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VALUED MODULES OVER SKEW POLYNOMIAL RINGS I

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Abstract. We introduce a notion of valued module which is suitable to study valued fields of positive characteristic. Then we built-up a robust theory of henselianity in the language of valued modules and prove Ax-Kochen Ershov type results.

§1. Introduction.

A valued abelian group, is an abelian group M together with a linearly ordered set Δ and a function $v : M \to \Delta$, such that

a. $v(x \pm y) \geqslant \min\{v(x), v(y)\}$,
b. $v(x) = \infty \Leftrightarrow x = 0$,

where ∞ is the maximum of Δ.

A module such that the underlying abelian group is valued will be called a valued module. Depending on the context one wants to study, various compatibility assumptions relating the valuation v and the action of scalars are considered in the literature. The first article on the subject that the author is aware of is due to Fleischer ([5]); it extends the notion of Krull valuation to modules where $v(x.r) \geqslant v(x)$ for any scalar r and proves the equivalence of maximality and (pseudo-) convergence of pseudo-Cauchy series. In Rohwer's thesis ([14]) modules are also used to understand the model theory of valued fields of positive characteristic or more generally valued difference fields. In more recent articles of Point and Bélair [1] (resp. Maalouf [9]) valued modules (resp. vector spaces) are studied.

Similarly as in [14], we are interested in modules which come from the theory of valued fields in characteristic $p > 0$. The rings that we will consider are unitary and arise as K-subalgebras of the endomorphisms ring of a field K generated by a distinguished endomorphism φ over K.

Definition 1.1. Let K be a field and φ a ring endomorphism of K. The ring $K[t;\varphi]$ is given as a set by formal sums $\sum_{i\in\mathbb{N}} t^i a_i$ where t is the indeterminate, $a_i \in K$ and $\{i \in \mathbb{N} \mid a_i \neq 0\}$ is finite. The addition is defined term by term and the multiplication obeys the (non)commutation rule: $at = t\varphi(a)$. More precisely,

1. $\sum_{i\in\mathbb{N}} t^i a_i + \sum_{i\in\mathbb{N}} t^i b_i = \sum_{i\in\mathbb{N}} t^i (a_i + b_i)$.
2. $\left(\sum_{i\in\mathbb{N}} t^i a_i\right)\left(\sum_{i\in\mathbb{N}} t^i b_i\right) = \sum_{i\in\mathbb{N}} t^i c_i$ where $c_i = \sum_{k+l=i} \varphi^l(a_k) b_l$.

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Most often for fixed K and φ we will refer to this ring $K[t;\varphi]$ as R. Any field K of characteristic of $p > 0$ has a natural module structure over its ring of additive polynomials: these are polynomials of the form $\sum_{i=0}^{n} a_i X^{p^i}$, where the ring operations are given by usual addition and (right-)composition. This ring is isomorphic to the ring $K[t;Frob]$ where $Frob$ is the map $x \mapsto x^p$. It is widely believed that serious problems in the model theory of valued fields in positive characteristic arise from additive polynomials (see for example [18]). Studying and axiomatizing the underlying ultrametric module structure can be considered as an abstract analysis that isolates this phenomenon. Note also that any valued field together with an arbitrary automorphism inherits an analogue module structure.

After the famous Ax-Kochen and Ershov (A-K, E) theorem which provides a corrected form of a conjecture of Artin,[1] there were similar theorems permitting to understand the first order theories of some other class of valued fields (see [17]) for a survey). Note that our understanding of positive characteristic valued fields are very limited relative to our understanding in the case of characteristic 0, this is a in general due to problems arising from the existence of the additive polynomials. One of the aim of this article is to initiate techniques which isolates phenomena related to additive polynomials. Here, we establish (A-K, E) type results by proving a relative elimination of quantifiers for modules called *henselian divisible valued R-modules*. We first observe the theory of that divisible valued modules is not suitable to reach our aim (cf. counter Example 3.26 and the remark which follows). Thus we add an extra assumption on the valuation, which comes from an interpretation of Hensel's lemma as a local inversion theorem. More precisely, in this context, there is an abstract analogue of the maximal ideal of a valuation ring; call this $M_{>\theta}$ and then our assumption asserts that the multiplication by some special elements of R (we will call them *separable polynomials*) induces a bijection $M_{>\theta} \to M_{>\theta}$.

Fix $R = K[K;\varphi]$ as above for given φ and K. In our model theoretical setting valued R-modules are considered in the two sorted first order language

$$L := \{0, +, -, (.r)_{r\in R}\} \cup \{<, \tau, \infty, (R_n)_{n\in\mathbb{N}}\} \cup \{v\},$$

where

1. the *module sort* language is $L_{Mod}(R) := \{0, +, -, \{.r\}_{r\in R}\}$ (which is the usual language of modules),
2. the *value set sort* language is $L_V := \{<, \tau, \infty, (R_n)_{n\in\mathbb{N}}\}$ with τ a unary function symbol and R_n a unary predicate symbol for each n,
3. v is a unary function symbol to be interpreted as the valuation.

We describe now the content of our article.

In Section 2, we consider pure R-modules and describe the completions of the theory of divisible R-modules as follows:

THEOREM 1.2. *Any complete theory of nonzero divisible R-modules admits elimination of quantifiers and is obtained by specifying for each irreducible $q \in R$, the number of elements annihilated by q.*

[1] Artin conjectured that the field $\mathbb{Q}_p$ is C_2 (i.e., every homogeneous polynomial of degree d with $> d^2$ variables has a nontrivial zero) for any prime p. This is false by a result of Terjanian (see [16]).

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In the Section 3, we introduce valued R-modules (M, Δ, v) which by definition satisfy the following compatibility properties:

1. $v(x.\lambda) = v(x)$ for all unit λ in R.
2. The map $x \to v(x.t)$ factors through Δ, that is, it induces a strictly increasing function $\tau : \Delta \to \Delta$ such that $v(x.t) = \tau(v(x))$ for all $x \in M$. Moreover, there exists at most one value $\theta \in \Delta$, such that, if $r \in R\backslash\{0\}$ and $x \in M$ with $v(x) \neq \theta$, then $v(x.r) = \tau^k(v(x))$ for some $k \in \mathbb{N}$ depending only on r and $v(x)$.

Note that any valued field (U, v) of characteristic $p > 0$ inherits naturally a valued $K[t; Frob]$-module structure where K is trivially valued subfield of U subfield and the function τ is $\gamma \mapsto p\gamma$.

We then investigate the L-theory of *henselian* and *henselian divisible* R*-modules*. We have the following main theorem which enables us to recover the theory of a divisible henselian valued R-module from the theory of its value set (in L_V) and from the theory of its torsion submodule (in $L_{Mod}(R)$) which, in some sense, plays the role of the *residue field*.

THEOREM 1.3 (A-K,E $\equiv$). *Let (F, v) and (G, w) be two nonzero henselian divisible modules such that F_{tor} and G_{tor} are elementary equivalent as $L_{Mod}(R)$-structures and $v(F)$ and $v(G)$ are elementary equivalent in the language L_V. Then (F, v) and (G, v) are elementary equivalent as L-structures.*

This article corresponds to Chapters Two and Three of author's thesis (cf. [11]).

§2. Divisible R-modules. First, we quickly summarize some basic algebraic facts about the rings over which we will consider our modules. One can refer to [2] Section 2.1 for more details.

Let K be a field and φ a *field endomorphism* of K. Set $R := K[t; \varphi]$, the ring of φ-twisted polynomials in the variable t (see Definition 1.1 in Introduction). We will mean by *polynomial* any element of this ring. Each $r \in R$ can be written uniquely as $\sum_{i \in \mathbb{N}} t^i a_i$, with $I := \{i \mid a_i \neq 0\}$ finite. The ring R is equipped with the degree function defined by $\deg(r) = \max I$ (with the convention that $\max \emptyset = -\infty$). We have $\deg(pq) = \deg(qp) = \deg(p) + \deg(q)$. Note that R is a domain, i.e., has no zero divisors. The set of units of R is $K^\times$ and together with the function deg, R is right euclidean (and hence right principal):[2] for all $q, q' \in R$ with $q' \neq 0$ there exists $r, r' \in R$ such that $q = q'r' + r$ where $\deg(r) < deg(q')$ and r is unique up to right multiplication by a unit. We say that q' divides q if q is a right multiple of q', that is if $r = 0$ above. For any nonzero a and b there exists a unique monic common divisor of maximal degree, denoted by $\gcd(a, b)$.

Furthermore, by the commutation rule $at = t\varphi(a)$, any nonzero element q can be written as

$$q = t^n q_1, \ldots, q_s, \tag{1}$$

where t and the q_i are irreducible. Note that R is in particular a right Ore ring, that is every two nonzero elements a and b have a nonzero common right multiple. Taking into account the degree function there is a common right multiple of least

[2]Note that this ring is also left euclidean whenever φ is an automorphism.

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degree unique up to a unit: we denote it $\mathrm{lcm}(a, b)$ as usual. As any Ore ring, R is embedded into a smallest (right-)division ring, called its skew field of fractions.

NOTATION 2.1. *We will denote by D the skew field of fractions of R.*

Any R-module that we will consider is a right R-module, and we will simply say R-module.

The following result is an easy observation as in the case of a torsion free divisible abelian groups:

LEMMA 2.2. *Any torsion free divisible module over a Ore ring S has a unique vector space structure over the skew field of fraction of S. In particular, every divisible torsion free R-module is a D-vector space.*

NOTATION 2.3. *Let M be an R-module. We denote by M_{tor} the set*

$$\{x \in M \mid \exists r \in R\backslash\{0\} \text{ such that } x.r = 0\}.$$

LEMMA 2.4. *If M is a divisible R-module then M_{tor} is a divisible submodule of M.*

PROOF. Since R is a Ore ring, M_{tor} is a submodule (cf. [3] Proposition 3.5). Moreover, if $0 \neq x \in M_{tor}$, with $q \in R\backslash\{0\}$ such that $x.q = 0$ then, for all $r \in R\backslash\{0\}$ and for all $y \in M$ such that $y.r = x$, we have $y.rq = 0$. Therefore $y \in M_{tor}$, which shows that M_{tor} is divisible. ⊣

REMARK 2.5. Any torsion free nonzero divisible R-module is infinite since it is a D-vector space.

Furthermore if M is a nonzero divisible R-module then for every $r \neq 0$, $M.r = M \neq 0$. Hence $M_{tor}.r \neq 0$ when $r \neq 0$ and $M_{tor} \neq 0$. In particular, M_{tor} is infinite if it is nonzero, since otherwise, the least common multiple of annihilators of nonzero elements of M_{tor} would annihilate M_{tor}.

Recall that a module I over a ring S is said to be injective if for any given inclusion of S-modules $N \subseteq M$, and any homomorphism $f : N \to I$, f admits an extension to M.

In the following lemma we state that in the case of R-modules the notions of injectivity and divisibility coincide, and we also summarize well-known facts about injective modules that we will frequently use in the rest of this article.

LEMMA 2.6. *Let S be any ring.*

1. *A divisible module over a right principal ring is injective, and an injective module over a domain is divisible. In particular for every R-module M, M is divisible if and only if it is injective.*
2. *Any injective S-module is a direct summand in every S-module which contains it.*
3. *Any S-module M has an extension $N \supseteq M$ maximal with the property*

   $$X \cap M \neq 0 \text{ for any nonzero submodule } X \subseteq N.$$

   N is called an injective hull of M and is unique up to an M-isomorphism of S-modules.
4. *Let $N \supseteq M$ be R-modules. If N is an injective hull of M and $x \in N\backslash\{0\}$, then there exists $r \in R$ such that $x.r \in M\backslash\{0\}$.*

PROOF. We refer to [7] (pp. 156–164). The last assertion is Exercise 3, page 164. ⊣

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By point 1. above, the notion *injective R-module* coincides with the notion of *divisible R-module*. For this reason we will use the expression *divisible closure*, instead of injective hull. Note that an R-module can have several divisible closures inside a given divisible R-module.

REMARK 2.7. 1. Let $M \subseteq E \subseteq N$ be R-modules, with E and N divisible. Then E is a divisible closure of M, if and only if for all $x \in E$ there exists a nonzero $r \in R$ such that $x.r \in M$. Moreover note that E can be different from

$$\{x \in N \mid \exists r \in R,\ r \neq 0 \ \&\ x.r \in M\}.$$

In fact, E is the unique divisible closure of M inside E of $M + N_{tor}$ (see also Definition 3.34 and Lemma 3.35).

2. By Lemma 2.6, divisible closures of an R-module are isomorphic as pure modules. When we will consider valued modules we will see that the various divisible closures are generally neither isomorphic nor even elementary equivalent as valued modules (see Proposition 3.26 and Corollary 3.27).

The following observation is essential:

LEMMA 2.8. *Let M be an R-module, $A \subseteq M$ a submodule of M and $x \in M \backslash A$ such that $x.r \in A$ for some $r \in R \backslash \{0\}$. Then there exists $q \in R$ of minimal degree such that $x.q \in A$. For such a q and for all r such that $x.r \in A$, q divides r.*

PROOF. Let $I := \{r \in R \mid x.r \in A\}$; I is a right ideal, therefore principal. Hence there is a generator of I, that we can take as q. ⊣

DEFINITION 2.9. Any polynomial q as above is called a minimal polynomial of x over A. In particular, for all $x \in M_{tor} \backslash \{0\}$, there exists monic polynomials of minimal degree such that $x.q = 0$. It is called a minimal polynomial of x.

REMARK 2.10. A minimal polynomial of x over A divides any minimal polynomial of x. If $x \in M \backslash A$ and $x.r \in A$ with r irreducible then r is a minimal polynomial of x over A. Minimal polynomials are unique up to a unit.

DEFINITION 2.11. For any $r \in R$ and any R-module M, the annihilator set of r in M, is the set $ann_M(r) := \{x \in M \mid x.r = 0\}$ and its elements are called zeros or roots of r.

NOTATION 2.12. *For an R-module M, let η_M be the cardinal-valued function defined for all $q \in R$, by $\eta_M(q) := |ann_M(q)|$.*

PROPOSITION 2.13. *Let M and N be divisible R-modules satisfying $\eta_M \leqslant \eta_N$. Suppose $\eta_M(q)$ is finite for all nonzero q and $f : A \to N$ is an R-module embedding where $A \subseteq M_{tor}$ is a submodule. Then f extends to an R-module embedding of M_{tor} into N_{tor}. In particular if $ann_M(q) = ann_N(q) < \infty$ for all nonzero $q \in R$, then M_{tor} and N_{tor} are isomorphic.*

PROOF. Note that the range of f is in N_{tor}. Let $x \in A \backslash M_{tor}$ and q be a minimal polynomial of x over A. Write $q = q_1, \ldots, q_k$ where the q_i are irreducible. Since q is a minimal polynomial of x over A, $z := x.q_1, \ldots, q_{k-1} \notin A$. Set $r := q_k$. Since r is irreducible it is a minimal polynomial of z over A.

CLAIM. There exists $y \in N_{tor} \backslash f(A)$ such that $y.r = f(z.r)$.

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PROOF OF THE CLAIM. Set $b := f(z.r)$. Since N_{tor} is divisible it contains some b_1 such that $b_1.r = b$. Suppose $b_1 \in f(A)$. Then $a_1 := f^{-1}(b_1) \in A$. It follows that $(a_1 - z).r = 0$. Note that $(a_1 - z) \notin A$ and r has the same number of zeros in A and $f(A)$. Since $\eta_M(r)$ is finite and $\eta_M(r) \leqslant \eta_N(r)$ by hypothesis, there exists $b_0 \in N \backslash f(A)$ such that $b_0.r = 0$. Now $b_0 + b_1 \notin f(A)$ and $(b_0 + b_1).r = b$.†

Take such an y, then since r is irreducible it is a minimal polynomial of y over A. Now it is easy to check that the map $A + z.R \to f(A) + y.R$ sending y to z is an isomorphism of R-modules.

Using Zorn Lemma we can extend f to M_{tor}. ⊣

2.1. Complete theories of divisible R-modules. Let S be a ring. Let $L_{Mod}(S) := \{0, +, -\} \cup \{.s \mid s \in S\}$ be the language of (right-)S-modules. An equation is an $L_{Mod}(S)$-formula $\gamma(x_1, \ldots, x_m)$ of the form

$$\gamma(\overline{x}) : x_1.r_1 + \cdots + x_m.r_m = 0.$$

A primitive positive (p.p.) formula $\phi(x_1, \ldots, x_m)$ of the language of S-modules is a formula of the form

$$\exists \overline{y} \ \ (\gamma_1(\overline{x}, \overline{y}) \wedge \cdots \wedge \gamma_n(\overline{x}, \overline{y})),$$

where the γ_i are equations. For instance, by choosing $\gamma_1(x, y) = x.1 + (-y).q = 0$ and $\gamma_2(x, y) = x.r + y.0 = 0$, the formula $\phi(x) : \exists y \ \ (y.q = x \wedge x.r = 0)$ is a p.p. formula. Note also that a p.p. formula can be written as

$$\exists y_1, \exists y_2, \ldots, \exists y_k \ \ (y_1, \ldots, y_k).A = (x_1, \ldots, x_n).B,$$

where A and B are matrices with coefficients in S. Such formulas define subgroups, called p.p. subgroups. It is a well known fact that modulo a complete theory of S-modules every formula is equivalent to a boolean combination of p.p. formulas. See [13] for more details. Moreover:

PROPOSITION 2.14. *The complete theory of a S-module $\langle M, +, -, 0, \{.s\}_{s \in S} \rangle$ is given by the p.p. indexes (these are cardinalities in $\mathbb{N} \cup \{\infty\}$ of relative quotients of p.p. subgroups of M).*

In our case we have a simpler description of p.p. definable sets.

PROPOSITION 2.15. *Modulo the theory of divisible R-modules, any p.p. formula is equivalent to a conjunction of atomic formulas. Consequently, a p.p. formula with only one free variable is equivalent to one of the form $x.q = 0$ for some $q \in R$. It follows also that any completion of the theory of divisible R-modules eliminates quantifiers.*

PROOF. This follows directly from [13], Proposition 15.40. ⊣

COROLLARY 2.16. *If M is a divisible R-module such that $M_{tor} \neq 0$ then M_{tor} is an elementary substructure of M.*

PROOF. First observe that M_{tor} is a pure submodule of M since it is divisible (Lemma 2.4). Thus, by [15], it is enough by to show that M_{tor} and M are elementary equivalent. To obtain this we will show that for p.p. formulas $\phi(x)$ and $\psi(x)$ with $\phi(M) \subseteq \psi(M)$, we have $\psi(M)/\phi(M) = \psi(M_{tor})/\phi(M_{tor})$. By the above proposition $\phi(x)$ and $\psi(x)$ are, respectively, equivalent to $x.r = 0$ and $x.q = 0$ for some $r, q \in R$. If r and q are both nonzeros, then there is nothing to show. If $r = 0$ then $\phi(M) = M = \psi(M)$ necessarily. Hence it is enough to consider the nontrivial case where $r \neq 0$ & $q = 0$. But $M_{tor}/\phi(M_{tor})$ is infinite in this case. In fact, if not,

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then choosing a set of representatives $x_1, \ldots, x_n$ of M modulo $\phi(M)$ and nonzero polynomials $r_1, \ldots, r_n$ such that $x_i.r_i = 0$, we have $M_{tor}.\operatorname{lcm}(r, r_1, \ldots, r_n) = 0$. This is impossible by Remark 2.5. ⊣

To prove Theorem 1.2 we will study relative quotients of p.p. subgroups of a divisible R-module M. They are of the form $ann_M(r)/ann_M(s)$ by elimination of quantifiers.

Set $K_0 := \{x \in K \mid \varphi(x) = x\}$. Remark that every annihilator set $ann_M(r)$ is a K_0-vector space. Until the end of this section, the symbol "$\simeq$" denotes isomorphism between K_0-vector spaces. The following lemmas establish a generalization of Lemmas 2.9 and 2.10 from [6] where it is assumed that φ is onto, which means R is both right and left euclidean.

LEMMA 2.17. *Let M be a divisible R-module. For all $0 \neq q = q_1, \ldots, q_k \in R$,*

$$|ann_M(q)| = \prod_{i=1}^{k} |ann_M(q_i)|,$$

where $|ann_M(q_i)|$ is the cardinality of $ann_M(q_i)$.

PROOF. If $q = sr$ then multiplication by s induces an one to one K_0-linear transformation

$$ann_M(q)/ann_M(s) \longrightarrow ann_M(r)$$

which is in fact also onto by divisibility of M. The result clearly follows by induction on k. ⊣

LEMMA 2.18. *Let T_0 be a complete theory of nonzero divisible R-modules. For all $s, q \in R\backslash\{0\}$ such that $T_0 \models \forall x\,(x.s = 0 \rightarrow x.q = 0)$, there exists $r \in R$ such that*

$$\big(ann_N(q)/ann_N(s)\big) \simeq ann_N(r)$$

for all $N \models T_0$.

PROOF. Let s and q be as in the statement of the lemma. If s divides q then the result follows by the proof of the above lemma. Also if $ann_N(s) = 0$ the assertion is trivial. Hence we assume that $ann_N(s) \neq 0$ for a model (hence for all models) N of T_0. If s is irreducible, then s is (up to a unit) the minimal polynomial of a nonzero root of q hence divides q. Now, consider the general case. We proceed by induction: Suppose that for all nonzero polynomials h, g such that $ann_N(h) \supseteq ann_N(g) \neq 0$ and g can be written as a product of $n-1$ irreducible polynomials, there exist h', g', r' such that $g' = h'r'$, and

$$ann_N(g)/ann_N(h) \simeq ann_N(g')/ann_N(h') \simeq ann_N(r')$$

for all $N \models T_0$. Consider the case $s = s_1, \ldots, s_n$ and set $s' = s_2, \ldots, s_n$. Since $ann_N(s) \neq 0$, s_1 or s' has a nonzero root in N.

CASE 1. $ann_N(s_1) \neq 0$. First, s_1 divides q since it is irreducible and hence is -up to a unit- the minimal polynomial of any nonzero $x \in ann_N(s_1) \subseteq ann_N(q)$. Thus we have $q = s_1 r$ and, by the fact that N is divisible, $ann_N(s') \subseteq ann_N(r)$ for some $r \in R$. If $ann_N(s') = 0$ then we have: $ann_N(q)/ann_N(s) \simeq ann_N(q)/ann_N(s_1) \simeq ann_N(r)$. Otherwise, by induction hypothesis there are s'_t and r' such that s'_t divides r' and

$$ann_N(r')/ann_N(s'_t) \simeq ann_N(r)/ann_N(s').$$

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Now the map "$x \mapsto x.s_1 \bmod ann_N(s')$" establishes a morphism of K_0-vector spaces from $ann_N(q)$ onto $ann_N(r)/ann_N(s')$ with kernel $ann_N(s)$.

CASE 2. $ann_N(s_1) = 0$. Then, $ann_N(s') \neq 0$,

$$ann_N(s).s_1 = ann_N(s') \subseteq ann_N(q).s_1$$

and the action of s_1 induces an isomorphism K_0-vector spaces:

$$ann_N(q) \to ann_N(q).s_1.$$

Therefore

$$ann_N(q)/ann_N(s) \simeq ann_N(q).s_1/ann_N(s').$$

By Lemma 2.15 the p.p. formula $\exists y \;\; y.s_1 = x \wedge y.q = 0$, defining $ann_N(q).s_1$, is equivalent to a formula of the form: $x.q_1 = 0$ for some $q_1 \in R$. By applying the induction hypothesis to (s', q_1) we get r as required.

It is clear from the proof that r does only depend on T_0, s, and q. ⊣

PROOF OF THEOREM 1.2. The theorem follows from 2.14, 2.16, and 2.18. ⊣

We get by quantifier elimination the following consequence of Theorem 1.2.

COROLLARY 2.19. *Suppose $\eta_M = \eta_N$. Let f be a partial isomorphism between submodules $A \subseteq M$ and $B \subseteq N$. Then f admits an extension to an embedding of the divisible closure of $A + M_{tor}$ inside M, into N with image the divisible closure of $N_{tor} + B$.*

In the rest of this section we will describe the *building blocks* of divisible R-modules and improve Theorem 1.2.

NOTATION 2.20. *Let T_d denote the theory of divisible R-modules.*

DEFINITION 2.21. A module M is said to be indecomposable if there are no submodules N_1, N_2, both nonzeros, such that $M = N_1 \oplus N_2$.

LEMMA 2.22. *Let N_1 and N_2 be indecomposable models of T_d and $q \in R$ an irreducible polynomial such that $ann_{N_i}(q)$ is nontrivial for $i = 1, 2$. Then N_1 and N_2 are isomorphic.*

PROOF. Let $a \in ann_{N_1}(q)$ and $b \in ann_{N_2}(q)$. Then N_1 is a divisible closure of $a.R$ and N_2 is a divisible closure of $b.R$. So it suffices to show that $a.R$ is isomorphic to $b.R$. In fact, the map $a.r \mapsto b.r$ is an isomorphism of R-module since for all $r \in R\backslash\{0\}$, $a.r = 0$ if and only if q divides r, if and only if $b.r = 0$. ⊣

LEMMA 2.23. *Let N be an indecomposable model of T_d, $r \in R$ a irreducible polynomial such that $ann_N(r) \neq 0$. Then, for all irreducible $q \in R$ we have: $ann_N(q) \neq 0$ if and only if, there exist $\lambda, \mu \in K\backslash\{0\}$ such that $q = \lambda r \mu$; in which case $ann_N(r) \simeq ann_N(q)$.*

PROOF. Let $0 \neq a \in ann_N(r)$ and $0 \neq b \in ann_N(q)$. Then N is a divisible closure of both $a.R$ and $b.R$. Let r_0 be of minimal degree such that $0 \neq a.r_0 \in b.R$. So r_0 divides r. Since r is irreducible, $a.r_0 \neq 0$ implies $r_0 \in K$. Therefore $a \in b.R$, i.e., $a = bs$ for some $s \in R\backslash\{0\}$; take such an s of minimal degree. Then s divides q. Since q is irreducible and $bs \neq 0$, $s \in K$. Moreover, the fact that $b.sr = 0$ implies

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q divides sr. Since $s \in K$, sr is also irreducible. Therefore $\deg(q) = \deg(r)$ and $sr = qv$ with $s, v \in K$.

Conversely, suppose λ, μ are as in the statement. Then the map $x \mapsto x.\lambda$ establishes an isomorphism of K_0-vector space between $ann_N(q)$ and $ann_N(r)$. ⊣

DEFINITION 2.24. Let $r, q \in R$ be irreducible polynomials. We say that r and q are K-conjugate if there exist $\lambda, \mu \in K\backslash\{0\}$ such that $q = \lambda r \mu$.

REMARK 2.25. K-conjugation is an equivalence relation.

NOTATION 2.26. *For the rest of this section we fix a set $\mathcal{P}$ of representatives of irreducible polynomials modulo K-conjugation. Note that the map by* 2.17, *the map η_M is entirely determined by its restriction on $\mathcal{P}$ for any divisible R-module M.*

LEMMA 2.27. *Let $r \in \mathcal{P}$ and N be an indecomposable model of T_d containing a nonzero root of r. Then for all $q, s \in R\backslash\{0\}$ such that $ann_N(q) \supseteq ann_N(s)$, if $|ann_N(q)/ann_N(s)|$ is finite, then it is equal to $|ann_N(r)|^k$, for some $0 \neq k \in \mathbb{N}$.*

PROOF. By Lemma 2.18 there exists $r' \in R$ such that

$$ann_N(q)/ann_N(s) \simeq ann_N(r').$$

Now, $r' = r_0, \ldots, r_n$ with the r_i irreducible. We have on one hand

$$|ann_N(r')| = \prod_{i \leqslant n} |ann_N(r_i)|$$

by 2.17, and on the other hand, each $|ann_N(r_i)| \in \{|ann_N(r)|, 1\}$ by Lemma 2.23. ⊣

PROPOSITION 2.28. *Any divisible R-module is the direct sum of indecomposable divisible submodules.*

PROOF. Any injective module over a noetherian ring is the direct sum of indecomposable submodules (see [8] Corollary 7.3) and every direct summand of a divisible R-module is divisible. ⊣

NOTATION 2.29. *Let $M \models T_d$ and $q \in R$ be an irreducible polynomial. We denote by M_q the sum of all divisible indecomposable submodules of M_{tor} containing at least one nonzero root of q.*

PROPOSITION 2.30. *One has $M_{tor} = \oplus_{q \in \mathcal{P}} M_q$.*

PROOF. It is enough to see that if $N \subseteq M_{tor}$ is a divisible indecomposable submodule of M_{tor} then N is a divisible closure of some nonzero element $a \in M$ annihilated by an irreducible polynomial q. First, since N is indecomposable, it is a divisible closure of any of its nonzero elements. Then, if q_x is the minimal polynomial of a nonzero $x \in N$, q_x can be written as $q_1, \ldots, q_k$, where the q_i are irreducible. Hence $a := x.(q_1, \ldots, q_{k-1})$ is annihilated by q_k. ⊣

Hence we obtain the following improvement of Theorem 1.2.

COROLLARY 2.31. *Completions of $T_d \cup \{\exists x \ x \neq 0\}$ are obtained by specifying for each $q \in \mathcal{P}$, $|ann(q)| \in \{|K_0|^n \mid n \in \mathbb{N}\} \cup \{\infty\}$, and they admit elimination of quantifiers.*

2.2. The case of $K[t; Frob]$-modules. Let K be a field of characteristic $p > 0$ and $Frob$ be the Frobenius endomorphism of K. We say that K is p-closed if every polynomial of the form $X^{p^n} + X^{p^{n-1}} a_{n-1} + \cdots + X^p a_1 + X a_0 + b \in K[X]$ has a

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root in K. This amounts to say that K is divisible as a $K[t; Frob]$-module. We have also the following result which was first shown by Whaples (cf. [19], Theorem 1) and then a more elementary proof was provided by Delon in her thesis (cf. [4], Section 4.1).

PROPOSITION 2.32. *A field of characteristic $p > 0$ is p-closed if and only if it has no extension of degree divisible by p.*

COROLLARY 2.33. *If L is a field of characteristic $p > 0$, p-closed, then every algebraic extension of L is p-closed.*

THEOREM 2.34. *Let $K \subseteq F$ be an extension of fields of characteristic $p > 0$. Then the torsion submodule F_{tor} of the $K[t; Frob]$-module F is equal to the algebraic closure of K in F. In addition, two p-closed algebraic extensions F_1 and F_2 of K are elementary equivalent as $K[t; Frob]$-modules if and only if F_1 and F_2 are isomorphic as $K[t; Frob]$-modules.*

PROOF. Any algebraic element over K is a zero of an additive polynomial by the fact that any polynomial divides an additive polynomial (see [12]). This shows the first assertion.

Since F_1 and F_2 are p-closed, by Corollary 2.16, their theories as $K[t; Frob]$-modules are given by the theories of their torsion submodules. Since F_1 and F_2 are algebraic over K, $F_1 = F_{1_{tor}}$ and $F_2 = F_{2_{tor}}$. By Proposition 2.13, they are isomorphic. ⊣

COROLLARY 2.35. *Consider a field K of characteristic $p > 0$, p-closed and the following tower of extensions*:

$$K \subseteq L \subsetneq M \subseteq K^{alg}.$$

Then $L \not\equiv M$ as $K[t; Frob]$-modules.

§3. Valued Modules. We are going to generalize the following situation:

Let (U, v) be a valued field of equal characteristic $p > 0$ with $K \subseteq U$ a subfield, on which the valuation v is trivial. We consider the valued group $(U, +, v)$ together with the right action of the ring $R = K[t; Frob]$ of additive polynomials over K. Consider $\tau : v(U) \to v(U)$, $\gamma \mapsto p\gamma$. The valued module structure of (U, v) is the two-sorted structure

$$\langle K, +, (.r)_{r\in R}, v(U), \leqslant, \tau, v, \infty\rangle,$$

where $\langle U, +, (.r)_{r\in R}\rangle$ is the (right)-R-module structure on $(U, +)$ and $\langle v(U), \leqslant, \tau, \infty\rangle$ is the ordered value set structure of $v(U)$ equipped with τ.

Since v is trivial on K, for all $x \in U$ and $a \in U$, we have $v(x^{p^m}a) = v(x^{p^m}) = p^m v(x)$ and if $v(x) \neq 0$ then we have $v(x^{p^m}) \neq v(x^{p^n})$ whenever $n \neq m$. Hence by ultrametric inequality, for an additive polynomial $P(X) := \sum_{i\in I} X^{p^i} a_i$ $(a_i \neq 0)$ over K, for all x with $v(x) \neq 0$ we have

$$v(P(x)) = \min_{i\in I}\{p^i v(x)\} = \begin{cases} \tau^{\min I}(v(x)) & \text{if } v(x) > 0, \\ \tau^{\max I}(v(x)) & \text{if } v(x) < 0. \end{cases}$$

3.1. τ-chains. Let L_{V_0} be the language $\{\leqslant, \tau, \infty\}$ where $\leqslant$ is a binary relation symbol, τ an unary function symbol, and ∞ a constant symbol.

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DEFINITION 3.1. A τ-chain is a L_{V_0}-structure Δ satisfying the following axioms:

1. Δ is linearly ordered by $\leqslant$ with ∞ being its maximum,
2. τ is strictly increasing on Δ and $\tau(\infty) = \infty$,
3. $\forall \gamma \neq \infty \forall \delta \quad (\tau(\gamma) \leqslant \gamma \wedge \delta < \gamma) \to \tau(\delta) < \delta$.

EXAMPLE 3.2. For all integer $n \geqslant 1$ the structure $\langle \mathbb{Z} \cup \{\infty\}, \leqslant, x \mapsto nx \rangle$ satisfies the above axioms.

Note that Axiom 3 of Definition 3.1 implies its dual:

$$\forall \gamma \, \forall \delta \neq \infty \quad (\tau(\gamma) \geqslant \gamma \wedge \gamma < \delta) \to \tau(\delta) > \delta. \tag{2}$$

We denote:

- $\Delta_+ := \{\gamma \in \Delta \mid \tau(\gamma) > \gamma\} \cup \{\infty\}$,
- $\Delta_- := \{\gamma \in \Delta \mid \tau(\gamma) < \gamma\}$.

Thus, Δ_+ is a final segment and Δ_- is an initial segment of Δ. In addition, by Axiom 3 there exists at most one fixed point of τ other than ∞: if it exists it is the unique point θ of Δ such that $\Delta_- < \theta < \Delta_+$. We define:

- the predicate $\delta < \theta$, saying $\delta \in \Delta_-$ and,
- the predicate $\delta > \theta$, saying $\delta \in \Delta_+$.

Similarly the expressions $\delta \leqslant \theta$ or $\delta \geqslant \theta$, does not mean that the cut (Δ_-, Δ_+) is realized in Δ.

3.2. (K-trivially) valued modules. Recall that a valued abelian group is a structure given by

- an abelian group M,
- a linear order Δ with a maximum element ∞,
- a surjective map $v : M \to \Delta$ such that
  - for all $x \in M$, $v(x) = \infty$ if and only if $x = 0$,
  - for all $x, y \in M$, $v(x \pm y) \geqslant \min\{v(x), v(y)\}$.

These axioms imply that for all $x, y \in M$, $v(x \pm y) = \min\{v(x), v(y)\}$, whenever $v(x) \neq v(y)$, and $v(x) = v(-x)$.

In a valued group (G, v), the valuation v induces a topology, a basis of which is given by the *open balls*: these are subsets of G of the form $\{x \in G \mid v(x - a) > \gamma\}$, where $a \in G$ (center) and $\gamma \in v(G)$ (radius). We define *closed balls* as usual by changing $>$ to $\geqslant$.

NOTATION 3.3. *In (G, v), for all $\gamma \in v(G)$ we denote by $G_{\geqslant \gamma}$ (respectively, $G_{>\gamma}$) the closed ball (respectively, the open ball) centered at 0 with radius γ.*

For the rest of this article we fix a field K and we let $R := K[t; \varphi]$ with $\varphi \in \mathrm{End}(K, +, \times, 1, 0)$.

DEFINITION 3.4. Let (M, v) be a valued abelian group with $v : M \to \Delta$. A K-trivially valued module structure on (M, v) is given by a right R-module structure on M such that

1. the function $x \mapsto x.t$ is injective,
2. Δ is a τ-chain,
3. $\forall x \in M$, $v(x.\lambda) = v(x)$ for all $\lambda \in K^\times$,
4. $\forall x \in M$, $v(x.t) = \tau(v(x))$.

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The name K-trivial comes from Axiom 3. In [11] we consider also different ("nontrivial") actions of K but in the present article we will only deal with K-trivially valued modules. Therefore we permit our self to omit the expression K-trivial and say only *valued module* until the end of this article.

REMARK 3.5. When we deal with a valued module (M, v), we write $M_{>\theta}$ or $M_{\geqslant\theta}$ independently of the existence of θ.

REMARK 3.6. One has $v(x) > \theta$ if and only if $v(x.t) > v(x)$, and $v(x) < \theta$ if and only if $v(x.t) < v(x)$.

DEFINITION 3.7. A polynomial $r \in R$ such that t does not divide r is called separable.

REMARK 3.8. Any irreducible $r \neq t$ is separable. An R-module is divisible if and only if it is divisible by separable polynomials and by t, in which case we will say t-divisible for short.

NOTATION 3.9. *Using Equation* 1 *we observe that any nonzero polynomial* $q \in R$ *can be written in the form* $q = t^n s$ *where* s *separable. We set*

$$\deg_{is}(q) := n.$$

LEMMA 3.10. *Let* (M, v) *be a valued module and* $r = \sum_{i\in I} t^i a_i \in R\backslash\{0\}$. *Then, for all* $x \in M$, *we have* $v(x.r) \geqslant \min_{i\in I}\{v(x.t^i a_i)\} = \min_{i\in I}\{\tau^i(v(x))\}$. *In addition for all* $x \in M$,

1. *if* $v(x) > \theta$ *then* $v(x.r) = \tau^{\deg_{is}(r)}(v(x)) = \min_{i\in I}\{\tau^i(v(x))\}$,
2. *if* $v(x) < \theta$ *then* $v(x.r) = \tau^{\deg(r)}(v(x)) = \min_{i\in I}\{\tau^i(v(x))\}$,
3. *if* $v(x.r) > min_{i\in I}\{\tau^i(v(x))\}$ *and* $x \neq 0$ *then* $v(x) = \theta$.

PROOF. Let $x \in M\backslash\{0\}$. Since (M, v) is a valued abelian group we have: $v(\sum x.t^i a_i) \geqslant \min_{i\in I} v(x.t^i a_i)$. Then we have $v(x.t^i a_i) = v(x.t^i) = \tau^i(v(x))$ by Axioms 3 and 4 of Definition 3.4.

1. If $v(x) > \theta$ then $v(x.t) > v(x)$. Now,
$$v(x.t^i a_i) = v(x.t^i) > v(x.t^j) = v(x.t^j a_j)$$
whenever $i > j$. Hence, $v(x.r) = v(\sum x.t^i a_i) = \min_i\{v(x.t^i)\} = \tau^k(v(x))$, where $k = \deg_{is}(r)$.
2. In this case we have $v(x.t) < v(x)$ and hence $v(x.t^i a_i) < v(x.t^j a_j)$ whenever $i > j$. Thus, $v(x.r) = v(\sum x.t^i a_i) = \min_i\{v(x.t^i\} = \tau^n(v(x))$, where $n = \deg(r)$.
3. By points 1. and 2. above, x can only have valuation θ. ⊣

COROLLARY 3.11. *The subsets* $M_{>\theta}$ *and* $M_{\geqslant\theta}$ *are* R*-submodules of* M.

DEFINITION 3.12. Let (G, v) be an abelian valued group. A subgroup of G is called convex if it is the inverse image under v of a nonempty final segment of $v(G)$.

LEMMA 3.13. *A (closed or open) ball centered at* 0 *is a convex subgroup. If* (M, v) *is a valued module then* $M_{\geqslant\theta}$ *and* $M_{>\theta}$ *are convex subgroups of* M.

PROOF. A ball centered at 0 with radius γ is the inverse image under v of the final segment $(\gamma, \infty]$ or $[\gamma, \infty]$. Hence $M_{\geqslant\theta}$ and $M_{>\theta}$ are convex subgroups if $\theta \in v(M)$. Otherwise, 3.5, $M_{\geqslant\theta} = M_{>\theta} = v^{-1}(\Delta_+)$. ⊣

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LEMMA 3.14. *If H is a convex subgroup of an abelian valued group (G, v) then the quotient G/H is valued by v_H, defined by*:

$$v_H(g + H) := \begin{cases} v(g) \text{ if } g \notin H, \\ \infty \text{ otherwise.} \end{cases}$$

PROOF. It is sufficient to verify that v_H is well defined. Let I_H be the final segment of $v(G)$ such that $H = v^{-1}(I_H)$. If $g \notin H$ and $g' \notin H$ are two representatives of the same class, then $v(g) \notin I_H$, $v(g') \notin I_H$, and $g - g' \in H$. Thus $v(g - g') \in I_H$ and $v(g - g') > v(g)$ necessarily. It follows that $v(g) = v(g')$. ⊣

COROLLARY 3.15. *Let (M, v) be a valued module. Then $M/M_{\geqslant\theta}$ is canonically equipped with a valued module structure.*

DEFINITION 3.16. A valued module (M, v) will be called regular if, for all $x \in M$ and nonzero $r = \sum_{i\in I}^{n} t^i a_i \in R$, $(a_i \neq 0)$, we have

$$v(x.r) = \min_i\{v(x.t^i)\} = \min_i\{\tau^i(v(x))\}.$$

If the above equality holds for a pair (x, r) we say that x is regular for r (otherwise irregular for r) and x is said to be regular if it is regular for all r (otherwise irregular).

REMARK 3.17. By 3.10, x is irregular if and only if $v(x) = \theta$ and $v(x.r) \in M_{>\theta}$ for some nonzero r.

REMARK 3.18. Regular modules are necessarily torsion free.

The following lemma follows directly by Corollary 3.15 and by Lemma 3.10.

LEMMA 3.19. *Let (M, v) be a valued module. The submodule $(M_{>\theta}, v)$ and the quotient $\left(M/M_{\geqslant\theta}, v_{M_{\geqslant\theta}}\right)$ are regular valued modules.*

DEFINITION 3.20. We say that a sequence of submodules $(A_i)_{i\in I}$ of M is valuation independent if, for any sequence $(x_i)_{i\in I}$ with $x_i \in A_i$ and for all finite J with $J \subseteq I$, we have

$$v\left(\sum_{i\in J} x_i\right) = \min\{v(x_i) \mid i \in J\}.$$

REMARK 3.21. If $(A_i)_{i\in I}$ are torsion free and valuation independent then any sequence $(x_i)_{i\in I}$ with $x_i \in A_i$, is R-linearly independent.

FACT 3.22. *Let (M, v) be a valued module, $A \subseteq M$ be a regular submodule and B a submodule of M_{tor}. Then A and B are valuation independent.*

PROOF. Let $x \in M_{tor}$, and $a \in A$. Without loss of generality we may assume both x and a are nonzero. If $v(a - x) > v(x) = v(a)$ then $v(a) = \theta$ and since a is regular $v(a.r) = \theta$. But now, for some nonzero r such that $x.r = 0$ we have $\theta = v(a.r) < v((a - x).r)$, a contradiction. ⊣

3.3. Henselian valued modules.

DEFINITION 3.23. A valued module is said to be henselian if it satisfies the following axiom scheme:

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H : $\forall x\ v(x) > \theta \to \exists y\ x = y.r \wedge v(y) > \theta$
for all separable $r \in R$.

We denote by T_h the theory of henselian valued modules.

LEMMA 3.24. *Let $(K \subseteq U, v)$ be an extension of valued fields of characteristic $p > 0$, where v is trivial on K. Then (U, v) canonically inherits a valued $K[t; Frob]$-module structure. In addition, if v is henselian on the field U, then (U, v) is a henselian valued module. Moreover, if U is perfect, then the maximal ideal $\mathcal{M}_U$ associated to v is a divisible $K[t; Frob]$-module.*

PROOF. As usual we interpret $x.t$ as x^p, τ as the map $\tau : v(U) \to v(U)$, $\gamma \mapsto p\gamma$, and θ as $0 \in v(U)$ which makes (U, v) a valued module. Suppose now that (U, v) is henselian as a valued field. Let $q = t^n a_n + \cdots + a_0$ be separable and $y \in U$ such that $v(y) > 0$. Let Q be the associated additive polynomial to q: i.e., $Q(X) = a_n X^{p^n} + \cdots + a_0 X$. Since $a_n, \ldots, a_0 \in K$, they have common valuation 0, and since q is separable, $a_0 \neq 0$. Then, by setting $F(X) := Q(X) - y$, we have $F'(0) = a_0$ and $F(0) = -y$. Since v is henselian, there exists $z \in U$ of valuation > 0 such that $F(z) = 0$. Thus $Q(z) = y$ and $v(z) > 0$. In other words, in the language of valued modules, we have $z.q = y$ and $v(z) > \theta$.

Finally, note that $U_{>\theta} = \mathcal{M}_U$ and if U is perfect, then $U_{>\theta}$ is divisible by t. Since by the above paragraph it is also divisible by separable polynomials, it is divisible. ⊣

LEMMA 3.25. *Any torsion free henselian valued module is regular.*

PROOF. Suppose for a contradiction that we have a henselian torsion free module (M, v), with an irregular element $x \in M$. Then for some nonzero $r \in R$, $v(x.r) > \theta$ and $v(x) = \theta$. Write $r = t^n s$ with s separable. By henselianity there exists $y \in M$, of valuation $> \theta$ such that $y.s = x.r$. Then $y - x.t^n$ is nonzero but annihilated by s. This is a contradiction. ⊣

It is a trivial fact that algebraically closed valued fields are henselian. We could view the notion of divisibility for valued modules as an analogue of the notion of algebraic closedness for fields. But there exist divisible valued modules which are not henselian as it is showed in the following proposition.

PROPOSITION 3.26. *There exist nonhenselian divisible valued modules.*

PROOF. Let U be an algebraically closed field of characteristic $p > 0$, v a non-trivial valuation on U and K a trivially valued subfield. We consider U with its $K[t; Frob]$ valued module structure.

Let $y \in U$, of valuation > 0. Then, by 3.10, y is not a torsion element. Since 1 is annihilated by $(t - 1)$, it is a torsion element (indeed $1.t = 1^p = 1$). Consider, the submodule $A := (1 + y).R$. It is torsion free. Set $x = (1 + y).(t - 1) \in A$. Since $1.(t-1) = 0$, we have also $x = y.(t-1)$. On the other hand, since U is algebraically closed, it is divisible as an R-module. Hence U contains a divisible closure of A. Let the module B be such a closure. By Lemma 2.6 (point 4) B is a torsion free module. Thus $1 \notin B$, hence $y \notin B$: B can not be henselian since y is the unique element of U of valuation > 0 such that $y.(t - 1) = x$. ⊣

The following example shows that there exists valued modules which have nonelementary equivalent (as valued modules) divisible closures.

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EXAMPLE 3.27. Take x as above. Since $v(x) > \theta$ and $U_{>\theta}$ is divisible, $x.R$ has a divisible closure inside $U_{>\theta}$, which is necessarily henselian. This can not be elementary equivalent to B, since B is not henselian.

3.4. Henselian divisible valued modules. The results of this subsection contain the essential information that will be used to establish an Ax-Kochen and Ershov principle.

LEMMA 3.28. *If (M, v) is a henselian divisible valued module, $M_{\geqslant\theta}$ is divisible and $M_{>\theta}$ is divisible torsion free. Hence $M_{\geqslant\theta}$ is a direct summand in M, and $M_{>\theta}$ is a direct summand in $M_{\geqslant\theta}$.*

PROOF. Let $r \in R\backslash\{0\}$, $x \in M_{\geqslant\theta}$ and $y \in M$ such that $y.r = x$. Then by 3.10 $y \in M_{\geqslant\theta}$. Thus $M_{\geqslant\theta}$ is a divisible submodule of M.

Now we show that $M_{>\theta}$ is divisible. Let x be of valuation $> \theta$ and $r \in R$, nonzero. Write $r = t^n.r'$ with r' separable, then there exists, by axiom H, $y \in M_{>\theta}$ such that $x = y.r'$. By t-divisibility of M, we have $y = y'.t^n$ for some $y' \in M_{>\theta}$, hence $y'.r = x$. Moreover $M_{>\theta}$ is torsion free since it is a regular submodule. ⊣

COROLLARY 3.29. *Let (M, v) be a henselian divisible valued module. If $x \in M$ is irregular for r, then there exists a unique couple $(x_{tor}, x_{>\theta})$, with $x_{tor}.r = 0$ and $x_{>\theta} \in M_{>\theta}$ such that*

$$x = x_{tor} + x_{>\theta}.$$

As a consequence,

1. *for all $x \in M$ and $q = \sum_{i\in I} t^i a_i \in R\backslash\{0\}$, there exists regular $y \in M$ such that $x = y.q$ and hence $v(x) = \tau^k(v(y))$, where $k = \deg_{is}(q)$ if $v(x) > \theta$, $k = \deg(q)$ if $v(x) < \theta$, and $k = 0$ if $v(x) = \theta$; in any case*
$$v(x) = \min_{i\in I}\{v(y.t^i a_i)\} = \min_{i\in I}\{\tau^i(v(y))\}.$$
2. *$0 \neq x \in M$ is regular for r (resp. regular) if and only if for all $a \in ann_M(r)$ (resp. for all $a \in M_{tor}$), $v(x - a) = \min\{v(x), \theta\}$.*

PROOF. Take an irregular $x \in M$. Let $r \in R\backslash\{0\}$ be such that $x.r \in M_{>\theta}$ (given by 3.17). Since $M_{>\theta}$ is divisible, $x.r = y.r$ for a unique $y \in M_{>\theta}$ and y is necessarily regular. Since $M_{tor} \cap M_{>\theta} = 0$ the couple $(x - y, y)$ is unique as required. Then it follows from 3.10 that $v(y.r) = \tau^{\deg_{is}(r)}(v(y))$. The first consequence mentioned follows then by 3.10.

The second consequence follows from the observation that,

$$v(x - a) > \min\{v(x), \theta\}\ (a \in ann_M(r))$$

if and only if $v(x) = \theta$ and $v(x - a) > \theta$, if and only if

$$v(x.r) = v((x - a).r) > \tau(\theta) = \theta.$$

That is, if and only if x is irregular for r. ⊣

THEOREM 3.30. *Let (M, v) be a henselian divisible valued module.*

1. *M_{tor} embeds in $M_{\geqslant\theta}/M_{>\theta}$ and for any such embedding the image of M_{tor} is a direct summand.*
2. *The R-modules $M_{\geqslant\theta}/(M_{tor} + M_{>\theta})$, $M_{>\theta}$ and $M/M_{\geqslant\theta}$ are torsion free and divisible.*

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3. *M can be written as a direct sum*:

$$M_{tor} \oplus M_\theta \oplus M_{>\theta} \oplus M_-,$$

where M_θ is isomorphic to $M_{\geqslant\theta}/(M_{tor} + M_{>\theta})$ and (M_-, v) is isomorphic as a valued module to $(M/M_{\geqslant\theta}, v_{M_{\geqslant\theta}})$. This decomposition is valuation independent and each member of this decomposition, except M_{tor}, is a regular D-vector space.

PROOF. 1. Since all nonzero elements of M_{tor} are of valuation θ, the canonical surjection $(M_{\geqslant\theta} \to M_{\geqslant\theta}/M_{>\theta})$ induces an embedding of M_{tor} in $M_{\geqslant\theta}/M_{>\theta}$. Since M_{tor} is divisible its image in $M_{\geqslant\theta}/M_{>\theta}$ is a direct summand.
2. The fact that $M_{>\theta}$ is torsion free and divisible is given by 3.28. The divisibility of $M/M_{\geqslant\theta}$ is induced by the divisibility of M, it is torsion free by 3.19 since it is a regular valued module. Also $M_{\geqslant\theta}/(M_{tor} + M_{>\theta})$ is divisible by divisibility of M. We will now show that $M_{\geqslant\theta}/(M_{tor} + M_{>\theta})$ is torsion free. Let $x \in M_{\geqslant\theta}$ such that $x.r \in M_{tor} + M_{>\theta}$ for some $r \neq 0$. Since M_{tor} is divisible, we have $x.r - z'.r = (x - z').r \in M_{>\theta}$ for some $z' \in M_{tor}$. If $x - z' \notin M_{>\theta}$ then $x - z'$ is irregular. In this case, by Corollary 3.29, $x - z' \in M_{tor} + M_{>\theta}$ and hence $x \in M_{tor} + M_{>\theta}$.
3. Since $M_{\geqslant\theta}$ and $M_{>\theta}$ are both divisible, $M \simeq M_{\geqslant\theta} \oplus M/M_{\geqslant\theta}$ and

$$M_{\geqslant\theta} \simeq M_{>\theta} \oplus M_{\geqslant\theta}/M_{>\theta}$$

and we get $M_{\geqslant\theta}/M_{>\theta} \simeq M_{tor} \oplus \big(M_{\geqslant\theta}/(M_{tor} + M_{>\theta})\big)$.

Take M_- such that $M_- \oplus M_{\geqslant\theta} = M$. If $x \in M_-\backslash\{0\}$ then $v(x) < \theta$ and by the definition of the quotient valuation $v_{M_{\geqslant\theta}}$, (M_-, v) and $(M/M_{\geqslant\theta}, v_{M_{\geqslant\theta}})$ are isomorphic as valued modules.
It remains to show that this decomposition is valuation independent. For this it is enough to see that M_{tor} and M_θ are valuation independent. Suppose that there exist $x \in M_{tor}$, $y \in M_{\geqslant\theta}\backslash(M_{tor} + M_{>\theta})$ with $v(x - y) > \theta$. Then, for some $r \in R\backslash\{0\}$ annihilating x, we have $v((x - y).r) = v(y.r) > \theta$: this is impossible since $M_{\geqslant\theta}/(M_{tor} + M_{>\theta})$ is torsion free. ⊣

REMARK 3.31. Take a decomposition

$$M = M_{tor} \oplus M_\theta \oplus M_{>\theta} \oplus M_-$$

then the regular elements $M_\theta \oplus M_{tor} \oplus M_-$ are all except the ones lying in M_{tor}.

COROLLARY 3.32. *Let N be a regular R-module. For all $x, y \in N$ and in particular, for all $i \in \{\theta, >\theta, -\}$, and $x, y \in M_i\backslash\{0\}$ the following are equivalent*:

- $v(x - y) > v(x) = v(y)$,
- *there is some $r \in R\backslash\{0\}$ such that $v(x.r - y.r) > v(x.r) = v(y.r)$,*
- *for all $r \in R\backslash\{0\}$ we have $v(x.r - y.r) > v(x.r) = v(y.r)$.*

PROOF. Follows by the fact that each M_i is regular. ⊣

REMARK 3.33. Let $(K \subseteq U, v)$ be an extension of fields of characteristic $p > 0$, where U is algebraically closed and $|K|^+$-saturated and v is trivial on K. Then every nonzero element of the set U_θ realizes the generic type of the valuation ring of U, i.e., the type $\{x \in \mathcal{O}_U \wedge x \notin B \mid B \subsetneq \mathcal{O}_U \text{ is } K\text{-definable}\}$. Note that here definable means definable in the language $\{0, 1, +, -, \times, \mathcal{O}\}$ of valued fields.

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3.5. Embedding theorems and A-K,E principles. Let (M, v) be a valued module. If A is a submodule of M, Corollary 3.27 shows that the various divisible closures of A in M, while being isomorphic as R -modules, may not be elementary equivalent as valued modules. This is the most important phenomenon to which we will pay attention.

DEFINITION 3.34. Let (M, v) be a divisible valued module and $A \subseteq M$ a submodule of M. Then we define

$$\hat{A} := \{y \in M \mid y.r \in A \text{ for some } r \in R\backslash\{0\}\}.$$

LEMMA 3.35. *The submodule $\hat{A}$ is the unique divisible closure of $A + M_{tor}$ inside M. In particular, for all $r \in R\backslash\{0\}$ and all $x \in M$, if $x.r \in \hat{A}$ then $x \in \hat{A}$. Moreover, if (M, v) is henselian then $\hat{A}$ is henselian.*

PROOF. By construction, $\hat{A}$ is a divisible submodule and all divisible submodules of M containing A and M_{tor} contains $\hat{A}$. This gives the uniqueness. Also by the definition of $\hat{A}$, if $x \in \hat{A}$ and $r \in R\backslash\{0\}$, $\hat{A}$ contain all y such that $x = y.r$ since $\hat{A}$ contains M_{tor}. Now, if (M, v) is henselian and $x \in \hat{A}$ of valuation $> \theta$, then $\hat{A}$ contains the unique element $y_{>\theta} \in M_{>\theta}$ such that $y_{>\theta}.r = x$. This shows that $\hat{A}$ is henselian. ⊣

REMARK 3.36. The set $v(\hat{A})$ is an L_{V_0}-substructure of $v(M)$, it is the closure of $v(A)$ under the function τ^{-1}.

REMARK 3.37. Consider a henselian divisible valued module (M, v). Then,

1. $M_{\geqslant\theta}/M_{>\theta} = 0$ or is infinite,
2. if $\gamma \neq \theta$ and r is nonzero with $\deg_{is}(r) = k$ and $\deg(r) = n$ then
   (*a*) if $\gamma > \theta$ the multiplication by r

$$.r : M_{\geqslant\gamma} \to M_{\geqslant\tau^k(\gamma)}$$

   is a bijection inducing a bijection

$$M_{\geqslant\gamma}/M_{>\gamma} \to M_{\geqslant\tau^k(\gamma)}/M_{>\tau^k(\gamma)}.$$

   (*b*) if $\gamma < \theta$ the multiplication by r

$$.r : M_{\geqslant\gamma} \to M_{\geqslant\tau^n(\gamma)}$$

   is a bijection inducing a bijection

$$M_{\geqslant\gamma}/M_{>\gamma} \to M_{\geqslant\tau^n(\gamma)}/M_{>\tau^n(\gamma)}.$$

PROOF. If $\theta \in v(M)$ then M_θ or M_{tor} is nonempty, divisible and embedded into $M_{\geqslant\theta}$. The second assertion follows from divisibility and regularity of any element of valuation $\neq \theta$. ⊣

– For each $n \in \mathbb{N}$, we add a unary predicate R_n into our language L_{V_0} and let L_V be this enrichment of L_{V_0}. We denote $L_0 := L_{Mod}(R) \cup L_{V_0} \cup \{v\}$ and $L := L_{Mod}(R) \cup L_V \cup \{v\}$. In every valued module (M, v) and for all $\gamma \in v(M)\backslash\{\infty\}$, $R_n(\gamma)$ will be interpreted by the equivalence

$$R_n(\gamma) \Leftrightarrow |M_{\geqslant\gamma}/M_{>\gamma}| \geqslant n.$$

PROPOSITION 3.38. *Let (M, v) and (N, w) be divisible henselian valued modules such that $\eta_M = \eta_N$ and N is $|M|^+$-saturated. Let $(A, \Delta_1) \subseteq (M, v(M))$ and $(B, \Delta_2) \subseteq (N, v(N))$ be L-substructures of M and N, respectively, which are L-isomorphic via*

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$\mathbf{f} = (f, f_v)$. *Then there is an L-embedding extending* $\mathbf{f}$ *to* $(\hat{A}, v(\hat{A}) \cup \Delta_1)$ *to* $(N, v(N))$, *denoted by* $\hat{\mathbf{f}} = (\hat{f}, \hat{f}_v)$, *having image* $(\hat{B}, v(\hat{B}) \cup \Delta_2)$.

PROOF. STEP 1. By Remark 3.36 and the fact that τ is strictly increasing, f_v extends uniquely to an L_{V_0}-embedding

$$\hat{f}_v : v(\hat{A}) \cup \Delta_1 \to w(\hat{B}) \cup \Delta_2.$$

From now on we may assume that $v(\hat{A}) \subseteq \Delta_1$

The steps 2,3 and 4 below are dedicated to extend the module sort isomorphism to $\hat{f}$ such that $\hat{\mathbf{f}}$ be an L_0-embedding.

Set $C = (\hat{A})_- \oplus (\hat{A})_\theta \oplus M_{tor}$ such that

$$\hat{A} = C \oplus (\hat{A})_{>\theta}$$

and take E similarly such that

$$\hat{B} = E \oplus (\hat{B})_{>\theta}.$$

Denote by $\pi_C(A)$ (respectively, by $\pi_E(B)$) the projection of A on C (respectively, the projection of B on E).

STEP 2. We will define the partial isomorphism of R-modules $h : \pi_C(A) \to \pi_E(B)$ such that $w \circ h = f_v \circ v \restriction_{\pi_C(A)}$.

First, note that $A \cap (\hat{A})_{>\theta} \neq \emptyset$ if and only if there exist $a \in A$ such that $v(a) > \theta$. Now, denote by $f_{>\theta}$ the restriction of f on $A \cap (\hat{A})_{>\theta}$. Since $(\hat{A})_{>\theta}$ is a D-vector space, $f_{>\theta}$ extends uniquely to $\hat{f}_{>\theta} : (\hat{A})_{>\theta} \to (\hat{B})_{>\theta}$. We define h on $\pi_C(A)$: Let $c \in \pi_C(A)$ and let $a_{>\theta}$ be such that $a_{>\theta} \in (\hat{A})_{>\theta}$ and $c + a_{>\theta} \in A$. Then we set

$$h(c) := f(c + a_{>\theta}) - \hat{f}_{>\theta}(a_{>\theta}).$$

We check that this definition depends only on c, hence h is a well defined map; moreover, h is injective and have its image in E:

Let $d \in (\hat{A})_{>\theta}$ be such that $c + d \in A$. Note that $d - a_{>\theta} \in A \cap (\hat{A})_{>\theta}$. We have

$$\begin{aligned} f(c+d) - \hat{f}_{>\theta}(d) - \Big(f(c + a_{>\theta}) - \hat{f}_{>\theta}(a_{>\theta})\Big) \\ = f(d - a_{>\theta}) - \hat{f}_{>\theta}(d - a_{>\theta}) = f(d - a_{>\theta}) - f(d - a_{>\theta}) = 0 \end{aligned}$$

hence h is well-defined. By the definition of h, h is a morphism of R-modules. Moreover h is injective: If for some $d \in (\hat{A})_{>\theta}$, $c+d \in A$ and $f(c+d) - \hat{f}_{>\theta}(d) = 0$ then $\hat{f}_{>\theta}(d) \in B$ hence $d \in A$ and $\hat{f}_{>\theta}(d) = f_{>\theta}(d)$. It follows that $c \in A$ and $f(c) = 0$ hence $c = 0$. We will now check that the image of h is exactly $\pi_E(B)$. Let $y \in E$ and $z \in B_{>\theta}$ be such that $y + z \in B$. Let $d := \hat{f}^{-1}_{>\theta}(z)$. Since $\hat{f}_{>\theta}$ uniquely extends $f_{>\theta}$ we have $\hat{f}^{-1}(y + z) - d \in C$ with image equal to y under h.

Now h satisfies $w(h(c)) = \hat{f}_v \circ v(c)$ for all $c \in \pi_C(A)$: let $c \in \pi_C(A)$ and $\gamma = v(c)$. If c is a torsion element then $f(c)$ is also a torsion element hence both have valuation θ (respectively, in C and in E). Suppose $c \notin M_{tor}$. Let r the minimal polynomial c over A. We have $\hat{f}_v(v(c.r)) = w(f(c.r)) = w(h(c.r) = w(h(c).r)$. Recall by Remark 3.31 that c is regular. Since $h(c) \in E$ and is not a torsion element, 3.31 $h(c)$ is regular for r. Since $\hat{f}_v$ commutes with τ we have

$$\hat{f}_v(v(c)) = w(h(c))$$

as required.

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STEP 3. Note that the R-module C is the unique divisible closure of $\pi_C(A)+M_{tor}$, since clearly we have $C=\widehat{\pi_C(A)}$. By Corollary 2.19, h extends to an embedding of C into E, denoting again by h this extension, h satisfies $w(h(x))=f_v(v(x))$ for all $x\in C$, exactly as in the previous step.

STEP 4. Set $\hat{f}:=h\oplus\hat{f}_{>\theta}$ and $\hat{\mathbf{f}}:=(\hat{f},\hat{f}_v)$. We show now that $\hat{\mathbf{f}}$ is an L_0-embedding. Note that $v(\hat{A})=v(C)\cup v(\hat{A})_{>\theta}$. Let h_v be the restriction of $\hat{f}_v$ to $v(C)$ and $\hat{f}_{v>\theta}$ be the restriction of f_v to $v(\hat{A})_{>\theta}$. Then by the above construction, (h,h_v) and $(\hat{f}_{>\theta},\hat{f}_{v>\theta})$ are both L_0-embeddings. Since the sums $C\oplus(\hat{A})_{>\theta}$ and $E\oplus(\hat{B})_{>\theta}$ are valuation independent we have $w(h(x)+\hat{f}_{>\theta}(x))=\min\{w(h(x),w(\hat{f}_{>\theta}(x))\}$ for all $x\in\hat{A}$. Hence $\hat{\mathbf{f}}$ is an L_0-embedding.

STEP 5. In order to finish the proof, we prove that $\hat{\mathbf{f}}$ preserves the predicates R_n. By Remark 3.36, $v(\hat{A})$ is the closure of $v(A)$ by τ^{-1} and by Remark 3.37 each quotient $\hat{A}_{\geqslant\gamma}/\hat{A}_{>\gamma}$ is entirely determined by a quotient of the form $A_{\geqslant\tau^k(\gamma)}/A_{>\tau^k(\gamma)}$. This finishes the proof. ⊣

NOTATION 3.39. *For $a\in M$, where M is valued by v, and $\gamma\in v(M)$, we denote by $M_\gamma(a)$ the quotient of the closed ball $\{u\in M\mid v(u-a)\geqslant\gamma\}$ given by the equivalence relation*

$$u\sim u'\Leftrightarrow v(u-u')>\gamma.$$

Note that

$$M_\gamma(a)=|M_{\geqslant\gamma}/M_{>\gamma}|.$$

PROPOSITION 3.40. *Let (M,v) and (N,w) be as before, A be a submodule of M such that $v(A)=v(M)$ and $\mathbf{f}=(f,f_v):(A,v)\to(N,w)$ an L-embedding. Suppose that N is $|M|^+$-saturated. Then we can extend $\mathbf{f}$ to an L-embedding of M into N.*

PROOF. Note first that we can extend $\mathbf{f}$ on $\hat{A}$ by Proposition 3.38. So we assume $\hat{A}=A$ with image $f(A)=B$ and we may also assume $B=\hat{B}$. Take $x\in M\backslash A$. It suffices to extend $\mathbf{f}$ to $A\oplus x.R$. In that case, by the above proposition, we can extend $\mathbf{f}$ to the divisible closure (unique by the fact that $A\supseteq M_{tor}$) of $A\oplus x.R$. Thus, by transfinite induction, $\mathbf{f}$ extends to M.

For all $a\in A$, we set $\gamma_a=f_v(v(x-a))$ and we define the set of formulae of the variable Y

$$p(Y):=\{\gamma_a=w(Y-b)\mid b=f(a),\ a\in A\}.$$

We first show that if $y\in N$ realizes $p(Y)$, then we can extend $\mathbf{f}$ to an L-embedding : $A\oplus x.R\to N$ sending x to y. Take $y\in N$ realizing $p(Y)$. Since A is divisible and $x\notin A$ then $y\notin B$, in particular for all $r\in R\backslash\{0\}$, $y.r\notin B$. We set $\tilde{f}(a+x.r)=f(a)+y.r$. Since $v(A)=v(M)$ it suffices to see that $\tilde{f}$ is compatible with f_v, i.e., to check that one has $f_v(v(a-x.r))=w(f(a)-y.r)$, for all $a\in A$ and $r\in R\backslash\{0\}$.

CLAIM. For $a\in A$ and $r\neq 0$, $x-a$ is regular for r if and only if $y-f(a)$ is regular for r.

PROOF OF THE CLAIM. If $y-f(a)$ is irregular for some r, then for some $b\in ann_N(r)\subseteq B$,

$$w(y-f(a)-b)>\theta$$

by 3.10 (Consequence 2). Hence $v(x-a-f^{-1}(b))>\theta$ by the choice of y. This means $x-a$ is irregular by 3.10. The converse can be proven in the same way.

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Let $a \in A$, $r \in R\backslash\{0\}$. By Corollary 3.29 (point 1), we can take a' such that $x.r - a = (x - a').r$ and $x - a'$ regular for r. Hence $f(a) = f(a').r$ and $f(a') - y$ is regular for r by the Claim above. Since f_v commutes with τ we have $f_v(v(a - x.r)) = w(f(a) - y.r)$ as required. The Claim is now proved.

It remains to prove that $p(Y)$ is realized in N. We will show that it is finitely consistent. Let $\alpha = \{a_1, \ldots, a_m\} \subseteq A$. We will find $y \in N$ such that for all $1 \leqslant i \leqslant m$, $\gamma_i := f_v(v(x - a_i))$ is equal to $w(y - f(a_i))$. Let $\sim$ be the equivalence relation on α defined by $a_i \sim a_j$ if and only if $\gamma_i = \gamma_j$. We will first observe that, without loss of generality, we can assume that there is only one class under this equivalence relation: In fact, it is enough to consider $\beta \subseteq \alpha$ be the equivalence class of $\gamma := \max\{\gamma_i\}_{i \in \{1,\ldots,m\}}$. If $a \in \beta$ and $a' \notin \beta$ then

$$f_v(v(x - a')) = f_v(v(x - a + a - a')) = f_v(v(a - a')) = w(f(a) - f(a')) \\ = w(f(a) - y + y - f(a')) = w(f(a') - y).$$

Set $\delta := f_v^{-1}(\gamma)$, with the assumption above, it is the common value of the $v(x - a)$ for $a \in \alpha$. Now, we can choose an element $y \in N$ such that for all $a \in \alpha$

$$w(y - f(a)) = \gamma$$

since $|M_\delta(a)| \leqslant |N_\gamma(f(a))|$, f_v preserves the predicate R_n and N is $|M|^+$-saturated. ⊣

To prove Theorem 1.3 we recall the following general result.

PROPOSITION 3.41. *Let $\mathcal{L}$ be a language containing a constant symbol, $\mathcal{T}$ an $\mathcal{L}$-theory and Θ a set of formulas of $\mathcal{L}$ closed under boolean combinations and containing all quantifier free formulas. Suppose that for all $M, N \models \mathcal{T}$ where N is $|M|^+$-saturated and for all substructure $A \subseteq M$, whenever $f : A \to N$ is an $\mathcal{L}$-embedding preserving formulas of Θ, there exists an $\mathcal{L}$-embedding $g : M \to N$ which extends f and preserves Θ. Then every formula of $\mathcal{L}$ is equivalent to a formula of Θ.*

PROOF. If Θ is the set of all quantifier-free formulas this is a well known fact (cf. [10] Proposition 4.3.28). The general case reduces to this case after adding to the language the predicates P_ϕ for each $\phi \in \Theta$ and enriching the theory $\mathcal{T}$ by the set of sentences $\{\forall x \ \ P_\phi(x) \leftrightarrow \phi(x) \mid \phi \in \Theta\}$. ⊣

Let $(F, v(F))$ be a nonzero henselian divisible R-module. Let T_v be the complete theory of $v(F)$ in the language L_V and Tor_F be the $L_{Mod}(R)$-theory of nonzero divisible R-modules with the extra statements of the form $\eta_F(r) = n$ or sets of statements expressing $|\eta_F(r)| = \infty$ where

$$\eta_F(r) = |\{x \in F \mid x.r = 0\}|.$$

By Theorem 1.2 this is the complete theory of (the pure module) F and also of F_{tor} if it is nonzero by Lemma 2.4 (if $F_{tor} = 0$, Tor_F is again complete since it is the theory of infinite D-vector spaces in which case for all nonzero r, $\eta_F(r) = 1$ and $T_h \models \mathrm{Tor}_F$). Similarly to the case of valued fields, Tor_F can be seen as *the residual theory* of (F, v) since F_{tor} embeds in $F_{\geqslant\theta}/F_{>\theta}$.

We set $\mathbf{T} := T_h \cup T_v \cup \mathrm{Tor}_F$ and Θ_v the set of L-formulas of the form:

$$\varphi(\bar{x}) \wedge Q\bar{y}_1 \ \ \psi(\bar{y}_1, \bar{y}_2, v(t_1(\bar{x})), \ldots, v(t_k(\bar{x}))),$$

where:

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– T_h is the theory of henselian valued R-modules,

– φ is quantifier-free in $L_{Mod}(R)$, ψ is quantifier-free in L_V and Q is a sequence of quantifiers over $\bar{y}_1$,

– $\bar{x}$ is a tuple of variables of the module sort, $\bar{y}_1, \bar{y}_2$ are tuples of variables of the value set sort, and the t_i are $L_{Mod}(R)$-terms.

Theorem 1.3 follows now by the following theorem:

THEOREM 3.42. *The theory* **T** *is complete and eliminates quantifiers on the module sort*: *any formula of L is equivalent modulo* **T** *to a formula of* Θ_v.

PROOF. Take $(M, v), (N, w) \models \mathbf{T}$ such that N is $|M|^+$-saturated. Let $\mathbf{f} = (f, f_v) : (A, \Delta) \to N$ be an L-embedding preserving formulas of Θ_v where (A, Δ) is a substructure of (M, v). Note that the condition $\theta \in v(M)$ is described by the L_{V_0}-theory of $v(F)$. Furthermore, since f_v preserves in particular all L_V-formulas, f_v is partial elementary and extends to an elementary embedding of $v(M)$ into $v(N)$ by saturation hypothesis. This extension will denoted as f_v as well. Hence we may assume $\Delta = v(M)$ (but v is not necessarily surjective).

On the other hand by Proposition 3.38, there is an L-embedding $\hat{\mathbf{f}} = (\hat{f}, \hat{f}_v)$ extending $\mathbf{f}$ to $(\hat{A}, v(M))$. Note that any element $\gamma \in v(\hat{A}) \backslash \{\theta\}$ is $v(A)$-definable by a formula of the form $\tau^k(\gamma) = \gamma_a$ for some integer k and some $\gamma_a \in v(A)$. This implies that $\hat{\mathbf{f}}$ preserves the formulas from Θ_v. Thus from now on we can assume that $A = \hat{A}$.

– Extending $\mathbf{f}$ to an (U, v) such that $v(U) = v(M)$. Let $\gamma \in v(M) \backslash A$, $x \in M$ of valuation γ and $y \in N$ such that $w(y) = f_v(\gamma)$. Since $v(A)$ is closed by τ and τ^{-1}, we have, for all nonzero $r \in R$, $v(x.r) \notin v(A)$. It follows that $v(a + x.r) = \min\{v(a), v(x.r)\}$ for all $r \in R$ and $a \in A$. Note that x is regular. In fact, either M is torsion-free and hence M is regular by Lemma 3.25, or, $M_{tor} \subseteq A$ and $\gamma \neq \theta$. Thus $v(x.r) = \tau^k(v(x))$ for some integer k depending only on γ and r. Taking into account that f_v is an elementary function, the map g defined by

$$g(a + x.r) := f(a) + y.r \quad (r \in R, a \in A)$$

yields that $\mathbf{g} = (g, f_v)$ is a partial L-isomorphism between $(A \oplus x.R, v)$ and $(B \oplus y.R, w)$ and $\mathbf{g}$ preserves the set Θ_v by the above discussion. By Proposition 3.38, $\mathbf{g}$ extends to $\widehat{A + x.R}$. Hence, by using the same argument, we can extend $\mathbf{g}$ to a model U such that $v(U) = v(M)$.

Now, by Proposition 3.40 we can extend $\mathbf{g}$ to M and hence we get quantifier elimination up to Θ_v by Theorem 3.41.

It follows that any L-sentence is equivalent modulo **T** to an L_V-sentence. Since by the definition of **T** any two models have L_V-elementary equivalent value sets, they are in fact L-elementary equivalent. This gives the completeness of **T** as required. ⊣

COROLLARY 3.43. *Theorem* 1.3 *is a consequence of the completeness of* **T**. *Moreover*,

- (A-K,E $\preceq$): *Let* $(M \subseteq N, v)$ *be an extension of nonzero henselian divisible valued modules such that the inclusion* $(M_{tor} \subseteq N_{tor})$ *is* $L_{Mod}(R)$*-elementary and the inclusion* $(v(M) \subseteq v(N))$ *is* L_V*-elementary. Then the inclusion* $(M \subseteq N, v)$ *is L-elementary.*

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