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WEIGHTED COMPOSITION OPERATORS BETWEEN LORENTZ SPACES

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Abstract

We investigate the boundedness, compactness, invertibility and Fredholmness of weighted composition operators between Lorentz spaces. It is also shown that the classes of Fredholm and invertible weighted composition maps between Lorentz spaces coincide when the underlying measure space is nonatomic.

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1. Introduction

Let (X, Σ, μ) be a σ-finite and complete measure space. Suppose f is a complex-valued Σ-measurable function on X. The distribution function, μ_f, of f is defined by

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\} \quad \text{for every } \lambda \geq 0.$$

Let f^* be the decreasing arrangement of f defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\} \quad \text{for all } t \geq 0$$

(with the usual convention that $\inf \varnothing = \infty$). For $t > 0$, let

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)\, ds$$

be the maximal function of f. Assume $1 < p < \infty$. We define

$$\|f\|_{pq} = \begin{cases} \left[\dfrac{q}{p} \displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \, \frac{dt}{t} \right]^{1/q} & \text{if } 1 \leq q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p} f^{**}(t) & \text{if } q = \infty. \end{cases}$$

The *Lorentz space*, denoted by $L(p, q, \mu)$, consists of all (equivalence classes of) Σ-measurable functions f on X for which $\|f\|_{pq} < \infty$. It is a Banach space under the

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norm $\|\cdot\|_{pq}$, which is written as $\|\cdot\|_{L(p,q,\mu)}$. Moreover, in view of

$$\frac{p-1}{p}\|f\|_{L(p,p,\mu)} \le \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p} = \left[\int_0^\infty (f^*(t))^p \, dt\right]^{1/p} \le \|f\|_{L(p,p,\mu)} \tag{1.1}$$

[4, pages 43 and 219], $L(p,p,\mu)$ coincides with the usual Lebesgue space $L^p(\mu)$ (equipped with the norm $\|\cdot\|_{L^p(\mu)}$). We denote the space of all essentially bounded functions on X by $L^\infty(\mu)$. Lorentz spaces arise in many classical and important applications, such as the Marcinkiewicz interpolation theorem, Hilbert transforms and wavelet analysis. The books [4, 5] and the paper [7] give an extensive introduction to Lorentz spaces and other related topics.

Suppose that (Y,Γ,ν) is another σ-finite and complete measure space. We define ν_g, g^* and g^{**} similarly for a complex-valued Γ-measurable function g on Y. Analogously, we may define $L(p,q,\nu)$ and $\|\cdot\|_{L(p,q,\nu)}$ for $1 < p < \infty$ and $1 \le q \le \infty$.

Let $\varphi : Y \to X$ be a point mapping such that $\varphi^{-1}(E) \in \Gamma$ for all $E \in \Sigma$. Assume that φ is also nonsingular, which means the measure defined by $\nu\varphi^{-1}(E) := \nu(\varphi^{-1}(E))$, for $E \in \Sigma$, is absolutely continuous with respect to μ. Let $u : Y \to \mathbb{C}$ be a Γ-measurable function. The functions u and φ induce a *weighted composition operator* uC_φ from $L(p,q,\mu)$ into the linear space of all Γ-measurable functions on Y by

$$(uC_\varphi f)(y) := u(y)\, f(\varphi(y)) \quad \text{for every } f \in L(p,q,\mu) \text{ and } y \in Y.$$

The nonsingularity of φ guarantees that uC_φ is a well-defined mapping of equivalence classes of functions. When $u \equiv 1$, the corresponding operator, C_φ, is called a *composition operator*, and when $(X,\Sigma,\mu) = (Y,\Gamma,\nu)$ and $\varphi(x) = x$ for $x \in X$, the corresponding operator, M_u, is called a *multiplication operator*. Note that $uC_\varphi = M_u \circ C_\varphi$. If uC_φ maps $L(p,q,\mu)$ into $L(p,q,\nu)$, it follows from the closed graph theorem that uC_φ is also bounded.

Weighted composition operators have been studied on various function spaces, such as Lebesgue spaces, Orlicz spaces, Hardy spaces and Bergman spaces. Composition and multiplication operators on Lorentz spaces have also been considered in [1, 2, 8, 9]. We shall investigate operator-theoretic properties of weighted composition maps between Lorentz spaces. In particular, we

(a) obtain some necessary or sufficient conditions for a weighted composition operator to be bounded or compact (Theorems 2.1, 2.3, 3.1 and 3.3),
(b) characterise the invertibility of weighted maps (Theorem 4.1), and
(c) show that Fredholm weighted composition maps are precisely the invertible ones when the underlying measure space is nonatomic (Theorem 4.3).

2. Bounded weighted composition operators

Let $(B_1,\|\cdot\|_{B_1})$ and $(B_2,\|\cdot\|_{B_2})$ be normed spaces. A linear operator $T : B_1 \to B_2$ is said to be bounded if there is a constant $M \ge 0$ such that $\|Tx\|_{B_2} \le M\|x\|_{B_1}$ for all $x \in B_1$. Boundedness of multiplication and composition operators on Lorentz

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spaces has been characterised in [1, Theorem 2.1] and [2, Theorem 2.1], [8, Theorem 2.3], respectively. In [3, Theorems 2.1 and 2.2], Arora *et al.* studied bounded operators of the form $f \mapsto (uf) \circ \varphi$ for $f \in L(p, q, \mu)$. A result therein is also valid for weighted composition operators between Lorentz spaces, as we now show. In what follows, we denote the Radon–Nikodym derivative of $\nu\varphi^{-1}$ with respect to μ by $d(\nu\varphi^{-1})/d\mu$.

THEOREM 2.1. *If $u \in L^\infty(\nu)$ and $d(\nu\varphi^{-1})/d\mu \in L^\infty(\mu)$, then uC_φ is a bounded weighted composition operator from $L(p, q, \mu)$ into $L(p, q, \nu)$.*

PROOF. Suppose $\|u\|_{L^\infty(\nu)} > 0$ (otherwise, uC_φ is just the zero operator). Choose any $f \in L(p, q, \mu)$. Note that

$$\begin{aligned} \nu_{uC_\varphi f}(\lambda) &= \nu\{y \in Y : |u(y)||f(\varphi(y))| > \lambda\} \;\le\; \nu\{y \in Y : \|u\|_{L^\infty(\nu)}|f(\varphi(y))| > \lambda\} \\ &= \nu\varphi^{-1}\left\{x \in X : |f(x)| > \frac{\lambda}{\|u\|_{L^\infty(\nu)}}\right\} \;\le\; \alpha\mu\left\{x \in X : |f(x)| > \frac{\lambda}{\|u\|_{L^\infty(\nu)}}\right\}, \end{aligned}$$

where $\alpha := \|d(\nu\varphi^{-1})/d\mu\|_{L^\infty(\mu)}$. If $\alpha = 0$, then $\nu_{uC_\varphi f}(0) = 0$ and $uC_\varphi f = 0$ ν-almost everywhere (a.e.) on Y. Assume $\alpha > 0$. Since

$$\begin{aligned} (uC_\varphi f)^*(t) &= \inf\{\lambda > 0 : \nu_{uC_\varphi f}(\lambda) \le t\} \\ &\le \inf\left\{\lambda > 0 : \mu\left\{x \in X : |f(x)| > \frac{\lambda}{\|u\|_{L^\infty(\nu)}}\right\} \le \frac{t}{\alpha}\right\} \\ &= \|u\|_{L^\infty(\nu)} \inf\left\{\lambda > 0 : \mu\{x \in X : |f(x)| > \lambda\} \le \frac{t}{\alpha}\right\} = \|u\|_{L^\infty(\nu)} f^*\left(\frac{t}{\alpha}\right) \end{aligned}$$

for every $t \ge 0$, we have

$$\begin{aligned} (uC_\varphi f)^{**}(t) &= \frac{1}{t}\int_0^t (uC_\varphi f)^*(s)\,ds \;\le\; \frac{\|u\|_{L^\infty(\nu)}}{t}\int_0^t f^*\left(\frac{s}{\alpha}\right)ds \\ &= \frac{\alpha\|u\|_{L^\infty(\nu)}}{t}\int_0^{t/\alpha} f^*(s)\,ds \;=\; \|u\|_{L^\infty(\nu)} f^{**}\left(\frac{t}{\alpha}\right). \end{aligned}$$

If $1 \le q < \infty$, then

$$\begin{aligned} \|uC_\varphi f\|^q_{L(p,q,\nu)} &= \frac{q}{p}\int_0^\infty (t^{1/p}(uC_\varphi f)^{**}(t))^q\,\frac{dt}{t} \;\le\; \frac{q\|u\|^q_{L^\infty(\nu)}}{p}\int_0^\infty \left(t^{1/p} f^{**}\left(\frac{t}{\alpha}\right)\right)^q\frac{dt}{t} \\ &= \frac{q\alpha^{q/p}\|u\|^q_{L^\infty(\nu)}}{p}\int_0^\infty (t^{1/p} f^{**}(t))^q\,\frac{dt}{t} \;=\; \alpha^{q/p}\|u\|^q_{L^\infty(\nu)}\|f\|^q_{L(p,q,\mu)}, \end{aligned}$$

that is,

$$\|uC_\varphi f\|_{L(p,q,\nu)} \le \alpha^{1/p}\|u\|_{L^\infty(\nu)}\|f\|_{L(p,q,\mu)}.$$

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If $q = \infty$, then

$$\|uC_\varphi f\|_{L(p,\infty,\nu)} = \sup_{t>0} t^{1/p}(uC_\varphi f)^{**}(t)$$
$$\leq \alpha^{1/p}\|u\|_{L^\infty(\nu)} \sup_{t>0}\left(\frac{t}{\alpha}\right)^{1/p} f^{**}\left(\frac{t}{\alpha}\right) = \alpha^{1/p}\|u\|_{L^\infty(\nu)}\|f\|_{L(p,\infty,\mu)}.$$

These estimates show that uC_φ is a bounded operator from $L(p,q,\mu)$ into $L(p,q,\nu)$. □

We remark that the conditions $u \in L^\infty(\nu)$ and $d(\nu\varphi^{-1})/d\mu \in L^\infty(\mu)$ are not necessary for the boundedness of uC_φ. This is shown by the following example.

EXAMPLE 2.2. Let $X = Y = (0,1]$ and $\Sigma = \Gamma$ be the σ-algebra of Borel sets in $(0,1]$ with the Lebesgue measure μ. Take $u(x) = x^{-1/4}$, $\varphi(x) = \sqrt{x}$, so $d(\mu\varphi^{-1})/d\mu = 2x$. By the inequalities in (1.1),

$$\|uC_\varphi f\|^2_{L(2,2,\mu)} \leq 4\|uC_\varphi f\|^2_{L^2(\mu)} = 4\int_0^1 x^{-1/2}|f(\sqrt{x})|^2\,dx = 8\int_0^1 |f(x)|^2\,dx$$
$$= 8\|f\|^2_{L^2(\mu)} \leq 8\|f\|^2_{L(2,2,\mu)}$$

for every $f \in L(2,2,\mu)$. Thus, uC_φ is bounded on $L(2,2,\mu)$, but u is unbounded on X.

Similarly, by choosing $u(x) = \sqrt{x}$ and $\varphi(x) = x^2$, one may also show that they induce a bounded operator uC_φ on $L(2,2,\mu)$, whereas $d(\mu\varphi^{-1})/d\mu$ $(= 1/(2\sqrt{x}))$ is unbounded on X.

The converse of Theorem 2.1 partially holds under additional assumptions on u and φ.

THEOREM 2.3. *Suppose that φ is one-to-one and there exists a constant $c > 0$ such that $d(\nu\varphi^{-1})/d\mu \geq c$ μ-a.e. on X. If uC_φ is a bounded operator from $L(p,q,\mu)$ into $L(p,q,\nu)$, then $u \in L^\infty(\nu)$.*

PROOF. Assume to the contrary that the sets $F_n := \{y \in Y : |u(y)| > n\}$ have positive ν-measures for all $n \in \mathbb{N}$. Let $E_n := \varphi(F_n)$. With $\nu\varphi^{-1}(E_n) = \nu(F_n) > 0$, the nonsingularity of φ ensures that $\mu(E_n) > 0$. Since $|u\chi_{\varphi^{-1}(E_n)}| \geq n\chi_{\varphi^{-1}(E_n)}$ on Y,

$$\begin{aligned}(uC_\varphi\chi_{E_n})^*(t) &= \inf\{\lambda > 0 : \nu\{y \in Y : |u(y)||(\chi_{E_n} \circ \varphi)(y)| > \lambda\} \leq t\}\\ &\geq \inf\{\lambda > 0 : \nu\{y \in Y : n|\chi_{E_n}(\varphi(y))| > \lambda\} \leq t\}\\ &= n\inf\{\lambda > 0 : \nu\{y \in Y : |\chi_{E_n}(\varphi(y))| > \lambda\} \leq t\}\\ &= n\inf\{\lambda > 0 : \nu\varphi^{-1}\{x \in X : |\chi_{E_n}(x)| > \lambda\} \leq t\}\\ &\geq n\inf\{\lambda > 0 : c\mu\{x \in X : |\chi_{E_n}(x)| > \lambda\} \leq t\} = nc{\chi_{E_n}}^*(t)\end{aligned}$$

for all $t \geq 0$. Thus,

$$(uC_\varphi\chi_{E_n})^{**}(t) = \frac{1}{t}\int_0^t (uC_\varphi\chi_{E_n})^*(s)\,ds \geq \frac{nc}{t}\int_0^t {\chi_{E_n}}^*(s)\,ds = nc{\chi_{E_n}}^{**}(t)$$

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and so

$$\|uC_\varphi \chi_{E_n}\|_{L(p,q,\nu)} \geq nc\|\chi_{E_n}\|_{L(p,q,\mu)}.$$

This contradicts the boundedness of uC_φ. □

3. Compact weighted composition operators

Let B_1 and B_2 be two normed spaces. A linear operator $T : B_1 \to B_2$ is compact if it maps bounded subsets of B_1 into relatively compact subsets of B_2. In other words, T is compact if and only if it maps every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in B_1 onto a sequence $\{Tx_n\}_{n\in\mathbb{N}}$ in B_2 which has a convergent subsequence. Compactness of operators with the form $f \mapsto (uf) \circ \varphi$, where $f \in L(p, q, \mu)$, has been investigated in [3, Theorems 3.1–3.3]. Following the ideas in the proofs of [8, Theorem 3.1] and [9, Theorem 2], which completely characterise compact composition operators on Lorentz spaces, we obtain some necessary or sufficient conditions for a weighted composition operator to be compact. In what follows we adopt the following decomposition of (X, Σ, μ):

$$X = \left(\bigcup_{i\in\mathbb{N}} A_i\right) \cup B,$$

where $\{A_i\}_{i\in\mathbb{N}}$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_i, is nonatomic. This decomposition is unique if equality of two Σ-measurable sets is interpreted as meaning that their symmetric difference is of zero μ-measure. The σ-finiteness of (X, Σ, μ) ensures that $(0 <) \mu(A_i) < \infty$ for every $i \in \mathbb{N}$. If $X = B$, then (X, Σ, μ) is said to be nonatomic. The following facts are also useful.

(a) Let E be a nonatomic set in Σ with $\mu(E) > 0$. Then

  (i) there is a sequence of pairwise disjoint (or decreasing) Σ-measurable subsets of E, $\{E_n\}_{n\in\mathbb{N}}$, such that $\mu(E_n) > 0$ for all n and $\lim_{n\to\infty} \mu(E_n) = 0$;
  (ii) for every real number α satisfying $0 < \alpha < \mu(E)$, there is a set $E_\alpha \in \Sigma$ such that $E_\alpha \subset E$ and $\mu(E_\alpha) = \alpha$.

(b) A Σ-measurable function f is constant μ-a.e. on an atom M in (X, Σ, μ). In view of this, we may consider an atom M as a 'point' at which f takes a constant value denoted by $f(M)$.

THEOREM 3.1. *Let uC_φ be a weighted composition operator from $L(p, q, \mu)$ into $L(p, q, \nu)$. Suppose $u \in L^\infty(\nu)$. Then uC_φ is compact if both the following conditions are satisfied:*

(i) $\nu\varphi^{-1}(B) = 0$;
(ii) $\lim_{i\to\infty} \nu\varphi^{-1}(A_i)/\mu(A_i) = 0$, *where we understand that the limit takes the value zero when the number of atoms in (X, Σ, μ) is finite.*

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PROOF. For every $n \in \mathbb{N}$, define $T_n f = \sum_{i=1}^{n} f(A_i) u C_\varphi \chi_{A_i}$ for $f \in L(p,q,\mu)$. Direct computation yields

$$\|\chi_{A_i}\|_{L(p,q,\mu)} = \begin{cases} \left(\dfrac{p}{p-1}\right)^{1/q} \mu(A_i)^{1/p} & \text{if } 1 \le q < \infty, \\ \mu(A_i)^{1/p} & \text{if } q = \infty. \end{cases}$$

Therefore, $\chi_{A_i} \in L(p,q,\mu)$ and T_n is a finite-rank operator from $L(p,q,\mu)$ into $L(p,q,\nu)$. If $\|u\|_{L^\infty(\nu)} = 0$, then uC_φ is the zero operator and thus compact. Suppose $\|u\|_{L^\infty(\nu)} > 0$. Choose any $f \in L(p,q,\mu)$. Since

$$(uC_\varphi f - T_n f)(y) = \begin{cases} u(y)\, f(\varphi(y)) & \text{if } y \in \varphi^{-1}(B), \\ 0 & \text{if } y \in \bigcup_{i=1}^{n} \varphi^{-1}(A_i), \\ \sum_{i>n} f(A_i) u(y) \chi_{\varphi^{-1}(A_i)}(y) & \text{if } y \in \bigcup_{i>n} \varphi^{-1}(A_i) \end{cases}$$

and $\nu\varphi^{-1}(B) = 0$, it follows that

$$\begin{aligned} \nu_{(uC_\varphi - T_n) f}(\lambda) &= \nu\{y \in Y : |(uC_\varphi f - T_n f)(y)| > \lambda\} \\ &= \nu\left\{y \in \bigcup_{i>n} \varphi^{-1}(A_i) : \left|\sum_{i>n} f(A_i) u(y) \chi_{\varphi^{-1}(A_i)}(y)\right| > \lambda\right\} \\ &= \sum_{i>n} \nu\{y \in \varphi^{-1}(A_i) : |f(A_i) u(y)| > \lambda\} \\ &\le \sum_{i>n} \nu\{y \in \varphi^{-1}(A_i) : \|u\|_{L^\infty(\nu)} |f(A_i)| > \lambda\} \\ &\le \sum_{i>n,\, |f(A_i)| > \lambda/\|u\|_{L^\infty(\nu)}} \nu\varphi^{-1}(A_i) \le \alpha_n \sum_{i>n,\, |f(A_i)| > \lambda/\|u\|_{L^\infty(\nu)}} \mu(A_i) \\ &\le \alpha_n \sum_{|f(A_i)| > \lambda/\|u\|_{L^\infty(\nu)}} \mu(A_i) \le \alpha_n \mu\left\{x \in X : |f(x)| > \frac{\lambda}{\|u\|_{L^\infty(\nu)}}\right\}, \end{aligned}$$

where $\alpha_n := \sup_{i>n} \nu\varphi^{-1}(A_i)/\mu(A_i)$. We may further assume that $\alpha_n \ne 0$ for all $n \in \mathbb{N}$ (if $\alpha_N = 0$ for some $N \in \mathbb{N}$, then $\alpha_n = 0$ and $uC_\varphi - T_n$ is the zero operator whenever $n \ge N$). Then

$$\begin{aligned} ((uC_\varphi - T_n) f)^*(t) &= \inf\{\lambda > 0 : \nu_{(uC_\varphi - T_n) f}(\lambda) \le t\} \\ &\le \inf\left\{\lambda > 0 : \mu\left\{x \in X : |f(x)| > \frac{\lambda}{\|u\|_{L^\infty(\nu)}}\right\} \le \frac{t}{\alpha_n}\right\} \\ &= \|u\|_{L^\infty(\nu)} \inf\left\{\lambda > 0 : \mu\{x \in X : |f(x)| > \lambda\} \le \frac{t}{\alpha_n}\right\} \\ &= \|u\|_{L^\infty(\nu)} f^{**}\left(\frac{t}{\alpha_n}\right) \end{aligned}$$

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for every $t \geq 0$. Similarly to the proof of Theorem 2.1, we obtain

$$\|(uC_\varphi - T_n)\, f\|_{L(p,q,\nu)} \leq \alpha_n^{1/p} \|u\|_{L^\infty(\nu)} \|f\|_{L(p,q,\mu)}.$$

Since $\lim_{n\to\infty} \alpha_n = \lim_{i\to\infty} \nu\varphi^{-1}(A_i)/\mu(A_i) = 0$,

$$\|uC_\varphi - T_n\| = \sup_{f\neq 0} \frac{\|(uC_\varphi - T_n)\, f\|_{L(p,q,\nu)}}{\|f\|_{L(p,q,\mu)}} \leq \alpha_n^{1/p} \|u\|_{L^\infty(\nu)} \to 0 \quad \text{as } n \to \infty.$$

As the limit of a sequence of finite-rank operators, uC_φ is compact. □

EXAMPLE 3.2. Let $X = Y = (\bigcup_{i\in\mathbb{N}}(i, i+1)) \cup \mathbb{N}$. Let μ be the Lebesgue measure on $\bigcup_{i\in\mathbb{N}}(i, i+1)$ and $\mu(\{i\}) = i$ for every $i \in \mathbb{N}$. Take $u(x) = x^{-3/2}$ and define $\varphi : X \to X$ by $\varphi(x) = i$ if $i < x < i+1$ and $\varphi(x) = 1$ if $x \in \mathbb{N}$. The operator uC_φ is bounded on $L(2, 2, \mu)$, since

$$\begin{aligned}
\|uC_\varphi\, f\|^2_{L(2,2,\mu)} &\leq 4\|uC_\varphi\, f\|^2_{L^2(\mu)} = 4\Big[\sum_{i\in\mathbb{N}} |f(i)|^2 \int_i^{i+1} x^{-3}\, dx + \sum_{i\in\mathbb{N}} |f(1)|^2 i^{-3} \mu(\{i\})\Big] \\
&= 4\Big[\sum_{i\in\mathbb{N}} |f(i)|^2 \Big(\frac{1}{2i^2} - \frac{1}{2(i+1)^2}\Big) + |f(1)|^2 \sum_{i\in\mathbb{N}} \frac{1}{i^2}\Big] \\
&= 4\Big[\sum_{i\in\mathbb{N}} i|f(i)|^2 \Big(\frac{1}{2i^3} - \frac{1}{2i(i+1)^2}\Big) + \frac{\pi^2}{6}|f(1)|^2\Big] \\
&\leq 4\Big(1 + \frac{\pi^2}{6}\Big) \sum_{i\in\mathbb{N}} i|f(i)|^2 \leq 4\Big(1 + \frac{\pi^2}{6}\Big)\|f\|^2_{L^2(\mu)} \leq 4\Big(1 + \frac{\pi^2}{6}\Big)\|f\|^2_{L(2,2,\mu)}
\end{aligned}$$

for each $f \in L(2, 2, \mu)$. Note that $\mu\varphi^{-1}(\bigcup_{i\in\mathbb{N}}(i, i+1)) = \mu(\varnothing) = 0$. Moreover, when $i = 2, 3, \ldots$, we have $\mu\varphi^{-1}(\{i\})/\mu(\{i\}) = 1/i \to 0$ as $i \to \infty$. By Theorem 3.1, uC_φ is also compact.

The two conditions in Theorem 3.1 are also necessary for the compactness of uC_φ with another assumption on u.

THEOREM 3.3. *Let uC_φ be a weighted composition operator from $L(p, q, \mu)$ into $L(p, q, \nu)$. Suppose that there is a constant $c > 0$ such that $|u| \geq c$ ν-a.e. on Y. If uC_φ is compact, then*

(i) $\nu\varphi^{-1}(B) = 0$, *and*
(ii) $\lim_{i\to\infty} \nu\varphi^{-1}(A_i)/\mu(A_i) = 0$, *where we understand that the limit takes the value zero when the number of atoms in (X, Σ, μ) is finite.*

PROOF. Suppose that uC_φ is compact. We first prove that (i) holds. Otherwise, there exists a constant $\delta > 0$ such that $0 < \mu(E) < \infty$, where $E := \{x \in B : d(\nu\varphi^{-1})/d\mu \geq \delta\}$. Since E is nonatomic, we may exhibit a sequence $\{E_n\}_{n\in\mathbb{N}}$ of Σ-measurable subsets of E for which

$$E_{n+1} \subset E_n \quad \text{and} \quad \mu(E_n) = \frac{\mu(E)}{2^n} \quad \text{for every } n \in \mathbb{N}.$$

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Let $f_n := (1/\|\chi_{E_n}\|_{L(p,q,\mu)})\chi_{E_n}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L(p,q,\mu)$, because $\|f_n\|_{L(p,q,\mu)} = 1$. If $m, n \in \mathbb{N}$ with $n > m$, then

$$\begin{aligned}
&(uC_\varphi f_n - uC_\varphi f_m)^*(t)\\
&\quad = \inf\{\lambda > 0 : \nu\{y \in Y : |u(y)||f_n(\varphi(y)) - f_m(\varphi(y))| > \lambda\} \le t\}\\
&\quad \ge \inf\{\lambda > 0 : \nu\{y \in Y : |f_n(\varphi(y)) - f_m(\varphi(y))| > \lambda/c\} \le t\}\\
&\quad = \inf\{\lambda > 0 : \nu\varphi^{-1}\{x \in X : |f_n(x) - f_m(x)| > \lambda/c\} \le t\}\\
&\quad = \inf\{\lambda > 0 : \nu\varphi^{-1}\{x \in E_m : |f_n(x) - f_m(x)| > \lambda/c\} \le t\}\\
&\quad \ge \inf\{\lambda > 0 : \alpha\mu\{x \in E_m : |f_n(x) - f_m(x)| > \lambda/c\} \le t\}\\
&\quad \ge \inf\left\{\lambda > 0 : \mu\left\{x \in E_m \setminus E_n : \left|\frac{\chi_{E_n}(x)}{\|\chi_{E_n}\|_{L(p,q,\mu)}} - \frac{\chi_{E_m}(x)}{\|\chi_{E_m}\|_{L(p,q,\mu)}}\right| > \frac{\lambda}{c}\right\} \le \frac{t}{\alpha}\right\}\\
&\quad = \inf\left\{\lambda > 0 : \mu\left\{x \in E_m \setminus E_n : \left|\frac{\chi_{E_m\setminus E_n}(x)}{\|\chi_{E_m}\|_{L(p,q,\mu)}}\right| > \frac{\lambda}{c}\right\} \le \frac{t}{\alpha}\right\}\\
&\quad = \inf\left\{\lambda > 0 : \mu\left\{x \in X : \left|\frac{\chi_{E_m\setminus E_n}(x)}{\|\chi_{E_m}\|_{L(p,q,\mu)}}\right| > \frac{\lambda}{c}\right\} \le \frac{t}{\alpha}\right\}\\
&\quad = \frac{c}{\|\chi_{E_m}\|_{L(p,q,\mu)}} \inf\left\{\lambda > 0 : \mu\{x \in X : |\chi_{E_m\setminus E_n}(x)| > \lambda\} \le \frac{t}{\alpha}\right\}\\
&\quad = \frac{c}{\|\chi_{E_m}\|_{L(p,q,\mu)}}{\chi_{E_m\setminus E_n}}^*\left(\frac{t}{\alpha}\right)
\end{aligned}$$

for every $t \ge 0$. Similarly to the proof of Theorem 2.1,

$$\begin{aligned}
\|uC_\varphi f_n - uC_\varphi f_m\|_{L(p,q,\nu)} \ge \frac{c\alpha^{1/p}}{\|\chi_{E_m}\|_{L(p,q,\mu)}}\|\chi_{E_m\setminus E_n}\|_{L(p,q,\mu)} &= c\alpha^{1/p}\left[\frac{\mu(E_m \setminus E_n)}{\mu(E_m)}\right]^{1/p}\\
&= c\alpha^{1/p}(1 - 2^{m-n})^{1/p} \ge c\left(\frac{\alpha}{2}\right)^{1/p}.
\end{aligned}$$

This shows that $\{uC_\varphi f_n\}_{n\in\mathbb{N}}$ does not have any convergent subsequence, contradicting the compactness of uC_φ.

It remains to prove (ii). If (ii) were not true, then (passing to a subsequence if necessary) there is a constant $\delta > 0$ such that

$$\frac{\nu\varphi^{-1}(A_n)}{\mu(A_n)} \ge \delta \quad \text{for all } n \in \mathbb{N}.$$

Let $g_n := (1/\|uC_\varphi\chi_{A_n}\|_{L(p,q,\nu)})\chi_{A_n}$. Since $|uC_\varphi\chi_{A_n}| \ge c\chi_{\varphi^{-1}(A_n)}$, $(uC_\varphi\chi_{A_n})^{**} \ge c{\chi_{\varphi^{-1}(A_n)}}^{**}$ and $\|uC_\varphi\chi_{A_n}\|_{L(p,q,\nu)} \ge c\|\chi_{\varphi^{-1}(A_n)}\|_{L(p,q,\nu)}$ and so

$$\|g_n\|_{L(p,q,\mu)} = \frac{\|\chi_{A_n}\|_{L(p,q,\mu)}}{\|uC_\varphi\chi_{A_n}\|_{L(p,q,\nu)}} \le \frac{\|\chi_{A_n}\|_{L(p,q,\mu)}}{c\|\chi_{\varphi^{-1}(A_n)}\|_{L(p,q,\nu)}} = \frac{1}{c}\left[\frac{\mu(A_n)}{\nu\varphi^{-1}(A_n)}\right]^{1/p} \le \frac{1}{c}\left(\frac{1}{\delta}\right)^{1/p}$$

for each $n \in \mathbb{N}$, that is, $\{g_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L(p,q,\mu)$. Suppose $m, n \in \mathbb{N}$ with $m \ne n$. Note that the functions $uC_\varphi g_m$ and $uC_\varphi g_n$ have disjoint supports. Thus,

$$\|uC_\varphi g_m + uC_\varphi g_n\|_{L(p,q,\nu)} = \|uC_\varphi g_m - uC_\varphi g_n\|_{L(p,q,\nu)}.$$

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Now,

$$\begin{aligned}1 &= \frac{\|uC_\varphi \chi_{A_m}\|_{L(p,q,\nu)}}{\|uC_\varphi \chi_{A_m}\|_{L(p,q,\nu)}} = \frac{1}{2}\|uC_\varphi g_m + uC_\varphi g_m\|_{L(p,q,\nu)} \\ &= \frac{1}{2}\|uC_\varphi g_m + uC_\varphi g_m - uC_\varphi g_n + uC_\varphi g_n\|_{L(p,q,\nu)} \\ &\le \frac{1}{2}(\|uC_\varphi g_m + uC_\varphi g_n\|_{L(p,q,\nu)} + \|uC_\varphi g_m - uC_\varphi g_n\|_{L(p,q,\nu)}) \\ &= \|uC_\varphi g_m - uC_\varphi g_n\|_{L(p,q,\nu)}.\end{aligned}$$

This shows that $\{uC_\varphi g_n\}_{n\in\mathbb{N}}$ does not have a convergent subsequence either. Hence uC_φ is not compact. □

Without the assumption that u is bounded away from zero on Y, a necessary condition for uC_φ to be compact is $\lim_{i\to\infty} \|u\chi_{\varphi^{-1}(A_i)}\|_{L(p,q,\nu)}/\mu(A_i)^{1/p} = 0$. In fact, Theorem 3.3 does not necessarily hold if this assumption on u is removed. This is illustrated by the following example.

EXAMPLE 3.4. Let $X = Y = \mathbb{N}$, $\Sigma = \Gamma = \mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and μ be the counting measure on $\mathcal{P}(\mathbb{N})$. Then $L^p(\mu)$ is just the sequence space l^p consisting of all p-summable complex sequences. Let $\{w_n\}_{n\in\mathbb{N}}$ be a fixed complex sequence such that $\lim_{n\to\infty} w_n = 0$. The unilateral weighted right shift operator T_R on l^p given by

$$T_R(x_1, x_2, \ldots) := (0, w_1x_1, w_2x_2, \ldots), \quad \text{for } \{x_n\}_{n\in\mathbb{N}} \in l^p,$$

can be realised as a weighted composition operator uC_φ, where

$$u(n) := \begin{cases} w_{n-1} & \text{if } n = 2, 3, \ldots, \\ 0 & \text{if } n = 1 \end{cases} \quad \text{and} \quad \varphi(n) := \begin{cases} n-1 & \text{if } n = 2, 3, \ldots, \\ 1 & \text{if } n = 1. \end{cases}$$

From [10, Example 3.10], T_R is compact. However,

$$\frac{\mu\varphi^{-1}(\{i\})}{\mu(\{i\})} = \begin{cases} 1 & \text{if } n = 2, 3, \ldots, \\ 2 & \text{if } n = 1. \end{cases}$$

The following corollary is a particular case of [8, Theorem 3.2].

COROLLARY 3.5. *When (X, Σ, μ) is nonatomic, the only compact composition operator from $L(p, q, \mu)$ into $L(p, q, \nu)$ is the zero operator.*

PROOF. If C_φ is compact, then it follows from Theorem 3.3 that

$$\nu\{y \in Y : f(\varphi(y)) \neq 0\} = \nu\varphi^{-1}\{x \in X : f(x) \neq 0\} \le \nu\varphi^{-1}(X) = \nu\varphi^{-1}(B) = 0$$

for each $f \in L(p, q, \mu)$. Thus, C_φ is the zero operator. □

4. Fredholm and invertible weighted composition operators

In this section assume that $1 < p < \infty$ and $1 \le q < \infty$. We first investigate invertible weighted composition operators from $L(p, q, \mu)$ onto $L(p, q, \nu)$. The following result

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generalises [2, Theorem 4.9] and [1, Theorem 2.3], which characterise the invertibility of composition and multiplication operators on Lorentz spaces.

THEOREM 4.1. *If uC_φ is a weighted composition operator from $L(p,q,\mu)$ into $L(p,q,\nu)$, then uC_φ is invertible if and only if both the following conditions are satisfied:*

(i) *there exists a constant $\alpha > 0$ such that $\|u\chi_{\varphi^{-1}(E)}\|_{L(p,q,\nu)} \geq \alpha\mu(E)^{1/p}$ for every set $E \in \Sigma$ with $\mu(E) < \infty$;*
(ii) *for each set $F \in \Gamma$, there is a set $G \in \Sigma$ such that $\varphi^{-1}(G) = F$.*

PROOF. Assume that uC_φ is invertible. By the bounded inverse theorem, there is a constant $c > 0$ such that

$$\|uC_\varphi f\|_{L(p,q,\nu)} \geq c\|f\|_{L(p,q,\mu)} \quad \text{for all } f \in L(p,q,\mu).$$

Taking $f = \chi_E$, where $E \in \Sigma$ and $\mu(E) < \infty$, gives

$$\|u\chi_{\varphi^{-1}(E)}\|_{L(p,q,\nu)} = \|uC_\varphi\chi_E\|_{L(p,q,\nu)} \geq c\|\chi_E\|_{L(p,q,\mu)} = c\left(\frac{p}{p-1}\right)^{1/q}\mu(E)^{1/p}.$$

Thus, (i) follows. To prove (ii), we first claim that $u \neq 0$ ν-a.e. on Y. Otherwise, there exists a Γ-measurable set S with $0 < \nu(S) < \infty$ and $u = 0$ on S. Since uC_φ is surjective, $uC_\varphi f = \chi_S$ for a function $f \in L(p,q,\mu)$. This equality, however, is not valid with the choice of S.

Choose an arbitrary set $F \in \Gamma$ with $\nu(F) < \infty$. Let g be the function in $L(p,q,\mu)$ such that $uC_\varphi g = \chi_F$, that is, $C_\varphi g = (1/u)\chi_F$. Let $\mathcal{E} := \{\varphi^{-1}(E) : E \in \Sigma\}$. Since $C_\varphi g$ is $\mathcal{E}$-measurable, so is $(1/u)\chi_F$. Write $Y = \bigcup_{i\in\mathbb{N}} F_i$, where $\{F_i\}_{i\in\mathbb{N}}$ is an increasing sequence of Γ-measurable sets with finite ν-measures. Since $1/u = \lim_{i\to\infty}(1/u)\chi_{F_i}$ on Y, it follows that $1/u$, and hence u, is also $\mathcal{E}$-measurable. Therefore, χ_F is $\mathcal{E}$-measurable for each $F \in \Gamma$ with $\nu(F) < \infty$.

We now prove the converse. Note that the condition in (i) may be expressed as

$$\|uC_\varphi\chi_E\|_{L(p,q,\nu)} \geq \alpha\left(\frac{p-1}{p}\right)^{1/q}\|\chi_E\|_{L(p,q,\mu)} \tag{4.1}$$

for all $E \in \Sigma$ satisfying $\mu(E) < \infty$. Choose any $f \in L(p,q,\mu)$ and approximate $|f|$ with an increasing sequence of simple functions. In conjunction with (4.1), this implies

$$\|uC_\varphi f\|_{L(p,q,\nu)} \geq \alpha\left(\frac{p-1}{p}\right)^{1/q}\|f\|_{L(p,q,\mu)}.$$

In particular, uC_φ is injective. It remains to show that uC_φ is also surjective, which is equivalent to the injectivity of uC_φ^* (the adjoint operator of uC_φ). Suppose that $uC_\varphi^*\phi$ is the zero functional for $\phi \in L(p',q',\nu)^*$ (the dual space of $L(p,q,\nu)$), where $1/p + 1/p' = 1/q + 1/q' = 1$. By [7, page 262], there is a function $h \in L(p',q',\nu)$ such that

$$\int_Y h(uC_\varphi f)\,d\nu = 0 \quad \text{for each } f \in L(p,q,\mu).$$

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If $G \in \Sigma$ and $\mu(G) < \infty$, then $\int_{\varphi^{-1}(G)} hu\, d\nu = 0$. In light of (ii) and the σ-finiteness of (X, Σ, μ), we indeed have

$$\int_F hu\, d\nu = 0 \quad \text{for all } F \in \Gamma.$$

We shall show that $h = 0$. To this end, it suffices to prove that $u \neq 0$ ν-a.e. on Y. Otherwise, the set $N := \{y \in Y : u(y) = 0\}$ has positive ν-measure. Using (ii) again, one may find a set $M \in \Sigma$ for which $\varphi^{-1}(M) \subset N$ and $0 < \mu(M) < \infty$. Then, from (i),

$$0 = \|u\chi_N\|_{L(p,q,\nu)} \geq \|u\chi_{\varphi^{-1}(M)}\|_{L(p,q,\nu)} \geq \alpha\mu(M)^{1/p} > 0,$$

which is absurd. Hence ϕ is the zero functional. □

We now consider the Fredholmness of weighted maps between Lorentz spaces. Let B_1, B_2 be Banach spaces and $T : B_1 \to B_2$ be a bounded linear operator. Then T is said to be Fredholm if $\operatorname{ran}(T)$ is closed in B_2 and the dimensions of $\ker(T)$ and $B_2/\operatorname{ran}(T)$ are finite, where $\ker(T)$ and $\operatorname{ran}(T)$ are the kernel and the range of T, respectively. It will be shown that when (X, σ, μ) is nonatomic, the two classes of Fredholm and invertible weighted composition operators are the same. A similar result was also obtained for composition operators in [2, Theorem 4.4]. We introduce some notation.

Let $\varphi^{-1}\Sigma$ be the relative completion of the σ-algebra generated by $\{\varphi^{-1}(E) : E \in \Sigma\}$, that is,

$$\varphi^{-1}\Sigma := \{\varphi^{-1}(E)\Delta F : E \in \Sigma \text{ and } \nu(F) = 0\}.$$

If g is a nonnegative Γ-measurable function on Y, then the measure defined by $S \mapsto \int_S g\, d\nu$ for $S \in \varphi^{-1}\Sigma$ is absolutely continuous with respect to ν. Thus, there exists a unique (ν-a.e.) nonnegative $\varphi^{-1}\Sigma$-measurable function on Y, denoted by $E(g)$, with

$$\int_S g\, d\nu = \int_S E(g)\, d\nu \quad \text{for each } S \in \varphi^{-1}\Sigma.$$

The function $E(g)$ is called the conditional expectation of g with respect to $\varphi^{-1}\Sigma$.

LEMMA 4.2. *Suppose that (X, Σ, μ) is nonatomic. Let uC_φ be a weighted composition operator from $L(p, q, \mu)$ into $L(p, q, \nu)$. Then*

(a) *the dimension of* $\ker uC_\varphi$ *is either zero or infinite, and*
(b) *the codimension of* $\overline{\operatorname{ran}(uC_\varphi)}$ *in $L(p, q, \nu)$ is either zero or infinite, where* $\overline{\operatorname{ran}(uC_\varphi)}$ *is the norm-closure of* $\operatorname{ran}(uC_\varphi)$ *in $L(p, q, \nu)$.*

PROOF. We first prove (a). If $\dim\ker uC_\varphi \neq 0$, then there exists a nonzero function $f \in L(p, q, \mu)$ such that $uC_\varphi f = 0$. Since the set $E := \{x \in X : |f(x)| > 0\}$ is of positive μ-measure and E is nonatomic, one may choose a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint Σ-measurable sets in E with $0 < \mu(E_n) < \infty$. Let $f_n := f\chi_{E_n}$ for $n \in \mathbb{N}$. The f_n are nonzero and linearly independent. Since $|f_n| \leq |f|$,

$$\|uC_\varphi f_n\|_{L(p,q,\nu)} \leq \|uC_\varphi f\|_{L(p,q,\nu)} = 0,$$

that is, $f_n \in \ker uC_\varphi$. Hence $\dim\ker uC_\varphi = \infty$.

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To prove (b), suppose that $\dim L(p,q,\nu)/\overline{\operatorname{ran}(uC_\varphi)} \neq 0$. In view of the fact that

$$\dim L(p,q,\nu)/\overline{\operatorname{ran}(uC_\varphi)} = \dim \ker uC_\varphi{}^*,$$

there is a nonzero function $g \in L(p',q',\nu)$, where $1/p + 1/p' = 1/q + 1/q' = 1$, such that

$$\int_Y g(uC_\varphi f)\,d\nu = 0 \quad \text{for all } f \in L(p,q,\mu).$$

Then

$$\int_Y E(|g|)\,d\nu = \int_Y |g|\,d\nu > 0,$$

so that the $\varphi^{-1}\Sigma$-measurable set $F := \{y \in Y : E(|g|) \geq \delta\}$ has positive ν-measure for some constant $\delta > 0$. By the definition of $\varphi^{-1}\Sigma$, we have $F = \varphi^{-1}(E)$ for a Σ-measurable set E. From the nonatomicity of (X,Σ,μ) and the lemma in [6], there is a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint Σ-measurable sets in E such that $0 < \nu\varphi^{-1}(E_n) < \infty$. The functionals $\phi_n \in L(p',q',\nu)^*$ represented by $g\chi_{\varphi^{-1}(E_n)}$, $n \in \mathbb{N}$, are all nonzero because

$$\int_Y |g\chi_{\varphi^{-1}(E_n)}|\,d\nu = \int_{\varphi^{-1}(E_n)} |g|\,d\nu = \int_{\varphi^{-1}(E_n)} E(|g|)\,d\nu \geq \delta\nu\varphi^{-1}(E_n) > 0.$$

These functionals are also linearly independent, since the sets $\{\varphi^{-1}(E_n)\}_{n\in\mathbb{N}}$ are pairwise disjoint. Moreover,

$$\phi_n(uC_\varphi f) = \int_Y g\chi_{\varphi^{-1}(E_n)}(uC_\varphi f)\,d\nu = \int_Y g(uC_\varphi f\chi_{E_n})\,d\nu = 0$$

for every $f \in L(p,q,\mu)$, that is, $\phi_n \in \ker uC_\varphi{}^*$. Thus, $\dim \ker uC_\varphi{}^* = \infty$. □

THEOREM 4.3. *Suppose that (X,Σ,μ) is nonatomic. Let uC_φ be a weighted composition operator from $L(p,q,\mu)$ into $L(p,q,\nu)$. Then uC_φ is Fredholm if and only if uC_φ is invertible.*

PROOF. If uC_φ is invertible, then it is Fredholm. Conversely, suppose that uC_φ is Fredholm. By Lemma 4.2, the dimensions of both $\ker uC_\varphi$ and $L(p,q,\nu)/\operatorname{ran}(uC_\varphi)$ are zero. Thus, uC_φ is injective and $\operatorname{ran}(uC_\varphi) = L(p,q,\nu)$, that is, uC_φ is surjective. This proves the invertibility of uC_φ. □

It was claimed in [2, Theorem 4.2] that the dimension of $\ker C_\varphi^*$ (i.e. the codimension of $\overline{\operatorname{ran}(C_\varphi)}$ in $L(p,q,\nu)$) is either zero or infinite, *without* assuming the nonatomicity of (X,Σ,μ). However, this is false in general. For example, the composition operator C_φ on l^2 induced by $\varphi(n) = 1$ if $n = 1,2$ and $\varphi(n) = n-1$ if $n = 3,4,\ldots$ has closed range and $\dim \ker C_\varphi^* = \dim l^2/\operatorname{ran}(C_\varphi) = 1$. The fallacy in the proof of this theorem is that if g is the function representing a nonzero functional in $\ker C_\varphi^*$, there may *not* exist a sequence $\{F_n\}_{n\in\mathbb{N}}$ of pairwise disjoint measurable sets such that $0 < \nu(F_n) < \infty$ and $F = \bigcup_{n\in\mathbb{N}} F_n$, where F is the support of g in Y. With the above example, the functional represented by the sequence $(1,-1,0,0,\ldots)$ is in $\ker C_\varphi^*$ but the support of this sequence in $\mathbb{N}$ is finite (the support consists of 1 and 2 only).

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