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Aperiodic Lorentz gas: recurrence and ergodicity

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Abstract. We prove that the general (i.e. possibly aperiodic) Lorenz gas in two dimensions, with finite horizon and non-degenerate geometrical features, is ergodic if it is recurrent. We also give examples of aperiodic recurrent gases.

1. *Introduction*

By Lorentz gas (LG) we mean the free motion of a point particle in the plane subject to elastic collisions against a fixed array of dispersing scatterers. Each scatterer $\mathcal{O}_\alpha$ is an open, bounded, connected, simply connected, strictly convex domain of $\mathbb{R}^2$, with smooth boundary and is labeled by the variable $\alpha \in \mathcal{I}$. The scatterers are assumed to be pairwise disjoint. A LG is called *periodic* when $\bigcup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$ is left invariant by G, a discrete group of translations in the plane, whose fundamental domain $\mathbb{R}^2/G$ is compact.

Note the departure from the usual terminology, whereby 'Lorentz gas' means 'periodic Lorentz gas' (PLG). To avoid possible confusions let us recall that some literature calls 'Lorentz gas' the dynamical system defined on $\mathbb{R}^2/G$, in the periodic case; common practice, however, has nowadays christened such systems *Sinai billiards*. Also, but more seldom, 'Lorentz gas' denotes the system of infinitely many non-interacting particles bouncing against scatterers (see, e.g., [**S2**]). In [**Sz**], Szász proposes to adopt that terminology and calls the system at hand the 'Lorentz process'.

The LG has a long and honorable history. It was first introduced in its periodic and three-dimensional version by Lorentz in 1905, to describe the motion of an electron in a crystal [**Lo**]. Later in the century it became extremely popular in statistical mechanics and ergodic theory. The scientific community was seeking a rather realistic model whose stochastic properties could be rigorously proven—in particular, Boltzmann's Ergodic Hypothesis (namely, ergodicity). It is because of this connection with statistical mechanics that it was renamed 'Lorentz *gas*'. At this point, the periodicity condition had lost its *raison d'être*, but still remained in virtually all the publications on the grounds of mathematical convenience.

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As a matter of fact, most of the literature in this area of research is actually concerned with Sinai billiards and many results on the PLG were generated as corollaries of theorems on the Sinai billiard. It is definitely outside the scope of this introduction to mention all that is known about these systems, but it is worthwhile to recall that they are named after Sinai, who in 1970 proved that they possess the K-property [**S1**]. The same result was extended to higher dimensions by Sinai and Chernov in 1987 [**SC**]. (A rather complete reference on billiards is [**T**]; the hasty reader may find a brief history of *dispersing* billiards in the introduction of [**L**].)

Turning to LGs, it is not surprising that much less work has been done on them: the relevant invariant measure for these systems is infinite and ordinary ergodic theory does not apply. Still it makes sense to pose the question of the stochastic properties; only one must utilize a more refined ergodic theory (as found, for example, in the beautiful book by Aaronson [**A**]). For instance, the most sound definition of ergodicity in infinite measure is the *indecomposability* of the system modulo the measure. That is, a dynamical system endowed with an invariant measure μ is ergodic when the only invariant subsets mod μ have measure zero or are complements of zero-measure sets (cf. Definition 4.1).

Another question that arises immediately for infinite-measure systems is that of recurrence (cf. Definition 2.1), because the Poincaré Recurrence Theorem does not hold. For the PLG this has baffled mathematicians for decades, and was only answered in the affirmative in the last few years, independently by Schmidt [**Sch**] and Conze [**Co**]. Both proofs use the Central Limit Theorem (for a suitable class of observables) obtained by Bunimovich and Sinai in 1981 [**BS**]. All these results assume *finite horizon*: that is, the free path between collisions must be bounded above.

Recurrence implies ergodicity (in the above sense) by a result of Simányi [**Si**].

We finish this short review of the PLG by mentioning a few other important properties that are not so central in the present work. Again for finite-horizon gases, the dynamics are diffusive and converge to a Brownian motion upon suitable rescaling [**BS**] (see also [**BSC**].) The finite-horizon condition appears to be physically relevant, as infinite-horizon PLGs are believed to have a super-diffusive behavior (the mean-square displacement at time t increasing like $t \log t$) [**B**]. A rather general version of the (local) Central Limit Theorem is contained in a recent work by Szász and Varjú [**SzV**].

In this paper we treat the general LG with finite horizon. Subject to very mild conditions on the curvature of the scatterers and their relative position (cf. (2.2) and (2.3)), we prove that recurrence implies ergodicity (Theorem 4.2) and we give several examples of recurrent aperiodic LGs (Propositions 5.3 and 5.4).

The exposition is as organized as follows: In §2 we lay down the necessary definitions and basic facts about billiard dynamics. In §3 we show that our systems are hyperbolic. In §4 we give the ergodicity result. In §5 we prove recurrence for a class of systems.

Throughout the paper we use the theory of hyperbolic systems with singularities without really reviewing it for the reader. Rather, we try to be as precise as possible as to what part of the theory we are applying, or we need to modify, at any given time. Standard references include [**CFS, SC, LW**].

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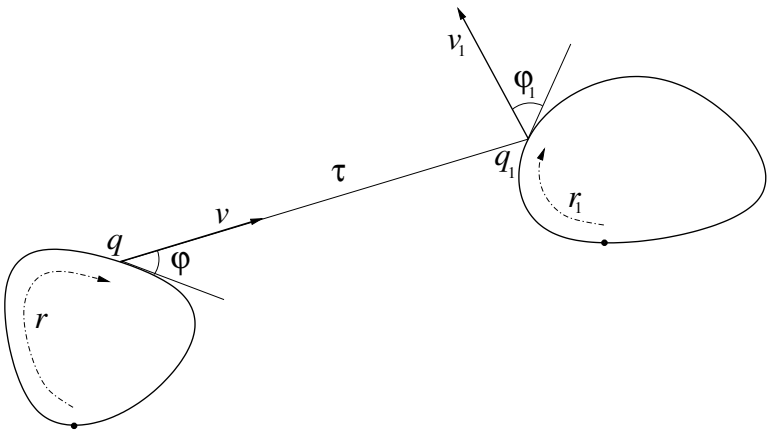


FIGURE 1. Basic definitions for the billiard map.

2. *Preliminaries*

We study the billiard dynamics on $\mathbb{R}^2 \setminus \bigcup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$ (notation introduced in §1). A material point moves with constant velocity until it hits the boundary of its allowed domain. The point is then instantaneously scattered by the obstacle according to the Fresnel law: the angle of incidence equals the angle of reflection. It is clear that in this process the modulus of the velocity never changes (a consequence of the conservation of energy for this singular Hamiltonian system). So this modulus is conventionally fixed to be 1. The dynamical system thus defined is called the *billiard flow* on $\mathbb{R}^2 \setminus \bigcup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$.

Since for most of the time the dynamics is trivial, we take the point of view of discrete dynamical systems and consider the Poincaré cross section corresponding to collisions with the obstacles. More precisely, we restrict our phase space to points of the type $x = (q, v)$, with $q \in \partial\mathcal{O}_\alpha$, for some α, and $v \in \mathrm{T}_q\mathbb{R}^2 = \mathbb{R}^2$, such that $|v| = 1$ and v points outwardly with respect to $\mathcal{O}_\alpha$. Any such point is also called a *line element* and the whole set is denoted by $\mathcal{M}$. Given an $x \in \mathcal{M}$ we define $Tx =: x_1 =: (q_1, v_1)$ to be first line element, in the forward (flow-)trajectory of x, that belongs to $\mathcal{M}$ (see Figure 1). T is then the Poincaré map induced by the flow on the cross section $\mathcal{M}$ and the T-orbit of x is the sequence of snapshots of the flow-trajectory of x, taken immediately after each collision. (In this work we will try to be as consistent as possible and use the term 'orbit' for an orbit of a map, such as T, and 'trajectory' for an orbit of the billiard flow.)

A convenient way to parameterize the phase space is to consider, for each α, the set $\mathcal{M}_\alpha := S_{L_\alpha} \times [0, \pi]$, where S_L is the circle of circumference L (i.e. the interval $[0, L]$ after identification of the endpoints). A pair $(r, \varphi) \in \mathcal{M}_\alpha$ represents the unit vector, based on $q \in \partial\mathcal{O}_\alpha$, that forms an angle φ with the clockwise tangent vector to $\partial\mathcal{O}_\alpha$ in q; and q is determined as the point whose clockwise distance along $\partial\mathcal{O}_\alpha$ (from a certain fixed origin) is r—again see Figure 1. L_α is thus the length of the closed curve $\partial\mathcal{O}_\alpha$. With a forgivable abuse of notation, then, the phase space is rewritten as $\mathcal{M} = \bigsqcup_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha$ (the symbol denoting disjoint union).

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We endow $\mathcal{M}$ with the measure μ defined by $d\mu(r,\varphi) = \sin\varphi\, dr\, d\varphi$. This is the physically relevant measure because it is induced on $\mathcal{M}$ by the Liouville measure for the billiard flow (which, by the way, is the uniform measure both in the position and in the momentum coordinates). Hence μ is preserved by T [**CFS**]. Obviously $\mu(\mathcal{M}) = \infty$. From a mathematical point of view, this is perhaps the most significant feature of the Lorentz gas.

It is well known that systems of the billiard type are discontinuous: any line element x whose trajectory is tangent to the next obstacle, say $\mathcal{O}_\beta$, is a point of discontinuity for T, because other line elements arbitrarily close to x may hit or miss $\mathcal{O}_\beta$, which causes Tx to end up in completely different regions of the phase space. The discontinuity set is then given by $\mathcal{S} := \mathcal{S}^+ := T^{-1}\partial\mathcal{M}$. Analogously, $\mathcal{S}^- := T\partial\mathcal{M}$ is the discontinuity set of T^{-1}. It is not hard to deduce that the differential of T blows up at $\mathcal{S}$ (see below). Therefore $\mathcal{S}$ is usually called the *singularity set*. It is made up of smooth curves (*singularity lines*) with the property that the endpoints of any curve belong to other such curves or $\partial\mathcal{M}$ [**S1, SC, LW**].

For all elements in $\mathcal{M} \setminus \mathcal{S}$, the differential of T is known [**LW**, §14], [**L**, §3] and, however, easily computed with the help of Figure 1 and a little patience. Denote by $\tau = \tau(x) = \tau(r,\varphi)$ the *free path* of the line element x. Also, if $x \in \mathcal{M}_\alpha$, call $k_\alpha(r)$ the curvature of $\partial\mathcal{O}_\alpha$ in the base-point of x: this is a smooth function by hypothesis and positive by convention. Also set $(r_1,\varphi_1) := x_1 = Tx$; $k = k_\alpha(r)$; $k_1 = k_\beta(r_1)$ (if $x_1 \in \mathcal{M}_\beta$). Then

$$DT_x = \begin{bmatrix} -\dfrac{\sin\varphi}{\sin\varphi_1} - \dfrac{k\tau}{\sin\varphi_1} & \dfrac{\tau}{\sin\varphi_1} \\ k + k_1\dfrac{\sin\varphi}{\sin\varphi_1} + \dfrac{kk_1\tau}{\sin\varphi_1} & -1 - \dfrac{k_1\tau}{\sin\varphi_1} \end{bmatrix}. \tag{2.1}$$

Before introducing the class of systems that we treat in this work we need a very important, although standard, definition.

Definition 2.1. The dynamical system $(\mathcal{M}, T, \mu)$ is said to be *recurrent* (in the sense of Poincaré) if, for every measurable set A, the orbit of μ-almost every $x \in A$ returns to A at least once (and thus infinitely many times, due to the invariance of μ).

We need not require the above property to hold for both the forward and backward orbits because our systems are reversible, i.e. there exists a map I on $\mathcal{M}$ such that $I^2 = \mathrm{id}$ and $T^{-1}IT = I$. (This map is, of course, $I(r,\varphi) = (r, \pi - \varphi)$.)

Since the Poincaré Recurrence Theorem fails here ($\mu(\mathcal{M}) = \infty$), one cannot take Definition 2.1 for granted and indeed there are obvious cases of non-recurrent systems—think of a LG with finitely many scatterers. (In fact, these open billiards are usually studied only on the recurrent part of their phase space [**LM, St**].)

Let us accept the abuse of notation whereby $k(x)$ means $k_\alpha(r)$, for $x = (r,\varphi) \in \mathcal{M}_\alpha$. Then we denote by $\mathcal{X}$ the class of all *recurrent* LGs for which there exist four positive numbers, k_m, k_M, τ_m, τ_M, such that, for all $x \in \mathcal{M}$,

$$k_m \le k(x) \le k_M, \tag{2.2}$$

$$\tau_m \le \tau(x) \le \tau_M. \tag{2.3}$$

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The first condition is of a physical nature: we want to model a gas, something that does not look much different in different parts of the plane. In fact, this condition and the convexity of the scatterers imply that no $\mathcal{O}_\alpha$ can be too large or too small; more precisely

$$2k_M^{-1} \le \operatorname{diam}(\mathcal{O}_\alpha) \le 2k_m^{-1}, \tag{2.4}$$

$$2\pi k_M^{-1} \le L_\alpha \le 2\pi k_m^{-1}. \tag{2.5}$$

Condition (2.3) is primarily dictated by mathematical feasibility, as we will see later. However, the existence of an upper bound for the free path (which goes by the name of *finite-horizon condition*) does make a difference in the dynamics, as recalled in §1.

For a given x, define $\mathcal{C}(x) := \{(dr,\, d\varphi) \in \mathrm{T}_x\mathcal{M} \mid dr\, d\varphi \le 0\}$; that is, the second and fourth quadrant of $\mathrm{T}_x\mathcal{M}$ in the $\{\partial/\partial r, \partial/\partial\varphi\}$ basis. This is usually called the *unstable cone* at x [**LW, L**]. Take $u = (dr,\, d\varphi) \in \mathcal{C}(x)$. Any such vector is called *unstable*. The checkered sign configuration of (2.1) implies that $DT_x u =: (dr_1,\, d\varphi_1) \in \mathcal{C}(Tx)$ and

$$(dr_1)^2 \ge \left(\frac{\sin\varphi}{\sin\varphi_1}\right)^2 \left(1 + \frac{k\tau}{\sin\varphi}\right)^2 dr^2, \tag{2.6}$$

$$(d\varphi_1)^2 \ge \left(1 + \frac{k_1\tau}{\sin\varphi_1}\right)^2 d\varphi^2. \tag{2.7}$$

Therefore, if we define the metric

$$\|u\|_x^2 = \|(dr, d\varphi)\|_{(r,\varphi)}^2 := \sin^2\varphi\, dr^2 + d\varphi^2, \tag{2.8}$$

and set $\lambda := 1 + k_m\tau_m > 1$, we obtain from (2.2) and (2.3)

$$\|DTu\|_{x_1} \ge \lambda\|u\|_x. \tag{2.9}$$

We call (2.8) the *increasing norm* for unstable vectors.

3. *Hyperbolicity*

In this section we see how to construct local stable and unstable manifolds (LSUMs) for almost every point of $\mathcal{M}$ and how to prove that they are absolutely continuous with respect to μ. Although the subject is very standard, it is useful for later purposes to state the definition of LSUM.

For $n \in \mathbb{N} \cup \{\infty\}$, let us denote

$$\mathcal{S}_n^{\pm} := \bigcup_{i=0}^{n-1} T^{\mp i}\mathcal{S}^{\pm}.$$

Definition 3.1. Given a point $x \in \mathcal{M} \setminus \mathcal{S}_\infty^{+(-)}$, we define a local (un)stable manifold $W^{s(u)}$ for T at x to be a C^1 curve containing x in its (one-dimensional) interior, not intersecting $\mathcal{S}_\infty^{+(-)}$ and such that the following apply.

(a) The tangent spaces to $W^{s(u)}$ are included in the pull-backwards (or push-forwards) of all future (past) stable (unstable) cones. (For example, in the unstable case, this means: for all $y \in W^u$, $\mathrm{T}_y W^u \subset \bigcap_{n\ge 0} DT^n_{T^{-n}y}\mathcal{C}(T^{-n}y)$.)

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(b) For all $y \in W^{s(u)}$, $d(T^n y, T^n x) \to 0$, as $n \to +\infty(-\infty)$.
(c) If $W_0^{s(u)}$ is another such manifold, then so is $W^{s(u)} \cap W_0^{s(u)}$.

Condition (c) is a sort of uniqueness property. For this reason we refer to any LSUM as *the* LSUM at x and denote it by $W^{s(u)}(x)$. One readily sees that $T^{-1}W^u(x)$ is a LUM at $T^{-1}x$ and $TW^s(x)$ is a LSM at Tx. This earns the two collections of LSUMs the name '*local invariant foliations*'.

Condition (a) states that the tangent vectors to, say, $W^u(x)$ are unstable 'at all orders'. This and (2.9) imply that if $y \in W^u(x)$ then, for all $n \leq 0$, $d_\parallel(T^n y, T^n x) \leq d_\parallel(y, x)\lambda^n$, where $d_\parallel$ is the distance in the $\|\cdot\|$-norm. We therefore say that $W^u(x)$ is *exponentially unstable* in the increasing metric. We will see later that this property holds for the Riemannian metric too.

We do not recall the other basic notions of hyperbolic theory, such as the definition of absolutely continuous foliation. (The reader who would like to review the fundamentals of Pesin's theory, especially in the context of billiards, can consult [**KS, SC**] and the references therein.)

Even concerning proofs, we only show the details of a couple of lemmas that are specific to our system, and simply sketch the remaining customary arguments—mainly to convince the reader that the infiniteness of μ is no real trouble. It will also be noted that the recurrence property is *not* needed here.

The first observation one has to make to prove that a certain billiard is hyperbolic is that the singularities are not a big obstruction. The lemma that does the job is the following.

LEMMA 3.2. *For almost every $x \in \mathcal{M}$, these exists a $C_0 = C_0(x)$ such that, for all $n \neq 0$,*

$$d_\parallel(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \geq C_0|n|^{-4}.$$

This is in turn based on the next lemma, whose formulation needs two extra definitions. Let $\mathcal{S}_\alpha^\pm := \mathcal{S}^\pm \cap \mathcal{M}_\alpha$. Also, for $A \subseteq \mathcal{M}$ and $\varepsilon > 0$, set

$$A_{(\varepsilon)} := \{x \in \mathcal{M} \mid d_\parallel(x, A) \leq \varepsilon\}. \tag{3.1}$$

LEMMA 3.3. *There exists a constant C_1,* independent of $\alpha \in \mathcal{I}$, *such that, for $\varepsilon > 0$ small enough,*

$$\mu((\mathcal{S}_\alpha^+ \cup \mathcal{S}_\alpha^- \cup \partial\mathcal{M}_\alpha)_{(\varepsilon)}) \leq C_1\varepsilon.$$

Proof of Lemma 3.3. Obviously we can treat $(\mathcal{S}_\alpha^\pm)_{(\varepsilon)}$ and $(\mathcal{M}_\alpha)_{(\varepsilon)}$ separately. For the latter case the assertion is immediate since $(\mathcal{M}_\alpha)_{(\varepsilon)} = \{(r, \varphi) \in \mathcal{M}_\alpha \mid \varphi \leq \varepsilon \text{ or } \varphi \geq \pi - \varepsilon\}$ and L_α is bounded above. (Remember that $\mathcal{M}_\alpha$ is a cylinder based on the circle of length L_α.)

As for $\mathcal{S}_\alpha^+$, it is easy to see that it is composed of a finite number of strictly increasing smooth curves γ and this number does not depend on α. In fact, any scatterer 'seen' from $\mathcal{O}_\alpha$ can only generate two tangent elements (r, φ) for any given r and, since $\mathcal{O}_\alpha$ is convex, φ must grow as r increases (cf. Figure 1). But, the number of accessible scatterers (we will also call them *nearest neighbors*) is bounded above, due to the finite-horizon condition

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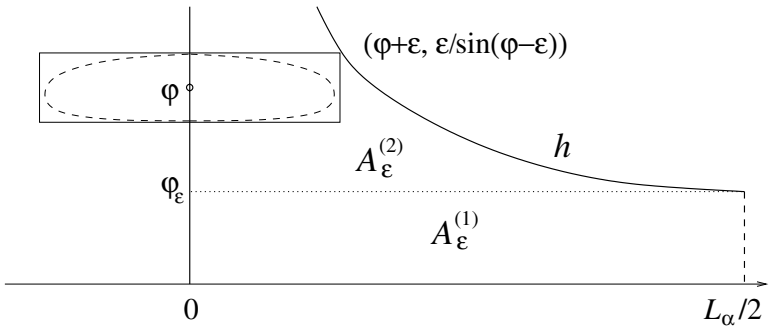


FIGURE 2. The set A_ε as in the proof of Lemma 3.3.

and (2.4). Note that the length of each γ cannot exceed $L_\alpha + \pi$, which is less than some universal constant by (2.5).

Analogous reasoning works for $\mathcal{S}_\alpha^-$, except that the curves are strictly decreasing.

So we only need prove that $\mu(\gamma_{(\varepsilon)}) \le C_2\varepsilon$, with C_2 independent of α. This is essentially contained in [**SC**, Lemma 2], but since the terminology used there is rather different from that of the present work, we give a simple 'analytical' proof here.

First of all, if γ keeps away from $\partial\mathcal{M}_\alpha$, the result is obvious as $d_\parallel$ is equivalent to the Riemannian distance d there. So it suffices to look at, say, $\gamma \cap \{\varphi \le \delta\}$, for some $\delta > 0$. Since $k_\alpha(r)$ is bounded below, γ can only intersect $\{\varphi = 0\}$ transversally. Therefore, without loss of generality, we can focus on the case $\gamma = \{0\} \times [0, \delta]$. Given $x = (r, \varphi) \in \mathcal{M}_\alpha$, the ball in the unstable metric $B_\parallel(x, \varepsilon)$ is more or less an ellipse of semiaxes $\varepsilon/\sin\varphi$ and ε, and is certainly contained in the rectangle $[r - \varepsilon/\sin(\varphi-\varepsilon), r + \varepsilon/\sin(\varphi - \varepsilon)] \times [\varphi - \varepsilon, \varphi + \varepsilon]$ (note that this might exceed $\mathcal{M}_\alpha$).

Hence, with the help of Figure 2, we see that the 'right' part of $\gamma_{(\varepsilon)}$ is contained in $A_\varepsilon := \{(r, \varphi) \in \mathcal{M}_\alpha \mid 0 \le \varphi \le \delta; 0 \le r \le h(\varphi)\}$, with

$$h(\varphi) = \begin{cases} \dfrac{\varepsilon}{\sin(\varphi - 2\varepsilon)} & \varphi \in [2\varepsilon, \delta], \\ \infty & \varphi \in [0, 2\varepsilon). \end{cases} \tag{3.2}$$

However, this is evidently an overestimate, as the right part of $\gamma_{(\varepsilon)}$ only comprises values of r within $[0, L_\alpha/2]$. Set φ_ε so that $h(\varphi_\varepsilon) = L_\alpha/2$. This means that

$$\varphi_\varepsilon = \arcsin\left(\frac{2\varepsilon}{L_\alpha}\right) + 2\varepsilon \simeq 2\varepsilon\left(\frac{1}{L_\alpha} + 1\right). \tag{3.3}$$

(We say that $a \simeq b$ if both quantities depend on ε and $a/b \to 1$, as $\varepsilon \to 0^+$.) Discarding the exceeding part of A_ε, we split the remaining part into two pieces

$$A_\varepsilon^{(1)} = [0, L_\alpha/2] \times [0, \varphi_\varepsilon], \tag{3.4}$$

$$A_\varepsilon^{(2)} = \{\varphi_\varepsilon \le \varphi \le \delta; 0 \le r \le h(\varphi)\}. \tag{3.5}$$

One can estimate $\mu(A_\varepsilon^{(1)})$ from above using the Lebesgue measure; (3.3) and (2.5) ensure

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that this is less than some $C_3\varepsilon$. On the other hand, for δ small,

$$\frac{\mu(A_\varepsilon^{(2)})}{\varepsilon} = \int_{\varphi_\varepsilon}^{\delta} \frac{\sin\varphi}{\sin(\varphi-2\varepsilon)}\,d\varphi \simeq \int_{\varphi_\varepsilon}^{\delta} \frac{\varphi}{\varphi-2\varepsilon}\,d\varphi = \int_{\varphi_\varepsilon-2\varepsilon}^{\delta-2\varepsilon}\left(1+\frac{2\varepsilon}{y}\right)dy = \\ = \delta - \varphi_\varepsilon + 2\varepsilon[\log(\delta-2\varepsilon) - \log(\varphi_\varepsilon-2\varepsilon)] \le 2\delta, \tag{3.6}$$

as $\varepsilon \to 0^+$, since from (3.3), $\varphi_\varepsilon - 2\varepsilon \simeq 2\varepsilon/L_\alpha$. □

Proof of Lemma 3.2. Without loss of generality we will only consider forward semiorbits. Denote

$$\mathcal{N}_n(\alpha) := \{\beta \in \mathcal{I} \mid \exists x \in \mathcal{M}_\alpha, k \le n, \text{ such that } T^k x \in \mathcal{M}_\beta\}. \tag{3.7}$$

In other words, $\mathcal{N}_n(\alpha)$ is the index set of all *kth-nearest neighbors* of $\mathcal{O}_\alpha$, for $k \le n$. By the same line of reasoning as in the previous proof, it is clear that

$$\#\mathcal{N}_n(\alpha) \le C_4\, n^2, \tag{3.8}$$

C_4 not depending on α. Consider an $x \in \mathcal{M}_\alpha$. If the inequality $d_\parallel(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \le n^{-4}$ is verified only for a finite number of n, then $C_0(x)$ can be found so that $d_\parallel(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \ge C_0\, n^{-4}$ for all $n > 0$. This does *not* happen if and only if, $x \in \mathcal{M}_\alpha$ and

$$T^n x \in \bigcap_{m\ge1} \bigcup_{n\ge m} ((\mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M})_{(n^{-4})}). \tag{3.9}$$

However, the arguments above about the phase-space accessibility of orbits starting in $\mathcal{M}_\alpha$ show that (3.9) is equivalent to

$$x \in \mathcal{M}_\alpha \cap \bigcap_{m\ge1} \bigcup_{n\ge m} T^{-n}\Big(\bigcup_{\beta\in\mathcal{N}_n(\alpha)} (\mathcal{S}_\beta^+ \cup \mathcal{S}_\beta^- \cup \partial\mathcal{M}_\beta)_{(n^{-4})}\Big). \tag{3.10}$$

The measure of the above right-hand side is zero by Borel–Cantelli, as the measure of the individual terms decreases like n^{-2} (by Lemma 3.3, (3.8) and the invariance of μ with respect to T). Taking the union over $\alpha \in \mathcal{I}$ completes the proof. □

The existence and absolute continuity of the local invariant foliations is a local matter. More precisely, for $x \in \mathcal{M}$, it has to do with the behavior of DT on a sequence of balls centered at $T^n x$ and of shrinking radii, as $|n| \to \infty$.

Say we want to consider LUMs. Lemma 3.2 implies that, for some $C_5 > 0$, no element of the sequence $\{B_\parallel(T^n x, C_5\lambda^{-|n|})\}_{n\le0}$ can intersect the singularities or the boundary of the phase space. So, inside those balls, we have a *bona fide* Pesin's theory. The invariance of the unstable cones ($DT\mathcal{C}(x) \subset \mathcal{C}(Tx)$) and (2.9) provide the stretching mechanism for *unstable curves* (curves whose tangent vectors are everywhere unstable). And the fact that $DT^{-n}\mathcal{C}(T^n x)$ collapses to a line (in this case, an easy consequence of (2.9) and the invariance of μ) proves that the process of pushing forward unstable curves 'from the past' has a limit, which is precisely the LUM $W^u(x)$. To make things even simpler, we are dealing with a *uniformly hyperbolic* system (with respect to $\|\cdot\|$) and standard results [**KH**, §6] show that $W^u(x)$ is as regular as T is (meaning, away from $\mathcal{S}^+$).

Condensing the arguments for the absolute continuity in the same way is impossible. It suffices to say that the major task is to control the *distortion coefficient* of T along the

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sequence of shrinking balls. The distortion coefficient is a certain logarithmic derivative of $|DTu|$, for some vector u, with $|u| = 1$, and the requirement is that it grows less than exponentially as $n \to -\infty$. General results on finite-horizon billiards show that this quantity can increase at most like a negative power of $\sin\varphi_n$, with $(r_n, \varphi_n) := y_n \in B_\parallel(T^n x, C_5\lambda^n)$ (we are simplifying a bit here—see [**L**, §7]). However, $\sin\varphi_n \geq C_6 \min\{\varphi_n, \pi - \varphi_n\} = C_6\, d_\parallel(y_n, \partial\mathcal{M})$ and Lemma 3.2 says, in particular, that this distance does not approach zero faster than $|n|^{-4}$. Which verifies the requirement.

Remark 3.4. The above also proves that the LSUMs are exponentially (un)stable with respect to the Riemannian metric. In fact $|\cdot|_{(r,\varphi)} \leq (\sin\varphi)^{-1}\|\cdot\|_{(r,\varphi)}$ and in Definition 3.1(b) the polynomial growth of $(\sin\varphi)^{-1}$ is tamed by the exponential contraction of $d_\parallel(T^n y, T^n x)$. Hence, the exponential rate is any $\lambda' < \lambda$.

None of the above assertions depend on the infiniteness of μ, or the non-compactness of the billiard. To describe it in lay terms, the particle has no way of knowing whether it is traveling in a plane or in a torus, by only looking at small neighborhoods of its orbit.

4. *Ergodicity*

For dynamical systems with an infinite invariant measure, the many available definitions of ergodicity fail to be equivalent. The definition that is usually retained is the following [**A**].

Definition 4.1. A dynamical system $(\mathcal{M}, T, \mu)$ is ergodic if every $A \subseteq \mathcal{M}$, measurable and invariant mod μ (i.e. $\mu(T^{-1}A \Delta A) = 0$), has either zero measure or full measure (i.e. $\mu(\mathcal{M} \setminus A) = 0$).

Definition 4.1 does not require the system to be recurrent, but it usually makes little sense if there is a dissipative part (thus causing the whole system to be dissipative). We do not have that problem here.

The goal of this section is to prove the following result.

THEOREM 4.2. *Let the LG $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}}$ belong to $\mathcal{X}$. The associated dynamical system $(\mathcal{M}, T, \mu)$ is ergodic.*

The essential tool to work out Theorem 4.2 is the *local ergodicity theorem*, also known as the *fundamental theorem* for hyperbolic billiards. We state it in a rather technical guise that is convenient for our purposes.

THEOREM 4.3. *Let A be a full-measure subset of $\mathcal{M}$. Then, for any x_0 that possesses a semiorbit (i.e., $x_0 \in \mathcal{M} \setminus \mathcal{S}^+_\infty$, or $x_0 \in \mathcal{M} \setminus \mathcal{S}^-_\infty$), there is a neighborhood U of x_0 with the following property.*

Almost every two points $x', x'' \in U$ are connected by a finite alternating sequence of LSUMs, $W^s(x_1), W^u(x_2), W^s(x_3), \ldots, W^u(x_m)$, with $x_1 := x'$ and $x_m := x''$. Each LSUM intersects the next transversally in a point of A. Also $x_1, \ldots, x_m \in A$.

We do not give the complete proof of the local ergodicity theorem. As the name suggests, most of it is strictly local and does not require μ to be finite [**LW**, §§8–12]. The technical conditions that are called for are all verified for dispersing billiards; this has been known since [**S1**].

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The non-local part in the proof of Theorem 4.3 is, in general, the most laborious and for this reason has its own name, the *tail bound* [**LW**, §13]. We state it here in the form of a lemma and prove it in detail.

First we assume that the LSUMs are maximal, i.e. for every x for which a LUM exists, we take $W^u(x)$ to be the union of all the LUMs at x. This implies that, for almost every x, $\partial W^u(x) \subset \partial\mathcal{M} \cup \mathcal{S}^-_\infty$. (It might also happen that $\partial W^u(x)$ intersects the closure of $\mathcal{S}^-_\infty$—which is typically the whole $\mathcal{M}$—but standard reasoning shows that this cannot occur for more than a null-measure set of x; see [**L**, Rk. 8.9].) The analogous statement holds of course for $W^s(x)$.

If $y \in \partial W^u(x) \cap T^m\mathcal{S}^-$, for some $m \geq 0$, we say that $W^u(x)$ is *cut* by $T^m\mathcal{S}^-$ at y. The radius of $W^u(x)$ is defined as the minimum distance, *along* $W^u(x)$ *and in the increasing metric*, between x and the two endpoints of $W^u(x)$.

LEMMA 4.4. *For every $x_0 \in \mathcal{M}$, there is a neighborhood U_0 of x_0 and a $\delta_0 > 0$ such that, for all $\eta > 0$, there exists M that verifies*

$$\mu\left(\left\{x \in U_0 \,\middle|\, W^u(x) \text{ has radius } < \delta \text{ because it is cut by } \bigcup_{m=M+1}^{\infty} T^m\mathcal{S}^-\right\}\right) \leq \eta\delta,$$

for every $\delta \leq \delta_0$.

Proof of Lemma 4.4. Let $Y = Y(U_0, \delta, M)$ be the set in the statement of the lemma and, for all $x \in Y$, set $m(x)$ to be the smallest integer such $T^m\mathcal{S}^-$ cuts $W^u(x)$ within a distance δ of x. If we define $Y_m := \{x \in Y \mid m(x) = m\}$, then

$$\mu(Y) = \sum_{m>M} \mu(Y_m) = \sum_{m>M} \mu(T^{-m}Y_m). \tag{4.1}$$

The first equality holds because the Y_m are the level sets of the function m; the second equality is due to the invariance of μ.

For $x \in Y_m$, denote by $y = y(x) \in \partial W^u(x) \cap T^m\mathcal{S}^-$ the point where the cut occurs. By the exponential contraction of $W^u(x)$, the portion of $T^{-m}W^u(x)$ between $T^{-m}x$ and $T^{-m}y \in \mathcal{S}^-$ has length less than $\delta\lambda^{-m}$ (in the increasing metric). So $d_\parallel(T^{-m}x, \mathcal{S}^-) < \delta\lambda^{-m}$.

Let $\alpha \in \mathcal{I}$ so that $x_0 \in \mathcal{M}_\alpha$, and select any $U_0 \subseteq \mathcal{M}_\alpha$. Following the line of reasoning of Lemma 3.2, $T^{-m}Y_m \subset \bigcup_{\beta\in\mathcal{N}_m(\alpha)} \mathcal{M}_\beta$. This and the above imply that

$$T^{-m}x \in \bigcup_{\beta\in\mathcal{N}_m(\alpha)} (\mathcal{S}^-_\beta)_{(\delta\lambda^{-m})}, \tag{4.2}$$

whose measure, by Lemma 3.3 and (3.8), is $\leq C_1C_4m^2\delta\lambda^{-m}$. Therefore, from (4.1),

$$\frac{\mu(Y)}{\delta} \leq C_1C_4 \sum_{m>M} m^2\lambda^{-m} \to 0, \tag{4.3}$$

as $M \to +\infty$. □

LEMMA 4.5. *Fix an $\alpha \in \mathcal{I}$ and denote by T_α the first return map induced by T on $\mathcal{M}_\alpha$ (this is well defined almost everywhere by recurrence): $(\mathcal{M}_\alpha, T_\alpha, \mu)$ is ergodic.*

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Proof. We proceed in three steps.

(1) *The LSUMs relative to T are also LSUMs relative to T_α.* Let us check this. In Definition 3.1, (c) has nothing to do with T or T_α. (a) is obvious since $T_\alpha^k y = T^{n_k} y$, for some $n_k < 0$. For (b) it is sufficient to prove that if $y \in W^{s(u)}(x)$ then the (future or past) return times to $\mathcal{M}_a$ are the same for y and x. However, this follows from (b) itself as, by definition, $d_\parallel(\mathcal{M}_\alpha, \mathcal{M}_\beta) = \infty$ when $\alpha \neq \beta$; therefore, the sequence of scatterers hit is the same for y and x.

(2) *Any x_0 that possesses a semiorbit has a neighborhood U contained in one ergodic component.* This is Hopf's idea. Take a measurable f, continuous (and thus uniformly continuous) on $\mathcal{M}_\alpha$, and consider its forward and backward ergodic averages with respect to T_α:

$$f^\pm(x) := \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} (f \circ T_\alpha^{\pm k})(x). \tag{4.4}$$

By a standard application of Birkhoff's Theorem, the set

$$A := \{x \in \mathcal{M} \mid f^+(x),\, f^-(x) \text{ exist and coincide}\} \tag{4.5}$$

has full measure. Let us use it in Theorem 4.3. In the resulting neighborhood U, almost every two points x', x'' are connected by a curvilinear polyline made up of LSUMs. Call y the intersection point between $W^s(x_1)$ and $W^u(x_2)$. From step (1) and the uniform continuity of f, $f^+(y) = f^+(x_1)$ and $f^-(y) = f^-(x_2)$. However, $f^+(y) = f^-(y)$ because $y \in A$. Proceeding this way, we conclude that $f^+(x') = f^+(x'')$; that is, f^+ is almost everywhere constant in U. This continues to hold for every $f \in L^1(\mathcal{M})$, via Birkhoff's Theorem and customary density arguments.

(3) *There is only one ergodic component for $(\mathcal{M}_\alpha, T_\alpha, \mu)$.* The set of points that fail to have a semiorbit is $\mathcal{S}_\infty^+ \cap \mathcal{S}_\infty^-$. However, as explained in §3, $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$ are countable unions of smooth curves, strictly increasing and strictly decreasing, respectively. Therefore, there can be at most one point of intersection for each pair of increasing–decreasing curves; that is, at most countably many points. Hence, $\mathcal{M}_\alpha \setminus (\mathcal{S}_\infty^+ \cap \mathcal{S}_\infty^-)$ is path-connected. □

Now it is simple to prove the main result of §4.

Proof of Theorem 4.2. We claim that the ergodic decomposition of T is coarser than the partition $\{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{I}}$. In fact the existence of A, B, invariant subsets, such that $\mu(A \cap \mathcal{M}_\alpha), \mu(B \cap \mathcal{M}_\alpha) > 0$, for some $\alpha \in \mathcal{I}$, contradicts Lemma 4.5.

Therefore, if there were more than one ergodic component, there would be two nearest neighbors $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, such that $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are contained in two different invariant sets. However, this is absurd since one can always find $A \subset \mathcal{M}_\alpha$, $\mu(A) > 0$, such that $T(A) \subset \mathcal{M}_\beta$. □

5. *Recurrence*

We recalled in §1 that a finite-horizon periodic LG is recurrent (as in Definition 2.1) [**Co, Sch**] and thus belongs to $\mathcal{X}$ (conditions (2.2) and (2.3) being trivially satisfied). Here we present other examples from the class $\mathcal{X}$.

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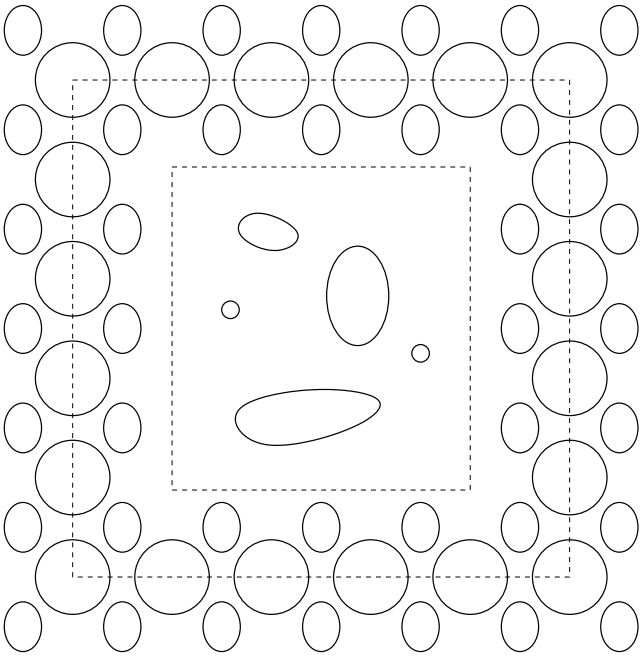


FIGURE 3. A finite modification of a periodic LG. The inner dashed square encloses the scatterers associated to $\mathcal{I}_2$; the outer square corresponds to $\mathcal{J}$.

Definition 5.1. The LG $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}}$ is called a *finite modification* of the finite-horizon PLG $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}_P}$ if $\mathcal{I} = (\mathcal{I}_P \setminus \mathcal{I}_1) \cup \mathcal{I}_2$, where:

(a) $\mathcal{I}_1$ is a finite subset of $\mathcal{I}_P$;

(b) $\mathcal{I}_2$ is the index set of a finite LG such that $d(\mathcal{O}_\alpha, \mathcal{O}_\beta) > 0$ for any $\alpha \in \mathcal{I}_2$, $\beta \in \mathcal{I}_P \setminus \mathcal{I}_1$.

Remark 5.2. Calling the above a 'perturbation' does not seem appropriate, as the modification can be arbitrarily large, provided it is finite.

An example of a finite modification of a PLG is illustrated in Figure 3. Since it is clear that Definition 5.1 implies (2.2) and (2.3), we focus on recurrence.

PROPOSITION 5.3. *A finite modification of a PLG is recurrent.*

Proof. Let us introduce $\mathcal{J} := \{\beta \in \mathcal{I} \mid \beta \in \mathcal{N}_1(\alpha), \text{ for some } \alpha \in \mathcal{I}_2\}$. This is $\mathcal{I}_2$ plus the indices of the nearest neighbors to the scatterers labeled by $\mathcal{I}_2$ (cf. Figure 3). Also denote $\mathcal{M}_\mathcal{J} := \bigsqcup_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha$. We proceed in three steps.

(1) *Almost every $x \in \mathcal{M}_\mathcal{J}$ returns to $\mathcal{M}_\mathcal{J}$.* First of all, for *every* $x \in \mathcal{M}_{\mathcal{I}_2}$ (notation understood), $Tx \in \mathcal{M}_\mathcal{J}$. Then consider the set $\{x \in \mathcal{M}_{\mathcal{J}\setminus\mathcal{I}_2} \mid T^n x \notin \mathcal{M}_\mathcal{J}, \forall n > 0\}$. The forward orbit of every point there lies in the 'periodic part' of the plane. We use the recurrence of a PLG to conclude that this set has measure zero.

(2) *Denoting by $T_\mathcal{J}$ the return map to $\mathcal{M}_\mathcal{J}$ (well defined almost everywhere by step (1)), $T_\mathcal{J}$ is recurrent.* This is the Poincaré Theorem applied to $(\mathcal{M}_\mathcal{J}, T_\mathcal{J}, \mu)$.

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(3) *T is recurrent.* It suffices to prove that $\mathcal{M}_{\mathcal{J}}$ is a global cross section (mod μ) for $T : \mathcal{M} \longrightarrow \mathcal{M}$. However, a positive-measure subset of $\mathcal{M}$ whose orbit never intersects $\mathcal{M}_{\mathcal{J}}$ would contradict the ergodicity of a PLG. □

If the systems above seem too close to those already known, we provide further examples of LGs whose aperiodicity is not limited to a compact region of $\mathbb{R}^2$.

PROPOSITION 5.4. *There are LGs in the class $\mathcal{X}$ that are not finite modifications of PLGs.*

Proof. We present an algorithm to construct such gases explicitly. Start with a PLG $\mathcal{L}_0 := \{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}_0}$ and denote by T_0 the corresponding billiard map. Furthermore, fix an index $\alpha \in \mathcal{I}_0$ and a point $O \in \mathcal{O}_\alpha$ which will be our origin from now on.

For $x \in \mathcal{M}_\alpha$, denote by $n_1 = n_1(x, T_0) \in \mathbb{N} \cup \{+\infty\}$ the first return time of x to $\mathcal{M}_\alpha$, with respect to T_0. Indicating by $q(x)$ the point in $\mathbb{R}^2$ corresponding to x, we define

$$A_0(R) := \left\{ x \in \mathcal{M}_\alpha \,\middle|\, \sup_{0<k<n_1} \operatorname{dist}(q(T_0^k x), O) \le R \right\}, \tag{5.1}$$

where dist is the distance in $\mathbb{R}^2$. Since T_0 is recurrent, $A_0(R) \nearrow \mathcal{M}_a$ mod μ, as $R \to +\infty$. Hence, there is an R_0 such that $\mu(\mathcal{M}_\alpha \setminus A_0(R_0)) \le \varepsilon_0$. Here $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\varepsilon_k \searrow 0$, for $k \to +\infty$.

Now modify $\mathcal{L}_0$ in the annulus of radii R_0 and $R_0 + \rho$ (ρ is a sufficiently large fixed quantity). This means that one can modify the shape, position and number of scatterers within that annulus. Call the resulting LG $\mathcal{L}_1 := \{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}_1}$ and the resulting map T_1. This is a finite modification of a PLG. Furthermore, denote by $T_{\alpha,k}$ the first return map to $\mathcal{M}_a$ relative to T_k; it is obvious that $(T_{\alpha,1})|_{A_0(R_0)} = (T_{\alpha,0})|_{A_0(R_0)}$.

It is clear that this process can be repeated recursively. Starting with $\mathcal{L}_k$, a finite modification of a PLG, we define $A_k(R)$ as in (5.1), with T_k in the place of T_0. By Proposition 5.3, $A_k(R) \nearrow \mathcal{M}_a$ mod μ, as $R \to +\infty$. We then choose $R_k > R_{k-1} + \rho$ such that

$$\mu(\mathcal{M}_\alpha \setminus A_k(R_k)) \le \varepsilon_k \tag{5.2}$$

and make a modification in the annulus of radii R_k and $R_k + \rho$, to obtain $\mathcal{L}_{k+1}$ and T_{k+1}. We emphasize that this modification can be made so that the curvature of the scatterers and the free path among them stay bounded above and below, to comply with (2.2) and (2.3). Evidently,

$$A_k(R_k) = A_{k+1}(R_k) \subseteq A_{k+1}(R_{k+1}), \tag{5.3}$$

$$(T_{\alpha,k+1})|_{A_k(R_k)} = (T_{\alpha,k})|_{A_k(R_k)}. \tag{5.4}$$

Since we never change the LG twice in the same region of the plane, this modification process has a limit, which we call $\bar{\mathcal{L}}$, with the corresponding maps $\bar{T}$ for the LG and $\bar{T}_\alpha$ for the returns to $\mathcal{M}_a$. In other words, from (5.2)–(5.4), $\bar{T}_\alpha : \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\alpha$ is well defined almost everywhere via $(\bar{T}_\alpha)|_{A_k(R_k)} = (T_{\alpha,k})|_{A_k(R_k)}$.

By construction, *all* points of $\bigcup_k A_k(R_k)$ come back to $\mathcal{M}_a$. So, as far as $\bar{T}$ is concerned, $\mathcal{M}_a$ is contained in B, the *conservative* part of $\bar{\mathcal{M}}$ (i.e. the complement of its dissipative part [**A**]). B is obviously $\bar{T}$-invariant.

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We claim that the LSUMs of $\bar{T}$ that intersect B are wholly contained in B, mod μ. (The LSUMs exist almost everywhere and are absolutely continuous by §3—no recurrence was used there.) More precisely, this means that there is a set B_0, $\mu(B \Delta B_0) = 0$, such that, calling B_1 the union of all the LSUMs based in B_0, we have $\mu(B \Delta B_1) = 0$.

In fact, take $x \in B \cap \mathcal{M}_\beta$, $\beta \in \bar{\mathcal{I}}$; passing maybe to a full-measure subset B_0, we can assume that x returns to $\mathcal{M}_\beta$ infinitely many times, both in the past and in the future. Then any $y \in W^s(x)$, say, returns to $\mathcal{M}_\beta$ infinitely many times in the future (Step 1 of the proof of Lemma 4.5). Therefore, save for a null-measure set of exceptions, y cannot be part of a wandering set and thus belongs to B.

At this point, Theorem 4.3 proves that the decomposition $\{B, \bar{\mathcal{M}} \setminus B\}$ is coarser than $\{\mathcal{M}_\alpha\}_{\alpha \in \bar{\mathcal{I}}}$. That $B = \bar{\mathcal{M}}$ mod μ follows from the invariance of B as in the proof of Theorem 4.2. □

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