THE REFRACTION OF A WAVE GROUP

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1. Introduction

The considerations set out in this communication arose originally in the discussion of the refraction of surface elastic waves at the edge of a continent*. In view, however, of their general application, it is perhaps worth while to state them apart from their special context.

In accordance with the general theory of wave propagation† it is to be expected that when a train of waves is refracted at the interface between two dispersive media, as for instance when a Rayleigh wave crosses the margin of a continent or when a water wave passes across a sandbank, the change in direction of a wave front will be determined by the ratio of the wave velocities, but that the time of transit of a group of waves will depend on the group velocities. A very short and elegant proof that this is so has been constructed by Dr H. Jeffreys, who has kindly allowed me to incorporate it in the paper previously cited. The energy associated with the disturbance likewise travels with the group velocity. The visible or measurable disturbance may be called the energy front or amplitude front‡, to distinguish it from the wave front. For seismic surface waves the form and progress of the amplitude front could be traced for any assigned period from the records of a close network of seismological stations. The fundamental distinction between the two fronts is that whereas an individual wave is advancing normally to the wave front with the wave velocity c (and incidentally varying in amplitude with the time), an element of the amplitude front advances with the group velocity C, not normal to the amplitude front, but normal to the wave front through the element in question.

2. The refracted amplitude front

To examine the matter more closely, consider a two-dimensional example in which a wave is incident at an angle θ_1 on the interface between two media, distinguished by suffixes 1 and 2. Let AM be a portion of the incident wave front as it meets the interface at A (Fig. 1). Suppose that the wave front reaches the

* R. Stoneley, *Monthly Notices Roy. Astr. Soc.*, Geophysical Supplement, 3 (1935), 262.

† Cf., e.g., Havelock, Cambridge Tract No. 17.

‡ J. B. Macelwane, *Bull. Nat. Res. Council*, 90 (Seismology), 110.

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interface at B a time δt after passing M, and let BN be the refracted wave front. Then, if $AB = h$, the angle θ_2 of refraction is given by

$$h/\delta t = c_1/\sin\theta_1 = c_2/\sin\theta_2. \tag{1}$$

Suppose, for simplicity, that AM is also the incident amplitude front. Then the energy associated with the wave group travels a distance MB in time $c_1\delta t/C_1$. In this time energy has travelled along AN a distance AE, equal to $c_1 C_2 \delta t/C_1$. Thus BE will be the new energy front (i.e. amplitude front).

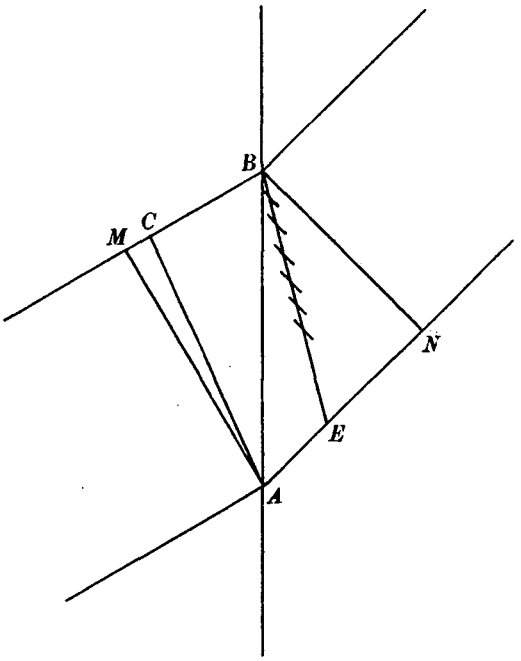


Fig. 1.

The apparent angle of refraction of the wave group is ABE; call this angle ϕ. Then from the triangle AEB

$$C_1 \sin\phi/c_1 C_2 \delta t = \cos(\theta_2 - \phi)\sin\theta_1/c_1\delta t. \tag{2}$$

From this equation, with the help of (1), we obtain

$$\begin{aligned}\cot\phi &= (C_1 - C_2\sin\theta_1\sin\theta_2)/C_2\sin\theta_1\cos\theta_2 \\ &= (c_2 C_1 - c_1 C_2\sin^2\theta_2)/c_1 C_2\sin\theta_2\cos\theta_2,\end{aligned} \tag{3}$$

where θ_2 is determined from equation (1).

Further, the amplitude front advances normally to itself with velocity

$$C_2\cos(\theta_2 - \phi).$$

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Now a wave group in the usual sense of the word consists of waves whose amplitude is a maximum at the middle of the group, and diminishes to zero towards the front and the rear of the group. To reconcile the difference in direction of the wave front and the amplitude front it is necessary to suppose that the amplitude of a wave varies along the crest of the waves, so that the amplitude front is arranged *en échelon*, as indicated in Fig. 1.

3. GENERALIZATION OF THE CLASSICAL THEORY OF GROUP VELOCITY

If the preceding argument is sound, it should be possible to predict theoretically the *échelon* wave pattern. It will be recalled that Stokes's kinematical explanation of group velocity* proceeds from the superposition of two wave trains of equal amplitude and slightly differing period and velocity. Then the equation

$$a\cos\{(\kappa+\delta\kappa)x-(\gamma+\delta\gamma)t\}+a\cos\{(\kappa-\delta\kappa)x-(\gamma-\delta\gamma)t\}$$
$$=2a\cos(\kappa x-\gamma t).\cos(\delta\kappa.x-\delta\gamma.t), \qquad (4)$$

indicates that the resulting disturbance may be regarded as a progressive wave of velocity $c=\gamma/\kappa$ and amplitude $2a\cos\delta\kappa.\{x-t\,d(\kappa c)/d\kappa\}$. Thus the amplitude varies with x and t in such a way that the visible groups (i.e. the places where the amplitude is nearly $2a$) advance with velocity $d(\kappa c)/d\kappa$. This, then, will be the group velocity C.

The appropriate extension is to consider two wave trains, both travelling before refraction in the same direction. After refraction the wave fronts will make angles $\theta-\delta\theta$ and $\theta+\delta\theta$ with the interface $x=0$, and their wave lengths will be $2\pi/(\kappa-\delta\kappa)$ and $2\pi/(\kappa+\delta\kappa)$; the wave velocity c is of course a function of κ. Then the amplitude of the superposed waves is given by

$$a\cos\{(\kappa-\delta\kappa)(x\cos(\theta-\delta\theta)+y\sin(\theta-\delta\theta))-(\gamma-\delta\gamma)t\}$$
$$+a\cos\{(\kappa+\delta\kappa)(x\cos(\theta+\delta\theta)+y\sin(\theta+\delta\theta))-(\gamma+\delta\gamma)t\}$$
$$=2a\cos\{\kappa x(\cos\theta+y\sin\theta)-\gamma t\}\times$$
$$\cos[\delta\kappa\{(x\cos\theta+y\sin\theta)+(y\cos\theta-x\sin\theta)\kappa(d\theta/d\kappa)-t\,d\gamma/d\kappa\}], \qquad (5)$$

approximating in the usual way to the first order in $\delta\kappa$. For an assigned value of θ, $d\theta/d\kappa$ is a function of κ.

The resulting disturbance is a train of progressive waves advancing in the θ direction, but having a variable amplitude, which is represented by a slowly varying harmonic variation, as before, but travelling in a direction inclined to that of the wave front. To simplify the interpretation, change the axes so that $\theta=0$. Then the wave pattern is given by the expression

$$2a\cos\kappa(x-ct)\cos\delta\kappa(x+y\kappa\,d\theta/d\kappa-t\,d(\kappa c)/d\kappa). \qquad (6)$$

* Stokes, *Math. and Phys. Papers*, 5, 362.

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The second harmonic factor represents a periodic fluctuation in amplitude, not along lines parallel to $x=0$, the direction of the wave crests, but parallel to the line

$$x+y\kappa d\theta/d\kappa=0. \tag{7}$$

The perpendicular distance between the groups is $2\pi/p\,\delta\kappa$, where

$$p^2=1+\kappa^2(d\theta/d\kappa)^2. \tag{8}$$

The *apparent* group velocity C', judged from the observed rate of progress of the *échelon* pattern normal to its amplitude front, is $p^{-1}d(\kappa c)/d\kappa$. The rate of progress of the disturbance in the x direction, that is to say, in the direction of the transport of energy, is, by (7), $C'\sec\{\tan^{-1}(\kappa\,dc/d\kappa)\}$, which in virtue of (8) is $C'p$, or $d(\kappa c)/d\kappa$. Thus the customary value of the group velocity is still to be employed, provided that it is taken to refer to the propagation of the disturbance (and presumably the transport of energy) normal to the *wave* front. The formula (8) is now consistent with the assumptions of § 1.

To see graphically how the formula (6) represents an *échelon* pattern, draw a set of lines parallel to $x=0$ at intervals $2\pi/\kappa$ (Fig. 2); these represent at the instant $t=0$ the crests of elementary waves that are advancing with velocity c in the direction Ox. Next draw a set of lines parallel to $x+\kappa y\,d\theta/d\kappa=0$ at intervals $2\pi/p\delta\kappa$; these lines (broken lines in Fig. 2) are the loci, at $t=0$, of the points of maximum amplitude. This enhancement of amplitude is shown in the diagram by a thickening of the wave crests. The thickenings, as connected by the broken lines, then represent the *échelon* form of the amplitude front.

It remains to show that the formula (7) for the angle between the wave front and the amplitude front agrees with the value derived from equations (2) and (3) of the previous section; this is equivalent to proving that $\tan(\theta_2-\phi)=-\kappa_2 d\theta_2/d\kappa_2$, for in Figs. 1 and 2 the angles between the wave front and the group front have opposite signs.

From equation (3) it is easy to show that

$$\tan(\theta_2-\phi)=\tan\theta_2\,(c_2C_1-c_1C_2)/c_2C_1. \tag{9}$$

Since the period is unaltered by refraction,

$$\kappa_1c_1=\kappa_2c_2, \tag{10}$$

so that

$$C_2/C_1=d\kappa_1/d\kappa_2. \tag{11}$$

Further, from (1) and (11), $\kappa_1\sin\theta_1=\kappa_2\sin\theta_2$, whence

$$\frac{d\kappa_1}{\kappa_1}+\cot\theta_1 d\theta_1=\frac{d\kappa_2}{\kappa_2}+\cot\theta_2 d\theta_2, \tag{12}$$

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and in the present problem we must put $d\theta_1 = 0$. Finally, by substituting from (10), (11) and (12) in (9) we get

$$\tan(\theta_2 - \phi) = \tan\theta_2\left(1 - \frac{\kappa_2 d\kappa_1}{\kappa_1 d\kappa_2}\right) = \tan\theta_2\left(1 - 1 - \kappa_2 \cot\theta_2 \frac{d\theta_2}{d\kappa_2}\right) = -\kappa_2 d\theta_2/d\kappa_2,$$

so that the two modes of approach to the problem are equivalent.

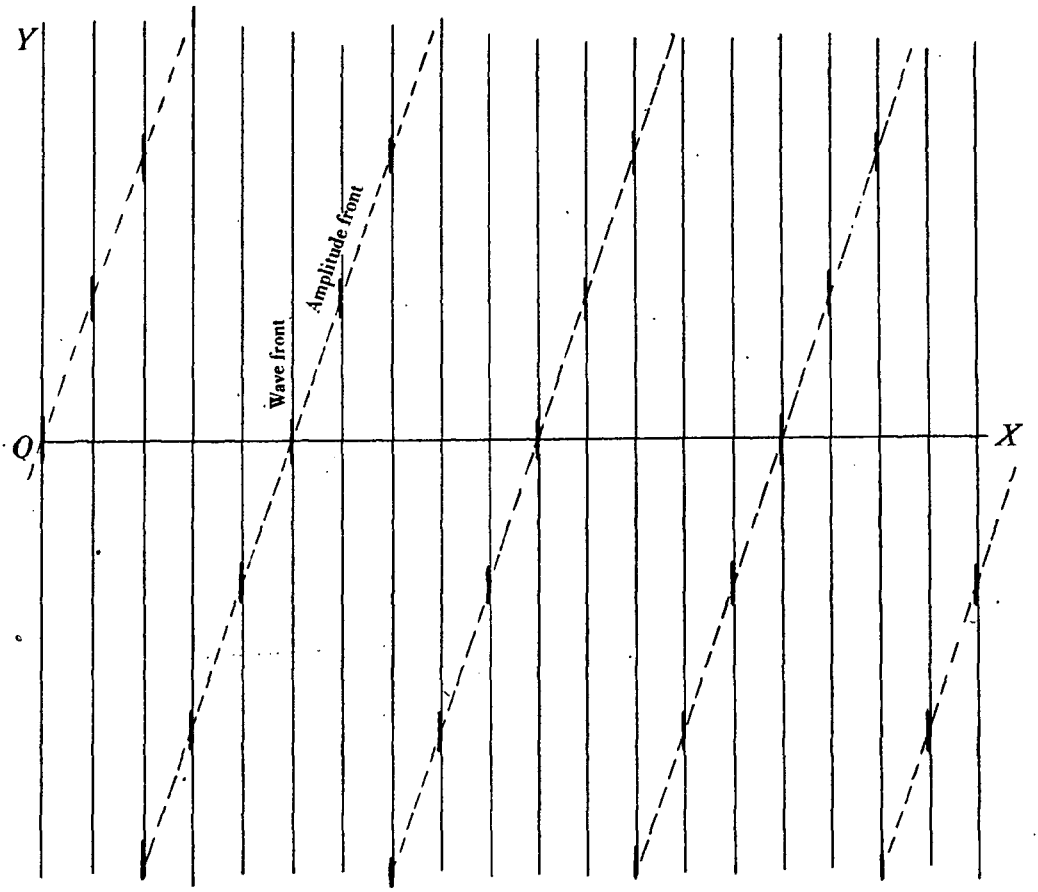


Fig. 2.

4. THE REFRACTION OF AN *ÉCHELON* SYSTEM

From the mode of deriving the *échelon* wave system from two systems of plane progressive waves it is evident that the *échelon* property is transitive for successive refractions at a plane interface, so that for each medium the angle χ between the wave front and the amplitude front satisfies a relation of the form $\cot\chi = -\kappa d\theta/d\kappa$, in which the angle of incidence enters only through $d\theta/d\kappa$.

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The corresponding law of refraction of the amplitude front is easily established. Let AM, AC respectively represent the wave front and amplitude front (Fig. 1) incident at angles θ_1, ϕ_1, and let BN, BD be the corresponding points after refraction. The suffixes 1 and 2 relate to the first and second media respectively. Then as before

$$c_1/\sin\theta_1 = c_2/\sin\theta_2, \tag{13}$$

and since the time taken by the group to travel along CB is the same as that required for AD,

$$CB/C_1 = AD/C_2, \tag{14}$$

so that

$$\frac{\sin\phi_1}{C_1\cos(\theta_1-\phi_1)} = \frac{\sin\phi_2}{C_2\cos(\theta_2-\phi_2)}, \tag{15}$$

which is the required formula for ϕ_2.

Writing $\theta_1-\phi_1=\chi_1$, $\theta_2-\phi_2=\chi_2$, we derive for the angle χ_2 between the refracted wave front and amplitude front:

$$(1-\cot\theta_2\cot\chi_2)/C_2 = (1-\cot\theta_1\cot\chi_1)/C_1, \tag{16}$$

in which, of course, θ_2 is regarded as having been found from (13).

By differentiating logarithmically the relation $\kappa_1\sin\theta_1=\kappa_2\sin\theta_2$ it is easy to show from the preceding formula (16) that the relation $\cot\chi_1 = -\kappa_1 d\theta_1/d\kappa_1$ implies $\cot\chi_2 = -\kappa_2 d\theta_2/d\kappa_2$, as the theory predicts.

5. Generalization of the Rayleigh theory

In one respect the analysis of § 3 is unsatisfactory; it indicates an infinitely repeated pattern with the wave groups occurring at a space interval inversely proportional to $\delta\kappa$. Now $\delta\kappa$ is arbitrary in this argument, and there is no obvious physical reason for assigning any particular magnitude to it. The origin of the difficulty is that Stokes's original treatment is an over-simplification, and the same criticism applies to the present extension; none the less, the simplicity that results is an ample justification for retaining Stokes's illustration.

The appropriate generalization was made by the late Lord Rayleigh* who considered, not merely two superposed wave trains, but the aggregate of wave trains, of which only those in the neighbourhood of a certain value κ_0 make an appreciable contribution to the disturbance at a given point (x) at a given time (t); the value of κ_0 is determined by the group-velocity relation $x - t\,d(\kappa c)/d\kappa = 0$, in which c is a function of κ. In place of the recurrent groups of waves there is one single group, constituted of waves of length $2\pi/\kappa$ as before; the other groups may be considered to have been destroyed by mutual interference. In the present problem, the *échelon* system determined by the condition that the disturbance travels with velocity $d(\kappa c)/d\kappa$ in the direction of the elementary refracted waves

* *Scientific Papers*, 1, 540. See also H. Jeffreys, *Operational Methods*, Cambridge Tract No. 23 (2nd Ed.), 88.

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will be the same, in form and position, for all small variations $\delta\kappa$; the other *échelons* given by formula (6) will disappear when the disturbances corresponding to a continuous variation of κ and θ with a small range are superposed.

To put the matter quantitatively, let P be a point (x, y) in medium 2. Then we may represent the disturbance received at P by the real part of

$$\int_0^\infty \int_{-\infty}^\infty f(\kappa_2, \eta) \exp\{iF(\kappa_2, \eta)\}\, d\kappa_2\, d\eta, \tag{17}$$

in which f is a slowly varying function of κ_2, and possibly of η as well, and

$$F(\kappa_2, \eta) = \kappa_1 \eta \sin\alpha + \kappa_2 \{x^2 + (y-\eta)^2\}^{\frac{1}{2}} - \gamma t. \tag{18}$$

In (18) α is the constant value of θ_1 for all values of κ_1; κ_2 is connected with κ_1 by (10), and η is the distance along the y-axis AB from some arbitrary origin, say A (fig. 1) at which the track representing an elementary contribution to the integral (17) meets the interface. The axis of x is drawn from A into medium 2. This integral may be evaluated approximately by the method of stationary phase; for the parts of the ranges of integration in which the contribution from the integrand is appreciable, the variation of f is neglected*.

The first order conditions are that F shall be stationary in respect of both κ_2 and η. Accordingly

$$\eta \sin\alpha \,.\, d\kappa_1/d\kappa_2 + r_2 - \gamma' t = 0, \tag{19}$$

$$\kappa_1 \sin\alpha - \kappa_2 (y-\eta)/r_2 \quad = 0, \tag{20}$$

in which r_2 is the distance from $(0, \eta)$ to (x, y) and accents denote differentiation with respect to κ_2. Equation (20) is equivalent to Snell's law $\kappa_1 \sin\alpha = \kappa_2 \sin\theta_2$, and thus defines the track of a refracted ray of given period. When this track is assigned, (19) indicates, in view of equation (11), that the disturbance travels *along the rays* with the appropriate group velocity.

The elimination of η between (19) and (20) should lead to a formula connecting the position of a point (x, y) with the time at which there is an appreciable disturbance there. We have, in fact,

$$\eta = y - x\tan\theta_2\,; \quad r_2 = x\sec\theta_2\,. \tag{21}$$

By differentiating (20) with respect to κ_2 we obtain a formula equivalent to (12), which with (19) and (21) leads to

$$x\cos\theta_2 + y\sin\theta_2 + (y\cos\theta_2 - x\sin\theta_2)\,\kappa_2\, d\theta_2/d\kappa_2 = \gamma' t, \tag{22}$$

shewing that the *échelon* pattern and group velocity of the disturbance is exactly that of one of the series of groups implied by (5).

The amplitude of the part of the disturbance that has a wave-length $2\pi/\kappa_0$ can be found approximately from

$$f(\kappa_0, 0) \exp\{iF(\kappa_0, 0)\} \int_{-\infty}^\infty \int_{-\infty}^\infty \exp \tfrac{1}{2} i\,(F_{\kappa\kappa} k^2 + 2F_{\kappa\eta} k\eta + F_{\eta\eta}\eta^2)\, dk\, d\eta, \tag{23}$$

* Cf. Jeffreys, *loc. cit.*

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where subscripts denote derivatives, $k = \kappa - \kappa_0$, and the origin is so chosen that, for an assigned (x, y), $\kappa = \kappa_0$ corresponds to $\eta = 0$ for the ray that passes through (x, y).

By evaluating the repeated integral in (23) it is readily found that the amplitude of the disturbance, for large values of t, is

$$2\pi f(\kappa_0, 0) \sec\theta_2 / |\kappa_0^2 (d\theta_2/d\kappa_0)^2 + \kappa_0 \gamma''/\gamma'|^{\frac{1}{2}}.$$

The phase depends on the sign of γ''.

6. Geophysical implications

Apart from its more obvious application to the interpretation of water-wave patterns, the present result must be borne in mind in comparing the progress of earthquake waves over the surface of the earth. The difference between continent and ocean entails only a small correction to the times of arrival of P and S waves, but, as is well known, causes large differences in the velocity of a given type of surface wave. Some information can be derived about the structure of a continent if the times of arrival of surface waves of an assigned period are compared for a number of stations on the same continent; the isochronous lines would map out the advance of this particular period. The group velocity, however, would not necessarily be the normal velocity of this amplitude front, and a separate estimate, based on the refraction of waves at the continental margin (if this had occurred) would be needed to allow for the angle between the two fronts. It would be of considerable interest to compare the value of $d\theta/d\kappa$ found from equation (7) with that corresponding to the structure at present assigned to the earth's crust. The investigation could be carried out for the passage of Love waves from a Pacific epicentre across Eurasia or the United States provided that the angle between the wave front and amplitude front was not too small.

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