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Quasi-Markovian transformations

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Abstract. Let (f, α) be the process given by an endomorphism f and by a finite partition $\alpha = \{A\}_{i=1}^{s}$ of a Lebesgue space. Let $E(f, \alpha)$ be the set of densities of absolutely continuous invariant measures for skew products with the base (f, α). We prove that a process (f, α) is Markovian if and only if $E(f, \alpha) \subset \{g : g = \sum_{i=1}^{s} 1_{A_i} \otimes g_i\}$. If f is the Lasota–Yorke or Misiurewicz type map with the Markovian partition α, then (f, α) is quasi-Markovian, i.e. $E(f, \alpha) \subset \{g : \operatorname{supp} g = \bigcup_{i=1}^{s} A_i \times B_i\}$. Moreover, we give the characterization of quasi-Markovian processes.

0. *Introduction*

Let f be a positive non-singular measure-preserving transformation of a probabilistic Lebesgue space $(X, \mathcal{A}, p)$. Let $\alpha = \{A_i\}_{i=1}^{s}$ be a finite measurable partition of X and let $a(x)$ be the mapping from X to $\{1, \ldots, s\}$ defined by $a(x) = i$ when $x \in A_i$, $i = 1, \ldots, s$. The pair (f, α) will be called the process. This is a stationary stochastic process given by the sequence of random variables $a(f^n(x))$, $n = 0, 1, \ldots$. The process (f, α) is the object of our investigations. Therefore we restrict our considerations to the factor transformation $f | \bigvee_{i=0}^{\infty} f^{-i}\alpha$ which from now on will be denoted by f.

Let $\{T_i\}_{i=1}^{s}$ be a finite family of positive and negative non-singular transformations of a probability space $(Y, \mathcal{B}, m)$. The process (f, α) and the family $\{T_i\}_{i=1}^{s}$ define the skew product transformation

$$T(x, y) = (f(x), T_{a(x)}y).$$

Let P_T denote the Frobenius–Perron operator for T, i.e.

$$P_T g = \frac{d}{dp \times m} \int_{T^{-1}(\cdot)} g \, dp \times m, \quad \text{for } g \in L_1(p \times m).$$

We say that a function $g \in L_1(p \times m)$ is a density if $g \geq 0$ and $\|g\|_1 = 1$.

Let

$$\begin{aligned} E(f, \alpha) \quad = \quad & \{g : \text{there exists a probabilistic Lebesgue space } (Y, \mathcal{B}, m) \text{ and a} \\ & \text{family of positive and negative non-singular transformations } \{T_i\}_{i=1}^{s} \\ & \text{such that } P_T g = g \text{ and } g \text{ is a density}\}. \end{aligned}$$

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In fact $E(f,\alpha)$ is the class of densities of absolutely continuous invariant measures (a.c.i.m.) for skew products with the base (f,α). We are interested in relations between the structure of the set $E(f,\alpha)$ and the process (f,α). Morita [**11**] proved that if (f,α) is a Bernoulli process then $E(f,\alpha)\subset\{g : g(x,y)=g(y)\}$. This property of the Bernoulli processes has been applied independently by other authors (see, for instance, [**1**], [**3**], [**13**]). Kowalski [**5**] showed that if (f,α) is a Markov process then

$$E(f,\alpha)\subset\left\{g : g(x,y)=\sum_{i=1}^{s}1_{A_i}(x)g_i(y)\right\}. \tag{0.1}$$

In §1 we prove the validity of the inverse implications. In §2 we generalize the above results as follows.

Definition. We say that the process (f,α) is quasi-Markovian (q.m.p.) if

$$E(f,\alpha)\subset\left\{g : \operatorname{supp} g=\bigcup_{i=1}^{s}A_i\times B_i\right\}.$$

Here $\operatorname{supp} g=\{(x,y) : g(x,y)>0\}$.

We obtain the following results about the q.m.p.

1. If (f,α) is q.m.p. then for every T and every T-eigenfunction $H\in L_1(\mu)$,
$$H1_{D_\mu}=\sum_{i=1}^{s}1_{A_i}\otimes g_i,\quad p\times m \text{ a.e.}$$
Here μ is a T-a.c.i.m. and $D_\mu=\operatorname{supp}\mu$.
2. If (f,α) and (g,β) are q.m.p. then $(f\times g,\alpha\times\beta)$ is also q.m.p.
3. If (f,α) is a q.m.p. then the partition α is Markovian (i.e. if $p(f(A_i)\cap A_j)>0$ then $p(A_j-f(A_i))=0$ for $i,j=1,\dots,s$).
4. By using the notion of a quasi-eigenfunction for f we get the description of eigenfunctions for the skew product T where T_i, $i=1,\dots,s$, are commuting automorphisms or elements of some compact group.

In §3 it is shown that the Lasota–Yorke and Misiurewicz type transformations with the Markovian partition are q.m.p. By Conclusion 6 we observe that if α generates $\mathcal{A}$ then there are no non-trivial q.m.p. in the class of automorphisms. However, the skew products in which the base transformation and the fibre transformations are automorphisms have been considered by many authors (the Bernoulli base has been considered by Kalikow [**4**], Meilijson [**8**] and Thouvenot [**17**]). The results of our paper imply directly the description of eigenfunctions for skew products as above when the process $(\bar f,\bar\alpha)$ is the natural extension to an automorphism of some q.m.p.

1. *Markovian processes*

We restrict our considerations to eigenfunctions of the eigenvalue 1, which are elements of the set $D=\{g : g\in L_1(p\times m), g\geq 0, \|g\|_1=1\}$, where the space $(Y,\mathcal{B},m)$ is fixed. According to condition (0.1) we consider T-a.c.i.m. of the form $\sum_{i=1}^{s}p_i\times m_i$, where $p_i(A)=p(A_i\cap A)$ for $A\in\mathcal{A}$, $A_i\in\alpha$ and $\{m_i\}_{i=1}^{s}$ is a family of probabilistic measures which are absolutely continuous on $\mathcal{B}$. We will denote by m_iT_i the measure defined as follows: $m_iT_i(B)=m_i(T_i^{-1}B)$ for $B\in\mathcal{B}$, $i=1,\dots,s$.

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LEMMA 1. *The skew product T preserves the measure $\mu = \sum_{i=1}^{s} p_i \times m_i$ if and only if*

$$\sum_{i=1}^{s} p(A_i|f^{-1}A)m_i T_i = m_r \quad \text{for every } A \subset A_r,\ r = 1, \ldots, s. \tag{1.1}$$

This result is an easy consequence of the definition of the T invariance.

Conclusion 1. If the system of measures $\{m_i T_i\}_{i=1}^{s}$ is linearly independent and T preserves the measure $\sum_{i=1}^{s} p_i \times m_i$, then (f, α) is a Markovian process.

Proof. By putting the set A_r as A in (1.1) and by using linear independence of the measures $\{m_i T_i\}_{i=1}^{s}$ we get

$$p(A_i|f^{-1}A) = p(A_i|f^{-1}A_r) \quad \text{for } A \subset A_r, r = 1, \ldots, s. \qquad \square$$

Conclusion 2. If the system of measures $\{mT_i\}_{i=1}^{s}$ is independent and T preserves the measure $p \times m$, then (f, α) is the Bernoulli process.

Proof. By (1.1) we get $m = \sum_{i=1}^{s} p(A_i|f^{-1}A)mT_i$ and $m = \sum_{i=1}^{s} p(A_i)mT_i$. The linear independence of the set $\{mT_i\}_{i=1}^{s}$ implies $p(A_i|f^{-1}A) = p(A_i)$, for every A. $\square$

Conclusion 3. If $s = 2$ and T is a generalized skew product, i.e. T preserves the measure $p \times m$ and T_1 does not preserve m, then (f, α) is a Bernoulli process.

Conclusion 4. If f is an automorphism, then T preserves the measure $\sum_{i=1}^{s} p_i \times m_i$, where $m_i \neq m_j$ for $i \neq j$, if and only if $f(\alpha) = \alpha$ and $m_k = m_j T_j$ for every pair (k, j) such that $f(A_j) = A_k$.

Proof. From (1.1) we conclude that if $f^{-1}A_k \cap A_j \neq \emptyset$ then $m_k = m_j T_j$. This gives $f^{-1}A_k = A_j$. $\square$

LEMMA 2. *Let (f, α) be a fixed process. If for any skew product T every T-a.c.i.m. has the form $\sum_{i=1}^{s} p_i \times m_i$ $(p \times m_1)$, then (f, α) is a Markov (Bernoulli) process.*

Proof. Our proof is based on the construction of the skew product T such that the assumptions of Conclusion 1 (2, respectively) hold. In order to attain this claim we define the following family of transformations:

$$T_i(x) = 4(x - j/4s) + (i-1)/s, \quad \text{when } x \in [j/4s, (j+1)/4s),$$
$$\text{for } j = 0, 1, \ldots, 4s-1 \text{ and } i = 1, \ldots, s.$$

The skew product T defined by the above family possesses T-a.c.i.m. (see [**10**; Theorem 2.1]). By assumption, this measure has the form $\sum_{i=1}^{s} p_i \times m_i$. Let $dm_i/dm = f_i$. Then $dm_i T_i/dm = P_{T_i} f_i, i = 1, \ldots, s$. By the definition of T_i, supp $P_{T_i} D \subset [(i-1)/s, i/s]$. Therefore the system of measures $\{m_i T_i\}_{i=1}^{s}$ is linearly independent. From this and Conclusion 1 we obtain our statement. The proof of the case when the invariant measure has the form $p \times m_1$ is similar. $\square$

By using Lemma 1 from [**5**] and Theorem 3.1 from [**11**] we can reformulate the above lemma as follows.

THEOREM 1. *A process (f, α) is Markovian (Bernoullian) if and only if*

$$E(f, \alpha) \subset \left\{ g : g = \sum_{i=1}^{s} 1_{A_i} \otimes g_i \right\}, \quad (E(f, \alpha) \subset L_1(m)).$$

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2. *Quasi-Markovian processes*

We present the first result characterizing q.m.p.

THEOREM 2. *If (f, α) is q.m.p., then for every T and for every T-eigenfunction H from $L_1(\mu)$*

$$H1_{D_\mu} = \sum_{i=1}^{s} 1_{A_i} \otimes g_i, \quad p \times m \text{ a.e.} \tag{2.1}$$

Here μ is any T-a.c.i.m. and $D_\mu = \operatorname{supp} \mu$.

Proof. Let $T^{-1}A = A$, μ a.e. Then $T(A \cap D_\mu) = A \cap D_\mu$, $p \times m$ a.e. Let $C = A \cap D_\mu$. Then $P_T g_1 = g_1$ and $g_1 \in D$, where $g_1 = 1_C g/p \times m(C)$ and $g = d\mu/dp \times m$. Therefore by the definition of q.m.p. $\operatorname{supp} g_1 = C = \bigcup_{i=1}^{s} A_i \times B_i$. Hence if H is an eigenfunction of eigenvalue 1 then the equality (2.1) holds. Now we define the following skew product:

$$\bar{T}(x, y, z, w) = (f(x), \bar{T}_{a(x)}(y, z, w)),$$

where

$$\bar{T}_{a(x)}(y, z, w) = (T_{a(x)}(y), f(z), T_{a(z)}(w)).$$

By the definition $\bar{T} = T \times T$. Hence, if H is an eigenfunction for T, then $H(x, y)\bar{H}(z, w)$ is an eigenfunction for $\bar{T}$ of eigenvalue 1. This implies

$$H(x, y)\bar{H}(z, w)1_{D_\mu}(x, y)1_{D_\mu}(z, w) = \sum_{i=1}^{s} 1_{A_i}(x) g_i(y, z, w), \quad p \times p \times m \times m \text{ a.e.}$$

Therefore $H(x, y)1_{D_\mu}(x, y) = \sum_{i=1}^{s} 1_{A_i}(x) g_i(y)$, $p \times m$ a.e. □

Let $(\bar{f}, \bar{\alpha})$ be the natural extension to the automorphism of the process (f, α). Let $\{T_i\}_{i=1}^{s}$ be a family of automorphisms of the space $(Y, \mathcal{B}, m)$. We obtain two skew products:

$$\bar{T}(\bar{x}, y) = (\bar{f}(\bar{x}), T_{a(\bar{x})} y) \quad \text{and} \quad T(x, y) = (f(x), T_{a(x)} y).$$

Here $\bar{T}$ is the natural extension of T to the automorphism. Therefore (see [**14**]) the eigenfunctions for $\bar{T}$ and T are the same ones.

Conclusion 5. If $(\bar{f}, \bar{\alpha})$ is the natural extension to the automorphism of a q.m.p. and H is an eigenfunction for $\bar{T}$, then $H = \sum_{i=1}^{s} 1_{A_i} \otimes g_i$.

THEOREM 3. *If (f, α) and (g, β) are q.m.p., then $(f \times g, \alpha \times \beta)$ is also q.m.p.*

Proof. Let $A = \operatorname{supp} \mu$, where μ is a $T_{f \times g}$-a.c.i.m. By the definition $T_{f \times g}(x, y, z) = (f(x), g(y), T_{a(x)b(y)} z)$. We define two families of transformations:

$$T_{a(x)}(y, z) = (g(y), T_{a(x)b(y)} z) \quad \text{and} \quad T_{b(y)}(x, z) = (f(x), T_{a(x)b(y)} z).$$

We get two skew products with base (f, α) and (g, β) respectively. Therefore by the assumptions of our theorem

$$1_A(x, y, z) = \sum_{i=1}^{s} 1_{A_i}(x) 1_{D_i}(y, z) = \sum_{j=1}^{s} 1_{B_j}(y) 1_{C_j}(x, z).$$

Hence $1_A(x, y, z) = \sum_{i,j=1}^{s} 1_{A_i}(x) 1_{B_j}(y) 1_{E_{ij}}(z)$ which finishes the proof. □

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THEOREM 4. *If (f, α) is q.m.p., then the partition α is Markovian.*

Proof. Let T be the skew product transformation defined in the proof of Lemma 2 and let μ be T-a.c.i.m. The set D_μ is T invariant, i.e. $TD_\mu = D_\mu$, $p \times m$ a.e., and hence $1_{D_\mu} T = 1_{D_\mu}$, μ a.e. By assumption $D_\mu = \bigcup_{i=1}^{s} A_i \times B_i$ and therefore

$$\bigcup_{i=1}^{s} f(A_i) \times T_i B_i = \bigcup_{j=1}^{s} A_j \times B_j. \tag{2.2}$$

By definition of the family $\{T_i\}_{i=1}^{s}$ the sets $T_i B_i$, $i = 1, \ldots, s$, are pairwise disjoint and therefore by (2.2) we get

$$p(f(A_i) \cap A_j) > 0 \Rightarrow A_j \subset f(A_i). \qquad \square$$

Conclusion 6. If f is an automorphism and (f, α) is q.m.p. then f permutes the elements of the partition α.

Let

$$\alpha_f = \max\{\beta : \text{for every } B \in \beta \text{ if } f(A_i) \cap B \neq \emptyset \text{ for some } i, \text{ then } f(A_i) \subset B\}.$$

Here β denotes a partition of X.

LEMMA 3. *If (f, α) is q.m.p. and if f has a quasi-eigenfunction h, $|h| = 1$, whose quasi-eigenvalue is measurable with respect to α, then h is measurable with respect to α_f.*

Proof. Let $h \circ f = (\sum_{i=1}^{s} c_i 1_{A_i}) h$, where $|c_i| = 1$, and let $T_i z = \bar{c}_i z$ for $z \in \{w : |w| = 1\}$, $i = 1, \ldots, s$. The function $h \otimes z$ is the eigenfunction of eigenvalue 1 for the skew product T determined by $\{T_i\}_{i=1}^{s}$. By assumptions, $h \otimes z = \sum_{i=1}^{s} 1_{A_i} \otimes g_i$ and hence $h = \sum_{i=1}^{s} d_i 1_{A_i}$. Due to the equality

$$\sum_{k=1}^{s} d_k 1_{A_k} \circ f = \sum_{j=1}^{s} c_j d_j 1_{A_j}$$

we get $d_{k_1} = d_{k_2}$, when $A_{k_1} \cup A_{k_2} \subset f(A_j)$ for some j, which proves our conclusion. $\square$

Finally, we consider families $\{T_i\}_{i=1}^{s}$ which are automorphisms of the space $(Y, \mathcal{B}, m)$.

THEOREM 5. *For every commuting family of automorphisms $\{T_i\}_{i=1}^{s}$ the eigenfunctions for T have the form $\sum_{i=1}^{s} 1_{A_i} \otimes g_i$ if and only if each quasi-eigenfunction f of modulus 1 whose quasi-eigenvalue is measurable with respect to α is also measurable with respect to α_f.*

Proof. Let G be the countable Abelian group generated by $\{T_i\}_{i=1}^{s}$ and let $\hat{G}$ be the compact group of characters. G has a unitary representation on $L_2(Y, m)$ of the form $U_g h(y) = h(g(y))$. This representation is a direct sum of cyclic sub-representations, namely

$$L_2(Y, m) = \bigoplus_{i=1}^{r} H_i \quad \text{and} \quad U_g = \bigoplus_{i=1}^{r} U_g^i, \quad \text{where } U_g^i = U_g | H_i, \, i = 1, \ldots, r, \, r \leq \infty.$$

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Let us fix one sub-representation U_g^j. By [**2**, p. 163] there exists a positive measure ν on $\hat{G}$ such that U_g^j is unitarily equivalent to the following representation:

$$V_g u(t) = \hat{\hat{g}}(t)u(t) \quad \text{for } u \in L_2(\hat{G}, \nu), \quad \text{where } \hat{\hat{g}}(t) = t(g).$$

Therefore the restriction $T|L_2(X, p) \times H_j$ is spectrally isomorphic to the operator V_j on $L_2(X, p) \times L_2(\hat{G}, \nu)$ such that

$$V_j H(x, t) = H(f(x), t)\left(\sum_{i=1}^{s} 1_{A_i} \otimes \hat{\hat{T}}_i\right).$$

Now, if $V_j H = \lambda H$ then for ν a.e. t

$$H_t \circ f = \left(\sum_{i=1}^{s} c_i 1_{A_i}\right) H_t \quad \text{for some } c_i, |c_i| = 1, i = 1, \ldots, s,$$

where $H_t(\cdot) = H(\cdot, t)$. Lemma 3 and the identity of our spectral isomorphism on $L_2(X, p)$ imply the assertion of our theorem. □

We now turn to the case of automorphisms which are not necessarily commutative. Let us assume that $T_i, i = 1, \ldots, s$, are elements of some separable compact group $G(Y)$ of automorphisms of (Y, m) such that its representation of the form $U_g h(y) = h(g(y))$ on $L_2(Y, m)$ is strongly continuous. By [**2** p. 169]

$$L_2(Y, m) = \bigoplus_{i=0}^{r} H_i, \quad \dim H_i < \infty \text{ and } H_i \text{ are } U_g \text{ invariant.}$$

Therefore every eigenfunction of T has the form $g = \sum_{t=1}^{s} b_t \otimes e_t$, where $b_t \in L_2(X, p)$, $t = 1, \ldots, d$, and $\{e_1, \ldots, e_d\}$ is the orthonormal basis of some invariant subspace of $L_2(Y, m)$. If $g \circ T = \lambda g$ then $\bar{b} \circ f(x) = \lambda W_{a(x)} \bar{b}(x)$, where W_i, $i = 1, \ldots, s$, are unitary matrices and $\bar{b} = (b_1, \ldots, b_d)$. The matrix λW_i is unitary as before and without loss of generality we may write

$$\bar{b} \circ f(x) = W_{a(x)} \bar{b}(x), \quad \text{where } W_i \in U(d), i = 1, \ldots, s. \tag{2.3}$$

Here $U(d)$ denotes the compact group of all complex $d \times d$ unitary matrices. After this introduction we may now formulate our theorem.

THEOREM 6. *For every family of automorphisms $\{T_i\}_{i=1}^s$ which are elements of some separable compact group $G(Y)$, the eigenfunctions for T have the form $\sum_{i=1}^s 1_{A_i} \otimes g_i$ if and only if the solutions of the equality $\bar{b} \circ f(x) = W_{a(x)} \bar{b}(x)$ where $W_i \in U(n)$, $i = 1, \ldots, s$, for every n, are vector functions with components measurable with respect to α_f.*

Proof. It suffices to show that every family $\{W_i\}_{i=1}^s$ of elements of $U(d)$ appears in (2.3), for $d = 1, 2, \ldots$. Let A_g be the natural d-dimensional unitary representation of $U(d)$; that is the representation taking each element of $U(d)$ into itself. Let μ be the Haar measure on $U(d)$ and let W_g denote the regular representation of $U(d)$, i.e.

$$W_g h(y) = h(gy) \quad \text{for } g \in U(d) \text{ and } h \in L_2(U(d), \mu).$$

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It follows, from the Peter–Weyl Theorem ([**2** p. 172]), that the regular representation W_g decomposes into an orthogonal direct sum of finite-dimensional irreducible unitary representations

$$W_g = \bigoplus_{j=1}^{r} \bigoplus_{k=1}^{d_j} W_g^{j,k}, \quad \text{where } W_g^{j,k} = W_g|H_{j,k} \text{ and } L_2(U(d), \mu) = \bigoplus_{j=1}^{r} \bigoplus_{k=1}^{d_j} H_{j,k}.$$

Furthermore, every irreducible unitary representation of G appears in this decomposition. Since A_g is irreducible it is unitarily equivalent to $W_g^{j,k}$ for some j, k. By putting $G(Y) = U(d) = Y$ we achieve our claim. □

The examples for the non-commutative case may be found in non-Abelian group extensions (see [**15**]).

3. *Examples of quasi-Markovian processes*

Let I be the unit interval and let $0 = a_0 < a_1 < \cdots < a_s = 1$. Denote by α the partition $\{I_i\}_{i=1}^s$, where $I_i = (a_{i-1}, a_i)$, and denote by A the set $\{a_i\}_{i=0}^s$. Let $f|I - A \to I$ be a continuous map which is strictly monotone on components of $I - A$.

Among the above transformations which possess an a.c.i.m. there are Lasota–Yorke and Misiurewicz transformations.

Let f be a Lasota–Yorke type transformation, i.e.

(a) the restriction f_i of f to (a_{i-1}, a_i) is C^1;
(b) $1/|f_i'|$ extends to a function of bounded variation on $[a_{i-1}, a_i]$, for $i = 1, \ldots, s$;
(c) $\inf |f'| > 1$.

Let p be an f invariant measure, which is absolutely continuous with respect to the Lebesgue measure (for existence see [**18**] or [**7**]). Assume that α is Markovian partition. Let

$$BV = \left\{ G : G \in L_1(\lambda \times m) \text{ and there exists } F \text{ such that } F = G \text{ a.e.} \right.$$
$$\left. \text{and } \int V F(\cdot, y)\, dm < \infty \right\}.$$

Denote by $E_\lambda(f, \alpha)$ the set we obtain after replacing the measure p by λ in the definition of $E(f, \alpha)$. By $p \ll \lambda$ we have $E(f, \alpha) \subset E_\lambda(f, \alpha)$.

LEMMA 4. *$E_\lambda(f, \alpha) \subset BV$.*

Proof. The density of BV in L_1 and Lemma 4 from [**5**] together with the stochastic ergodic theorem [**6**] applied to P_T imply our conclusion. □

THEOREM 7. *If f is the Lasota–Yorke type transformation with the Markovian partition α and f-a.c.i.m. p, then (f, α) is q.m.p.*

Proof. Lemma 4 implies Lemma 6(i) from [**5**] which, as a consequence, implies the theorem. □

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Now we will consider the Misiurewicz type transformations.

Before the definition of f we introduce the following notations (as in [**9**]): $\hat{I} = I \times \{+,-\} - \{(0,-),(1,+)\}$, $\hat{f} : \hat{I} \to \hat{I}$ given by $\hat{f}(x,\varepsilon) = (x',\varepsilon')$, where $x' = \lim_{y \to x^{\varepsilon}} f(y)$ and $\varepsilon' = \varepsilon$ if and only if $f'_{\varepsilon}(x) > 0$. If $x \in \hat{I}$ then $\check{x}$ is its first coordinate and $\underline{x}$ is its second coordinate. We set $\hat{A} = \hat{I} \cap (A \times \{+,-\})$. Now we make further assumptions on f:

(i) f is of class C^3;

(ii) $f' \neq 0$;

(iii) $Sf = f'''/f' - \frac{3}{2}(f''/f')^2 \leq 0$ (negative Schwarzian derivative);

(iv) for $x \in \hat{I}$, if $\hat{f}^p(x) = x$, then $|(\hat{f}^p)'(x)| > 1$ (here $\hat{f}'(x,\varepsilon) = f'_{\varepsilon}(x)$);

(v) $\hat{f}(\hat{A}) \subset \hat{A}$ (the Markovian property);

(vi) for every $a \in \hat{A}$ there exist constants δ, α_0, $\omega > 0$ and $u \geq 0$ such that $\alpha_0|x - \check{a}|^u \leq |f'(x)| \leq \omega|x - \check{a}|^u$ for every $x \in U_a$, where

$$U_a = \begin{cases} (\check{a}, \check{a} + \delta) & \text{if } \underline{a} = + \\ (\check{a} - \delta, \check{a}) & \text{if } \underline{a} = - \end{cases}$$

(vii) for $s \in \{2,3\}$ and $a \in \hat{A}$ there exist constants $\psi > 0$ and $v \geq u - s + 1$, such that $|f^{(s)}(x)| \leq \psi|x - \check{a}|^v$ for $x \in U_a$.

Now we introduce the following notations:

$$\hat{A}_1 = \hat{f}(\hat{A}), \quad \hat{A}_2 = \hat{A} - \hat{A}_1, \quad \xi(a) = u(a)/1 + u(a), \quad \eta(a) = 1 - \xi(a),$$

$$a_j = \hat{f}^j(a), \text{ for } j \in \mathbb{N}, \quad \text{and} \quad \psi_b(x) = \begin{cases} 1/|x - \check{b}| & \text{for } x > \check{b} \text{ if } \underline{b} = + \\ & \text{for } x < \check{b} \text{ if } \underline{b} = - \\ 0 & \text{elsewhere.} \end{cases}$$

As in [**16**] we define the functions φ_i: $I - A_1 \to \mathbb{R}^+$ for $i = 0, 1$ as follows:

$$\varphi_i(x) = \left[\sum_{n=0}^{\infty} \sum_{a \in \hat{A}_2} |(\hat{f}^n)'(a_1)|^{-\eta(a)} \psi_{a_{n+1}}^{i+\xi(a)}(x) \right] + 1.$$

Let β be the partition given by components of continuity of φ_i for $i = 0, 1$. By assumption (v), $\beta < \alpha$. For every $\rho : I - A_1 - C$ such that ρ is of class C^1 on all $J \in \beta$ we define

$$|\rho|_0 = \sup_{J \in \beta} \sup_J |\rho|/\varphi_0 \quad \text{and} \quad |\rho|_1 = \sup_{J \in \beta} \sup_J |\rho'|/\varphi_1.$$

As in [**16**] we denote by C_f^1 the set of $\rho : I - A_1 \to C$ being of class C^1 on all $J \in \beta$ such that $|\rho| = |\rho|_0 + |\rho|_1 < \infty$.

Let P_f denote the Frobenius–Perron operator for f with respect to λ. Then by [**16**] $P_f : C_f^1 \to C_f^1$ and $(C_f^1, |\ |)$ is a Banach space.

THEOREM A. ([**16**]) *Let f satisfy* (i)–(vii). *There exist constants t, $d > 0$ and $c < 1$ such that for $n \in \mathbb{N}$, $\rho \in C_f^1$,*

$$|P_f^n \rho|_1 \leq d(c^n|\rho|_1 + |\rho|_0) \quad \text{and} \quad |P_f^n \rho|_0 \leq t|\rho|_0.$$

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THEOREM B. ([**9**]) *Let f satisfy* (i)–(vi). *Then there exists a probabilistic f-invariant measure which is absolutely continuous with respect to the Lebesgue measure.*

Let $\{T_i\}_{i=1}^s$ be a family of positive and negative non-singular transformations of a Lebesgue space $(Y, \mathcal{B}, m)$ and let $T(x, y) = (f(x), T_{a(x)}y)$. We will denote by μ the product measure $\lambda \times m$. The Frobenius–Perron operator for T with respect to the measure μ is given by the formula

$$P_T G(x, y) = \sum_{i=1}^{s} P_i G(f_i^{-1}(x), y)|(f_i^{-1})'(x)|1_{f_i(I_i)}(x).$$

Here $G \in L_1(\mu)$, P_i denotes the Frobenius–Perron operator for T_i and $f_i = f|I_i$ for $i = 1, \ldots, s-1$.

For a function $G : I \times Y \to C$ such that $G(\cdot, y) \in C^1(I)$ for every y, we define

$$\|G\|_1 = \int \sup_{J\in\beta} \sup_J |G_x(x, y)|/\varphi_1(x)\, dm = \int |G(\cdot, y)|_1\, dm$$

and

$$\|G\|_0 = \int \sup_{J\in\beta} \sup_J |G(x, y)|/\varphi_0(x)\, dm = \int |G(\cdot, y)|_0\, dm.$$

Moreover, let P be the Frobenius–Perron operator for certain positively and negatively non-singular transformations h of Y. Positive non-singularity of h implies that

$$Pg(y) = \sum_{h(z)=y} g(z)/h'(z), \quad m \text{ a.e.},$$

where the summation is taken over all z such that $z \in h^{-1}(y)$ and $h'(z) > 0$. Here h' denotes the Jacobian of h (for the definition of Jacobian see [**12**]).

LEMMA 5.

$$\frac{\partial}{\partial x} PG(\varphi(x), y) = P\frac{\partial}{\partial x} G(\varphi(x), y), \quad m \text{ a.e.}$$

for $G \in C_1$ and $\varphi \in C^1$. Here

$$C_1 = \left\{ G : G \in C^1 \text{ and } \|G\|_\infty + \left\| \frac{\partial}{\partial x} G \right\|_\infty < \infty \right\}.$$

Proof. We have

$$PG(\varphi(x), y) = \sum_{h(z)=y} G(\varphi(x), z)/h'(z).$$

The assumptions about regularity of G and φ imply

$$\frac{\partial}{\partial x} \sum_{h(z)=y} G(\varphi(x), z)/h'(z) = \sum_{h(z)=y} \frac{\partial}{\partial x} G(\varphi(x), z)/h'(z). \qquad \square$$

LEMMA 6. *If $G \in C_1$, then $\|P_T^n G\|_1 \le d(c^n\|G\|_1 + \|G\|_0)$ and $\|P_T^n G\|_0 \le t\|G\|_0$.*

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Proof.

$$\begin{aligned}\|P_T^n G\|_1 &= \int \sup_{J\in\beta}\sup_J \left|\frac{\partial}{\partial x} P_T^n G(x,y)\right| / \varphi_1(x)\,dm \\ &= \int \sup_{J\in\beta}\sup_J \left|\frac{\partial}{\partial x}\Bigg(\sum_{i_1,\dots,i_n=1}^{s} P_{i_1}\circ\cdots\circ P_{i_n} G(f_{i_n}^{-1}\circ\cdots\circ f_{i_1}^{-1}(x),y)\right. \\ &\quad \left.\times |(f_{i_n}^{-1}\circ\cdots\circ f_{i_1}^{-1})'(x)| 1_{f^n(I_{i_1,\dots,i_n})}(x)\Bigg)\right| / \varphi_1(x)\,dm.\end{aligned}$$

By Lemma 5 and by properties of the Frobenius–Perron operator we have

$$\begin{aligned}\|P_T^n G\|_1 &\le \int \sup_{J\in\beta}\sup_J \sum_{i_1,\dots,i_n=1}^{s} \left|\frac{\partial}{\partial x}\right| G(f_{i_n}^{-1}\circ\cdots\circ f_{i_1}^{-1}(x),y) \\ &\quad |(f_{i_n}^{-1}\circ\cdots\circ f_{i_1}^{-1})'(x)| 1_{f^n(I_{i_1,\dots,i_n})}(x)| / \varphi_1(x)\,dm.\end{aligned}$$

Next, by using the argumentation as in the proof of Theorem A, we get

$$\|P_T^n G\|_1 \le \int d(c^n |G(\cdot,y)|_1 + |G(\cdot,y)|_0)\,dm = d(c^n \|G\|_1 + \|G\|_0).$$

The proof of the inequality $\|P_T^n G\| \le t\|G\|_0$ is similar. □

Now, let E be any closed interval such that $E \subset J$ for some $J \in \beta$. Let $\bigvee_E G = \int \bigvee_E G(\cdot,y)\,dm$. Then

$$\bigvee_E G = \int\int_E |G_x(x,y)|\,d\lambda\,dm \le \|G\|_1 \int_E \varphi_1\,d\lambda.$$

Conclusion 7.

$$\overline{\lim}_{n\to\infty} \bigvee_E P_T^n G \le d\|G\|_0 \int_E \varphi_1\,d\lambda \quad \text{for } G \in C_1.$$

For $G \in L_1(\mu)$ we define

$$\begin{aligned}\bigvee_E G &= \inf\left\{\int \bigvee_E G(\cdot,y)\,dm : F = G \text{ a.e.}\right\}, \\ \overline{B\bigvee} &= \left\{G : \bigvee_E G < \infty \text{ for every closed interval } E\right. \\ &\qquad \left.\text{such that } E \subset J \text{ for some } J \in \beta\right\} \\ \|G\|_0 &= \inf\{\|F\|_0 : F = G \text{ a.e.}\}.\end{aligned}$$

LEMMA 7. *If a sequence of functions $G_n : I \times Y \to R$ converges in μ to a function $G : I \times Y \to R$, then:*

(i) $\|G\|_0 \le \limsup_{n\to\infty} \|G_n\|_0$;

(ii) $\bigvee_E G \le \limsup_{n\to\infty} \bigvee_E G_n$.

Proof. (i) If $\limsup_{n\to\infty} \|G_n\|_0 = \infty$, the conclusion is obvious. Assume that $\limsup_{n\to\infty} \|G_n\|_0 < \infty$. Fix $\varepsilon > 0$ and let $\bar{G}_n$ be such that $\bar{G}_n = G_n$ a.e. and

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$\|\bar{G}_n\|_0 \le \|G_n\|_0 + \varepsilon$. There exists a subsequence $\bar{G}_{n_k}$ which is pointwise convergent to G a.e. Denote by D the set of points for which the sequence $\bar{G}_{n_k}$ does not converge. We put $\bar{G} = 1_{D^c} G$. Then $\bar{G} = G$ a.e. and

$$|\bar{G}(\cdot, y)|_0 \le \liminf_{k\to\infty} |\bar{G}_{n_k}(\cdot, y)|_0,$$

for every $y \in Y$. From this, by Fatou's lemma we obtain

$$\int |\bar{G}(\cdot, y)|_0 \, dm \le \liminf_{k\to\infty} \int |\bar{G}_{n_k}(\cdot, y)| \, dm$$

and consequently

$$\|G\|_0 \le \liminf_{k\to\infty} \|G_{n_k}\|_0 + \varepsilon.$$

Since ε is chosen arbitrarily, this gives

$$\|G\|_0 \le \liminf_{k\to\infty} \|G_{n_k}\|_0 \le \limsup_{n\to\infty} \|G_n\|_0.$$

(ii) See proof of Lemma 5 in [**19**]. □

LEMMA 8. *Let G be a density and $\|G\|_0 < \infty$. Then for every $\varepsilon > 0$ there exists $F_\varepsilon \in C_1$ such that $\|F_\varepsilon - G\|_{L_1} < \varepsilon$ and $\|F_\varepsilon\|_0 \le 2\|G\|_0$.*

Proof. Let M be a positive real number such that

$$\int_{A_M^c} G \, d\mu \le \frac{\varepsilon}{4} \quad \text{where} \quad A_M = \{(x, y) : G(x, y) \le M\}.$$

Let G_1 be such that $G_1 = 1_{A_M} G$ a.e., $G_1(b, y) \equiv 0$ for every $b \in A$ and $\|G_1\|_0 \le \frac{3}{2}\|G\|_0$. Moreover, let

$$g(y) = \max_{J\in\beta} \sup_{x\in J} \frac{|G_1(x, y)|}{\varphi_0(x)}.$$

Then

$$\bigwedge_{x,y} G_1(x, y) \le g(y)\varphi_0(x) \le g_1(y)\varphi_0(x), \quad \text{where } g_1(y) = g(y) + \tfrac{1}{4}\|G\|_0.$$

We take a partition $\{J_i \times B_j\}_{1\le i\le n, 1\le j\le m}$ of $I \times Y$ such that J_i, $1 \le i \le n$, are intervals and there exists a sequence $0 \le c_{ij} \le M$, $1 \le i \le n$, $1 \le j \le m$, such that the function

$$G_2(x, y) = \sum_{\substack{1\le i\le n\\ 1\le j\le m}} c_{ij} 1_{J_i}(x) 1_{B_j}(y)$$

satisfies the inequality $\|G_2 - G_1\|_{L_1} \le \varepsilon/4$. Moreover, if

$$d_{ij} = \sup_{J_1\times B_j} g_1(y)\varphi_0(x) \quad \text{and} \quad e_{ij} = \inf_{J_i\times B_j} g_1(y)\varphi_0(x)$$

then either $e_{ij} \ge M + \frac{1}{4}\|G\|_0$ or $d_{ij} - e_{ij} < \frac{1}{4}\|G\|_0$ for $1 \le i \le n$, $1 \le j \le m$. Let $\bar{c}_{ij} = \min\{e_{ij} + \frac{1}{4}\|G\|_0, c_{ij}\}$ for $1 \le i \le n$, $1 \le j \le m$. The function

$$G_3(x, y) = \sum_{\substack{1\le i\le n\\ 1\le j\le m}} \bar{c}_{ij} 1_{J_i}(x) 1_{B_j}(y)$$

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has the following properties:

$$\bigwedge_{x,y} |G_1(x,y)-G_3(x,y)| \le |G_1(x,y)-G_2(x,y)| \quad \text{and} \quad G_3(x,y) \le g_1(y)\varphi_0(x)+\tfrac{1}{4}\|G\|_0.$$

Finally, we replace 1_{J_i} by $f_i \in C^\infty$, $0 \le f_i \le 1_{J_i}$, such that the function

$$F_\varepsilon(x,y) = \sum_{\substack{1\le i\le n\\ 1\le j\le m}} \bar{c}_{ij} f_i(x) 1_{B_j}(y)$$

satisfies the inequality $\|F_\varepsilon - G_3\|_{L_1} \le \varepsilon/4$. □

THEOREM 8. *Let G be the density of T-a.c.i.m. If $\|G\|_0 < \infty$ then $G \in \overline{B\bigvee}$.*

Proof. Let $A_n F = (1/n)\sum_{k=0}^{n-1} P_T^k F$. By the stochastic ergodic theorem [**6**], for any $F \in L_1$, $A_n F$ converges in measure to $F^* \in L_1$ such that $P_T F^* = F^*$. Let $\varepsilon > 0$ be fixed. There exists $F_\varepsilon \in C_1$ such that $\|F_\varepsilon - G\|_{L_1} < \varepsilon/5$. Let $F_\varepsilon^* = \lim_{k\to\infty} A_{n_k} F_\varepsilon$ for some subsequence $\{n_k\}$ where the convergence is almost uniform and, hence, almost everywhere. By Conclusion 7 and Lemma 7(ii) $F_\varepsilon^* \in \overline{B\bigvee}$ and $\bigvee_E F_\varepsilon^* \le d\|F_\varepsilon\|_0 \int_E \varphi_1 \, d\lambda$. Let $\delta > 0$ be such that if $\mu(A) < \delta$ then $\int_A |F_\varepsilon^*| \, d\mu < \varepsilon/5$ and $\int_A G \, d\mu < \varepsilon/5$. Moreover, by the almost uniform convergence of $A_{n_k} F_\varepsilon$, there exist k_0 and U, $\mu(U) > 1-\delta$, such that $\int_U |A_{n_k} F_\varepsilon - F_\varepsilon^*| \, d\mu < \varepsilon/5$ for $k \ge k_0$. Therefore we get

$$\int_{I\times Y} |F_\varepsilon^* - G| \, d\mu < \int_U |F_\varepsilon^* - A_{n_k} F_\varepsilon| \, d\mu + \int_U |A_{n_k} F_\varepsilon - A_{n_k} G| \, d\mu + 2\varepsilon/5 < 4\varepsilon/5.$$

By Lemma 8 for $\varepsilon \to 0$ we can find F_ε such that $\|F_\varepsilon\|_0 \le 2\|G\|_0$. Hence $G = \lim_{\varepsilon\to 0} F_\varepsilon^*$ in L_1 and $\overline{\lim_{\varepsilon\to 0}} \bigvee_E F_\varepsilon^* \le c_1 \int_E \varphi_1 \, d\lambda$ which implies $G \in \overline{B\bigvee}$. □

Remark. If G is the density of T-a.c.i.m. then for every $\varepsilon > 0$ there exists the density F_ε^* of T-a.c.i.m. such that $\|F_\varepsilon^*\|_0 < \infty$ and $\|G - F_\varepsilon^*\|_{L_1} < \varepsilon$. Moreover, if G is the density of the ergodic T-a.c.i.m. then $\|G\|_0 < \infty$ and hence $G \in \overline{B\bigvee}$.

Proof. Let F_ε^* be the sequence constructed for G in the proof of Theorem 8. By Lemmas 6 and 7(i), $\|F_\varepsilon^*\|_0 \le t\|F_\varepsilon\|_0 < \infty$. If we assume the ergodicity of μ_G, where $d\mu_G/d\mu = G$, then $F_\varepsilon^*|\operatorname{supp} G = \text{constant}$ G for small ε. Therefore $\|G\|_0 < \infty$. □

THEOREM 9. *If f satisfies (i)–(vii) and p is f-a.c.i.m. then (f,α) is q.m.p.*

Proof. Let $P_T G = G$, where $G \in D$. At first we assume that $\|G\|_0 < \infty$. By $G \in \overline{B\bigvee}$ we may repeat the reasoning used in the Lasota–Yorke case. To obtain the result as in Conclusion 5(i) [**5**] we make the following observation. By Lemma 4.7 in [**9**] and by assumption (v), for any non-empty interval E there exist j and n such that $f^n(E) \supset I_j$. This property of f allows us to conclude that $\operatorname{supp} G = \bigcup_{i=1}^s A_i \times B_i$.

If $\|G\|_0 = \infty$ then we consider $D_G = \operatorname{supp} G$. By the Remark there exists a sequence G_n such that $\|G_n\|_0 < \infty$, $P_T G_n = G_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} G_n = G$ in L_1. Since $T D_G = D_G$, $p \times m$ a.e., the function $1_{D_G} G_n$ is the density of T-a.c.i.m. and, due

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to $1_{D_G} G_n \leq G_n$, $\|1_{D_G} G_n\|_0 < \infty$ for $n = 1, 2, \ldots$. According to $\lim_{n\to\infty} G_n = G$ in L_1 we have

$$D_G = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} \bigcap_{j=k}^{\infty} D_{G_j} \cap D_G.$$

The first part of the proof yields $D_G \cap D_{G_n} = \bigcup_{i=1}^{s} A_i \times B_i^n$. This implies that $D_G = \bigcup_{i=1}^{s} A_i \times B_i$ which completes the proof. □

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