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Robustly expansive codimension-one homoclinic classes are hyperbolic

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Abstract. We shall prove that C^1-robustly expansive codimension-one homoclinic classes are hyperbolic.

1. *Introduction*

Let M be a d-dimensional manifold and $\mathrm{Diff}^1(M)$ be the set of C^1 diffeomorphisms f on M endowed with the C^1 topology. Let p be a hyperbolic periodic point of f and $H(p)$ be its homoclinic class. The diffeomorphism f is α-expansive in $H(p)$ if $\mathrm{dist}(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$, with $x, y \in H(p)$, implies $x = y$. It is well known that hyperbolicity implies α-expansiveness for some $\alpha > 0$. But expansiveness alone does not guarantee hyperbolicity, even when one is dealing with a codimension-one expansive homoclinic class, as can be seen in [**PPV**, §2]. Moreover, there are examples of expansive codimension-one homoclinic classes that are not hyperbolic such that all their periodic orbits are hyperbolic.

To see this, consider a Smale horseshoe H in a plane and Λ a nontrivial minimal subset of H. In a complementary direction multiply H by a nonuniform contraction $\lambda(w)$, depending on the distance from w to Λ, and in such way that $\lambda(w) = 1$ for $w \in \Lambda$.

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The resulting homoclinic class is expansive, has all periodic points hyperbolic but it is not hyperbolic.

If we assume that expansiveness holds in a C^1 neighborhood of the homoclinic class then for all diffeomorphisms g C^1-near f the homoclinic class $H(p_g)$ of the continuation p_g of p is α-expansive. In [**PPV**] it was proved that robustly expansive homoclinic classes of a three-dimensional manifold are generically hyperbolic. We generalize this result in two ways. First, we drop the assumption $\dim(M) = 3$ and get that robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting. Second, we prove that robustly expansive codimension-one homoclinic classes with a dominated splitting are hyperbolic.

In order to state our results precisely, let us introduce some notations and definitions. The homoclinic class $H(p)$ of a hyperbolic periodic point p of $f \in \mathrm{Diff}^1(M)$ is the closure of all transverse intersections of the stable manifold $W^s(p)$ with the unstable manifold $W^u(p)$ of p. We say that $H(p)$ is a codimension-one homoclinic class if $\dim W^u(p) = 1$ or $\dim W^s(p) = 1$.

A compact f-invariant set K has a dominated splitting if $T_K M$ splits into a continuous Df-invariant direct sum $E \oplus F$ and there are $c > 0$, $0 < \lambda < 1$ such that, for all $x \in K$,

$$\|Df^n/E(x)\| \, \|Df^{-n}/F(f^n(x))\| \leq c\lambda^n$$

holds for all $n \geq 0$. We say that the splitting is a codimension-one splitting if $\dim(E(x)) = 1$ or $\dim(F(x)) = 1$ for all $x \in K$.

The following theorem is our first result.

THEOREM A. *Robustly expansive codimension-one homoclinic classes have a codimension-one dominated splitting $E \oplus F$.*

The corresponding theorem in [**PPV**, §4] assumed a generic hypothesis that we remove here.

Our next result is as follows.

THEOREM B. *Robustly expansive homoclinic classes with a codimension-one dominated splitting are hyperbolic.*

In proving the corresponding theorem in [**PPV**, §5] we profited from the density of C^2 diffeomorphisms in the C^1 topology by proving that the dominated splitting for a C^2 diffeomorphism near the original one is hyperbolic. If in addition the homoclinic class is germ expansive it is proved in [**SV**] that $H(p)$ is hyperbolic (for any codimension).

Next we sketch the proof of the results. To prove Theorem A we follow the same steps as in [**PPV**, §4], dropping the generic assumptions assumed there.

The main step in obtaining Theorem B is to prove that center-unstable manifolds for all point x in the homoclinic class $H(p)$ are true unstable manifolds, that is, center-unstable manifolds are dynamically defined (see Definition 3.4 below). To achieve this it is enough to obtain this property for periodic points homoclinically related to p. For this we use a result, Proposition 3.5, that controls the behavior of periodic points homoclinically related to p. Once this is settled, under the hypothesis that the dominated splitting $E \oplus F$ is not hyperbolic, the fact that center-unstable manifolds are dynamically defined allows us

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to obtain a hyperbolic periodic point q with arbitrarily large period as near $H(p)$ as we wish, and with arbitrarily small rate of contraction along the E-direction; see [**PPV**]. Then we prove that such hyperbolic periodic points are in fact in $H(p)$, contradicting uniform E-contraction in the period for periodic points in $H(p)$. As in [**Ma2**], this implies that the sub-bundle F is uniformly expanding, and then $E \oplus F$ is hyperbolic.

2. *Proof of Theorem A*

As already stated, the proof of Theorem A follows the same steps as in [**PPV**, §4]. Here we shall only indicate the modifications needed to achieve the proof in the codimension-one case, and leave the details for the reader.

We say that a hyperbolic periodic point $q \in H(p)$ is homoclinically related to p if

$$W^s(q) \cap W^u(p) \neq \emptyset \quad \text{and} \quad W^u(q) \cap W^s(p) \neq \emptyset,$$

where $W^j(q)$, $j = s$ or u, stands for the stable or unstable manifold of q. Denote by $H_r(p)$ the points homoclinically related to p. Then $H_r(p)$ is dense in $H(p)$. So, it suffices to construct the dominated splitting for periodic points in $H_r(p)$.

Step 1. We prove that there exists $\delta > 0$ such that if $q \in H_r(p)$ with period $\pi(q)$ and λ is a contracting eigenvalue of $Df^{\pi(q)}(q)$ then $|\lambda| < (1-\delta)^{\pi(q)}$. Similarly, if μ is an expanding eigenvalue of $Df^{\pi(q)}(q)$ then $|\mu| > (1+\delta)^{\pi(q)}$.

The proofs of these statements are analogous to the proofs of [**PPV**, Propositions 4.3, 4.4 and 4.6]. Indeed, as we are dealing with periodic points in $H_r(p)$ we can use transition matrices to achieve the results in the codimension-one case exactly as in the three-dimensional case.

Step 2. Next we prove that there exist $\bar{\gamma} > 0$ and $m_0 > 0$ such that, for $q \in H_r(p)$ with period $\pi(q) > m_0$,

$$\angle(E(q), F(q)) > \bar{\gamma}$$

holds, where $\angle(E(q), F(q))$ stands for the angle between $E(q)$ and $F(q)$.

The proof of this statement proceeds like that of [**PPV**, Proposition 4.9].

Using the expansiveness of f in $H(p)$ we obtain that there are only a finite number of periodic points with period bounded by m_0. Therefore, there exists $\gamma > 0$ such that, for all $q \in H_r(p)$,

$$\angle(E(q), F(q)) > \gamma.$$

Step 3. The previous steps allow us to prove [**PPV**, Lemma 4.12] in the codimension-one case, that is, we obtain the following lemma.

LEMMA 2.1. *There are a C^1 neighborhood $\mathcal{U}$ of f, $K \geq 2$, $0 < \lambda < 1$ such that for all $g \in \mathcal{U}$, for all $q \in H_r(p_g)$ with period $\pi(q)$,*

$$\|Dg^{-\pi(q)}/F(q)\| \leq K \quad \text{and} \quad \|Dg^{-\pi(q)n}/F(q)\| \leq K\lambda^{\pi(q)n},$$
$$\|Dg^{\pi(q)}/E(q)\| \leq K \quad \text{and} \quad \|Dg^{\pi(q)n}/E(q)\| \leq K\lambda^{\pi(q)n}.$$

Step 4. With the help of Lemma 2.1 we prove the existence of the dominated splitting as in [**PPV**, Theorem 4.13]. This result was also obtained in [**SV**] by a different proof for any codimension. On the other hand, the proof given in [**SV**] does not prove Lemma 2.1.

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3. *Proof of Theorem B*

Throughout we assume that $H(p)$ is robustly expansive with α as a constant of expansivity and that $H(p)$ has a codimension-one dominated splitting $T_{H(p)} = E \oplus F$ with $\dim(F) = 1$ and such that for all $x \in H(p)$,

$$\|Df^n/E(x)\| \, \|Df^{-n}/F(f^n(x))\| \leq C\,\lambda^n. \tag{1}$$

Taking a power f^k of f, we may (and do) assume that $C = 1$.

The splitting $E \oplus F$ defined on $T_{H(p)}M$ can be uniformly extended to a neighborhood $\mathcal{V}(H(p))$ of $H(p)$ (see [**PS**, p. 985]). Even though [**PS**] is for surface diffeomorphisms the results mentioned there are true for any dimension. We point out that this extension is not necessarily invariant. Nevertheless points whose f-orbits stay in $\mathcal{V} = \mathcal{V}(H(p))$, belonging or not to $H(p)$, have a dominated splitting.

Thus if $\mathcal{O}(x) \subset \mathcal{V}$ then (1) holds for x. Here $\mathcal{O}(x)$ stands for the orbit of x. We call $\mathcal{V}$ an *admissible* neighborhood of $H(p)$; see [**Ma1, PS**].

Let us assume that the local manifold $W^\sigma_\epsilon(x)$, $\sigma = s, u, cs, cu$ is an embedded topological k-disk, $k \geq 1$. Let Γ be the family of all parameterized continuous arcs $\gamma : [0, 1] \to W^\sigma_\epsilon(x)$ joining x with the boundary of $W^\sigma_\epsilon(x)$.

Definition 3.1. The size of $W^\sigma_\epsilon(x)$ is defined as $\inf\{\mathrm{diam}(\gamma) \,/\, \gamma \in \Gamma\}$.

The proof of the following lemma can be found in [**Ma1**]; see also [**PS**].

LEMMA 3.1. *Assume that $H(p)$ has a dominated splitting $E \oplus F$ and let $x \in M$ such that $\mathcal{O}(x) \subset \mathcal{V}$. There exists $\epsilon > 0$ such that the local center-unstable manifold $W^{cu}_\epsilon(x)$ is defined and is transverse to E and the local center-stable manifold $W^{cs}_\epsilon(x)$ is defined and is transverse to F. Such manifolds are of class C^1 and there exists $\epsilon_1 > 0$, depending on ϵ, such that their sizes are greater than ϵ_1. Moreover, there exists $\delta > 0$ such that*

$$\begin{gathered}
\text{if } y \in W^{cs}_\epsilon(x) \quad \text{and} \quad \mathrm{dist}(f^j(x),\, f^j(y)) \leq \delta,\ 0 \leq j \leq n, \\
\text{then } f^j(y) \in W^{cs}_\epsilon(f^j(x)), \quad 0 \leq j \leq n, \\
\text{if } y \in W^{cu}_\epsilon(x) \quad \text{and} \quad \mathrm{dist}(f^{-j}(x),\, f^{-j}(y)) \leq \delta,\ 0 \leq j \leq n, \\
\text{then } f^{-j}(y) \in W^{cs}_\epsilon(f^{-j}(x)), \quad 0 \leq j \leq n.
\end{gathered}$$

For $\epsilon' > 0$ small, the ϵ'-local unstable manifold of a hyperbolic periodic point $q \sim p$, $W^u_{\epsilon'}(q)$, is an arc tangential to F containing q in its interior. Analogously the ϵ'-local stable manifold of q, $W^s_{\epsilon'}(q)$, is a $(d-1)$-dimensional disk, $d = \dim(M)$, with q interior to this disk. Clearly there exists ϵ' small enough such that $W^u_{\epsilon'}(q) \subset W^{cu}_\epsilon(q)$ and $W^s_{\epsilon'}(q) \subset W^{cs}_\epsilon(q)$. In order to simplify notation we will assume that $\epsilon' = \epsilon$ in what follows.

The arc $W^u_\epsilon(q)$ is separated by q into two connected components $W^{u,+}_\epsilon(q)$ and $W^{u,-}_\epsilon(q)$ that, with the point q added, we shall call (local unstable) branches of $W^u_{\epsilon'}(q)$. Similarly, for any point $x \in H(p)$ we may speak of the local center-unstable branches $W^{cu,+}_\epsilon(x)$ and $W^{cu,-}_\epsilon(x)$ for the connected components of $W^{cu}_\epsilon(x)\backslash\{x\}$ with the point x added.

LEMMA 3.2. *Given $q \in H_r$, there exist $k \in \mathbb{Z}$ and $\delta_0 > 0$ such that if $W^{u,+}_\epsilon(q) \cap H(p) \neq \emptyset$ then $\mathrm{diam}(W^{u,+}_\epsilon(f^k(q))) \geq \delta_0$. A similar result holds for $W^{u,-}_\epsilon(q)$ if $W^{u,-}_\epsilon(q) \cap H(p) \neq \emptyset$.*

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Proof. Assume in what follows that $H(p)$ has points in, say, $W^{u,+}_\epsilon(q)$. Then there is a Cantor set of points $y \in W^{u,+}_\epsilon(q)$ belonging to $H(p)$ because, by [**PM**, λ-Lemma], $W^s(p)$ accumulates on $W^s(q)$ and $W^u(p)$ accumulates on $W^u(q)$. Backward iterates of these points by f approach the orbit of q.

Backward iterates of points in $H(p) \cap W^{u,+}_\epsilon(q)$ rest at a distance less than ϵ from the orbit of q. Therefore, by expansiveness, forward iterates must separate. Hence there is a first $k > 0$ such that $\operatorname{diam}(f^{k+1}(W^{u,+}_\epsilon(q))) \geq \epsilon$. Thus $\operatorname{diam}(f^j(W^{u,+}_\epsilon(q))) < \epsilon$ for $0 \leq j \leq k$ and therefore $f^j(W^{u,+}_\epsilon(q)) \subset W^{u,+}_\epsilon(f^j(q))$.

This proves the lemma with $\delta = \delta_0$ defined as $\delta_0 = \min\{\delta, \operatorname{diam}(f^{-1}(X)) = \delta$ and $\operatorname{diam}(X) \geq \epsilon\}$ where X is any subset of M. □

Let us reduce $\delta_0 > 0$ if it were necessary to have that if $\operatorname{dist}(y, H(p)) \leq \delta_0$, $y \in M$, then $y \in \mathcal{V}$, an admissible neighborhood of $H(p)$.

LEMMA 3.3. *Let $q \in H_r$. If $W^{u,-}_\epsilon(q) \cap H(p) = \emptyset$ then either there exists $k \in \mathbb{Z}$ such that*

$$\operatorname{diam}(W^{u,-}_\epsilon(f^k(q))) \geq \delta_0$$

or there is a periodic point q' which is an end-point of $W^{u,-}_\epsilon(q)$, of the same period as that of q, such that $\operatorname{dist}(f^j(q), f^j(q')) \leq \epsilon$, for all $j \in \mathbb{Z}$.

Proof. If $\operatorname{diam}(W^{u,-}_\epsilon(q)) \leq \delta_0$ then we will have the end-point q' $(q' \neq q)$ of $W^{u,-}_\epsilon(q)$ contained in $\mathcal{V}$ and therefore $\operatorname{Cl}(W^{u,-}(q))$ is tangential to the unstable F-cone $\mathcal{C}_F$ for all the iterates by f. Here $\operatorname{Cl}(A)$ stands for the closure of A. It follows that q' is periodic of the same period of q or of double the period of q, depending on the sign of the expanding eigenvalue of Df^τ_q. But if this eigenvalue were negative then $H(p)$ would have points different from q in both branches of $W^u_\epsilon(q)$ and, since Lemma 3.2 holds, we arrive at a contradiction. Hence the period of q' is the same of that of q, concluding the proof of Lemma 3.3. □

An analogous result holds if $W^{u,+}(q) \cap H(p) = \emptyset$.

Next we state a result by Pliss [**Pl1, Pl2**] that we shall use later. A proof of this may be found in [**Al**, Lemma 2.11].

Definition 3.2. We say that a pair of points $(x, f^n(x))$ contained in $H(p)$, $n > 0$, is a γ-string, $0 < \gamma < 1$, if

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| \leq \gamma^n.$$

We say that $(x, f^n(x))$ is a uniform γ-string if $(x, f^k(x))$ is a γ-string for all $0 \leq k \leq n$.

LEMMA 3.4. *Let $0 < \gamma_1 < \gamma_2 < 1$ and $(x, f^n(x))$ be a γ_1-string. There exist a positive integer $N = N(\gamma_1, \gamma_2, f)$, $c = c(\gamma_1, \gamma_2, f) > 0$ such that if $n \geq N$ then there exist numbers*

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_l \leq n$$

such that $(f^{n_r}(x), f^n(x))$ are uniform γ_2-strings for all $r = 1, 2, \ldots, l$, with $l \geq cn$. That is,

$$\prod_{i=n_r}^{j} \|Df/E(f^i(x))\| \leq \gamma_2^{j-n_r}, \quad r = 1, 2, \ldots, l;\ n_r \leq j \leq n.$$

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The numbers n_i in Lemma 3.4 are called γ_2-*hyperbolic times* for $(x,\ E)$. Analogously, replacing Df^n/E by Df^{-n}/F in the inequality above, we define hyperbolic times for $(x,\ F)$.

Definition 3.3. Given $0 < \gamma < 1$ and $0 < N \leq n$, we say that $(x,\ f^n(x))$ is an (N, γ)-obstruction if $(f^m(x),\ f^n(x))$ is not a γ-string for all $n - N \leq m \leq n$.

3.1. *Dynamically defined center-unstable manifolds.*

Definition 3.4. We say that the local center-unstable (center-stable) manifold $W^{cu}_\epsilon(x)$ ($W^{cs}_\epsilon(x)$) is dynamically defined if it is part of the local unstable (stable) manifold of x – that is, given $\epsilon > 0$, there exists $\epsilon' > 0$ such that $\epsilon' \to 0$ when $\epsilon \to 0$, and for all $y \in W^{cu}_\epsilon(x)$ ($y \in W^{cs}_\epsilon(x)$) we have that $\text{dist}(f^n(x),\ f^n(y)) \leq \epsilon'$ for all $n \leq 0$ (for all $n \geq 0$).

Next we shall prove that any hyperbolic periodic point of $H(p)$ has an iterate for which the center-stable and center-unstable manifolds are dynamically defined. The proof of this result is similar to one in [**Sa**].

PROPOSITION 3.5. *Let M be a compact manifold, $f : M \to M$ a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant subset with a dominated splitting $E \oplus F$, $\dim(F) = 1$. Assume that Lemma 2.1 holds. Then, given $\epsilon > 0$, there exist $N_1 \in \mathbb{Z}^+$, $\epsilon' > 0$, $\gamma > 0$, with $\epsilon' \to 0$ and $\gamma \to 0$ when $\epsilon \to 0$, and $0 < \lambda_1 < 1$ such that for all hyperbolic periodic points $q \in \Lambda$ of period greater than N_1, with $\dim(W^u(q)) = 1$, there exists $m = m(q) \geq 0$ such that:*

(1) *for all $n \geq 0$, $\prod_{j=1}^n \|Df/E(f^{m+j}(q))\| \leq \lambda_1^n$;*
(2) *$W^{cs}_\epsilon(f^m(q)) \subset W^s_{\epsilon'}(f^m(q))$;*
(3) *$f^{-n}(W^{cu,+}_\epsilon(f^m(q))) \subset W^{cu,+}_\gamma(f^{-n+m}(q))$, $n \geq 0$, where $W^{cu,+}_\epsilon(f^m(q))$ is a branch of the local center-unstable manifold of $f^m(q)$.*

Proof. Let us denote by Λ_1 the set of all hyperbolic periodic points in Λ with one-dimensional unstable manifold. From [**Ma3**, Lemma II.5] we may replace $\|Df^n/E(q)\| \leq \lambda^n$ by $\prod_{j=1}^n \|Df/E(f^j(q))\| \leq C\tilde{\lambda}^n$ with $C > 0$, $0 < \lambda \leq \tilde{\lambda} < 1$. In order to simplify notation we assume $\tilde{\lambda} = \lambda$ and $C = 1$. Let us also assume that there is a dominated splitting $E \oplus F$ satisfying equation (1) and that for any periodic point $q \in \Lambda_1$ the inequalities of Lemma 2.1 are valid. Let us consider λ_1 such that $0 < \lambda < \sqrt{\lambda} < \lambda_1 < 1$.

From Lemma 2.1, there exists k, $0 \leq k < \pi(q)$, such that $\|Df^{\pi(q)}/E(q)\| < \lambda^{\pi(q)}$ holds which, by [**Ma3**, Lemma II.5], implies that $\prod_{j=1}^{\pi(q)} \|Df/E(f^j(q))\| \leq \lambda^{\pi(q)}$. From this inequality the proof that, given $q \in \Lambda_1$, there exists m such that, for all $n \geq 0$, $\prod_{j=1}^n \|Df/E(f^{m+j}(q))\| \leq \lambda_1^n$ follows from Lemma 3.4 provided that the period $\pi(q)$ of q is greater than N_1 where $N_1 \geq N(\sqrt{\lambda}, \lambda_1)$, with the constant $N(\sqrt{\lambda}, \lambda_1)$ given by Lemma 3.4. To see this, let $1 \leq n_1 < n_2 < \cdots < n_k$ be a maximal sequence of hyperbolic times for $(q,\ E)$. Hence

$$\prod_{j=n_1}^{n} \|Df_{f^j(q)}|_E\| \leq \lambda_1^{n-n_1} \quad \text{for all } n : n_1 \leq n \leq \pi(q).$$

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Now consider the point $f^{n_1}(q)$ in the orbit of q and find a hyperbolic time for $(f^{n_1}(q), E)$, say $\hat{n}$. We have $n_1 < \hat{n} \leq \pi(q)$. Otherwise $n_1 + \pi(q) \geq \hat{n} > \pi(q)$ which, on account of q being periodic, would imply that n_1 is not the first hyperbolic time for (q, E). From this fact it is easy to see that

$$\prod_{j=n_1}^{n} \|Df_{f^j(q)}|_E\| \leq \lambda_1^{n-n_1} \quad \text{for all } n : n_1 \leq n \leq \pi(q) + n_1,$$

and therefore, again by periodicity of q, that for all $n \geq 0$,

$$\prod_{j=1}^{n} \|Df/E(f^{n_1+j}(q))\| \leq \lambda_1^n.$$

This concludes the proof of (1), letting m be n_1.

To prove (2) we assume that (1) holds with $m = 0$ in order to simplify notation. Let $\delta > 0$ be such that $(1+\delta)\lambda_1 < \lambda_2 < 1$. There exist $N_0 \in \mathbb{N}$ and $\lambda_3 < 1$ such that $C\lambda_2^n < \lambda_3^n < 1$ for all $n \geq N_0$. Let $\epsilon' > 0$ be such that if $\operatorname{dist}(x, y) < \epsilon'$ then

$$(1-\delta)\|Df_x|_{E^s}\| \leq \|Df_y|_{E^s}\| \leq (1+\delta)\|Df_x|_{E^s}\|,$$

and let $\epsilon > 0$ be such that if $\operatorname{dist}(x, y) < \epsilon$ then

$$\operatorname{dist}(f^j(x), f^j(y)) < \epsilon', \quad j = 0, 1, \ldots, N_0.$$

Clearly one can choose $\epsilon' \to 0$ when $\epsilon \to 0$. Thus for all $x \in W_\epsilon^{cs}(q)$ and for $0 \leq n \leq \pi(q)$,

$$\prod_{j=1}^{n} \|Df_{f^j(x)}|_{E^s}\| \leq \prod_{j=1}^{n} (1+\delta)\|Df_{f^j(q)}|_{E^s}\| \leq C(\lambda_1(1+\delta))^n = C{\lambda_2}^n.$$

In particular, for $\pi(q) > N_0$ we obtain

$$\prod_{j=1}^{\pi(q)} \|Df_{f^j(x)}|_{E^s}\| \leq \prod_{j=1}^{\pi(q)} (1+\delta)\|Df_{f^j(q)}|_{E^s}\| \leq C(\lambda_1(1+\delta))^{\pi(q)} = C{\lambda_2}^{\pi(q)} < \lambda_3^{\pi(q)}.$$

Let N_1 be greater than N_0. Hence for all hyperbolic periodic points q of period $\pi(q)$ greater than N_1 and any arc γ joining q with a point $y \in W_\epsilon^{cs}(q)$, applying the mean value theorem, the length of $f^n \circ \gamma$ satisfies $\ell(f^n \cdot \gamma) \leq \lambda_3^n \cdot \ell(\gamma)$ for all $n \geq N_0$, from which $W_\epsilon^{cs}(q) \subset W_{\epsilon'}^s(q)$ follows, concluding the proof of (2).

Let us now prove (3). Given $\gamma > 0$ and q a hyperbolic periodic point, choose coherently one branch of the local center-unstable manifold of the points of the orbit of q, that is, such that $f^{-1}(W_\epsilon^{cu,+}(f^j(q))) \cap W_\epsilon^{cu,+}(f^{j-1}(q))$ is an open subset of $W_\epsilon^{cu,+}(f^{j-1}(q))$ for all $j = 0, 1, \ldots, \pi(q) - 1$, and let us define

$$\epsilon^+(q) = \sup\{\epsilon > 0 : f^{-n}(W_\epsilon^{cu,+}(q)) \subset W_\gamma^{cu,+}(f^{-n}(q)), n \geq 0\},$$

and

$$\epsilon^+(\mathcal{O}(q)) = \sup\{\epsilon^+(f^j(q)) : j = 1, \ldots, \pi(q)\}.$$

It is enough to prove that $\inf\{\epsilon^+(\mathcal{O}(q)) : q \in \Lambda_1\} > 0$.

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Since q is hyperbolic, $\epsilon^+(f^j(q)) > 0$ for all $j = 0, 1, \ldots, \pi(q) - 1$, and so $\epsilon^+(\mathcal{O}(q)) > 0$. Arguing by contradiction, let us suppose that there is a sequence $\{p_k\}$ of points in Λ_1 such that $\epsilon^+(\mathcal{O}(p_k)) \to 0$ as $k \to \infty$. Suppose without loss of generality that $\epsilon^+(\mathcal{O}(p_k)) = \epsilon^+(p_k)$. Since

$$f^{-n}(W^{cu,+}_{\epsilon(p_k)}(p_k)) \subset W^{cu,+}_{\epsilon(f^{-n}(p_k))}(f^{-n}(p_k)) \quad \text{for all } n \geq 0$$

and that we may assume that $\epsilon^+(p_k) < \gamma$, then there exist $t > 0$ and $m_t > \pi(p_k)$ such that, setting $\delta_t = \epsilon(p_k) + t$, we get $f^{-j}(W^{cu,+}_{\delta_t}(p_k)) \subset W^{cu,+}_{\gamma}(f^{-j}(p_k))$ for $0 \leq j \leq m_t$ and $W^{cu,+}_{\gamma}(f^{-m_t}(p_k))$ coincides with $f^{-m_t}(W^{cu,+}_{\delta_t}(p_k))$.

Let us set $q_k = f^{-m_t}(p_k)$ and l_k to $W^{cu,+}_{\gamma}(q_k) = f^{-m_t}(W^{cu,+}_{\delta_t}(p_k))$. Hence we obtain that $f^{\pi(p_k)}(l_k) \subset l_k$. This implies that $\|Df^{-n}/F(f^n(z))\| \geq \lambda_1^n$ for all $n \geq 0$, for all $z \in l_k$. Otherwise $f^{-n}(f^n(l_k)) \subsetneq l_k$. Therefore, $\|Df^n/E(z)\| \leq \lambda_1^n$ for all $n \geq 0$ for all $z \in l_k$. Hence, as in the proof of (2), there exists $\gamma' > 0$ such that, for $z \in l_k$, $W^{cs}_{\gamma}(z) \subset W^{s}_{\gamma'}(z)$. From Lemma 3.1 the size of $W^{cs}_{\gamma}(z)$ is bounded away from zero. Since, by (1), the angle between the center-stable manifold and the center-unstable manifold of points $z \in l_k$ is bounded away from zero, we conclude that the stable manifold $W^s(l_k)$ of l_k has volume bounded away from zero. Since $W^s(l_k) \cap W^s(l_{k'}) = \emptyset$ for $k \neq k'$ and M has finite volume, the last fact leads to a contradiction. This proves that for a given $\gamma > 0$ there exists $\epsilon^+(\gamma) > 0$ such that (3) holds.

To complete the proof of the proposition we have to find m such that (1), (2) and (3) hold at the same time. Assume then that, for some j, (1) holds for $f^j(q)$. Hence (2) holds too. If it were the case that $\epsilon^+(f^j(q)) = \epsilon^+(\mathcal{O}(q))$ which is, by definition, greater or equal than $\epsilon^+(\gamma)$, then we are done. Otherwise define

$$\begin{aligned} m = \min\{0 \leq n \leq \pi(q) : \epsilon^+(f^{-n}(f^j(q))) \geq \epsilon^+(\gamma); \text{ and} \\ \epsilon^+(f^{-n}(f^j(q))) \geq \epsilon^+(f^{-h}(q)),\ 0 \leq h \leq n\}. \end{aligned}$$

Then it is easy to see that (1), (2) and (3) are satisfied for $f^{-m+j}(q)$. □

Remark 3.6. Since the choice of the branch $W^{cu,+}_{\epsilon}(q)$ is arbitrary, the existence of $\gamma > 0$ and $\epsilon(\gamma)$ such that Proposition 3.5(3) holds is valid for both branches. On the other hand, we cannot assume that m is the same for both branches.

COROLLARY 3.7. *Let $\{p_k\}$ be a sequence of hyperbolic periodic points, $p_k \sim p$ converging to x and satisfying Proposition 3.5. Then given λ_2, $0 < \lambda_1 < \lambda_2$, and n_1, a λ_1-hyperbolic time for x, there exists $k_0 > 0$, depending on λ_2, such that, for all $k \geq k_0$, n_1 is a λ_2-hyperbolic time for p_k.*

Proof. Since n_1 is a λ_1-hyperbolic time for x, E,

$$\prod_{n_1}^{n-1} \|Df/E(f^j(x))\| \leq \lambda_1^{n-n_1} \quad \text{for all } n \geq n_1.$$

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Let $I = \inf\{\|Df/E(y)\|,\ y \in H(p)\}$. If the corollary does not hold, there would be a subsequence of $\{p_k\}$, which for simplicity we also denote by p_k, such that for all $k > 0$, there exists m_k such that

$$\prod_{j=n_1}^{m_k-1} \|Df/E(f^j(p_k))\| > \lambda_2^{m_k-n_1}.$$

Then

$$\lambda_2^{m_k-n_1} < \prod_{j=n_1}^{m_k-1} \|Df/E(f^j(p_k))\| = \frac{\prod_{j=0}^{m_k-1} \|Df/E(f^j(p_k))\|}{\prod_{j=0}^{n_1-1} \|Df/E(f^j(p_k))\|} < \frac{\lambda_1^{m_k}}{I^{n_1}},$$

implying that

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{m_k} < \left(\frac{\lambda_2}{I}\right)^{n_1} \quad \text{for all } k \geq k_0. \tag{2}$$

Since $p_k \to x$ as $k \to \infty$ we obtain $m_k \to \infty$, and so equation (2) leads to a contradiction. □

Next we complement the results stated above, establishing other properties for the center-unstable manifolds of periodic points homoclinically related to p.

Definition 3.5. A point $x \in H(p)$ is a border point if there exists a neighborhood N of x, homeomorphic to a ball, such that $N \backslash W_\epsilon^{cs}(x)$ has two connected components and there is only one component of $N \backslash W_\epsilon^{cs}(x)$ with x in its boundary and containing points of $H(p)$ accumulating on the point x.

Remark 3.8. For any nontrivial homoclinic class $H(p)$, p hyperbolic, there is a dense subset in $H(p)$ of hyperbolic periodic points homoclinically related to p, and which are not border points.

LEMMA 3.9. *Assume that $x \in H(p)$ is such that $\ell(f^n(W_\epsilon^{cu,+}(x))) \leq \delta_0$ for all $n \geq 0$. Then x is a border point of $H(p)$.*

Proof. Set $I = W_\epsilon^{cu,+}(x)$ and assume that $\ell(f^n(I)) \leq \delta_0$ for all n. Then, using equation (1), we obtain that there is $\epsilon > 0$ such that every $w \in I$ has a local stable manifold $W_\epsilon^s(w)$, that is, that the local center-unstable manifold is contained in the stable one.

Now we prove that x is a border point of $H(p)$. Indeed, since $\ell(f^n(I)) \leq \delta_0$ for all n, we obtain that $\text{dist}(f^n(y),\ f^n(x)) \leq \delta_0$ for all $n \geq 0$ and $y \in I$. Assume that there exists $y \in H(p) \cap W_\epsilon^s(I)$, $y \neq x$. Then there exists $z \in W^s(p) \cap W^u(p)$ with $\text{dist}(z,\ y) \leq \epsilon$, and we conclude, on account of the domination, that

$$\text{dist}(f^n(z),\ f^n(x)) \leq 2\,\delta_0 \quad \text{for all } n \geq 0. \tag{3}$$

Since $z \in W^s(p)$, then $\text{dist}(f^n(z),\ p) \to 0$ as $n \to \infty$, and this, together with (3), gives $\text{dist}(f^n(x),\ p) \leq 2\,\delta_0$ for $n \geq n_0$. Thus $x \in W^s(p)$ and since p is hyperbolic, by the λ-lemma we obtain $\ell(f^m(I)) > \delta_0$ for some $m > 0$, a contradiction. This proves the lemma. □

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THEOREM 3.10. *For all $x \in H(p)$, $W^{cu}_\epsilon(x)$ is dynamically defined.*

Let $\delta_0 > 0$ and $\mathcal{V}$ be an admissible neighborhood of $H(p)$. Theorem 3.10 is a consequence of the following lemma.

LEMMA 3.11. *There exists $\nu > 0$ such that for any periodic point $q \sim p$, with q not a border point, $\ell(W^{u,\pm}_\epsilon(q)) \geq \nu$.*

Proof. Assume that the thesis does not hold. Then there is a sequence of periodic points q_k homoclinically related to p and with q_k not a border point, such that $W^{u,+}_\epsilon(q_k)$ or $W^{u,-}_\epsilon(q_k)$ has length less than $1/k$. Assume, for instance, that this happens to $W^{u,+}_\epsilon(q_k)$. Then, since q_k is periodic, there exists $m_k > 0$ such that $\ell(f^{-m_k}(W^{u,+}_\epsilon(q_k))) = \epsilon$ and so $f^{-m_k}(W^{u,+}_\epsilon(q_k)) = W^{u,+}_\epsilon(f^{-m_k}(q_k))$. Set $f^{-m_k}(q_k) = p_k$ and note that $m_k \to \infty$ as $k \to \infty$. By Proposition 3.5 we may assume that

$$W^{cu,+}_\epsilon(p_k) \subset W^{u,+}_\epsilon(p_k), \quad W^{cs}_\epsilon(p_k) \subset W^s_\epsilon(p_k) \quad \text{and}$$
$$\prod_{j=1}^{j=n} \|Df/E(f^j(p_k))\| \leq \lambda^n \quad \text{for all } n \geq 0. \tag{4}$$

Taking into account that $H(p)$ is compact, let us assume that $x = \lim_{k\to\infty} p_k$. Moreover, taking subsequences, we may also assume that $W^{u,\pm}_\epsilon(p_k)$ converge to arcs $I^\pm = I^\pm(x)$ with x at their common boundary. Next we prove that I^+ is contained in $W^u_\epsilon(x)$. Indeed, if this were not the case, there would exist $n_0 > 0$ and $y \in I^+$ such that $\operatorname{dist}(f^{-n_0}(x), f^{-n_0}(y)) = \varphi > \epsilon$. Let $k > 0$ be so great that $y_k \in W^{u,+}_\epsilon(p_k)$ is so close to y that $\operatorname{dist}(f^{-j}(y), f^{-j}(y_k)) < (\varphi - \epsilon)/3$ and $\operatorname{dist}(f^{-j}(x), f^{-j}(p_k)) < (\varphi - \epsilon)/3$ for all $j = 0, 1, \ldots, n_0$. Then $\operatorname{dist}(f^{-n_0}(y_k), f^{-n_0}(p_k)) > \epsilon$, which is absurd. Thus we may write $W^{u,+}_\epsilon(x)$ instead of I^+. The same holds for I^- and we write $W^{u,-}_\epsilon(x) = I^-$.

Since for all $j \in [0, m_k]$ we obtain that $\ell(f^j(W^{u,+}_\epsilon(p_k))) \leq \epsilon$ and $m_k \to \infty$ when $k \to \infty$, we see that $\ell(f^n(W^{u,+}_\epsilon(x))) \leq \epsilon$ for all $n \geq 0$.

To conclude the proof we need the following result that we shall prove later in §3.2. Recall that $\omega(x)$ is the set of points $w \in M$ such that there exists $n_j \to +\infty$ such that $f^{n_j}(x) \to w$ as $j \to \infty$.

THEOREM 3.12. *If $f/H(p)$ is C^1-robustly expansive and there exists $x \in H(p)$ constructed as in Lemma 3.11 such that $\ell(f^n(W^{cu,+}_\epsilon(x))) \leq \delta_0$ for all $n \geq 0$, then $\omega(x)$ is a periodic orbit.*

Returning to the proof of Lemma 3.11, on account of the above theorem, $x \in W^s(q)$ for some $q \in H(p)$, q periodic. Taking a positive iterate by f of x, we may assume that $x \in W^s_\epsilon(q)$.

Moreover, since $W^{u,+}_\epsilon(p_k)$ has length bounded away from zero and the angle between the sub-bundles E and F is bounded away from zero, we claim that $W^{u,+}_\epsilon(p_k) \cap W^s_\epsilon(q) \neq \emptyset$. Indeed, to prove this, observe that since $\ell(f^n(W^{u,+}_\epsilon(x))) \leq \epsilon$ for all $n \geq 0$ then, from equation (1), we get that $W^{cs}_\epsilon(x)$ is a true stable manifold. Therefore there are also local stable manifolds for all points $y \in I(x)$. Take $r > 0$ small such that $W^s_\epsilon(x)$ locally separates the ball at x with radius r, $B(x, r)$. Then in the region of $B(x, r) \backslash W^s_\epsilon(x)$ containing $I(x)$ we cannot have points of the sequence p_k accumulating in x. Otherwise, since for points

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in that region their local center-stable manifolds are local stable ones, we will have points $y \in I(x)\backslash\{x\}$ belonging to $H(p)$. Just take $y = W^s_\epsilon(p_{k_0}) \cap I(x)$ for a suitable k_0 and then using that $W^{u,+}_\epsilon(p_k) \to I(x)$, in the Hausdorff sense, we will obtain $y \in H(p)$. But if $y \in I(x)$ we have shown that, for all $n \in \mathbb{Z}$, $\mathrm{dist}(f^n(x), f^n(y)) \leq \epsilon$. But this contradicts the fact that $f/H(p)$ is expansive if ϵ is less than an expansivity constant of $f/H(p)$. Therefore the points p_k accumulate x from the opposite region to that containing $I(x)$ and eventually $W^{u,+}_\epsilon(p_k)$ will cut $W^s_\epsilon(x) \subset W^s_\epsilon(q)$. Since $p_k \sim p$ it follows that $W^u(p)$ cuts $W^s_\epsilon(q)$, proving the claim.

If q is hyperbolic, by Hayashi's connecting lemma we may C^1-perturb f, obtaining a diffeomorphism g so that $W^s(p_g) \cap W^u(q_g) \neq \emptyset$ and still having $W^u(p_g) \cap W^s(q_g) \neq \emptyset$.

Since periodic points homoclinically related to p_g cannot have a weak expanding eigenvalue (see [**PPV**, §4]), the same is true for q_g, and therefore for q. As $x \in W^s(q) \cap I(x)$, by the λ-lemma, there is n_0 such that $\ell(f^n(I(x))) \geq \epsilon$ for all $n \geq n_0$. Thus our assumption that there is a sequence $\{q_k\}$ of periodic points homoclinically related to p such that $\ell(W^{u,+}_\epsilon(q_k)) \leq 1/k$ leads to a contradiction.

Hence there exists $\nu > 0$ such that $\ell(W^{u,\pm}(p')) \geq \nu$ for all periodic points p' homoclinically related to p, concluding the proof of Lemma 3.11 in this case.

If q is not hyperbolic then, by [**Fr**, Lemma 1.1], we may perturb Df_q/F to obtain hyperbolicity of q without losing the intersection between $W^{u,+}_\epsilon(p_k)$ and $W^s_\epsilon(q)$ and still having p_k accumulating in $W^s_\epsilon(q)$. So, q is still a point of the homoclinic class (of the perturbed diffeomorphism) and we can reason as above to achieve the same result. □

Next we extend Lemma 3.11 for all $x \in H(p)$.

PROPOSITION 3.13. *For all $x \in H(p)$, $W^u_\epsilon(x)$ is a nontrivial set and both branches of $W^u_\epsilon(x)$ have length greater than $\nu > 0$.*

Proof. Let $x \in H(p)$. Since periodic points homoclinically related to p are dense in $H(p)$, p_k is homoclinically related to p such that $p_k \to x$. Taking a converging subsequence, in the Hausdorff sense, of $W^{u,+}_\epsilon(p_k)$ and $W^{u,-}_\epsilon(p_k)$, we conclude that $W^{cu,+}_\epsilon(x)$ and $W^{cu,-}_\epsilon(x)$ are defined. As both $W^{u,+}_\epsilon(p_k)$ and $W^{u,-}_\epsilon(p_k)$ are tangential to F and locally separated by $W^{cs}_\epsilon(p_k)$, $W^{cu,+}_\epsilon(x) \neq W^{cu,-}_\epsilon(x)$. Moreover, both branches are tangential to F and have length greater than ν since this holds for $W^{u,\pm}_\epsilon(p_k)$. Also $\ell(f^{-n}(W^{cu,\pm}_\epsilon(x))) \leq \epsilon$, $n \geq 0$, because $W^{cu,\pm}(p_k)$ converges to $W^{cu,\pm}(x)$. Hence, $W^{cu,\pm}_\epsilon(x)$ are in fact part of the local unstable manifold of x. □

LEMMA 3.14. *For all $x \in H(p)$, $W^{cu}_\epsilon(x)$ is dynamically defined, that is,*

$$\ell(f^{-n}(W^{cu,\pm}_\epsilon(x))) \xrightarrow[n\to\infty]{} 0.$$

Proof. If it were the case that

$$\ell(f^{-n}(W^{cu,+}_\epsilon(x))) \not\to 0 \quad \text{when } n \to +\infty$$

for all $x \in H(p)$ then there would exist $x \in H(p)$, $n_k \to +\infty$, and $\rho > 0$ such that

$$\ell(f^{-n_k}(W^{cu,+}_\epsilon(x))) \geq \rho.$$

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By compactness of $H(p)$ we may assume that

$$z = \lim_{k\to+\infty} f^{-n_k}(x) \quad \text{and} \quad I(z) = \lim_{k\to+\infty} f^{-n_k}(W_\epsilon^{cu,+}(x)),$$

the last limit in the Hausdorff metric for compact subspaces of M. Thus, from the construction of $I(z)$,

$$\rho \le \ell(f^n(I(z))) \le \epsilon, \tag{5}$$

for all $n \ge 0$. Moreover, $I(z)$ is tangential to F, part of $W_\epsilon^{cu}(z)$ and therefore an $\epsilon - E$-interval, that is, an interval I such that $I \subset \bigcap_{n\ge 0} f^{-n}(U)$, where U is an admissible neighborhood, $f^n(I)$ is always transversal to the E direction and $\ell(f^n(I)) \le \epsilon$ for all $n \ge 0$; see [**PS**, Definition 2]. Since $I(z) \subset W_\epsilon^{cu}(z)$ and $W_\xi^{cu}(y)$ is a one-dimensional submanifold for all small $\xi > 0$, $I(z) \supset W_\xi^{cu,+}(z)$ for some small ξ. Therefore, for all $n \ge 0$, $\ell(f^n(I(z)))$ is bounded away from zero.

As in [**PS**, §3.2, Proposition 3.1] we obtain that $\omega(z)$ is a periodic orbit or a periodic circle tangential to F. (Although in [**PS**] it is assumed that f is C^2, this is not used in that part of the proof of [**PS**, Proposition 3.1].) As a robustly expansive homoclinic class cannot have a periodic circle or an attracting Cantor set in $H(p)$ included in a periodic circle tangential to F, we conclude that $\omega(z)$ is a periodic orbit. Using this fact together with (5), we arrive at a contradiction reasoning as in Lemma 3.11. □

Let $\nu > 0$ be the number given in Proposition 3.13.

LEMMA 3.15. *For all $0 < \xi \le \nu$, there exists $m_0 = m_0(\xi) > 0$ such that for all $x \in H(p)$, $\ell(f^n(W_\xi^{u,\pm}(x))) \ge \nu$ for all $n \ge m_0$.*

Proof. Given $0 < \xi \le \nu$, arguing as in the proof of Lemma 3.11, we see that, given $x \in H(p)$, there exists $m_0(x, \xi) > 0$ such that

$$\ell(f^{m_0(x,\xi)-j}(W_\xi^{u,+}(x))) \le \epsilon, \quad j = 1, \ldots, m_0(x, \xi),$$

but

$$\ell(f^{m_0}(W_\xi^{u,+}(x))) > \epsilon.$$

Let $\xi \ge c' > 0$ be such that

$$f^{m_0(x,\xi)}(W_{c'}^{u,+}(x)) = W_\epsilon^{u,+}(f^{m_0(x,\xi)}(x)).$$

Hence $\ell(W_\epsilon^{u,+}(f^{m_0(x,\xi)}(x))) > \nu$. For $n = m_0(x, \xi) + 1$ either

$$f^n(W_{c'}^{u,+}(x)) = W_\epsilon^{u,+}(f^n(x)) \quad \text{or} \quad \ell(f^n(W_{c'}^{u,+}(x))) > \epsilon.$$

In the first case $\ell(f^n(W_{c'}^{u,+}(x))) > \nu$ by Proposition 3.13. In the second case it is certainly also true that $\ell(f^n(W_{c'}^{u,+}(x))) > \nu$. In this latter case we choose $0 < c'' < c'$ such that $f^n(W_{c''}^{u,+}(x)) = W_\epsilon^{u,+}(f^n(x))$. By induction on $n \ge m_0(x, \xi)$ we conclude that $\ell(f^n(W_\xi^{u,+}(x))) > \nu$ for all $n \ge m_0(x, \xi)$.

By compactness of $H(p)$ there exists a common m_0 for all $x \in H(p)$. This concludes the proof. □

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3.2. *Proof of Theorem 3.12.* The proof of Theorem 3.12 will be done in several steps. We shall assume in this subsection that $x \in H(p)$ is such that $\ell(f^n(W_\epsilon^{cu,+}(x))) \le \delta_0$ for all $n \ge 0$, where δ_0 is such that the δ_0-neighborhood of $H(p)$ is admissible.

LEMMA 3.16. *There exists $\epsilon > 0$ small such that $W_\epsilon^{cs}(x) \subset W_\epsilon^s(x)$.*

Proof. Let p_k be as in Lemma 3.11. By (4) of that lemma,

$$\prod_{j=1}^{j=n} \|Df/E(f^j(p_k))\| \le \lambda^n \quad \text{for all } n \ge 0.$$

Since $p_k \to x$ the same holds for x. Thus $W_\epsilon^{cs}(x) \subset W_\epsilon^s(x)$. □

Pick some $\gamma \in \mathbb{R}$ such that $0 < \sqrt{\lambda} < \gamma < 1$ and let $0 \le n_1 < \cdots < n_l < \cdots$ be the maximal sequence of hyperbolic times for Df/E along $\mathcal{O}^+(x)$ associated with that choice of γ. Thus, for $i = 1, 2, \ldots,$

$$\prod_{j=n_i}^{n} \|Df/E(f^j(x))\| < \gamma^{n-n_i} \quad \text{for } n \ge n_i.$$

LEMMA 3.17. *Assume that there exists $L > 0$ such that, for all i, $n_{i+1} - n_i \le L$. Then $E/\omega(x)$ is a uniformly contracting sub-bundle.*

Proof. Let $z \in \omega(x)$. Then there exists $n_r \to +\infty$ such that $f^{n_r}(x) \to z$. By the definition of δ_0 we know that if $\operatorname{dist}(y, z) < \delta_0$ then

$$\frac{\|Df/E(z)\|}{\|Df/E(y)\|} \le 1 + c.$$

For $r > 0$ large enough let $J_r > 0$ be such that $\operatorname{dist}(f^j(z), f^j(f^{n_r}(x))) < \delta_0$ for all $j \in [-J_r, J_r]$. Then $J_r \to +\infty$ when $r \to +\infty$. Let $K = \sup\{\|Df_y\| : y \in M\}$. For a given r let $i = i(r)$ be such that $n_i < n_r \le n_{i+1}$. Then $n_{i+1} - n_r < L$ and therefore

$$\begin{aligned}
\prod_{j=0}^{n-1} \|Df/E(f^j(z))\| &\le (1+c)^n \prod_{j=0}^{n-1} \|Df/E(f^{n_r+j}(x))\| \\
&\le (1+c)^n K^{n_{i+1}-n_r} \prod_{j=0}^{n-1-n_{i+1}+n_r} \|Df/E(f^{n_{i+1}+j}(x))\| \\
&\le [(1+c)K]^{n_{i+1}-n_r}(1+c)\lambda_2]^{n-1-n_{i+1}+n_r} \\
&\le [(1+c)K]^L \lambda_3^{n-1-L} \quad \text{for all } n \le J_r.
\end{aligned}$$

Since $J_r \to +\infty$ we conclude that $E/\omega(x)$ is uniformly contracting. □

LEMMA 3.18. *Let $p_k \to x$, $p_k \sim p$ as in Lemma 3.11. Then we may assume that there exists $\gamma > 0$ such that, for a subsequence of $\{p_k\}$, $\ell(W_\epsilon^{u,+}(p_k)) = \epsilon$ and $\ell(W_\epsilon^{u,-}(p_k)) > \gamma$.*

Proof. We already know that $\ell(W_\epsilon^{u,+}(p_k)) = \epsilon$. Let $\gamma' > 0$ be so small that if $x \in H(p)$ and $\ell(W_\epsilon^{u,\pm}(x)) \le \gamma'$ then $\ell(f^{-j}(W_\epsilon^{u,\pm}(x))) \le \epsilon$ for all $j = 0, \ldots, N$, where N is given by Lemma 3.4.

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Assume that the thesis is false and that there is no $\gamma > 0$ such that $\ell(W^{u,-}_\epsilon(p_k)) \geq \gamma$ for some subsequence of $\{p_k\}$. Hence there exists $m_{1,k} > 0$ such that $\ell(f^{-m_{1,k}}(W^{u,-}_\epsilon(p_k))) = \epsilon$ and, since $f^{-j}(W^{u,-}_\epsilon(p_k)) \subset W^{u,-}_\epsilon(f^{-j}(p_k))$ for all $j \geq 0$, then $f^{-m_{1,k}}(W^{u,-}_\epsilon(p_k)) = W^{u,-}_\epsilon(f^{-m_{1,k}}(p_k))$. From the choice of γ', there exists k_0 such that, for $k \geq k_0$, $m_{1,k} > N$. Therefore there can be no (p_k, F) hyperbolic time between $-(m_k + m_{1,k})$ and $-m_k$. For, if there exists a (q_k, F) hyperbolic time between $-m_j$ and $-(m_j + m_{1,k})$, say $-\hat{n}$, then, since $\ell(W^{u,-}_\epsilon(f^{-\hat{n}}(q_k))) \leq \epsilon$ and $f^{-\hat{n}+m_k}(W^{u,-}_\epsilon(f^{-m_k}(q_k))) \subset W^{u,-}_\epsilon(f^{-\hat{n}}(q_k))$, then it cannot occur that $\ell(f^{-m_{1,k}}(W^{u,-}_\epsilon(p_k))) = \epsilon$. Similarly, there can be no (p_k, F) hyperbolic time between $-m_k$ and 0 (recall that $f^{m_k}(p_k) = q_k$ is such that $\ell(W^{u,+}_\epsilon(q_k)) < 1/k$).

Thus there are no (q_k, F) hyperbolic times between $-(m_k + m_{1,k})$ and 0.

For $k > 0$ given, we replace q_k with $p_k = f^{-m_k}(q_k)$, $W^{u,+}_\epsilon(q_k)$ with $W^{u,-}_\epsilon(p_k)$, and m_k with $m_{1,k}$. In this way we may repeat the same arguments given above, obtaining a convergent subsequence from $f^{-m_{1,k}}(p_k)$. Now $\ell(W^{u,-}_\epsilon(f^{-m_{1,k}}(p_k))) = \epsilon$, and if, for some $\gamma > 0$, $\ell(W^{u,+}_\epsilon(f^{-m_{1,k}}(p_k))) \geq \gamma$ for infinitely many k then we are done. Otherwise we may repeat the above arguments and find $m_{2,k}$ such that $\ell(W^{u,+}_\epsilon(f^{-(m_{1,k}+m_{2,k})}(p_k))) = \epsilon$ and try again to find a subsequence from $\{f^{-(m_{1,k}+m_{2,k})}(p_k)\}$ with good sizes for both branches of the local unstable manifolds.

Repeating this reasoning again, in case that in alternate form both unstable branches have small sizes going to 0 with $k \to \infty$, we find $m_{1,k}, m_{2,k}, \ldots, m_{i,k}$ such that $m_{j,k} > N$ for all $j = 1, 2, \ldots, i$ and $m_{j,k} \to \infty$ when $k \to \infty$. Let us set $m_{0,k} = m_k$. Given q_k, there has to exist some $i = i(q)$ such that

$$\ell(W^{u,+}(f^{-(m_{0,k}+m_{1,k}+\cdots+m_{i,k})}(q_k))) \geq \gamma \quad \text{and} \quad \ell(W^{u,-}(f^{-(m_{0,k}+m_{1,k}+\cdots+m_{i,k})}(q_k))) \geq \gamma$$

with at least one of these lengths equal to ϵ. Otherwise we can find i such that $m_{0,k} + m_{1,k} + \cdots + m_{i,k} > N \cdot i > \pi(q_k)$, implying that there are no (q_k, F) hyperbolic times in the whole orbit of q_k, contradicting Lemma 2.1.

Renaming points and putting $p_k = f^{-(m_{0,k}+m_{1,k}+\cdots+m_{i,k})}(q_k)$, we conclude that there exist a sequence of points p_k such that there is a (q_k, E) hyperbolic time for them and both branches of their local unstable manifold have length bounded away from zero. Moreover, one of the branches has length ϵ and there are no (q_k, F) hyperbolic times between $-(m_{0,k} + m_{1,k} + \cdots + m_{i,k})$ and 0. Since there are only two different unstable branches for p_k, we may assume that $\ell(W^{u,+}_\epsilon(p_k)) = \epsilon$. On account of compactness we may assume the sequence $\{p_k\}_{k\in\mathbb{N}}$ converging to a point $x \in H(p)$ as $k \to \infty$ such that $\ell(f^n(W^{u,+}_\epsilon(x))) \leq \epsilon$ for all $n \geq 0$. This concludes the proof of the lemma. □

LEMMA 3.19. *Assume that* $\ell(f^n(W^{cu,+}_\epsilon(x))) \not\to 0$ *when* $n \to +\infty$. *Then* $\omega(x)$ *is a periodic orbit.*

Proof. If $\ell(f^n(W^{cu,+}_\epsilon(x)) \not\to 0$, as in the proof of [**PS**, Proposition 3.1], we may conclude, on account of equation (1), that there is a subsequence $f^{n_j}(W^{cu,+}_\epsilon(x))$ converging to an arc L tangential to F. We remark that the hypothesis of f being of class C^2 is not necessary for this part of the proof given in [**PS**, Proposition 3.1]. Moreover, again by the domination property (equation (1)) and the boundedness of the lengths of the forward iterates of $W^{cu,+}_\epsilon(x)$, if $y \in L$ then $W^s_\epsilon(y)$ has uniform size. Thus, as $W^s_\epsilon(y)$ is tangential

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to E, $W^s_\epsilon(L) = \bigcup_{y\in L} W^s_\epsilon(y)$ is a neighborhood of L. As in [**PS**, p. 989] we may conclude that $\omega(x)$ is a periodic orbit or $\omega(x)$ is contained in a C^1 simple closed curve $\mathcal{C}$ invariant by f^m, some m, which attracts a neighborhood of itself.

If f were C^2 we would conclude, as in [**PS**], that $\omega(x)$ is a periodic orbit or $\omega(x)$ is the whole curve $\mathcal{C}$. As we do not assume that f is C^2 we proceed in a different way to get directly that $\omega(x)$ is a periodic orbit. The proof proceeds by contradiction. Assume that $\omega(x)$ is not a periodic orbit. We already know that $\omega(x) \subset \mathcal{C}$. By the attracting properties of $\mathcal{C}$, derived from equation (1), it follows that any point in a neighborhood of $\mathcal{C}$ is asymptotic to $\mathcal{C}$. But $x \in H(p)$ and therefore its omega-limit set is in $H(p)$. Take a point $z \in H(p)$ such that its forward and backward orbit is dense in $H(p)$. There is a residual set of such points in $H(p)$. Then z has to visit $\mathcal{C}$ because $\omega(x) \subset \mathcal{C}$. But as $\mathcal{C}$ attracts a neighborhood of itself we have that $z \in \mathcal{C}$ and therefore $H(p) \subset \mathcal{C}$. In particular, $p \in \mathcal{C}$ and the C^1 curve $\mathcal{C}$, which is tangential to F and transverse to E, has to self-accumulate (for instance) in p in the disk $W^s_\epsilon(p)$ tangential to E. But this is not possible since $\mathcal{C}$ is locally connected, and we conclude the proof of Lemma 3.19. □

From now until the end of this subsection we assume that

$$\ell(f^n(W^{cu,+}_\epsilon(x))) \to 0, \quad n \to +\infty. \tag{6}$$

As a consequence of Proposition 3.5 and Lemma 3.18, together with the fact that the angle between E and F is uniformly bounded away from 0, we obtain that for all sufficiently large k,

$$W^u(p_k) \cap W^s_\epsilon(x) = \{z_k\} \quad \text{and} \quad W^s(p_k) \cap W^u_\epsilon(x) = \{y_k\}. \tag{7}$$

CLAIM 3.1. $z_k, y_k \in H(p)$.

Proof. Fix k_1 and consider the intersections $z_{k_1,k} = W^u_\epsilon(p_{k_1}) \cap W^s_\epsilon(p_k)$. Then $z_{k_1,k} \to z_{k_1}$ as $k \to \infty$. As $z_{k_1,k} \in H(p)$ we conclude that $z_{k_1} \in H(p)$. In the same way we prove that $y_k \in H(p)$. □

If there are subsequences y_{k_i} and y_{k_j} of y_k with $y_{k_i} \in W^{cu,+}(x)$ for all i and $y_{k_j} \in W^{cu,-}(x)$ for all j, then by Lemma 3.2,

$$\ell(f^n(W^{cu,\pm}_\epsilon(x))) \not\to 0 \quad \text{as } n \to \infty, \tag{8}$$

contradicting equation (6). Assume then that $y_k \in W^{cu,-}_\epsilon(x)$ for all k.

Let $I_0 = [x, y]$ be the segment of maximal length, tangent to F and containing $W^{cu,+}_\epsilon(x)$, such that for all $I'_0 \subset I_0$ we have $\ell(f^n(I'_0)) \le \delta_0$ for all $n \ge 0$.

Then $\ell(f^n(I_0)) \le \delta_0$ for all $n \ge 0$ and $\ell(f^n(I_0)) \to 0$. Indeed, if $\ell(f^n(I_0)) \not\to 0$, Lemma 3.19 implies that $\omega(x)$ is a periodic orbit and we are done. Moreover, for all $J \supset I_0$, $\ell(f^n(J)) \not\to 0$, for otherwise it would also be the case that $\omega(x)$ is a periodic orbit.

Let $0 = j_0 < j_1 < j_2 < \cdots$ be a maximal sequence of hyperbolic times for (x, E). For each j_h consider the maximal interval I_{j_h} tangential to F such that $I_{j_h} \supset f^{j_h}(W^{cu,+}_\epsilon(x))$ and, for all $I'_{j_h} \subset I_{j_h}$, $\ell(f^n(I'_{j_h})) \le \delta_0$ for all $n \ge 0$. Note that $f^{j_{h+1}-j_h}(I_{j_h}) \subset I_{j_{h+1}}$ for all j_h.

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LEMMA 3.20. *Fixing $\delta \le \delta_0/2$, there exists H_0 such that, for all $h \ge H_0$, $\ell(f^n(I_{j_h})) \le \delta$ for all $n \ge 0$ or $\omega(x)$ is a periodic orbit.*

Proof. The proof proceeds by contradiction. If the conclusion does not hold, then since $f^{j_{h'}-j_h}(I_{j_h}) \subset I_{j_{h'}}$ for all $j_{h'} > j_h$, the volume $\mathrm{vol}(W^{cs}_\epsilon(I_{j_h}))$ would exceed ν_0 for some $\nu_0 > 0$. This implies that $W^{cs}_\epsilon(I_{j_h}) \cap W^{cs}_\epsilon(I_{j_{h'}}) \neq \emptyset$ for some $h' > h$. And this implies that $\omega(x)$ is a periodic orbit. □

Lemma 3.20 allows us to assume that $\ell(f^n(I_{j_h})) \le \delta$ for all $n \ge 0$, for all j_h.

Now let $W^{u,+}_\epsilon(p_k) \cap W^s_\epsilon(I_0) = \tilde{\beta}_k$. Observe that $\tilde{\beta}_k$ is an arc tangential to F. Pick an arc β_k, $\beta_k \supset \tilde{\beta}_k$ tangential to F joining z_k to a point $y_k \in W^{cs}_\epsilon(y)$, where y is the end-point of I_0 different from x.

LEMMA 3.21. *$\beta_k \subset W^{u,+}_{\epsilon+\delta}(p_k)$ or $\omega(x)$ is a periodic orbit.*

Proof. If this were not true, there would exist a subsequence of $\{p_k\}$, which we also denote $\{p_k\}$, such that $\ell(f^{-m_k}(\beta_k)) > \delta$ for some m_k but $\ell(f^{-n}(\beta_k)) \le \delta$ for all $0 \le n \le m_k$. Note that $m_k \to \infty$ as $k \to \infty$ because $\ell(f^{-n}(I_0))$ goes to zero as $n \to \infty$. By equation (1),

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| \, \|Df^{-1}/F(f^{j+1}(y_k))\| < \lambda^{m_k+1}. \tag{9}$$

Pick λ_1 such that $0 < \sqrt{\lambda} < \lambda_1 < 1$ and let $N(\sqrt{\lambda}, \lambda_1)$, $c(\sqrt{\lambda}, \lambda_1)$ be the numbers given in Lemma 3.4. Then, for $m_k > N$ there exists a hyperbolic time from $-n_k$ to 0 for E. For from (9),

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| < \sqrt{\lambda}^{m_k+1} \quad \text{or} \quad \prod_{j=-m_k}^{0} \|Df^{-1}/F(f^{j+1}(y_k))\| < (\sqrt{\lambda})^{m_k+1}.$$

But the last possibility cannot hold, otherwise there will be a hyperbolic time for F which will contradict the fact that $\ell(f^{-m_k}(\beta_k)) > \delta$. Thus

$$\prod_{j=-m_k}^{0} \|Df/E(f^j(y_k))\| < (\sqrt{\lambda})^{m_k+1}, \tag{10}$$

which implies the existence of a hyperbolic time $-n_k$ for E. We have that $-m_k \le -n_k < -m_k + N$, otherwise there will be a hyperbolic time for F between $-m_k$ and $-n_k$, and this again contradicts $\ell(f^{-m_k}(\beta_k)) > \delta$. On the other hand, since $y_k \in W^s_\epsilon(I_0)$,

$$\prod_{j=0}^{n} \|Df/E(f^j(y_k))\| < \lambda^{n+1} \quad \text{for all } n \ge 0. \tag{11}$$

Therefore, equations (10) and (11) imply that the ϵ-stable manifold of $f^{-m_k}(\beta_k)$ has volume bounded away from zero. As this holds for all k sufficiently large we eventually have that there exist k, k', $k \neq k'$ such that

$$W^s_\epsilon(f^{-m_k}(\beta_k)) \cap W^s_\epsilon(f^{-m_{k'}}(\beta_{k'})) \neq \emptyset.$$

This implies that $\omega(x)$ is a periodic orbit. So, if $\omega(x)$ is not a periodic orbit then $\beta_k \subset W^{u,+}_{\epsilon+\delta}(p_k)$. □

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Remark 3.22.

(1) Lemma 3.21 implies that if $\omega(x)$ is not a periodic orbit then

$$W^{u,+}_{\epsilon+\delta}(p_k) \cap W^{cs}_{\epsilon}(y) \neq \emptyset.$$

(2) Since $\ell(f^n(\beta_k)) \to 0$ as $n \to \infty$ we obtain from Lemma 3.2 that the arc β_k cannot have points in $H(p)$ different from its end-points.

(3) Since $z_k \in H(p)$ and we cannot have points in $\beta_k \cap H(p)$ different from its end-points we obtain that $W^u(p_k)$ crosses $W^s_\epsilon(I_0)$. To see this, it suffices to pick $f^{\pi(p_k)}(z_k)$ where $\pi(p_k)$ is the period of p_k.

Let y_k, z_k be, as above, the end-points of β_k. Our next target is to prove that $y_k \in H(p)$. For this we orient $W^{u,+}(p_k)$ from p_k and, as noted in Remark 3.22, there are no points of $H(p) \cap \beta_k$ different from its end-points and there are points $w \in H(p) \cap W^{u,+}(p_k)$ with $w > y_k$ in the ordering established. Let $w_k = \inf\{w \in H(p) \cap W^{u,+}(p_k),\ w > y_k\}$. Since $H(p)$ is closed, $w_k \in H(p)$.

LEMMA 3.23. $w_k = y_k$.

Proof. By definition, $w_k \geq y_k$. Assume, by way of contradiction, that $w_k > y_k$. Then $[z_k, w_k] \supset [z_k, y_k] = \beta_k$.

Assume that $\ell(f^n([z_k, w_k])) \leq \delta_0$ for all $n \geq 0$. Since $\ell(f^n([y_k, w_k])) \not\to 0$ we can choose a subsequence $n_j \to \infty$ such that $\ell(f^{n_j}([y_k, w_k])) > \delta$ for all j, for some $0 < \delta < \delta_0$. Let $z = \lim_{j\to\infty} f^{n_j}(y_k)$. As $\ell(f^n([z_k, y_k])) \to 0$ we also have that $z = \lim_{j\to\infty} f^{n_j}(z_k)$ which implies that $z \in H(p)$. Since $\delta < \ell(f^{n_j}([y_k, w_k])) < \delta_0$, $f^{n_j}([y_k, w_k]) \in \mathcal{V}(H(p))$, for all j, where $\mathcal{V}(H(p))$ is an admissible neighborhood of $H(p)$. This implies that $W^s_\epsilon(f^{n_j}(z_k))$ is well defined for all j. Then, for $j > j'$ sufficiently big $W^s_\epsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, w_k)) \neq \emptyset$. On the other hand, by Corollary 3.7, for infinitely many points $p_{k'}$ we have $W^s_\epsilon(f^{n_j}(p_{k'})) \to W^s_\epsilon(f^{n_j}(x))$ and this implies that $W^s_\epsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, w_k)) \in H(p)$. Hence, the definition of w_k implies that $W^s_\epsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([z_k, y_k)) = \emptyset$ and so $W^s_\epsilon(f^{n_j}(z_k)) \cap f^{n_{j'}}([y_k, w_k)) \neq \emptyset$, implying that there exists $u \in H(p) \cap f^{n_{j'}}([y_k, w_k))$, leading to a contradiction with the definition of w_k.

If $\ell(f^n([z_k, w_k])) > \delta_0$ for some $n \geq 0$ then we pick $w_{k(n)} \in [y_k, w_k]$ so that $\ell(f^n([z_k, w_{k(n)}])) \leq \delta_0$ and repeat the same argument replacing w_k by $w_{k(n)}$. □

Since $I_0 = [x, y]$ and $y_k \to y$ we get the following corollary.

COROLLARY 3.24. $y \in H(p)$.

To finish the proof of Theorem 3.12 we prove that $W^u_\epsilon(p_k)$ cuts $W^s_\epsilon(f^n([z_k, y_k]))$ for arbitrarily large n. Recall that, given $\nu > 0$, we say that a subset X of Y is ν-dense in Y if the ν neighborhood of X contains Y.

LEMMA 3.25. *There exists $n_j \to \infty$ with $j \to \infty$ such that*

$$W^s_\epsilon(f^{n_j}([z_k, y_k])) \cap W^u_\epsilon(p_k) \neq \emptyset.$$

Proof. Let us take a fixed multiple $n_0 > 0$ of $\pi(p_k)$ such that $f^{n_0}(W^{u,+}_\epsilon(p_k))$ is ν-dense in $H(p)$ with $\nu > 0$ but $\nu \ll \epsilon$. Since p_k is homoclinically related to p, by the λ-lemma,

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there exists such an n_0. Moreover, we may assume that the end-point e_k of $f^{n_0}(W^{u,+}_\epsilon(p_k))$ different from p_k lies in $W^s_\epsilon(p_k)$ and is at a distance less than ν from p_k. Take a small arc $\gamma(e_k, p_k) \subset W^s_\epsilon(p_k)$ joining e_k with p_k and such that $\gamma(e_k, p_k) \cap f^{n_0}(W^{u,+}_\epsilon(p_k)) = \{e_k, p_k\}$. Joining $f^{n_0}(W^{u,+}_\epsilon(p_k))$ with $\gamma(e_k, p_k)$, we obtain a circle $\mathcal{C}$ and we order it, for instance, with the (unique) order $\prec$ such that $p_k \prec z_k \prec e_k$. Without loss of generality we may assume that for infinitely many multiples n of $\pi(p_k)$, $W^{cs}_\epsilon(f^n(z_k)) \subset W^s_\epsilon(f^n(z_k))$. Now take the arc $[p_k, y_k] \subset W^u(p_k)$ and iterate it by multiples of $f^{\pi(p_k)}$. For the iterates n in which the center-stable manifold of $f^n(z_k)$ is a stable one, and for large multiples of $\pi(p_k)$, since $f^{n_0}(W^{u,+}_\epsilon(p_k))$ is ν dense in $H(p)$, then $W^s_\epsilon(f^n([z_k, y_k]))$ cuts $f^{n_0}(W^{u,+}_\epsilon(p_k))$. By Corollary 3.7 there exists $h_0 > 0$ such that, for all $h > h_0$, $W^{cs}_\epsilon(f^n(p_h)) \subset W^s_\epsilon(f^n(p_h))$ where $p_h \to x$ is the sequence of periodic points given in the proof of Lemma 3.11. Therefore

$$f^{n_0}(W^{u,+}_\epsilon(p_k)) \cap W^s_\epsilon(f^n([z_k, y_k]))$$

defines an arc $I_n = [z_{n,k}, y_{n,k}]$ contained in $f^{n_0}(W^{u,+}_\epsilon(p_k))$ whose end-points are in $H(p)$. Observe that for $n \neq n'$ we cannot have $I_n \cap I_{n'} \neq \emptyset$, otherwise we would obtain that $\omega(x)$ is a periodic orbit. Thus for different iterates $f^n([z_k, y_k])$ we obtain different I_n on the circle $\mathcal{C}$. Therefore there exists n_1 such that (in the order $\prec$ given in $\mathcal{C}$) there are infinitely many I_n preceding I_{n_1}. Iterating by f^{-n_1} we obtain that $W^{u,+}_\epsilon(p_k)$ contains infinitely many intervals I_n. □

COROLLARY 3.26. *$W^u_\epsilon(p_k)$ cuts both $W^s_\epsilon(f^{n_j}(z_k))$ and $W^s_\epsilon(f^{n_j}(y_k))$.*

Returning to the proof of Theorem 3.12, and keeping the previous notation, take $2(\epsilon + \delta) < \alpha$, where α is the expansiveness constant for $f/H(p)$. Then, since $z_{n,k}, y_{n,k} \in W^u_{\epsilon+\delta}(p_k)$, we obtain

$$\operatorname{dist}(f^m(z_{n,k}), f^m(y_{n,k})) \leq \ell(f^m([z_{n,k}, y_{n,k}])) \leq 2(\epsilon + \delta) < \alpha \quad \text{for all } m \leq 0. \tag{12}$$

On the other hand, since $z_{n,k} \in W^s_\epsilon(f^n(x))$ and $y_{n,k} \in W^s_\epsilon(f^n(y))$,

$$\begin{aligned}\operatorname{dist}(f^m(z_{n,k}), f^m(y_{n,k})) &\leq \operatorname{dist}(f^m(z_{n,k}), f^{m+n}(x)) + \operatorname{dist}(f^{m+n}(x), f^{m+n}(y)) + \operatorname{dist}(f^{m+n}(y), f^m(y_{n,k})) \\ &< \epsilon + \delta + \epsilon < \alpha \quad \text{for all } n \geq 0. \end{aligned} \tag{13}$$

Equations (12) and (13) give

$$\operatorname{dist}(f^m(z_{n,k}), f^m(y_{n,k})) \leq \alpha \quad \text{for all } n \in \mathbb{Z},$$

contradicting the expansiveness of $f/H(p)$. Thus if $\omega(x)$ is not periodic we lead to a contradiction.

Taken together, this concludes the proof of Theorem 3.12. □

4. *Proof of hyperbolicity of $H(p)$*

In this section we prove that $(E \oplus F)/H(p)$ is hyperbolic. For this, we shall prove first that E is uniformly contracting, and this is done following Mañé's proof of [**Ma2**, Theorem I.4].

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LEMMA 4.1. *There exist $c > 0$, a positive integer m, and a dense subset $\mathcal{D}$ of $H_r \subset H(p)$ such that, for $x \in \mathcal{D}$,*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^m/E(f^{jm}(x))\| \le -c.$$

Proof. H_r is dense in $H(p)$. By Lemma 2.1, for any periodic point $q \in H_r$,

$$\|Df^{\tau n}|_{E^s(q)}\| \le K\lambda^{\tau n} \quad \text{for all } n \ge 0. \tag{14}$$

Here τ is the period of q and $K > 0$, $0 < \lambda < 1$ are independent of the particular periodic point in H_r. Therefore from [**Ma3**, Lemma II.5] there exist $m > 0$, $C > 0$, and μ, $0 < \lambda \le \mu < 1$ such that

$$\prod_{j=1}^{[\tau/m]} \|Df^m_{f^{jm}(q)}|_{E^s}\| \le C\mu^{[\tau/m]}, \tag{15}$$

whenever (14) holds. Here $[\tau/m]$ is the greatest integer less than or equal to τ/m.

Taking logarithms in (15) gives

$$\frac{1}{[\tau/m]} \sum_{j=1}^{[\tau/m]} \log \|Df^m/E(f^{jm}(q))\| \le \frac{\log C}{[\tau/m]} + \log \mu. \tag{16}$$

Since periodic points in H_r of period τ greater than $m \cdot \log(C^2/\mu)/\log(1/\mu)$ are dense in $H(p)$, inequality (16) and the fact that $0 < \mu < 1$ imply the result. □

Lemma 4.1 ensures that for $x \in \mathcal{D}$, $\prod_{j=1}^{n} \|Df^m/E(f^{-jm}(x))\|$ converges to 0 exponentially fast as $n \to \infty$. Since if $f : H(p) \to H(p)$ is expansive then the same is true for f^m, we may (and do) assume that $m = 1$ in Lemma 4.1.

Let γ_0 be such that $0 < e^{-c} < \gamma_0 < 1$ where $c > 0$ is given by Lemma 4.1. It follows that for all $x \in \mathcal{D}$ there are infinitely many values of n satisfying

$$\prod_{j=1}^{n} \|Df/E(f^j(x))\| < \gamma_0^n.$$

Take γ_1, γ_2, γ_3, γ_4 such that

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1.$$

Let $N_0 = N(\gamma_3, \gamma_4)$ be given by Lemma 3.4.

LEMMA 4.2. *If $E|_{H(p)}$ is not a contracting bundle, then for all $\epsilon > 0$ there exist a compact invariant set $\Lambda(\epsilon) \subset H(p)$ and $N_1 = N_1(\epsilon)$ such that every $x \in \Lambda(\epsilon)$ has the following property: there exist x_0 arbitrarily near x, $n_0 = n_0(x_0, x, \epsilon) \ge 0$ and $y \in \Lambda(\epsilon)$ such that*

(a) $\operatorname{dist}(f^{n_0}(x_0), y) < \epsilon$;

(b) $(y, f^n(y))$ *is an (N, γ_2)-obstruction for all $n \ge N_1$; and*

(c) *if $n_0 > 0$, then $(x_0, f^{n_0}(x_0))$ is a uniform γ_4-string.*

Moreover $\Lambda(\epsilon)$ is the closure of its interior.

Proof. See [**Ma2**, Lemmas II.6 and II.7]. □

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Fix $0 < \gamma < \gamma_0$ and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be as in Lemma 4.2. Choose $k_0 \in (0, 1)$ such that $\gamma < k_0^2\gamma_1$ and $k_0^{-1}\gamma_4 < 1$, that is, $1 > k_0 > \max\{\gamma_4, \sqrt{\gamma/\gamma_1}\}$.

PROPOSITION 4.3. *If E is not contracting, for all $\epsilon > 0$ there exist sequences $\{x_i\} \subset \Lambda(\epsilon/4)$, ($\Lambda(\epsilon)$ as in Lemma 4.2) and $n_i > 0$ such that:*

(a) $\operatorname{dist}(f^{n_i}(x_i), x_{i+1}) < \epsilon$;

(b) $(x_i, f^{n_i}(x_i))$ *is a uniform γ_4-string, but if i is even $(x_i, f^{n_i}(x_i))$ is not a γ_1-string;*

(c) *if $K = \min\{\|Df/E(x)\| ; x \in H(p)\}$ then $\gamma_1^{n_i} K^{n_{i-1}} \geq (k_0\gamma_1)^{n_i+n_{i+1}}$ for all even i;*

(d) *there exist odd numbers k, l with $k > l$ such that $\operatorname{dist}(x_l, x_k) < \epsilon/2$.*

Proof. See [**Ma2**, p. 179] for the proof of (a), (b), (c). Item (d) follows from compactness. □

COROLLARY 4.4. *Given $\epsilon > 0$ there exist $k \geq 2$, a sequence $x_1, x_2, \ldots, x_k$ and uniform γ_4-strings $(x_i, f^{n_i}(x_i))$ that are not γ_1-strings, such that $\operatorname{dist}(f^{n_i}(x_i), x_{i+1}) < \epsilon$ and $\operatorname{dist}(f^{n_k}(x_k), x_1) < \epsilon$ for all $1 \leq i \leq k-1$.*

The following proposition is similar to a lemma proved in [**Li**] (see also [**Ma2**]). The proof there uses the fact that $Df|F$ is uniformly expanding. We use instead the fact that the diameters of the local unstable manifolds are uniformly bounded away from zero (Proposition 3.13).

PROPOSITION 4.5. *Let $(x_i, f^{n_i}(x_i))$, $i = 1, \ldots, k$, be a sequence of uniform γ-strings in $H(p)$ for $0 < \gamma < 1$. Given $\delta > 0$, there exist $\mu = \mu(\delta, \gamma) > 0$ and $N_0 > 0$ such that the following holds.*

Suppose that $\operatorname{dist}(f^{n_i}(x_i), x_{i+1}) < \mu$ for $i = 1, \ldots, k-1$, $\operatorname{dist}(f^{n_k}(x_k), x_1) < \mu$ and $n_1 + \cdots + n_k \geq N_0$. Then there exists a periodic point q of f with period $N = n_1 + \cdots + n_k$ such that $\operatorname{dist}(f^n(q), f^n(x_1)) < \delta$ for $0 \leq n \leq n_1$ and for $1 \leq i \leq k-1$ we have

$$\operatorname{dist}(f^{N_i+n}(q), f^n(x_{i+1})) < \delta$$

for $0 \leq n \leq n_{i+1}$, where $N_i = n_1 + \cdots + n_i$.

Proof. See [**PPV**, Lemma 5.15 and Proposition 5.16]. Although it is assumed there that f is C^2, this is not used in the proof. □

LEMMA 4.6. *Let $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1$ be as in Lemma 4.2. Moreover, let $0 < \gamma < \gamma_0$ and take $0 < k_0 < 1$ such that $\gamma < k_0^2\gamma_1$ and $k_0^{-1}\gamma_4 < 1$. Then the periodic point q given by Lemma 4.5 satisfies*

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^N.$$

Proof. See [**Ma2**, §II, pp. 179–181]. □

PROPOSITION 4.7. *We may construct q given in Proposition 4.5 such that $q \in H(p)$.*

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Proof. By Lemma 4.6, if $\delta > 0$ is small enough we ensure that $Df_q/E(q)$ contracts at a rate similar to that of $Df_{x_j}/E(x_j)$, $j = 1, 2, \ldots, k$, if $\text{dist}(f^j(q), f^j(x_j)) < \delta$ for all $j = 0, \ldots, N$, in the local center-stable manifold of q. More precisely,

$$\gamma^N < \prod_{n=1}^{N} \|Df/E(f^n(q))\|.$$

And for all $j \in [1, N]$,

$$\prod_{n=1}^{j} \|Df/E(f^n(q))\| < (k_0^{-1}\gamma_4)^j.$$

As q is periodic of period N this implies that its center-stable manifold is a stable manifold, of size similar to the center-stable manifold of x_j. Hence the local unstable manifold of a periodic point $q' \in H(p)$ close to x_1 intersects the local stable manifold of q. Therefore $W^u(p)$ accumulates in $W^u(q)$ or there would exist, by [**PS**, Lemma 3.3.1] another periodic point $\hat{q}$ in the unstable separatrix of $W^{cu}_\epsilon(q)$ cutting $W^{cs}_\epsilon(q')$ and between q and $w \in W^{cu}_\epsilon(q) \cap W^{cs}_\epsilon(q')$. Let Q be the periodic point closest to $H(p)$, in the linear ordering from q to w. Then $W^u_\epsilon(Q)$ is accumulated by $W^u(q')$ and therefore also by $W^u(p)$. If, for some $j = 1, \ldots, k$, $i = 0, \ldots, n_j$, $W^{cs}_\epsilon(f^i(x_j))$ is a true stable manifold then we are done. For in that case $W^s(p)$ would accumulate in $W^s(Q)$ and $W^u(p)$ in $W^u(Q)$ respectively and we obtain that $Q \in H(p)$, concluding the proof of Proposition 4.7.

As we cannot assume that, we have to proceed in a different way. Choose $\mathcal{V}_l(H(p))$, $l \geq 1$, a sequence of admissible neighborhoods of $H(p)$ such that

$$\text{Cl}(\mathcal{V}_{l+1}(H(p))) \subset \mathcal{V}_l(H(p)) \quad \text{and} \quad \bigcap_{l \geq 1} \mathcal{V}_l(H(p)) = H(p).$$

For any $\mathcal{V}_l(H(p))$ we may find q_l, like the point Q as above, such that q_l shadows the pseudo-orbit given by a sequence $(x_j, f^{n_j}(x_j))$, $j = 1, \ldots, k_l$, of uniform γ_4-strings that are not γ-strings as in Proposition 4.3. Take an accumulation point x of $\{q_l\}$. If there is a subsequence $\{q_{l_h}\}$ of $\{q_l\}$ such that the periods of q_{l_h} are uniformly bounded then $x \in H(p) \cap \text{Per}(f)$ and x is the desired periodic point. Otherwise the periods of q_l are unbounded and $x \in H(p)$ is a uniform γ_4-string for all $n \geq 0$. Therefore by arguments similar to those used in [**PS**, Corollary 3.3] we obtain a stable manifold for x that the unstable manifold of q_l will cut for $l \geq l_0$. Thus q_l is homoclinically related with p, completing the proof. □

Proof of Theorem B. By Propositions 4.5 and 4.7, and Lemma 4.6, if E is not a contracting sub-bundle then there exists $q \in H(p) \cap \text{Per}(f)$ such that Df_q contracts in E rather weakly in the period, that is, for any $0 < \zeta < 1$ we may find q such that $\zeta^{\pi(q)n} < \|Dg^{\pi(q)n}/E(q)\| < 1$, thus contradicting Lemma 2.1. Therefore E is a uniformly contracting bundle. This in turn implies, using the arguments of [**Ma3**, §II], that F is a uniformly expanding bundle. Thus $E \oplus F$ is a hyperbolic splitting. This implies that the same is true for g C^1-close to f. □

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