(S^{m-1}\tau'_M)$$

is defined as the formal dual of the operator d. We call δ the *divergence operator*.

We deduce Theorem 1.5 from the following.

LEMMA 2.1. *Let (M, g) be an Anosov manifold. If a function $u \in C^\infty(\Omega M)$ and a tensor field $f \in C^\infty(S^m\tau'_M)$ are connected by the kinetic equation*

$$Hu(x,\xi) = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}, \tag{2.1}$$

then the estimate

$$\|u\|^2_{H^1(\Omega M)} \le C(\|u\|^2_{L_2(\Omega M)} + \|\delta f\|_{L_2(S^{m-1}\tau'_M)} \cdot \|u\|_{L_2(\Omega M)}) \tag{2.2}$$

holds with some constant C independent of u and f.

The proof of Lemma 2.1 is accomplished in §6. Now, we demonstrate how Theorem 1.5 follows from this lemma.

Proof of Theorem 1.5. If the theorem is not true, there exists an infinite sequence of tensor fields $z_k \in Z(S^m\tau'_M)$ $(k = 1, 2, \dots)$ which is linearly independent mod$(P(S^m\tau'_M))$. Applying Theorem 2.2 of [**CSh**], we decompose every field z_k into potential and solenoidal parts

$$z_k = y_k + dv_k, \quad \delta y_k = 0. \tag{2.3}$$

Then the sequence $y_k \in Z(S^m\tau'_M)$ is linearly independent. Applying the smooth version of the Livshits Theorem [**LMM**], we find functions $w_k \in C^\infty(\Omega M)$ satisfying the kinetic equation

$$Hw_k = \langle y_k(x), \xi^m\rangle. \tag{2.4}$$

The sequence w_k is linearly independent, otherwise (2.4) would imply linear dependence of y_k. Orthogonalizing the sequence w_k, we construct a new sequence of functions $u_k \in C^\infty(\Omega M)$ such that

$$\|u_k\|_{L_2(\Omega M)} = 1, \quad (u_k, u_l)_{L_2(\Omega M)} = 0 \quad \text{for } k \ne l, \tag{2.5}$$

and every u_k is a linear combination of $w_1, \dots, w_k$. Equation (2.4) implies that

$$Hu_k = \langle f_k(x), \xi^m\rangle,$$

where $f_k \in C^\infty(S^m\tau'_M)$ is a linear combination of $y_1, \dots, y_k$. By (2.3) we have

$$\delta f_k = 0. \tag{2.6}$$

In view of (2.5) and (2.6), the estimate (2.2) acquires the following form for the pair u_k, f_k:

$$\|u_k\|_{H^1(\Omega M)} \le C\|u_k\|_{L_2(\Omega M)} = C.$$

In other words, the sequence u_k is bounded in $H^1(\Omega M)$. Since the imbedding $H^1(\Omega M) \subset L_2(\Omega M)$ is compact, the sequence u_k has a subsequence converging in $L_2(\Omega M)$. But this contradicts (2.5). □

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3. *The modifying tensor fields* $\overset{s}{\alpha}$ *and* $\overset{u}{\alpha}$

Given a manifold M and an open set $U \subset TM$, denote by $C(\beta_s^r M; U)$ the space of continuous semibasic tensor fields of degree (r, s) over U. The definition of a semibasic tensor field is given in [**Sh1**, Ch. 3], see also [**PSh**].

LEMMA 3.1. *Let (M, g) be an n-dimensional Anosov manifold. There exist continuous semibasic tensor fields $\overset{s}{\alpha} = (\overset{s}{\alpha}_{ij}(x,\xi)) \in C(\beta_2^0 M; T^0 M)$ and $\overset{u}{\alpha} = (\overset{u}{\alpha}_{ij}(x,\xi)) \in C(\beta_2^0 M; T^0 M)$ defined on $T^0 M = \{(x,\xi) \in TM \mid \xi \neq 0\}$ such that:*

(1) *the fields are symmetric,*

$$\overset{s}{\alpha}_{ij} = \overset{s}{\alpha}_{ji}, \quad \overset{u}{\alpha}_{ij} = \overset{u}{\alpha}_{ji},$$

and orthogonal to the vector ξ,

$$\xi^i \overset{s}{\alpha}_{ij}(x,\xi) = 0, \quad \xi^i \overset{u}{\alpha}_{ij}(x,\xi) = 0;$$

(2) *they are positively homogeneous of degree one in ξ,*

$$\overset{s}{\alpha}(x, t\xi) = t\overset{s}{\alpha}(x,\xi), \quad \overset{u}{\alpha}(x, t\xi) = t\overset{u}{\alpha}(x,\xi) \quad \text{for } t > 0;$$

(3) *the rank of the matrix $(\overset{s}{\alpha}_{ij} - \overset{u}{\alpha}_{ij})$ equals $n-1$ at every point $(x,\xi) \in T^0 M$;*

(4) *along every geodesic $\gamma : \mathbb{R} \to M$, the fields $\overset{s}{\alpha}{}^i_j(t) = (g^{ik}\overset{s}{\alpha}_{kj})(\gamma(t), \dot\gamma(t))$ and $\overset{u}{\alpha}{}^i_j(t) = (g^{ik}\overset{u}{\alpha}_{kj})(\gamma(t), \dot\gamma(t))$ are smooth and satisfy the Riccati equation*

$$\alpha' + \alpha^2 + R = 0, \tag{3.1}$$

where the prime denotes the covariant derivative and $R = R(t)$ is the curvature operator, $R^i_j = R^i_{kjl}\dot\gamma^k\dot\gamma^l$.

Before proving the lemma we recall some notions concerning the geodesic flow and Jacobi fields.

Let $\tau_M = (TM, \pi, M)$ be the tangent bundle of a Riemannian manifold (M, g). For every point $(x,\xi) \in TM$, we have the canonical isomorphism

$$T_{(x,\xi)}(TM) \cong T_x M \oplus T_\xi M, \quad v \mapsto (d\pi(v), Kv),$$

where $K : TTM \to TM$ is the *connection mapping*. The subspaces of $T_{(x,\xi)}(TM)$ corresponding to the summands on the right-hand side are called the *horizontal* and *vertical spaces*, respectively. This isomorphism defines the *Sasaki metric* on TM:

$$\langle v, w\rangle = \langle d\pi(v), d\pi(w)\rangle + \langle Kv, Kw\rangle.$$

The metric g determines the isomorphism of the tangent and cotangent bundles. The standard symplectic structure of the cotangent bundle, being shifted to TM by means of this isomorphism, is defined by the 2-form

$$\omega(v, w) = \langle d\pi(v), Kw\rangle - \langle d\pi(w), Kv\rangle.$$

The tangent space of the manifold ΩM at a point $(x,\xi) \in \Omega M$ can be distinguished by the equality

$$T_{(x,\xi)}(\Omega M) = \{v \in T_{(x,\xi)}(TM) \mid \langle Kv, \xi\rangle = 0\}. \tag{3.2}$$

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Let H be the geodesic vector field on TM generating the geodesic flow $G^t : TM \to TM$. Note that H is horizontal, $KH = 0$. For every vector $v \in T_{(x,\xi)}(TM)$, the vector field $Y_v(t) = d\pi \circ dG^t(v)$ is a Jacobi vector field along the geodesic $\gamma(t) = \exp_x t\xi$ with the covariant derivative $Y_v'(t) = K \circ dG^t(v)$.

Now, let (M, g) be an Anosov manifold. Given $(x, \xi) \in \Omega M$, two $(n-1)$-dimensional subspaces $X_s(x, \xi)$ and $X_u(x, \xi)$ of $T_{(x,\xi)}(\Omega M)$ are defined which are called the *stable* and *unstable spaces*, respectively. We will use the following properties of these spaces which follow from Proposition 1.7 and Theorem 3.2 of [**Eb**].

(i) The distributions $(x, \xi) \mapsto X_s(x, \xi)$ and $(x, \xi) \mapsto X_u(x, \xi)$ are continuous and invariant with respect to the geodesic flow, i.e.

$$dG^t(X_s(x, \xi)) = X_s(G^t(x, \xi)), \quad dG^t(X_u(x, \xi)) = X_u(G^t(x, \xi)).$$

(ii) Each of the spaces $X_s(x, \xi)$ and $X_u(x, \xi)$ is orthogonal to the vector $H(x, \xi)$. The space $T_{(x,\xi)}(\Omega M)$ splits into the (not orthogonal) direct sum of the three subspaces

$$T_{(x,\xi)}(\Omega M) = X_s(x, \xi) \oplus X_u(x, \xi) \oplus \{H(x, \xi)\}.$$

(iii) The restriction of the mapping $d\pi$ to each of the spaces $X_s(x, \xi)$ and $X_u(x, \xi)$ is an isomorphism onto $N_{(x,\xi)} = \{\eta \in T_xM \mid \langle \xi, \eta \rangle = 0\}$.

(iv) $X_s(x, \xi)$ and $X_u(x, \xi)$ are Lagrangian spaces, i.e. $\omega(v, w) = 0$ for $v, w \in X_s(x, \xi)$(respectively $X_u(x, \xi)$).

Proof of Lemma 3.1. Given $(x, \xi) \in \Omega M$, we define two endomorphisms $b_s(x, \xi)$ and $b_u(x, \xi)$ of the space $N_{(x,\xi)}$ as the following compositions:

$$b_s(x, \xi) : N_{(x,\xi)} \xrightarrow{(d\pi)^{-1}} X_s(x, \xi) \xrightarrow{K} N_{(x,\xi)},$$

$$b_u(x, \xi) : N_{(x,\xi)} \xrightarrow{(d\pi)^{-1}} X_u(x, \xi) \xrightarrow{K} N_{(x,\xi)}.$$

Note that $Kv \in N_{(x,\xi)}$ for every $v \in T_{(x,\xi)}(\Omega M)$ because of (3.2). This definition is equivalent to the following rule which is more comfortable for usage: two vectors $\eta, \zeta \in N_{(x,\xi)}$ are connected by the equality $b_s(x, \xi)\eta = \zeta$ if and only if there exists $v \in X_s(x, \xi)$ such that $d\pi(v) = \eta$ and $Kv = \zeta$. A similar rule holds for the operator $b_u(x, \xi)$.

The operators $b_s(x, \xi)$ and $b_u(x, \xi)$ possess the following properties.

(1) The operators $b_s(x, \xi)$ and $b_u(x, \xi)$ depend continuously on $(x, \xi) \in \Omega M$.

(2) The operators b_s and b_u are self-dual. Indeed, assume that $\eta_i \in N_{(x,\xi)}$ and $\zeta_i = b_s(x, \xi)\eta_i$ $(i = 1, 2)$. Then there are $v_i \in X_s(x, \xi)$ such that $d\pi(v_i) = \eta_i$ and $Kv_i = \zeta_i$. Therefore,

$$\begin{aligned}\langle b_s\eta_1, \eta_2\rangle - \langle \eta_1, b_s\eta_2\rangle &= \langle \zeta_1, \eta_2\rangle - \langle \eta_1, \zeta_2\rangle \\ &= \langle Kv_1, d\pi(v_2)\rangle - \langle d\pi(v_1), Kv_2\rangle = \omega(v_1, v_2) = 0\end{aligned}$$

since $X_s(x, \xi)$ is a Lagrangian space.

(3) The operator $b_s - b_u$ is non-degenerate. Indeed, let $b_s\eta = b_u\eta$ for some vector $\eta \in N_{(x,\xi)}$. There exist vectors $v \in X_s$ and $w \in X_u$ such that

$$d\pi(v) = \eta = d\pi(w), \quad Kv = b_s\eta = b_u\eta = Kw.$$

These relations imply that $v = w \in X_s \cap X_u = 0$. Consequently, $v = w = 0$ and $\eta = 0$.

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(4) Along every unit speed geodesic $\gamma : \mathbb{R} \to M$, each of the operator functions

$$b_s(t) = b_s(\gamma(t), \dot\gamma(t)) : N_{(\gamma(t),\dot\gamma(t))} \to N_{(\gamma(t),\dot\gamma(t))},$$
$$b_u(t) = b_u(\gamma(t), \dot\gamma(t)) : N_{(\gamma(t),\dot\gamma(t))} \to N_{(\gamma(t),\dot\gamma(t))}$$

satisfies the Riccati equation

$$b' + b^2 + R = 0. \tag{3.3}$$

Indeed, fix a geodesic γ and denote $N_t = N_{(\gamma(t),\dot\gamma(t))}$. Define the operator function $D(t) : N_t \to N_t$ as a solution to the Jacobi equation

$$D'' + RD = 0 \tag{3.4}$$

satisfying the initial conditions

$$D(0) = E \text{ (the identity)}, \quad D'(0) = b_s(0).$$

Establish the equality

$$D'(t) = b_s(t)D(t). \tag{3.5}$$

To this end, fix a vector $\eta \in N_0$ and denote by $\eta(t) \in N_t$ the result of parallel translation of η along γ. Then the vector function $Y(t) = D(t)\eta(t)$ is a Jacobi vector field along γ satisfying the initial conditions:

$$Y(0) = \eta, \quad Y'(0) = b_s(0)\eta.$$

On the other hand, if a vector $v \in X_s(\gamma(0), \dot\gamma(0))$ is such that $d\pi(v) = \eta$, then

$$D(t)\eta(t) = Y(t) = d\pi \circ dG^t(v), \quad D'(t)\eta(t) = Y'(t) = K \circ dG^t(v).$$

The vector $v_t = dG^t(v)$ belongs to $X_s(\gamma(t), \dot\gamma(t))$ because the distribution X_s is invariant with respect to the geodesic flow. The preceding equalities can be rewritten as follows:

$$D(t)\eta(t) = d\pi(v_t), \quad D'(t)\eta(t) = Kv_t, \quad v_t \in X_s(\gamma(t), \dot\gamma(t)).$$

These relations imply that

$$D'(t)\eta(t) = b_s(t)D(t)\eta(t),$$

which is equivalent to (3.5) because η is an arbitrary vector.

The operator $D(t)$ is non-degenerate for $|t|$ small enough, and (3.5) can be rewritten as follows: $b_s(t) = D'(t)D^{-1}(t)$. From this and the Jacobi equation (3.4), we see that $b_s(t)$ satisfies the Riccati equation (3.3) at least for $|t|$ small enough. Since the Riccati equation is invariant with respect to the shift $t \mapsto t + t_0$, it is satisfied for all t.

Given $(x, \xi) \in \Omega M$, we define the operators

$$\overset{s}{\alpha}(x, \xi) : T_xM \to T_xM, \quad \overset{u}{\alpha}(x, \xi) : T_xM \to T_xM$$

by the equalities

$$\overset{s}{\alpha}(x, \xi)|_{N_{(x,\xi)}} = b_s(x, \xi), \quad \overset{s}{\alpha}(x, \xi)\xi = 0;$$
$$\overset{u}{\alpha}(x, \xi)|_{N_{(x,\xi)}} = b_u(x, \xi), \quad \overset{u}{\alpha}(x, \xi)\xi = 0.$$

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We then extend the functions $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ to T^0M so that they are positively homogeneous in ξ:

$$\overset{s}{\alpha}(x, t\xi) = t\overset{s}{\alpha}(x, \xi), \quad \overset{u}{\alpha}(x, t\xi) = t\overset{u}{\alpha}(x, \xi) \quad \text{for} \quad t > 0.$$

We have thus constructed the semibasic tensor fields $\overset{s}{\alpha} = (\overset{s}{\alpha}{}^i_j(x, \xi)) \in C(\beta_1^1 M; T^0M)$ and $\overset{u}{\alpha} = (\overset{u}{\alpha}{}^i_j(x, \xi)) \in C(\beta_1^1 M; T^0M)$. The above-listed properties of b_s and b_u imply that the semibasic tensor fields $\overset{s}{\alpha}_{ij} = g_{ik}\overset{s}{\alpha}{}^k_j$ and $\overset{u}{\alpha}_{ij} = g_{ik}\overset{u}{\alpha}{}^k_j$ satisfy all the requirements in Lemma 3.1. □

4. *Smoothing the tensor fields $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$*

We would like to define two modified horizontal derivatives along the lines of the scheme of [**Sh1**, §8.2] (see also [**Sh3**]) with the modifying tensors $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ that are constructed in the previous section. Unfortunately, the fields $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ are merely continuous but not smooth. To construct a modified horizontal derivative, we need at least C^2-smoothness of the modifying tensor, since the definition of the curvature tensor presumes existence of second-order derivatives. Therefore, we have to smooth the tensor fields $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$. We will choose the approximate smooth tensor fields in such a way that they satisfy all statements of Lemma 3.1 with the following exception: the Riccati equation (3.1) is satisfied approximately.

First of all we discuss some questions of smoothing the sections of a vector bundle.

Let $\pi : E \to N$ be a smooth m-dimensional vector bundle over a compact manifold N. Choose a finite atlas $\{U_\iota, \varphi_\iota\}_{\iota=1}^I$ of the manifold N, a partition of unity $\{\mu_\iota\}_{\iota=1}^I$ subordinate to the atlas, and local trivializations $(e_1^\iota, \dots, e_m^\iota)$ of the bundle above U_ι (this means that $e_1^\iota, \dots, e_m^\iota \in C^\infty(E; U_\iota)$ and the vectors $e_1^\iota(x), \dots, e_m^\iota(x)$ constitute a basis for the fiber E_x above every point $x \in U_\iota$). Each section $\Gamma \in C^\infty(E)$ can be uniquely represented in the form

$$\Gamma(x) = \sum_{\kappa=1}^m \Gamma_\iota^\kappa(x) e_\kappa^\iota(x), \quad x \in U_\iota. \tag{4.1}$$

For $0 \le s < \infty$, let $C^s(E)$ be the space of sections Γ such that $(\mu_\iota \Gamma_\iota^\kappa) \circ \varphi_\iota^{-1} \in C^s(\mathbb{R}^n)$. The norm in the space is defined by

$$\|\Gamma\|_{C^s(E)} = \sum_{\iota=1}^I \sum_{\kappa=1}^m \|(\mu_\iota \Gamma_\iota^\kappa) \circ \varphi_\iota^{-1}\|_{C^s(\mathbb{R}^n)}.$$

Up to equivalence, the norm is independent of the choice of the atlas, the partition of unity, and trivializations.

Let $H \in C^\infty(\tau_N)$ be a smooth vector field on N. Choosing a connection on E, we can define the derivative $H\Gamma$ of a section Γ with respect to H. A section $\Gamma \in C(E)$ is said to be *differentiable with respect to H* if the derivative $H\Gamma$ exists and belongs to $C(E)$. Then the norm $\|H\Gamma\|_{C(E)}$ is well defined and is independent, up to equivalence, of the choice of the connection.

LEMMA 4.1. *Let $H \in C^\infty(\tau_N)$ be a smooth vector field on a compact manifold N which does not vanish at any point and let E be a smooth vector bundle over N. Fix a C-norm*

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on $C(E)$ and a connection on E. For a section $\Gamma \in C(E)$ differentiable with respect to H and for $\varepsilon > 0$, there exists a smooth section $\widetilde{\Gamma} \in C^\infty(E)$ such that

$$\|\Gamma - \widetilde{\Gamma}\|_{C(E)} < \varepsilon, \quad \|H(\Gamma - \widetilde{\Gamma})\|_{C(E)} < \varepsilon.$$

Proof. A chart (U, φ) of the manifold N is called a *straightening chart for the vector field* H if $\varphi_* H$ coincides with the coordinate vector field $\partial/\partial x^1$ on the range $\varphi(U) \subset \mathbb{R}^n$. There exists a finite atlas $\{U_\iota, \varphi_\iota\}_{\iota=1}^I$ of straightening charts [**Ar**]. Choose a partition of unity which is subordinate to the atlas and trivializations of E above U_ι. Given a section $\Gamma \in C(E)$ differentiable with respect to H, write down representation (4.1) for Γ. Then, for all ι and κ, $\widetilde{\Gamma}_\iota^\kappa = (\mu_\iota \Gamma^\kappa) \circ \varphi_\iota^{-1}$ is a continuous compactly supported function on $\mathbb{R}^n$ with the continuous derivative $\partial \widetilde{\Gamma}_\iota^\kappa / \partial x^1$.

Fix a non-negative function $\lambda \in C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \lambda \, dx = 1$ and put $\lambda_\delta(x) = \lambda(x/\delta)/\delta^n$ for $\delta > 0$. For all indices ι and κ, the function $\overset{\delta}{\Gamma}{}_\iota^\kappa = \widetilde{\Gamma}_\iota^\kappa * \lambda_\delta$ is C^∞-smooth on $\mathbb{R}^n$, and $\operatorname{supp} \overset{\delta}{\Gamma}{}_\iota^\kappa \subset \varphi_\iota(U_\iota)$ for δ small enough. The differences $\overset{\delta}{\Gamma}{}_\iota^\kappa - \widetilde{\Gamma}_\iota^\kappa$ and $\partial(\overset{\delta}{\Gamma}{}_\iota^\kappa - \widetilde{\Gamma}_\iota^\kappa)/\partial x^1$ tend to zero uniformly on $\mathbb{R}^n$ as $\delta \to 0$. Therefore, the section

$$\widetilde{\Gamma} = \sum_{\iota=1}^{I} \sum_{\kappa=1}^{m} \mu_\iota (\overset{\delta}{\Gamma}{}_\iota^\kappa \circ \varphi_\iota) e_\kappa^\iota$$

possesses all the desired properties for a sufficiently small $\delta > 0$. The lemma is proved. □

Given an Anosov manifold (M, g), denote by $\beta_2^0 M|_{\Omega M}$ the restriction of the bundle $\beta_2^0 M$ to the compact manifold $N = \Omega M$ and let E be the subbundle of $\beta_2^0 M|_{\Omega M}$ that consists of the semibasic tensors $f = (f_{ij})$ satisfying the conditions $f_{ij} = f_{ji}$ and $\xi^i f_{ij} = 0$. Let $\overset{s}{\alpha}, \overset{u}{\alpha} \in C(E)$ be the sections constructed in Lemma 3.1 and let H be the geodesic vector field on ΩM. Applying Lemma 4.1 to $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ and extending the resultant smooth fields to $T^0 M$ by homogeneity, we arrive at the following.

LEMMA 4.2. *Let (M, g) be an n-dimensional Anosov manifold. For every $\varepsilon > 0$, there exist smooth semibasic tensor fields $\overset{s}{a} = (\overset{s}{a}_{ij}(x, \xi)) \in C^\infty(\beta_2^0 M; T^0 M)$ and $\overset{u}{a} = (\overset{u}{a}_{ij}(x, \xi)) \in C^\infty(\beta_2^0 M; T^0 M)$ such that:*

(1) *the fields are symmetric,*

$$\overset{s}{a}_{ij} = \overset{s}{a}_{ji}, \quad \overset{u}{a}_{ij} = \overset{u}{a}_{ji},$$

and orthogonal to the vector ξ,

$$\xi^i \overset{s}{a}_{ij}(x, \xi) = 0, \quad \xi^i \overset{u}{a}_{ij}(x, \xi) = 0;$$

(2) *they are positively homogeneous of degree one in ξ,*

$$\overset{s}{a}(x, t\xi) = t \overset{s}{a}(x, \xi), \quad \overset{u}{a}(x, t\xi) = t \overset{u}{a}(x, \xi) \quad \text{for } t > 0;$$

(3) *the rank of the matrix $(\overset{s}{a}_{ij} - \overset{u}{a}_{ij})$ equals $n - 1$ at every point $(x, \xi) \in T^0 M$;*

(4) *along every unit speed geodesic $\gamma : \mathbb{R} \to M$ of M, the fields*

$$\overset{s}{a}{}_j^i(t) = (g^{ik} \overset{s}{a}_{kj})(\gamma(t), \dot\gamma(t)) \quad \text{and} \quad \overset{u}{a}{}_j^i(t) = (g^{ik} \overset{u}{a}_{kj})(\gamma(t), \dot\gamma(t))$$

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satisfy the inequality

$$|a' + a^2 + R| < \varepsilon$$

for all $t \in \mathbb{R}$.

5. *The modified horizontal derivatives*

Here we will use the vertical and horizontal derivatives $\overset{v}{\nabla}, \overset{h}{\nabla} : C^\infty(\beta_s^r M) \to C^\infty(\beta_{s+1}^r M)$ which are defined for a Riemannian manifold (M, g). The definition of the derivatives is presented in [**Sh1**, Ch. 3] (see also [**PSh**]). The metric g gives rise to the canonical isomorphisms $\beta_s^r M \cong \beta_0^{r+s} M \cong \beta_{r+s}^0 M$, and we will sometimes identify these bundles.

We now recall the construction of a modified horizontal derivative as exposed in [**Sh1**, Ch. 8] and in [**Sh3**]. A *modifying tensor* is a semibasic tensor field $a = (a^{ij}(x, \xi)) \in C^\infty(\beta_0^2 M)$ which is: symmetric, $a^{ij} = a^{ji}$; orthogonal to ξ, $\xi_i a^{ij}(x, \xi) = 0$; and positively homogeneous of degree one in ξ. Given such a field, the *modified horizontal derivative* is the first-order differential operator

$$\overset{a}{\nabla} : C^\infty(\beta_s^r M) \to C^\infty(\beta_s^{r+1} M)$$

defined in local coordinates as follows. For $u = (u^{i_1 \dots i_r}_{j_1 \dots j_s}) \in C^\infty(\beta_s^r M)$,

$$\begin{aligned}\overset{a}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} &= \overset{h}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} + a^{kp} \overset{v}{\nabla}_p u^{i_1 \dots i_r}_{j_1 \dots j_s} - \sum_{\alpha=1}^{r} \overset{v}{\nabla}_p a^{k i_\alpha} \cdot u^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} \\ &\quad + \sum_{\alpha=1}^{s} \overset{v}{\nabla}_{j_\alpha} a^{kp} \cdot u^{i_1 \dots i_r}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_s}. \end{aligned} \tag{5.1}$$

In contrast to the ordinary horizontal derivative, the metric tensor g is not parallel with respect to $\overset{a}{\nabla}$. This is just the reason why k is an upper index in definition (5.1). Nevertheless, we will also use the operator $\overset{a}{\nabla}_k = g_{kl} \overset{a}{\nabla}{}^l$.

Since the modifying tensor is orthogonal to ξ, the geodesic vector field, regarded as a differential operator $H : C^\infty(TM) \to C^\infty(TM)$, can be expressed in terms of $\overset{a}{\nabla}$ as follows:

$$Hu = \xi_k \overset{a}{\nabla}{}^k u \quad (u \in C^\infty(TM)). \tag{5.2}$$

The curvature tensor $\overset{a}{R} = (\overset{a}{R}_{ijkl}(x, \xi)) \in C^\infty(\beta_4^0 M)$ corresponding to $\overset{a}{\nabla}$ is defined by

$$\begin{aligned}\overset{a}{R}_{ijkl} &= R_{ijkl} + \overset{h}{\nabla}_l \overset{v}{\nabla}_j a_{ik} - \overset{h}{\nabla}_k \overset{v}{\nabla}_j a_{il} + a_{lp} \overset{v}{\nabla}{}^p \overset{v}{\nabla}_j a_{ik} - a_{kp} \overset{v}{\nabla}{}^p \overset{v}{\nabla}_j a_{il} \\ &\quad + \overset{v}{\nabla}{}^p a_{ik} \cdot \overset{v}{\nabla}_j a_{lp} - \overset{v}{\nabla}{}^p a_{il} \cdot \overset{v}{\nabla}_j a_{kp},\end{aligned}$$

where (R_{ijkl}) is the Riemannian curvature tensor. This formula implies that

$$\overset{a}{R}_{ijkl} \xi^j \xi^l = (Ha)_{ik} + a_i^l a_{kl} + R_{ijkl} \xi^j \xi^l. \tag{5.3}$$

Now, suppose that (M, g) is an Anosov manifold and let $\overset{s}{a}$ ($\overset{u}{a}$) be the semibasic tensor field constructed in Lemma 4.2. Setting $a = \overset{s}{a}$ ($\overset{u}{a}$) in (5.1), we define the

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modified horizontal derivative on $C^\infty(\beta_s^r M; T^0M)$ which will be denoted by $\overset{s}{\nabla}$ ($\overset{u}{\nabla}$). The corresponding curvature tensor will be denoted by $\overset{s}{R}$ ($\overset{u}{R}$).

Comparing (5.3) with item (4) of Lemma 4.2, we arrive at the following important conclusion: for all semibasic vector fields $v, w \in C^\infty(\beta_0^1 M)$, the estimates

$$|\overset{s}{R}_{ijkl}\xi^i\xi^k v^j w^l| < \varepsilon|v||w|, \quad |\overset{u}{R}_{ijkl}\xi^i\xi^k v^j w^l| < \varepsilon|v||w| \tag{5.4}$$

hold uniformly on ΩM.

Item (3) of Lemma 4.2 implies that the collection

$$\overset{s}{\nabla}{}^i u, \quad \overset{u}{\nabla}{}^i u \ (i = 1, \dots, n), \quad \xi^k \overset{v}{\nabla}_k u$$

is a full family of derivatives of a function $u \in C^\infty(T^0M)$, i.e. every first-order derivative of u is a linear combination of the members of this collection. This observation is specified by the following.

LEMMA 5.1. *For every function $u \in C^\infty(T^0M)$, the estimates*

$$|\overset{v}{\nabla} u - \langle \xi, \overset{v}{\nabla} u\rangle \xi| \le C(|\overset{s}{\nabla} u| + |\overset{u}{\nabla} u|), \tag{5.5}$$

$$|\overset{h}{\nabla} u| \le C(|\overset{s}{\nabla} u| + |\overset{u}{\nabla} u|) \tag{5.6}$$

hold on ΩM with some constant C independent of u.

Proof. It suffices to consider a function u whose support lies in the domain $U \subset T^0M$ of a local coordinate system. The semibasic vector field $y = \overset{v}{\nabla} u - \langle \xi, \overset{v}{\nabla} u\rangle \xi$ is orthogonal to ξ on ΩM. By the definition of the modified derivatives, we have

$$\overset{s}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{s}{a}{}^{ij} \overset{v}{\nabla}_j u, \quad \overset{u}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{u}{a}{}^{ij} \overset{v}{\nabla}_j u. \tag{5.7}$$

Substituting $\overset{v}{\nabla}_j u = y_j + \langle \xi, \overset{v}{\nabla} u\rangle \xi_j$ into these equalities and using orthogonality of $\overset{s}{a}$ and $\overset{u}{a}$ to ξ, we obtain

$$\overset{s}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{s}{a}{}^{ij} y_j, \quad \overset{u}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{u}{a}{}^{ij} y_j.$$

This implies that

$$(\overset{s}{a}{}^{ij} - \overset{u}{a}{}^{ij}) y_j = \overset{s}{\nabla}{}^i u - \overset{u}{\nabla}{}^i u. \tag{5.8}$$

By items (1) and (3) of Lemma 4.2, the operator $\overset{s}{a} - \overset{u}{a}$ is an automorphism of the space $N_{(x,\xi)} = \{\eta \in T_x M \mid \langle \xi, \eta\rangle = 0\}$. The right-hand side of (5.8) belongs to this space because of $\langle \xi, \overset{s}{\nabla} u - \overset{u}{\nabla} u\rangle = \xi_i \overset{s}{\nabla}{}^i u - \xi_i \overset{u}{\nabla}{}^i u = Hu - Hu = 0$. Therefore, equation (5.8) has a unique solution

$$y_i = c_{ij}(\overset{s}{\nabla}{}^j u - \overset{u}{\nabla}{}^j u)$$

with some coefficients c_{ij} that are smooth functions in U and are independent of u. From this we obtain (5.5). The estimate (5.6) follows from (5.5) and (5.7). The lemma is proved. □

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For a semibasic tensor field $u \in C^\infty(\beta_s^r M; T^0M)$, we will use the notation

$$\|u\|_{L_2}^2 = \int_{\Omega M} |u(x,\xi)|^2 \, d\Sigma(x,\xi), \quad \|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|\overset{h}{\nabla} u\|_{L_2}^2 + \|\overset{v}{\nabla} u\|_{L_2}^2,$$

where $d\Sigma(x,\xi) = d\omega_x(\xi) \wedge dV^n(x)$ is the symplectic volume form on ΩM. Lemma 5.1 has the following corollary.

COROLLARY 5.2. *The two norms* $\|u\|_{H^1}$ *and* $(\|\overset{s}{\nabla} u\|_{L_2}^2 + \|\overset{u}{\nabla} u\|_{L_2}^2 + \|u\|_{L_2}^2)^{1/2}$ *are equivalent to one another on the space of functions* $u(x,\xi) \in C^\infty(T^0M)$ *which are positively homogeneous in* ξ *with the same degree of homogeneity.*

Indeed, $\langle \xi, \overset{v}{\nabla} u \rangle = \lambda u$ if u is homogeneous of degree λ.

Remark. The numbers ε and C involved in (5.4)–(5.6) are independent of one another in the following sense: ε can be chosen arbitrarily small with a fixed value of C. Indeed, C is determined by the continuous fields $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ of Lemma 3.1, while ε depends on the degree of approximation of $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ by smooth fields.

6. *Proofs of Lemma 2.1 and Theorems 1.1 and 1.3*

Proof of Lemma 2.1. Let a tensor field $f \in C^\infty(S^m \tau_M')$ and a function $u \in C^\infty(\Omega M)$ satisfy (2.1). Extend u to a function on T^0M which is positively homogeneous of degree $m-1$ in ξ. Then equation (2.1) holds on T^0M. By (5.2), this equation can be rewritten in the form

$$Hu = \xi_k \overset{s}{\nabla}{}^k u = \xi_k \overset{u}{\nabla}{}^k u = \langle f(x), \xi^m \rangle. \tag{6.1}$$

For the modified horizontal derivative $\overset{s}{\nabla}$, the following *Pestov Identity* is valid (see [**Sh1**, §8.2] or [**Sh3**]):

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu \rangle = |\overset{s}{\nabla} u|^2 + \overset{s}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u, \tag{6.2}$$

where

$$v_i = \xi_i \overset{s}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j u - \xi_j \overset{v}{\nabla}_i u \cdot \overset{s}{\nabla}{}^j u, \tag{6.3}$$

$$w^i = \xi_j \overset{s}{\nabla}{}^i u \cdot \overset{s}{\nabla}{}^j u. \tag{6.4}$$

We transform the left-hand side of (6.2). From (6.1) we obtain

$$\overset{v}{\nabla}_k (Hu) = \overset{v}{\nabla}_k (\langle f(x), \xi^m \rangle) = m f_{k i_2 \dots i_m} \xi^{i_2} \dots \xi^{i_m}.$$

Since $\overset{s}{\nabla}{}^i \xi_j = 0$, this implies

$$\begin{aligned} 2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu \rangle &= 2m \overset{s}{\nabla}{}^i u \cdot f_{i i_2 \dots i_m} \xi^{i_2} \dots \xi^{i_m} \\ &= \overset{s}{\nabla}{}^i (2mu f_{i i_2 \dots i_m} \xi^{i_2} \dots \xi^{i_m}) - 2mu \overset{s}{\nabla}{}^i (f_{i i_2 \dots i_m} \xi^{i_2} \dots \xi^{i_m}). \end{aligned}$$

Introducing the notation

$$\widetilde{v}_i = 2mu f_{i i_2 \dots i_m} \xi^{i_2} \dots \xi^{i_m},$$

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we obtain

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \overset{s}{\nabla}{}^i \widetilde{v}_i - 2mu \overset{s}{\nabla}{}^i (f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}). \tag{6.5}$$

By the definition of the modified derivative, we have

$$\begin{aligned}
&\overset{s}{\nabla}{}^i (f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) \\
&\quad = \overset{h}{\nabla}{}^i (f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) + \overset{s}{a}{}^{ip}\overset{v}{\nabla}_p (f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) + \overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot (f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) \\
&\quad = (\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} + (m-1)\overset{s}{a}{}^{ip} f_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} + \overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.
\end{aligned}$$

Substituting this expression into (6.5), we obtain

$$\begin{aligned}
2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \overset{s}{\nabla}{}^i \widetilde{v}_i &- 2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} \\
&- 2m(m-1)\overset{s}{a}{}^{ip} u f_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} - 2m\overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot u f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.
\end{aligned}$$

Using this formula, we transform the Pestov Identity (6.2) to the form

$$\begin{aligned}
|\overset{s}{\nabla} u|^2 = \overset{s}{\nabla}{}^i(\widetilde{v}_i - v_i) - \overset{v}{\nabla}_i w^i &- 2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} \\
&- 2m(m-1)\overset{s}{a}{}^{ip} u f_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} \\
&- 2m\overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot u f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} + \overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u.
\end{aligned} \tag{6.6}$$

In view of item (4) of Lemma 4.2 and (5.4), the last three terms on the right-hand side of (6.6) can be estimated at a point $(x,\xi)\in\Omega M$ as follows:

$$|\overset{s}{a}{}^{ip} u f_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m}| \le C|u||f|, \quad |\overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot u f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}| \le C|u||f|,$$

$$|\overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u| < \varepsilon|\overset{v}{\nabla} u|^2$$

with $C = \|\overset{s}{a}\|_{C^1}$. Now, (6.6) gives the inequality

$$|\overset{s}{\nabla} u|^2 \le \overset{s}{\nabla}{}^i(\widetilde{v}_i - v_i) - \overset{v}{\nabla}_i w^i + C|u||f| + 2m|u||\delta f| + \varepsilon|\overset{v}{\nabla} u|^2, \tag{6.7}$$

which is valid on ΩM with some new constant C.

We integrate (6.7) over ΩM and transform the integrals of divergent terms by the Gauss–Ostrogradskiĭ formulas (Theorem 3.6.3 and formula (8.2.30) of [**Sh1**]). The integral of $\overset{s}{\nabla}{}^i(\widetilde{v}_i - v_i)$ vanishes because ΩM is a closed manifold. The integral of the second term is non-positive; indeed,

$$-\int_{\Omega M}\overset{v}{\nabla}_i w^i\, d\Sigma = -(n+2m-2)\int_{\Omega M}\langle w,\xi\rangle\, d\Sigma = -(n+2m-2)\int_{\Omega M}|Hu|^2\, d\Sigma \le 0,$$

where we have used the equality $\langle w,\xi\rangle = |Hu|^2$ which follows from (6.4). After integration, (6.7) thus gives us the inequality

$$\|\overset{s}{\nabla} u\|^2_{L_2} \le C(\|u\|_{L_2}\cdot\|f\|_{L_2} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2}) + \varepsilon\|u\|^2_{H^1}. \tag{6.8}$$

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We can repeat our arguments with $\overset{s}{\nabla}$ substituted for $\overset{u}{\nabla}$. In this way we obtain the following analog of (6.8):

$$\|\overset{u}{\nabla} u\|_{L_2}^2 \le C(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2}) + \varepsilon \|u\|_{H^1}^2. \tag{6.9}$$

With the help of Corollary 5.2, inequalities (6.8) and (6.9) give

$$\|u\|_{H^1}^2 \le CC'(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2}) + C'\|u\|_{L_2}^2 + C'\varepsilon \|u\|_{H^1}^2. \tag{6.10}$$

As we have emphasized, the number ε can be chosen arbitrarily small with some fixed value of C'. In particular, we may assume that $C'\varepsilon < 1$, and then (6.10) can be rewritten in the form

$$\|u\|_{H^1}^2 \le C(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2} + \|u\|_{L_2}^2) \tag{6.11}$$

with some new constant C independent of u.

The kinetic equation $Hu = \xi^i \overset{h}{\nabla}_i u = \langle f, \xi^m \rangle$ implies the estimate $\|f\|_{L_2} \le C\|\overset{h}{\nabla} u\|_{L_2} \le C\|u\|_{H^1}$ that allows us to rewrite (6.11) in the form

$$\|u\|_{H^1}^2 \le C(\|u\|_{L_2} \cdot \|u\|_{H^1} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2} + \|u\|_{L_2}^2). \tag{6.12}$$

Considering (6.12) as a quadratic inequality in $x = \|u\|_{H^1}$, one can easily see that it implies (2.2) with another constant C. The lemma is proved. □

Proof of Theorem 1.1. Assume that a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic. Applying the Livshits Theorem, we obtain a function $u \in C^\infty(\Omega M)$ that satisfies the kinetic equation

$$Hu(x, \xi) = f(x) \tag{6.13}$$

on ΩM. Extend u to T^0M so that $u(x, t\xi) = t^{-1}u(x, \xi)$ for $t > 0$. Then (6.13) is valid on T^0M. The left-hand side of the Pestov identity (6.2) vanishes identically in our case. After integration over ΩM, the identity gives

$$\|\overset{s}{\nabla} u\|_{L_2}^2 + (n-2)\|Hu\|_{L_2}^2 = \int_{\Omega M} \overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u \, d\Sigma. \tag{6.14}$$

Put $y^i = \overset{v}{\nabla}^i u - \langle \xi, \overset{v}{\nabla} u \rangle \xi^i$. Using symmetries of the curvature tensor, we obtain

$$\overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u = \overset{s}{R}_{ijkl} \xi^i \xi^k y^j y^l.$$

By (5.4) and (5.5), this implies the estimate

$$|\overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u| \le \varepsilon |y|^2 \le C\varepsilon(|\overset{s}{\nabla} u|^2 + |\overset{u}{\nabla} u|^2).$$

Combining this estimate with (6.14) yields

$$\|\overset{s}{\nabla} u\|_{L_2}^2 \le C\varepsilon(\|\overset{s}{\nabla} u\|_{L_2}^2 + \|\overset{u}{\nabla} u\|_{L_2}^2). \tag{6.15}$$

In the same way we derive the similar estimate

$$\|\overset{u}{\nabla} u\|_{L_2}^2 \le C\varepsilon(\|\overset{s}{\nabla} u\|_{L_2}^2 + \|\overset{u}{\nabla} u\|_{L_2}^2). \tag{6.16}$$

On choosing ε so as to have $C\varepsilon < 1/2$, inequalities (6.15) and (6.16) imply that $\overset{s}{\nabla} u = \overset{u}{\nabla} u = 0$. Consequently, $f = \xi_i \overset{s}{\nabla}{}^i u = 0$. The theorem is proved. □

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Proof of Theorem 1.3. In this case the kinetic equation looks like

$$Hu(x,\xi) = f_i(x)\xi^i, \tag{6.17}$$

and $\overset{v}{\nabla} Hu = f$. Therefore, the Pestov identity (6.2) has the form

$$2\langle \overset{s}{\nabla} u,\, f\rangle = |\overset{s}{\nabla} u|^2 + \overset{s}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u.$$

After integration over ΩM this gives

$$\|\overset{s}{\nabla} u\|^2_{L_2} - 2(\overset{s}{\nabla} u,\, f)_{L_2} + n\|Hu\|^2_{L_2} = \int_{\Omega M} \overset{s}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \, d\Sigma. \tag{6.18}$$

From (6.17) we obtain

$$\|Hu\|^2_{L_2} = \int_{\Omega M} f_i(x) f_j(x)\xi^i\xi^j \, d\Sigma(x,\xi)$$

$$= \int_M f_i(x) f_j(x) \left[\int_{\Omega_x M} \xi^i\xi^j \, d\omega_x(\xi) \right] dV^n(x) = \frac{1}{n}\|f\|^2_{L_2}.$$

Now, (6.18) takes the form

$$\|\overset{s}{\nabla} u - f\|^2_{L_2} = \int_{\Omega M} \overset{s}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \, d\Sigma.$$

Estimating the integral on the right-hand side by (5.4), we obtain

$$\|\overset{s}{\nabla} u - f\|^2_{L_2} \le \varepsilon C \|\overset{v}{\nabla} u\|^2_{L_2}. \tag{6.19}$$

Repeating our arguments with $\overset{u}{\nabla}$ substituted for $\overset{s}{\nabla}$, we derive the similar inequality

$$\|\overset{u}{\nabla} u - f\|^2_{L_2} \le \varepsilon C \|\overset{v}{\nabla} u\|^2_{L_2}. \tag{6.20}$$

Now, let ε in (6.19) and (6.20) tend to zero. The vector fields f and $\overset{v}{\nabla} u$ are independent of ε, and so is the constant C. The fields $\overset{s}{\nabla} u$ and $\overset{u}{\nabla} u$ tend respectively to

$$\overset{s}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{s}{\alpha}{}_i^p \overset{v}{\nabla}_p u, \quad \overset{u}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{u}{\alpha}{}_i^p \overset{v}{\nabla}_p u, \tag{6.21}$$

where $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ are the continuous tensors of Lemma 3.1. Passing to the limit in (6.19) and (6.20) as $\varepsilon \to 0$, we obtain

$$\overset{s}{\nabla}_i u = f_i = \overset{u}{\nabla}_i u. \tag{6.22}$$

Now, equalities (6.21) give

$$(\overset{s}{\alpha}{}_i^p - \overset{u}{\alpha}{}_i^p)\overset{v}{\nabla}_p u = 0. \tag{6.23}$$

In our case the function $u(x,\xi)$ is positively homogeneous of zero degree, and therefore

$$\xi^k \overset{v}{\nabla}_k u = 0. \tag{6.24}$$

In view of item (3) of Lemma 3.1, (6.23) and (6.24) imply that $\overset{v}{\nabla} u = 0$, i.e. the function u is independent of ξ: $u = u(x)$. Equalities (6.21) and (6.22) now take the form $f_i = \overset{h}{\nabla}_i u = \partial u/\partial x^i$. Therefore, f is an exact form: $f = du$. The theorem is proved. □

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References

[An] D. Anosov. *Geodesic Flows on Closed Riemannian Manifolds with Negative Curvature (Proc. Steklov Inst. of Math., 90)*. Nauka, Moscow, 1967 (in Russian).

[Ar] V. I. Arnold. *Ordinary Differential Equations*. Nauka, Moscow, 1984 (in Russian).

[CSh] C. B. Croke and V. A. Sharafutdinov. Spectral rigidity of a negatively curved manifold. *Topology* **37**(6) (1998), 1265–1273.

[Eb] P. Eberlein. When is a geodesic flow of Anosov type? I. *J. Diff. Geom.* **8** (1973), 437–463.

[Es] J.-H. Eschenburg. Horospheres and the stable part of the geodesic flow. *Math. Z.* **153** (1977), 237–251.

[F] P. Funk. Über eine geometrische Anwendung der Abelschen Integralgleichung. *Math. Ann.* **77** (1916), 129–135.

[G] L. W. Green. A theorem of E. Hopf. *Michigan Math. J.* **5** (1958), 31–34.

[GK] V. Guillemin and D. Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology* **19** (1980), 301–312.

[H] E. Hopf. Closed surfaces without conjugate points. *Proc. Natl Acad. Sci. U.S.A.* **34** (1948), 47–51.

[Hl] S. Helgason. *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions (Pure and Applied Mathematics, 113)*. Academic, Orlando–São Paulo, 1984.

[LMM] R. de la Llave, J. M. Marco and R. Moriyon. Canonical perturbation theory of Anosov Systems and regularity results for the Livsic cohomology equation. *Ann. Math.* **123** (1986), 537–611.

[M1] R. Michel. Les tenseurs symétriques dont l'énergie est nulle toutes les géodésiques des espaces symétriques de rang 1. *J. Mat. Pures Appl.* **45** (1974), 271–278.

[M2] R. Michel. Tenseurs symétriques et géodésiques. *C. R. Acad. Sci. Sér I Math.* **284** (1977), 1065–1068.

[PSh] L. N. Pestov and V. A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Siberian Math. J.* **29**(3) (1988), 427–441.

[Sh1] V. A. Sharafutdinov. *Integral Geometry of Tensor Fields*. VSP, Utrecht, The Netherlands, 1994.

[Sh2] V. A. Sharafutdinov. Finiteness theorem for the ray transform on a Riemannian manifold. *Inverse Problems* **11** (1995), 1039–1050.

[Sh3] V. A. Sharafutdinov Modified horizontal derivative and some of its applications. *Siberian Math. J.* **36**(3) (1995), 664–700.

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