# A METHOD OF SOLVING SECOND ORDER SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS USING THE MALLOCK MACHINE

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## 1. Introduction

Many problems in physics and mathematics lead to differential equations which must be solved numerically. Normally the best way of doing this is by means of a differential analyser if one is available. Otherwise, or if exceptional accuracy is required, one of the various methods depending on finite difference formulae must be used.

A very convenient method of this type for dealing with second order equations has been described by Hartree (1). The purpose of this note is to describe an adaptation of Hartree's method in which practically all the calculation is performed on the Mallock linear equation machine (2). This machine is entirely electrical in operation, and is intended for solving sets of linear algebraic equations involving up to ten unknowns. It is also capable of forming products, and sums of products, and the method to be described makes use of all these properties. The Mallock machine is available for use in the Mathematical Laboratory, Cambridge.

To solve a pair of second order simultaneous differential equations of the type to be considered in the next section on a differential analyser would require six integrators, and up to four input tables. Such a machine is not yet available in Cambridge. The method given here was originally developed in connexion with some work on the propagation of wireless waves in a stratified non-uniform refracting medium. It has the advantage that the necessary connexions on the machine can be made in a few minutes and that one operator only is required. Any desired degree of accuracy is possible.

## 2. Description of the Method

The method described by Hartree is applicable to equations of the type

$$\frac{d^2L}{ds^2} + \alpha L = 0, \tag{1}$$

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where $\alpha$ is a function of $s$. He remarks, however, that it can be extended to cover simultaneous equations of the type

$$\frac{d^2L}{ds^2}+\alpha L+\beta M = 0, \quad \frac{d^2M}{ds^2}+\gamma L+\rho M = 0, \tag{2}$$

where $\alpha$, $\beta$, $\gamma$, $\rho$ are functions of $s$. We deal with equations of the type (1).

The range of $s$ over which a solution is required is divided up into a number of integrals of length $\delta s$. The value of $L$ must be known at the beginning of the first two intervals, and its value at the beginning of the third is then to be calculated. The basic formula used is

$$\delta^2 L = (\delta s)^2(L''+\tfrac{1}{12}\delta^2 L''),$$

where $\delta^2$ and $\delta^4$ denote second and fourth order central differences respectively. If $L_1$, $L_2$, $L_3$ are the three consecutive values of the function, this may be written

$$L_1-2L_2+L_3 = (\delta s)^2[L_2''+\tfrac{1}{12}(L_1''-2L_2''+L_3'')], \tag{3}$$

i.e., from (1),

$$L_1-2L_2+L_3+(\delta s)^2[\alpha_2 L_2+\tfrac{1}{12}(\alpha_1 L_1-2\alpha_2 L_2+\alpha_3 L_3)] = 0, \tag{4}$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are the values of $\alpha$ at the beginning of the three intervals. This is a linear equation giving $L_3$ if $L_1$ and $L_2$ are known. Then by another application of the same formula, taking $L_2$ and $L_3$ for the new values of $L_1$ and $L_2$, the value $L_4$ of the function at the beginning of the next interval may be obtained, and in this way the whole solution may be traced out point by point. Hartree provides a convenient technical method of carrying out this process, together with the necessary controls for providing a running check on the arithmetic.

For solving a pair of simultaneous equations like (2) the method is similar, except that we then have in place of (4) a pair of simultaneous linear equations in the six quantities $L_1$, $L_2$, $L_3$ and $M_1$, $M_2$, $M_3$, and if $L_1$, $L_2$ and $M_1$, $M_2$ are known, $L_3$ and $M_3$ can be found. These equations are

$$L_1-2L_2+L_3+(\delta s)^2 \times [\alpha_2 L_2+\beta_2 M_2+\tfrac{1}{12}(\alpha_1 L_1+\beta_1 M_1-2\alpha_2 L_2-2\beta_2 M_2+\alpha_3 L_3+\beta_3 M_3)] = 0,$$

$$M_1-2M_2+M_3+(\delta s)^2 \times [\gamma_2 L_2+\rho_2 M_2+\tfrac{1}{12}(\gamma_1 L_1+\rho_1 M_1-2\gamma_2 L_2-2\rho_2 M_2+\gamma_3 L_3+\rho_3 M_3)] = 0.$$

The operation of evaluating the products $\alpha_1 L_1$, ... and the solution of the equations can be performed simultaneously on the Mallock machine. When the necessary connexions have been made on the plug board of the machine, the method of proceeding is as follows. From the initial conditions given in the problem the first two values of $L$ and $M$ are calculated, i.e. $L_1$, $L_2$; $M_1$, $M_2$. These are set on the machine together with the first three values of $\alpha$, $\beta$, ..., namely, $\alpha_1$, $\alpha_2$, $\alpha_3$; $\beta_1$, $\beta_2$, $\beta_3$; .... The machine then gives at once $L_3$ and $M_3$. The quantities

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are then all moved on one, that is to say, the old $L_2$ becomes the new $L_1$, the old $L_3$ the new $L_2$ and so on. The machine then gives the next values of $L$ and $M$. In this way the solution is plotted out.

As described, the method suffers from the serious drawback that the accuracy is limited to that of the Mallock machine, namely, three significant figures. A better method of working is to read off from the machine the values of $\delta^2 L_2$ and $\delta^2 M_2$ instead of $L_3$ and $M_3$. These quantities are also given to three-figure accuracy. $L_3$ and $M_3$ are then calculated mentally, the work being set out in the form of a difference table. In this way any desired degree of accuracy in $L$ and $M$ can be attained, it only being necessary to choose the interval $\delta s$ sufficiently small for the second differences to contain only three significant figures. As a means of checking the work the higher orders of differences, e.g. the third and fourth, of $L$ and $M$ should be formed. Any errors in $\delta^2 L$ and $\delta^2 M$ (caused, for example, by the wrong setting of a value of $\alpha, \beta, \ldots$) will then be shown up by irregularities in these higher differences. A specimen solution is given in the next section, showing the method of setting out the work.

## 3. EXAMPLES

The interval length should be chosen sufficiently small for the error in the calculated value of $\delta^2 L$ to be less than half a unit in the last decimal place retained. It does not, however, follow that the later values of $L$ will be given to the same degree of accuracy, since there is a tendency for errors introduced during the earlier stages to grow. In general, therefore, it will be necessary to begin the integration with a higher degree of accuracy than that finally required, and to be perfectly safe it may be necessary to allow one extra significant figure for every ten intervals $\delta s$ in the range.

Each equation must be considered on its merits, however, since some equations are more favourable than others in this respect.

As an example of the method a numerical solution of the following pair of equations in the range $s = 1$ to $s = 2$ is given below:

$$\frac{d^2L}{ds^2} - \frac{3}{s^2}L - \frac{3}{s}M = 0, \quad \frac{d^2M}{ds^2} - \frac{1}{s^3}L - \frac{1}{s^2}M = 0.$$

The length of the interval was taken as 0·1, this being chosen to give roughly four figure accuracy.

These equations can actually be solved in finite terms, and their general solution is

$$L = As^3 - B/s, \qquad M = As^2 - B/s^2.$$

In the numerical integration the initial values were chosen so as to make $A = \frac{1}{2}$ and $B = 0$.

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The following table gives the details of the integration. The figures in heavy type are obtained from the initial conditions. The numbers actually read off the machine are those in the columns marked $\delta^2 L$ and $\delta^2 M$. Since the solutions being obtained are polynomials, $\delta^3 L$ and $\delta^2 M$ are approximately constant.

| $s$ | $L$ | $\delta^2 L$ | $M$ | $\delta^2 M$ |
|---|---|---|---|---|
| 1·0 | **0·5000** | | **0·5000** | |
| | **0·1655** | | **0·1050** | |
| 1·1 | **0·6655** | 0·0330 | **0·6050** | 0·0100 |
| | 0·1985 | 0·0030 | 0·1150 | |
| 1·2 | 0·8640 | 0·0360 | 0·7200 | 0·0100 |
| | 0·2345 | 0·0029 | 0·1250 | |
| 1·3 | 1·0985 | 0·0389 | 0·8450 | 0·0100 |
| | 0·2734 | 0·0030 | 0·1350 | |
| 1·4 | 1·3719 | 0·0419 | 0·9800 | 0·0100 |
| | 0·3153 | 0·0030 | 0·1450 | |
| 1·5 | 1·6872 | 0·0449 | 1·1250 | 0·0100 |
| | 0·3602 | 0·0031 | 0·1550 | |
| 1·6 | 2·0474 | 0·0480 | 1·2800 | 0·0100 |
| | 0·4082 | 0·0029 | 0·1650 | |
| 1·7 | 2·4556 | 0·0509 | 1·4450 | 0·0100 |
| | 0·4591 | 0·0032 | 0·1750 | |
| 1·8 | 2·9147 | 0·0541 | 1·6200 | 0·0100 |
| | 0·5132 | 0·0029 | 0·1850 | |
| 1·9 | 3·4279 | 0·0570 | 1·8050 | 0·0100 |
| | 0·5702 | | 0·1950 | |
| 2·0 | 3·9991 | | 2·0000 | |

It will be seen that the error in $L$ is only about 1 part in 4000, while that in $M$ is not large enough to affect the last figure retained.

As a further example a table of sines constructed by solving the differential equation $d^2L/ds^2 + L = 0$ is given. The interval $\delta s$ was taken to correspond to 2°. The true values of $\sin s$ are given for comparison, and the way in which the error

| $s$ | $\sin s$ | |
|---|---|---|
| | Calculated | True value |
| 0 | 0·000,000,0 | 0 |
| 2 | 0·034,899,5 | 0·034,899,5 |
| 4 | 0·069,756,5 | 0·069,756,5 |
| 6 | 0·104,528,2 | 0·104,528,5 |
| 8 | 0·139,172,6 | 0·139,173,3 |
| 10 | 0·173,647,7 | 0·173,648,2 |
| 12 | 0·207,911,3 | 0·207,911,7 |
| 14 | 0·241,921,6 | 0·241,921,9 |
| 16 | 0·275,637,2 | 0·275,637,4 |
| 18 | 0·309,017,0 | 0·309,017,0 |
| 20 | 0·342,020,3 | 0·342,020,1 |
| 22 | 0·374,607,4 | 0·374,606,6 |
| 24 | 0·406,738,6 | 0·406,736,6 |
| 26 | 0·438,375,1 | 0·438,371,1 |
| 28 | 0·469,477,8 | 0·469,471,6 |
| 30 | 0·500,008,9 | 0·500,000,0 |

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increases with $s$ should be noted. In practice, if the integration was to be continued, the seventh figure would be dropped, and the interval length doubled. If the accuracy of the final solution was then insufficient it would be necessary to repeat the integration using a smaller interval throughout.

The final error is about 1 part in 50,000.

SUMMARY

The paper describes a method of integrating differential equations by means of the Mallock linear equation machine. The method, which is based on one described by Hartree, is applicable to pairs of second order simultaneous linear ordinary differential equations in two unknowns. Any required degree of accuracy in the solution can be obtained.

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