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Bessel function $J_\nu(z)$ of complex order and its zeros

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Abstract. Methods are developed for the computation of the complex zeros of $(\frac{1}{2}z)^{-\nu}J_\nu(z)$ when the index ν is an arbitrary complex number. These methods, which do not require an explicit knowledge of the $J_\nu(z)$, are susceptible to rapid numerical evaluation on a computer. Beyond the interest in the zeros in their own right, these methods now make feasible the use of the infinite product representation of $J_\nu(z)$ for the rapid computation of Bessel functions of complex order.

1. *Introduction.* In certain problems of mathematical physics it is proving of importance to develop an improved understanding of the Bessel Function

$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)} \tag{1·1}$$

when the index ν is variable, and in general complex.* Specifically for rapid computation of these functions it may often be advantageous to use the infinite product representation (12),

$$J_\nu(z) = [(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1)] \prod_{m=1}^{\infty} \{1-(z^2/j_{\nu,m}^2)\}, \tag{1·2}$$

provided one can find an easy method for computing the zeros, $\pm j_{\nu,m}$, of $(\frac{1}{2}z)^{-\nu}J_\nu(z)$. Or, as is often the case, the zeros are of interest in their own right. Nevertheless, although certain asymptotic formulae for the $j_{\nu,m}$ exist and extensive tabulations have been prepared as functions of m for particular values of ν, little has apparently been done in obtaining zeros as functions of variable order (and essentially nothing for complex order). Moreover, the methods for computing the $j_{\nu,m}$ are in general tedious and poorly adapted to the need for a technique to evaluate the zeros rapidly.† It is with regard to this problem that we are concerned here.

* In particular, the author's interest stems from the analysis of the instability of waves propagating in an anisotropic magnetized plasma, for which the function

$$\Upsilon_\nu(z) \equiv (\pi\nu)\csc(\pi\nu)J_\nu(z)J_{-\nu}(z)$$

is of fundamental importance. [For example, see (6).]

† A relatively recent reference on the computation of zeros, containing as well an extensive tabulation for integral and half-integral indices, is in (8). A bibliography of work prior to publication is included. Some very recent work (kindly brought to our attention by the referee) is found in (13). An extensive and modern bibliography is given in (4). See also (7). This latter reference contains graphs and limited tables of the $j_{\nu,m}$ as functions of real index ν. Slight extensions of the ranges and accuracy of these tables and graphs can be found in isolated circumstances (see the references in the bibliographies cited), but the available work seems surprisingly incomplete. As far as the author is aware, no tables involving complex ν are extant.

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Our methods will prove similar to that used by Euler,* and independently extended by Rayleigh (10) to the computation of the smallest zero of $(\frac{1}{2}z)^{-\nu} \times J_\nu(z)$ when ν is real. The present work, which applies for arbitrary complex ν, systematizes the previous method and new procedures are devised whereby the larger roots can also be obtained. Our general approach, in addition, appears to have possible application to the computation of zeros of other functions for which an infinite product representation can be obtained and for which the logarithmic derivative of the infinite product can be shown to satisfy a suitably simple equation.

2. *Formulae and derivations.* If we introduce the function

$$\sigma_\nu(z) \equiv J_{\nu+1}(z)/J_\nu(z) = -d \ln [(\tfrac{1}{2}z)^{-\nu} J_\nu(z)]/dz \tag{2·1}$$

then it follows from the infinite product representation, equation (1·2), that†

$$\sigma_\nu(z) = \sum_{m=1}^{\infty} 2z(j_{\nu,m}^2 - z^2)^{-1}. \tag{2·2}$$

Rewriting (2·1) in the form $z^{\nu+1}J_{\nu+1}(z) = z^{-\nu}J_\nu(z)\,[z^{2\nu+1}\sigma_\nu(z)]$, differentiating once using the identities

$$d[z^p J_p(z)]/dz = z^p J_{p-1}(z), \quad d[z^{-p}J_p(z)]/dz = -z^{-p}J_{p+1}(z),$$

and resubstituting from (2·1), we find that $\sigma_\nu(z)$ satisfies the Riccati equation‡

$$d\sigma_\nu/dz + [(2\nu+1)/z]\sigma_\nu - \sigma_\nu^2 - 1 = 0. \tag{2·3}$$

A second relation of which we also have need follows from the expansion (1·1), whereby we find that $\lim_{z\to 0} [z^{-1}\sigma_\nu(z)] = \frac{1}{2}(1+\nu)^{-1}$. Hence from (2·2),§

$$1 = \sum_{m=1}^{\infty} 4(\nu+1) j_{\nu,m}^{-2}. \tag{2·4}$$

Next, upon substituting from (2·2) and (2·4) into (2·3), we get

$$\sigma_\nu^2 = d\sigma_\nu/dz + [(2\nu+1)/z]\sigma_\nu - 1 = 4z^2 \sum_{m=1}^{\infty} \{(j_{\nu,m}^2 - z^2)^{-2} + (\nu+1)(j_{\nu,m}^2 - z^2)^{-1} j_{\nu,m}^2\}. \tag{2·5}$$

We then define $t = \frac{1}{4}z^2$ [as suggested by the expansion of $J_p(z)$] and $\alpha_m(\nu) = \frac{1}{4}j_{\nu,m}^2$. Thus, again substituting for σ_ν from (2·2) on the left of (2·5), we finally obtain

$$\left[\sum_{m=1}^{\infty} (\alpha_m - t)^{-1}\right]^2 - \sum_{m=1}^{\infty} (\alpha_m - t)^{-2} = (\nu+1) \sum_{m=1}^{\infty} [\alpha_m(\alpha_m - t)]^{-1}. \tag{2·6}$$

In the present notation, (2·4) also becomes

$$(\nu+1) \sum_{m=1}^{\infty} \alpha_m^{-1} = 1. \tag{2·7}$$

* Euler (3), quoted by Watson (12), p. 500. The latter credits the original idea to Waring (11), but Airey (1) attributes its introduction in 1629 to Girard (5).

† See (12). Watson chooses to define the sign of $j_{\nu,m}$ such that $-\frac{1}{2}\pi < \arg(j_{\nu,m}) \leqslant \frac{1}{2}\pi$ and then orders the pairs $\pm j_{\nu,m}$ for increasing m according to their increasing distance from the imaginary axis.

‡ Alternatively, substitution of $\sigma_\nu(z) = -d \ln u_\nu(z)/dz$ in (2·3) gives a linear second-order differential equation whose solution is $z^{-\nu}\mathscr{C}_{\pm\nu}(z)$, where $\mathscr{C}_p(z)$ is a cylinder function. Thus $\sigma_\nu(z) = \pm\, \mathscr{C}_{\pm\nu\pm1}(z)/\mathscr{C}_{\pm\nu}(z)$, where one or the other sign is to be used throughout.

§ Equation (2·4) was given by Rayleigh (10), p. 192.

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Except for the isolated circumstances when ν is a negative integer, all of the zeros α_m are distinct and non-vanishing,* and equation (2·6) may be expanded in Taylor's series about $t = 0$. If the coefficients of powers of t are then separately equated, one obtains the recursion relation

$$s^{(l)} = (\nu+l)^{-1} \sum_{k-1}^{l-1} s^{(k)} s^{(l-k)}, \tag{2·8}$$

where

$$s^{(l)}(\nu) \equiv \sum_{m=1}^{\infty} \alpha_m^{-l}. \tag{2·9}$$

Equation (2·8) summarizes the results obtained somewhat more laboriously by Rayleigh,† who evaluated $4^{-l}s^{(l)}$ through $l = 5$. Table 1 gives $s^{(l)}$ through $l = 10$.

Table 1. *Giving* $s^{(k)}(\nu) \equiv \sum_{m=1}^{\infty} [\alpha_m(\nu)]^{-k}$ *for* $1 \leqslant k \leqslant 10$. *All entries were checked analytically for* $\nu = \pm \frac{1}{2}$

	$s^{(k)}(\nu)$ = numerator/denominator	
k	Numerator	Denominator
1	1	$(\nu+1)$
2	1	$(\nu+1)^2\,(\nu+2)$
3	2	$(\nu+1)^3\,(\nu+2)\,(\nu+3)$
4	$5\nu+11$	$(\nu+1)^4\,(\nu+2)^2\,(\nu+3)\,(\nu+4)$
5	$2(7\nu+19)$	$(\nu+1)^5\,(\nu+2)^2\,(\nu+3)\,(\nu+4)\,(\nu+5)$
6	$2(21\nu^3+181\nu^2+513\nu+473)$	$(\nu+1)^6\,(\nu+2)^3\,(\nu+3)^2\,(\nu+4)\,(\nu+5)$ $\times(\nu+6)$
7	$4(33\nu^3+329\nu^2+1{,}081\nu+1{,}145)$	$(\nu+1)^7\,(\nu+2)^3\,(\nu+3)^2\,(\nu+4)\,(\nu+5)$ $\times(\nu+6)\,(\nu+7)$
8	$429\nu^5+7{,}640\nu^4+53{,}752\nu^3+185{,}430\nu^2$ $+311{,}387\nu+202{,}738$	$(\nu+1)^8\,(\nu+2)^4\,(\nu+3)^2\,(\nu+4)^2\,(\nu+5)$ $\times(\nu+6)\,(\nu+7)\,(\nu+8)$
9	$2(715\nu^6+16{,}567\nu^5+158{,}568\nu^4$ $+798{,}074\nu^3+2{,}217{,}079\nu^2$ $+3{,}212{,}847\nu+1{,}893{,}046)$	$(\nu+1)^9\,(\nu+2)^4\,(\nu+3)^3\,(\nu+4)^2\,(\nu+5)$ $\times(\nu+6)\,(\nu+7)\,(\nu+8)\,(\nu+9)$
10	$2(2{,}431\nu^8+80{,}425\nu^7+1{,}152{,}851\nu^6$ $+9{,}315{,}667\nu^5+46{,}240{,}675\nu^4$ $+143{,}917{,}279\nu^3+273{,}583{,}653\nu^2$ $+289{,}891{,}557\nu+130{,}934{,}438)$	$(\nu+1)^{10}\,(\nu+2)^5\,(\nu+3)^3\,(\nu+4)^2\,(\nu+5)^2$ $\times(\nu+6)\,(\nu+7)\,(\nu+8)\,(\nu+9)\,(\nu+10)$

In addition to (2·8) which resulted from an expansion about $t = 0$, further useful formulae may be obtained by expansions about other points in the complex t-plane. For since equation (2·6) holds for all finite t except at the isolated singular points

* When $\nu = -l$ is a negative integer, the first l roots vanish (i.e. $\alpha_{-l,m} = 0$ for $1 \leqslant m \leqslant l$) and all other roots are non-vanishing. (In this case, $\alpha_{-l,m} = \alpha_{l,m-l}$, $m > l$.) No other multiple roots can occur.

† Rayleigh (**10**), p. 192. These results are also given by Watson (**12**), p. 502, who gives in addition Cayley's result for $4^{-8}s^{(8)}$, which we have also checked by our method. Our attention has also been called by R. E. Shafer to his calculation for $l = 6$, 7 and 9, reported in (**2**). Some of the terms in the latter result for $l = 9$ are in error.

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$\alpha_m(\nu)$—all of which are simple poles—we may develop the terms of the equation in a Laurent series in $\epsilon_l \equiv t-\alpha_l$. Let us therefore define the (absolutely convergent) sums

$$T_l^{(k)} \equiv \sum_{m=1}^{\infty}{}' (\alpha_m-\alpha_l)^{-k}, \quad S_l^{(k)} \equiv \sum_{m=1}^{\infty}{}' \alpha_m^{-1}(\alpha_m-\alpha_l)^{1-k}, \tag{2·10}$$

for $k = 1, 2, 3, \ldots$, where the prime on the summation is to exclude the term $m = l$. Then from the coefficient of ϵ_l^{-1} in the expansion of (2·6), and from (2·7), respectively,*

$$T_l^{(1)} = (1+\nu)/2\alpha_l, \quad S_l^{(1)} = (1+\nu)^{-1}-\alpha_l^{-1}. \tag{2·11}$$

Next, upon separately equating the coefficients of the remaining (higher) powers of ϵ_l in (2·6), we get

$$(k+3)\,T_l^{(k+2)}+(1+\nu)\,S_l^{(k+2)} = \sum_{i=0}^{k} T_l^{(i+1)}T_l^{(k-i+1)} \quad (k = 0, 1, 2, \ldots). \tag{2·12}$$

Then, since the definitions (2·10) admit the identity

$$S_l^{(k+2)} = \alpha_l^{-1}\,[T_l^{(k+1)}-S_l^{(k+1)}] \quad (k = 0, 1, 2, \ldots).$$

$S_l^{(k+2)}$ may be eliminated from (2·12) entirely in terms of $S_l^{(1)}$ and the $T_l^{(m)}, \ldots$ $1 \leqslant m \leqslant (k+1)$. Thus

$$(k+1)\,T_l^{(k)}+(-\alpha_l)^{1-k}+(1+\nu)\,(-\alpha_l)^{-k} = \sum_{i=1}^{k-1}[(1+\nu)\,(-\alpha_l)^{-i}+T_l^{(i)}]\,T_l^{(k-i)} \quad (k = 2, 3, 4, \ldots).$$

When $k \geqslant 3$, a simplification may be obtained by combining the first and last terms in the sum, and it is also convenient to alter our notation at this point and define

$$R_l^{(k)}(\nu) \equiv \sum_{m=1}^{\infty}[\alpha_l/(\alpha_l-\alpha_m)]^k, \tag{2·13}$$

i.e. $R_l^{(k)} = (-\alpha_l)^k T_l^{(k)}$. Writing

$$x \equiv (1+\nu), \quad y \equiv \alpha_l-\tfrac{1}{2}(1+\nu)\,(3+\nu), \tag{2·14}$$

we find

$$\left.\begin{aligned} &R_l^{(1)} = -\tfrac{1}{2}x, \quad R_l^{(2)} = \tfrac{1}{3}(y+\tfrac{1}{4}x^2), \quad R_l^{(3)} = \tfrac{1}{4}y, \\ &R_l^{(k)} = (1+k)^{-1}\left\{y+\sum_{i=2}^{k-2}[x+R_l^{(k-i)}]\,R_l^{(i)}\right\} \quad (k = 4, 5, 6, \ldots). \end{aligned}\right\} \tag{2·15}$$

Explicit expressions for $R_l^{(k)}$ are given in Table 2 through $k = 8$.

3. *Computation of the* $\alpha_l(\nu)$. As k is increased, the sum defining $s^{(k)}$, equation (2·9), tends rapidly toward the single term containing that zero closest to the origin. Similarly, the quantities $R_l^{(k)}$ from (2·13) converge very quickly to just those (one or two) terms involving the zeros closest to α_l. When ν is real and positive, this means $s^{(k)}$ converges to α_1^{-k} and that $R_l^{(k)}$ is given very closely by $[\alpha_1/(\alpha_1-\alpha_2)]^k$ for $l = 1$ and by $\{[\alpha_l/(\alpha_l-\alpha_{l-1})]^k+[\alpha_l/(\alpha_l-\alpha_{l+1})]^k\}$ for $l \geqslant 2$. In general, however, there is no guarantee that the conventional ordering of the α_l (see footnote, p. 142) will certify that α_{l-1} and

* One should beware the temptation to substitute the result for $T_l^{(1)}$ from equation (2·11) into (2·7) and too hastily conclude that $0 = 1$. The reason is that the order of summations in the expression $\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}{}'(\alpha_m-\alpha_l)^{-1}$ cannot be changed with impunity.

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α_{l+1} are the two zeros closest to α_l, or that α_1 is closest to the origin. Nevertheless, in principle one can use the independent determination of $s^{(k)}$ from (2·8) to evaluate that zero of smallest modulus, and then work outward throughout the whole complex plane using equation (2·15). In practice, since the errors are cumulative, one eventually will switch to the use of asymptotic expansions (e.g. the McMahon series) for the computation of large zeros.*

Table 2. *Giving* $R_l^{(k)}(\nu) \equiv \sum_{m=1}^{\infty}{}' \{\alpha_l(\nu)/[\alpha_l(\nu)-\alpha_m(\nu)]\}^k$ *for* $1 \leqslant k \leqslant 8$. *Entries through* $k = 4$ *were checked analytically for* $\nu = \frac{1}{2}$. *The remaining entries were checked numerically by direct computation from equation* (2·15)

$$x \equiv (\nu+1), \quad y \equiv \alpha_l - \tfrac{1}{2}(\nu+1)(\nu+3)$$

k	$R_l^{(k)}(\nu)$
1	$-\frac{1}{3}x$
2	$\frac{1}{3}[y+\frac{1}{4}x^2]$
3	$\frac{1}{4}y$
4	$\frac{1}{5}\{\frac{1}{9}y^2+y[1+\frac{1}{3}x+\frac{1}{18}x^2]+\frac{1}{12}x^3+\frac{1}{144}x^4\}$
5	$\frac{1}{6}\{\frac{1}{8}y^2+y[1+\frac{7}{12}x+\frac{1}{24}x^2]+\frac{1}{12}x^3\}$
6	$\frac{1}{7}\{\frac{2}{135}y^3+y^2[\frac{47}{240}+\frac{1}{15}x+\frac{1}{90}x^2]+y[1+\frac{47}{60}x+\frac{1}{10}x^2+\frac{1}{30}x^3+\frac{1}{360}x^4]$ $+\frac{1}{12}x^3[+\frac{1}{60}x+\frac{1}{240}x^2+\frac{1}{4320}x^3]\}$
7	$\frac{1}{8}\{\frac{4}{135}y^3+y^2[\frac{19}{90}+\frac{4}{27}x+\frac{2}{135}x^2]+y[1+\frac{19}{20}x+\frac{23}{120}x^2+\frac{7}{135}x^3+\frac{1}{540}x^4]$ $+x^3[\frac{1}{12}+\frac{11}{360}x+\frac{1}{270}x^2]\}$
8	$\frac{1}{9}\{\frac{1}{525}y^4+y^3[\frac{29}{700}+\frac{2}{175}x+\frac{1}{525}x^2]+y^2[\frac{153}{700}+\frac{319}{1400}x+\frac{101}{2800}x^2+\frac{3}{350}x^3+\frac{1}{1400}x^4]\}$ $+y[1+\frac{153}{140}x+\frac{151}{504}x^2+\frac{457}{6300}x^3+\frac{29}{2520}x^4+\frac{3}{1400}x^5+\frac{1}{8400}x^6]$ $+x^3[\frac{1}{12}+\frac{107}{2520}x+\frac{29}{5040}x^2+\frac{2}{1575}x^3+\frac{1}{5600}x^4+\frac{1}{134400}x^5]\}$

Since the recursion formulae (2·8) and (2·15) can be easily programmed on a computer for truncation at a given level k, it is not necessary to solve these equations explicitly as has been done in Tables 1 and 2. Nevertheless, for applications in which speed is a primary concern it is of advantage to have explicit expressions worked out.

Finally, we should present a word of caution in the use of our formulae. Note that in addition to the possible scrambling of the indices l, a need for care in the use of our relations for complex ν is required in choosing the proper branch when taking the kth roots of expressions for $(\alpha_l-\alpha_m)^{-k}$. The precautions to be taken, however, are obvious and the total process is straightforward. A second complication also arises when ν approaches $-l$, where additional care also must be exercised in determining the first l roots, which lie approximately on a circle about the origin.† Lastly, our methods lose efficiency for large $|\nu|$, since the separation of the early roots is not large compared to the distance of the first zero from the origin. Thus, an evaluation of the smallest zero by means other than use of $s(\nu)$ is desirable. Nevertheless, equations (2·13)–(2·15) still should be most useful in obtaining the larger zeros.‡

* See, e.g. Olver (8), xvi.

† When $|\nu+l| \ll 1$, the first l roots converge to the l branches of $\alpha = [l!(l-1)!\,(\nu+l)]^{1/l}$.

‡ When $|\nu|$ is large, the Olver expansions (properly extended to the complex domain) appear most useful for obtaining the smallest zero, and perhaps, then, some of the larger zeros as well. See (9).

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4. *Conclusions.* We have developed a general procedure for the determination of the zeros of $(\frac{1}{2}z)^{-\nu}J_\nu(z)$, and thus of making the infinite product representation of $\Gamma(\nu+1)(\frac{1}{2}z)^{-\nu}J_\nu(z)$ more useful for the computation of Bessel functions of complex index. The method employs equations (2·7)–(2·9) for the computation of the smallest zero and equations (2·13)–(2·15) for the sequential computation of larger zeros. The recursion relations (2·8) and (2·15) are solved explicitly for the lower orders in Tables 1 and 2.

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