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WARING'S PROBLEM WITH PIATETSKI-SHAPIRO NUMBERS

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Abstract. In this paper, we investigate in various ways the representation of a large natural number as a sum of a fixed power of Piatetski-Shapiro numbers, thereby establishing a variant of the Hilbert–Waring problem with numbers from a sparse sequence.

§1. *Introduction.* In 1770, Waring asserted (cf. [14]) without proof that every natural number is the sum of at most four squares, nine cubes, 19 biquadrates, and so on. His assertion was proved in 1909 by Hilbert [6], whose proof implies, for each $k \geqslant 2$, the existence of the *least* number $g(k)$ such that every natural number is the sum of at most $g(k)$ positive kth powers (the exact value of $g(k)$ is now known for any given k). A subsequent problem to which many mathematicians have contributed is that of determining the *least* number $G(k)$ such that every *sufficiently large* integer can be represented as the sum of at most $G(k)$ kth powers of positive integers. It is known that $G(2) = 4$ (Lagrange, 1770) and $G(4) = 16$ [2]. As for other values of k, however, there are only upper bounds available, the trivial one being $G(k) \leqslant g(k)$. Over the years, these bounds have been progressively improved, and today there are considerably smaller upper bounds than $g(k)$ for large values of k.

In this work, motivated by the latter problem, we establish a variant of the Hilbert–Waring theorem with numbers from the set $\mathcal{A}_c$ of *Piatetski-Shapiro*† numbers defined by

$$\mathcal{A}_c = \{\lfloor m^c \rfloor : m \in \mathbb{N}\} \quad (c > 1).$$

More precisely, we study the problem of representing *every* sufficiently large integer $\mathcal{N}$ in the form

$$\mathcal{N} = n_1^k + \cdots + n_s^k, \quad \text{with } n_1, \ldots, n_s \in \mathcal{A}_c; n_i \geqslant 1, \tag{1}$$

for a fixed $k \geqslant 2$ and $c > 1$. In this setting it is natural to ask for the smallest number of summands $s = G_k(c)$ that would be needed for a given c. Indeed, for $k = 1$, this has been done by several authors (see [1, 3, 8, 10, 11]). Unfortunately, for $k \geqslant 2$, we cannot adapt the methods used therein for an *arbitrary* $c > 1$. Instead, given $s = s(k)$, we ask for the largest interval $\mathcal{I} \subset (1, \infty)$ such that

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† Piatetski-Shapiro was the first to prove an analog of the prime number theorem (cf. [9]) for numbers in $\mathcal{A}_c$, hence the name.

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for any fixed $c \in \mathcal{I}$, (1) holds for every large integer $\mathcal{N}$. Our first result in this direction is established by studying the number of such representations.

THEOREM 1. *For $k = 2, 3, 4$, set $t = t(k) = 2^{k-1}$, and for $k > 4$, let t be any integer such that the inequality*

$$\int_0^1 \left| \sum_{n \leqslant X} \mathrm{e}^{2\pi \mathrm{i} \alpha n^k} \right|^{2t} d\alpha < C X^{2t-k+\varepsilon} \tag{2}$$

holds for some constant $C = C(\varepsilon, k, t)$. Then, for any integer $s > 2t$, the number of representations $R_c(\mathcal{N})$ of a positive integer $\mathcal{N}$ as in (1) satisfies the asymptotic relation

$$R_c(\mathcal{N}) \sim \frac{\Gamma(1 + 1/(ck))^s}{\Gamma(s/(ck))} \mathfrak{S}(\mathcal{N}) \mathcal{N}^{s/(ck)-1} \quad (\mathcal{N} \to \infty),$$

where $\mathfrak{S}(\mathcal{N})$ is the singular series in the classical Waring problem, provided c is a fixed number satisfying $1 < c < 1 + (s - 2t)/d_k(s)$ with

$$d_k(s) = \begin{cases} 3s + 4 & \text{if } k = 2, \\ 15s & \text{if } k = 3, \\ 95s + 199 & \text{if } k = 4, \\ 2t\nu_0 + s(\nu_0 - 1) & \text{if } k \geqslant 5, \end{cases} \tag{3}$$

where $\nu_0 = \nu_0(k)$ in the last row is given by (12).

By [**12**, Theorems 4.3 and 4.6] the singular series satisfies $\mathfrak{S}(\mathcal{N}) \asymp 1$ for the values of s given in Theorem 1, which implies the existence of representations for all sufficiently large integers, as desired.

Remark 1. For $2 \leqslant k \leqslant 4$, the smallest exponent satisfying (2) is $t = 2^{k-1}$, and is a result of Hua's inequality. For larger values of k, one can take, for example, $t = \lceil s_1(k)/2 \rceil$ or $t = \lceil s_3(k)/2 \rceil$, where $s_i(k)$ is given by [**19**, Lemmas 10.2 and 10.4]. In particular, these results imply that (2) holds with $t = k^2 - k - 4$ for $k \geqslant 6$ and $2t \geqslant 2k^2 - k^{4/3} + O(k)$ for large k (cf. [**19**, Corollaries 1.6 and 1.7]). Lemmas 10.3 and 10.6 of [**18**] give further refinements of these results, and Theorems 1.3, 1.4 and Corollary 10.5 provide their corresponding upper bounds. It is worth mentioning that these refinements imply that for sufficiently large k, s *in Theorem 1 can be taken as small as* $1.543k^2$. In fact, these new results provide the smallest known values for t when k is small as well. For example, for $k = 5$, $t = 14$ beats the exponent $t = 16$ provided by Hua's inequality.

Theorem 1 is established via the Hardy–Littlewood circle method. As in the classical case, we show in Lemma 8 that major arcs determine the asymptotic behavior. In §3.2, minor arcs are shown to contribute negligibly in comparison

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provided the sequence determined by $\mathcal{A}_c$ is not too sparse. This is done by combining upper bound estimates for the supremum

$$\sup_{\alpha\in\mathfrak{m}}\left|\sum_{\substack{1\leqslant n\leqslant P\\ n\in\mathcal{A}_c}} e^{2\pi i\alpha n^k}\right|, \tag{4}$$

and the integral

$$\int_{\mathfrak{m}}\left|\sum_{\substack{1\leqslant n\leqslant P\\ n\in\mathcal{A}_c}} e^{2\pi i\alpha n^k}\right|^{2t} d\alpha \tag{5}$$

over the minor arcs $\mathfrak{m}$. The exponential sum in (4) is approximated in Lemma 5 by the weighted sum

$$\sum_{n\leqslant P} c^{-1}n^{1/c-1}e^{2\pi i\alpha n^k}.$$

The error in the approximation is handled by the van der Corput method (see Lemma 2) for $k = 2$ and 3, which works only for these two cases. For larger k, after a shifting argument is used in Lemma 5 we apply Weyl-type inequalities given by Lemmas 3 and 4. As for the integral in (5), looking at the main term in the asymptotic relation, one would ideally expect it to be bounded by a constant multiple of $P^{2t/c-k}$, or even $P^{2t/c-k-\varepsilon}$ for some small $\varepsilon > 0$ (at least for c not too large). Unfortunately, we are currently unable to achieve either of these bounds. Instead, we use an *analog of Hua's inequality* in Lemma 10 for $2 \leqslant k \leqslant 4$, which provides a slight saving over Hua's original inequality, thereby yielding a relatively wider range for admissible c. For $k > 4$, we use the estimate in (2). In either case, using these estimates in place of (5) comes at the cost of undesirable restrictions on the range of c, leading to (3) in Theorem 1.

Another way to estimate (5) worth mentioning proceeds by showing that it is bounded by $P^{k(k-1)/2}$ times the number of solutions to the system of equations

$$n_1^j + \cdots + n_t^j = n_{t+1}^j + \cdots + n_{2t}^j, \quad j = 1, \ldots, k, \tag{6}$$

with each $n_i \in \mathcal{A}_c \cap [1, P]$. As part of a rather general conjecture (the *restriction conjecture*), it is claimed that for $t > \frac{1}{2}k(k+1)$, the number of solutions to (6) is bounded by a constant multiple of $P^{t(1+1/c)-k(k+1)/2}$ for any $c \geqslant 1$. Wooley has very recently announced‡ that the conjectured upper bound for (6) holds for $t \geqslant k(k-1)$. Although this would imply the stronger bound $P^{(1+1/c)t-k+\varepsilon}$ for (5), the exponent $t = k(k-1)$ in this case is not small enough (in comparison to those in Remark 1) to produce as large of a range for c as Theorem 1 claims. Hence, this method proves to be weaker unless a different argument making genuine use of minor arcs can be applied to relate (5) to (6), such as the one in the proof of [**17**, Theorem 2.1] that uses the definition of minor arcs and *translation invariance* of integers. In fact, it is this result that ultimately leads

‡ At the *Analytic Number Theory Workshop* held at the University of Oxford during September 28–October 3, 2014.

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to the improvements mentioned in Remark 1. Unfortunately, since the set $\mathcal{A}_c$ does not enjoy this property (translation invariance), we are currently unable to adapt this technique.

In the next theorem, we show that the lower bound demands on the number of variables can be significantly reduced for large k by requiring only the existence of representations for all large $\mathcal{N}$, which in turn increases the admissible range of c.

THEOREM 2. *To every sufficiently large k, there corresponds a positive integer $t_0(k)$ satisfying*

$$t_0 \leqslant \frac{k}{2}\left(\log k + \log\log k + 2 + O\left(\frac{\log\log k}{\log k}\right)\right) \tag{7}$$

such that, for $s > 1$ and $t \geqslant t_0 + 1$, every sufficiently large integer can be represented as in (1) using $s + 2t$ variables from $\mathcal{A}_c$ provided

$$1 < c < 1 + \frac{s}{2t(2\nu_0 - 1) + s(\nu_0 - 1)}. \tag{8}$$

Remark 2. It is assumed that $2t$ of the variables in Theorem 2 are smooth. This in turn allows the use of smooth Weyl sums (which provide an extra saving of order $\log k/k$ compared to ordinary Weyl sums) and their mean values, which are combined in Lemma 11 to yield (7), reducing the number of variables significantly. Unfortunately, since we cannot obtain an estimate for the quantities

$$\sup_{\alpha\in\mathfrak{m}}\left|\sum_{\substack{1\leqslant n\leqslant P\\ n\in\mathcal{A}_c\\ n \text{ smooth}}} e^{2\pi i\alpha n^k}\right| \quad \text{and} \quad \int_{\mathfrak{m}}\left|\sum_{\substack{1\leqslant n\leqslant P\\ n\in\mathcal{A}_c\\ n \text{ smooth}}} e^{2\pi i\alpha n^k}\right|^{2t} d\alpha$$

that is as strong as the one for smooth Weyl sums, when estimating them we are forced to remove the condition that n be in $\mathcal{A}_c$, losing dependency on the sequence. As a result, the only saving on the range of c comes from the reduction on the number of variables; that is, we cannot currently do better than (8). In particular, taking $s = 2$ and $t = t_0 + 1$ gives an interval for c of length

$$\frac{1}{t_0(2\nu_0 - 1) + \nu_0 - 1} \sim \frac{2}{27k^3\log k} \quad (k \to \infty).$$

In the theorem below, instead of all sufficiently large integers, we require that *almost all* integers be represented by (1).

THEOREM 3. *For sufficiently large k, there is an integer $t_0(k) > 0$ satisfying (7) such that, for $s \geqslant 1$ and $t \geqslant \lceil (t_0 + 1)/2\rceil$, almost all integers can be represented as in (1) using $s + 2t$ variables from $\mathcal{A}_c$ provided (8) holds.*

As a final remark, we would like to mention two relevant research problems. One is to consider whether the range of c can be dramatically improved in the theorems above for *almost all* c. The other is to establish a Waring–Goldbach type of result using primes in $\mathcal{A}_c$, which we shall leave to another paper.

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§2. *Preliminaries and Notation.*

Notation. Throughout the paper, we assume that k, m, and n are natural numbers with $k \geqslant 2$, and p denotes a prime number. We write $n \sim N$ to mean that $N < n \leqslant 2N$. Furthermore, $c > 1$ is a fixed real number and we put $\delta = 1/c$.

For any subset $\mathcal{S}$ of integers and a real number x, $\mathcal{S}(x)$ denotes the subset $\mathcal{S} \cap [1, x]$ of $\mathcal{S}$, and $\#\mathcal{S}(x)$ the number of elements of $\mathcal{S}(x)$.

Given a real number x, we write $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$, $\{x\}$ for the fractional part of x, $\lfloor x \rfloor$ for the greatest integer not exceeding x, and $\lceil x \rceil$ for the least integer not smaller than x.

For any function f, we put

$$\Delta f(x) = f(-(x+1)^{\delta}) - f(-x^{\delta}) \quad (x > 0).$$

We recall that for functions F and real non-negative G the statements $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. If $F \geqslant 0$ also, then $F \gg G$ is equivalent to $G \ll F$. We also write $F \asymp G$ to indicate that $F \ll G$ and $G \ll F$. In what follows, any implied constants in the symbols $\ll$ and O may depend on the parameters c, ε, k, s, t, but are absolute otherwise. In a slight departure from convention, we shall frequently use ε (not ϵ) to mean a small positive number, possibly a different one each time.

2.1. *Preliminaries.* The characteristic function of the set $\mathcal{A}_c$ is given by

$$\lfloor -n^{\delta} \rfloor - \lfloor -(n+1)^{\delta} \rfloor = \begin{cases} 1 & \text{if } n \in \mathcal{A}_c, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Putting $\psi(x) = x - \lfloor x \rfloor - 1/2$, we obtain

$$\begin{aligned} \lfloor -n^{\delta} \rfloor - \lfloor -(n+1)^{\delta} \rfloor &= (n+1)^{\delta} - n^{\delta} + \Delta\psi(n) \\ &= \delta n^{\delta-1} + O(n^{\delta-2}) + \Delta\psi(n). \end{aligned} \tag{10}$$

The following result due to Vaaler gives an approximation to $\psi(x)$ (see, for example, [4, Appendix]).

LEMMA 1. *There exists a trigonometric polynomial*

$$\psi^*(x) = \sum_{1 \leqslant |h| \leqslant H} a_h e(hx) \quad (a_h \ll |h|^{-1})$$

such that for any real x,

$$|\psi(x) - \psi^*(x)| \leqslant \sum_{|h| < H} b_h e(hx) \quad (b_h \ll H^{-1}).$$

Next, we state three lemmas needed for exponential sum estimates. The proofs of first two can be found in [4, Theorem 2.8] and [12, Lemma 2.3], respectively.

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LEMMA 2. *Let q be a positive integer. Suppose that f is a real-valued function with $q+2$ continuous derivatives on some interval I. Suppose also that for some $\lambda > 0$ and for some $\alpha > 1$,*

$$\lambda \leqslant |f^{(q+2)}(x)| \leqslant \alpha\lambda$$

on I. Let $Q = 2^q$. Then

$$\sum_{n\in I} e(f(n)) \ll |I|(\alpha^2\lambda)^{1/(4Q-2)} + |I|^{1-1/(2Q)}\alpha^{1/(2Q)} + |I|^{1-2/Q+1/Q^2}\lambda^{-1/(2Q)}$$

where the implied constant is absolute.

LEMMA 3 (Weyl's inequality). *Let k be an integer with $k \geqslant 2$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$ such that $|\alpha_k - a/q| \leqslant q^{-2}$. Then*

$$\sum_{x\leqslant X} e(\alpha_1 x + \cdots + \alpha_k x^k) \ll X^{1+\varepsilon}(q^{-1} + X^{-1} + qX^{-k})^{2^{1-k}}.$$

LEMMA 4. *Let k be an integer with $k \geqslant 4$, and let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that there exists a natural number j with $2 \leqslant j \leqslant k$ such that, for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, one has $|\alpha_j - a/q| \leqslant q^{-2}$ and $q \leqslant X^j$. Then there exists a $\sigma(k)$ such that*

$$\sum_{x\leqslant X} e(\alpha_1 x + \cdots + \alpha_k x^k) \ll X^{1+\varepsilon}(q^{-1} + X^{-1} + qX^{-j})^{\sigma(k)}.$$

Remark 3. According to [19, Theorem 11.1], the exponent $\sigma(k)$ in Lemma 4 can be taken such that $\sigma(k)^{-1} = 2k(k-2)$. This is improved, for $k \geqslant 3$, in [18, Corollary 1.2] to $\sigma(k)^{-1} = 2(k^2 - 3k + 3)$. Better yet, [20, Theorem 1.2] claims that one can take $\sigma(k)^{-1} = 2(k^2 - 3k + 2)$.

LEMMA 5. *For $\delta < 1$,*

$$\sum_{n\in\mathcal{A}_c(P)} e(\alpha n^k) = \sum_{n\leqslant P} \delta n^{\delta-1} e(\alpha n^k) + O(P^{\theta(k)})$$

holds uniformly for $\alpha \in \mathbb{R}$, where

$$\theta(k) = \begin{cases} \dfrac{\delta}{2} + \dfrac{3}{8} & \text{if } k = 2 \text{ and } c < \dfrac{4}{3}, \\ \dfrac{7\delta}{15} + \dfrac{1}{2} & \text{if } k = 3 \text{ and } c < \dfrac{16}{15}, \\ \dfrac{(\delta+1)(\nu_0-1)}{2\nu_0 - 1} & \text{if } k \geqslant 4 \text{ and } c < \dfrac{\nu_0}{\nu_0 - 1}, \end{cases} \tag{11}$$

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and

$$\nu_0(k) = \begin{cases} 2^k(k+2) & \text{if } k = 4, 5, \\ \dfrac{3k+2}{k\sigma(3k/2)} & \text{if } 2 \mid k, k \geqslant 6, \\ \dfrac{3k+1}{(k-1)\sigma((3k-1)/2)} & \text{if } 2 \nmid k, k \geqslant 7, \end{cases} \tag{12}$$

in which $\sigma(k)$ can be taken to be any of the exponents in Remark 3.

Proof. By (9) and (10),

$$\sum_{n \in \mathcal{A}_c(P)} e(\alpha n^k) = \sum_{n \leqslant P} \delta n^{\delta-1} e(\alpha n^k) + \sum_{n \leqslant P} e(\alpha n^k) \Delta\psi(n) + O(\log P).$$

In order to bound the middle term on the right, we divide the range of summation $[1, P]$ into dyadic intervals of the form $(N, 2N]$. Applying Lemma 1 on each such interval, it is easy to show that (cf. [**4**, §4.6])

$$\sum_{n \sim N} e(\alpha n^k) \Delta\psi(n) \ll A(N) + B(N),$$

where

$$A(N) = H_N^{-1} \sum_{|h| < H_N} \left| \sum_{n \sim N} e(hn^\delta) \right|$$

and

$$B(N) = N^{\delta-1} \sum_{1 \leqslant |h| \leqslant H_N} \max_{N < N' \leqslant 2N} \left| \sum_{N < n \leqslant N'} e(\alpha n^k + hn^\delta) \right|. \tag{13}$$

Using the exponent pair $(1/2, 1/2)$ (cf. [**4**, Ch. 3]), we obtain the estimate

$$\sum_{n \sim N} e(hn^\delta) \ll |h|^{1/2} N^{\delta/2} + |h|^{-1} N^{1-\delta} \quad (h \neq 0).$$

From now on, we shall write

$$H_N = N^{1-\delta+\nu}$$

and choose ν optimally. Thus, we have obtained so far that

$$A(N) \ll N H_N^{-1} + H_N^{1/2} N^{\delta/2} + H_N^{-1} \log H_N N^{1-\delta} \ll N^{\delta-\nu} + N^{(1+\nu)/2}.$$

It remains to estimate $B(N)$. Put

$$f(x) = \alpha x^k + hx^\delta.$$

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For $x \sim N$,

$$|f^{(k+1)}(x)| \asymp |h|N^{\delta-k-1} = \lambda,$$

say. For $k = 2$ and $c < 4/3$, we choose $\nu = \frac{1}{2}(\delta - 3/4)$ and apply Lemma 2 with $q = 1$. In this case, we obtain

$$\begin{aligned} B(N) &\ll N^{\delta-1}\sum_{h}(N^{1/2+\delta/6}|h|^{1/6} + N^{3/4} + N^{1-\delta/4}|h|^{-1/4}) \\ &\ll N^{3/4+\nu} = N^{\delta-\nu}. \end{aligned}$$

The case $k = 3$ follows similarly. In either case, combining the estimates for $A(N)$ and $B(N)$ and summing over $N \leqslant P$ yields the desired estimate.

Assume that $k \geqslant 4$ and $N \gg 1$, and put $M = N(|h|N^{\delta})^{-1/(k_0+1)}$, where $k_0 \geqslant k+1$ is an integer to be chosen optimally. For each positive integer m with $m \leqslant M$,

$$\sum_{N<n\leqslant N'} e(f(n)) = \sum_{N<n\leqslant N'} e(f(n+m)) + O(M).$$

Thus, summing over $m \in [1, M]$,

$$\sum_{N<n\leqslant N'} e(f(n)) \ll \frac{1}{M}\sum_{N<n\leqslant N'}\left|\sum_{m\leqslant M} e(f(n+m))\right| + M.$$

Let $R_j(x) = (1+x)^{\delta} - F_j(x)$, where $F_j(x) = \sum_{0\leqslant i\leqslant j}\binom{\delta}{i}x^i$ is the jth Taylor polynomial of $(1+x)^{\delta}$. Then, taking $x = m/n$,

$$\begin{aligned} f(n+m) &= \alpha(n+m)^k + hn^{\delta}(F_{k_0}(m/n) + R_{k_0}(m/n)) \\ &= P_{k_0}(m) + hn^{\delta}R_{k_0}(m/n) \end{aligned}$$

where $P_{k_0}(x)$ is a polynomial of degree k_0 whose $(k+1)$th coefficient is, say, c_{k+1}, where

$$c_{k+1} = hn^{\delta-k-1}\binom{\delta}{k+1}.$$

Noting that $R'_{k_0}(x) \ll |x|^{k_0}$ uniformly for $|x| \leqslant M/N$, we derive by partial integration that

$$\sum_{m\leqslant M} e(f(n+m)) \ll (1 + \underbrace{|h|N^{\delta}(M/N)^{k_0+1}}_{\leqslant 1})\max_{M'\leqslant M}\left|\sum_{1\leqslant m'\leqslant M'} e(P_{k_0}(m'))\right|.$$

Note that

$$\begin{aligned} \sum_{1\leqslant m'\leqslant M'} e(P_{k_0}(m')) &= \int_0^1 \sum_{1\leqslant m\leqslant M} e(P_{k_0}(m)+\gamma m)\sum_{1\leqslant m'\leqslant M'} e(-\gamma m')\,d\gamma \\ &\ll \sup_{\gamma\in[0,1)}\left|\sum_{1\leqslant m\leqslant M} e(P_{k_0}(m)+\gamma m)\right|\log M. \end{aligned}$$

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By conjugating the last sum above if necessary, one can assume that $c_{k+1} > 0$. Note that c_{k+1} has the rational approximation $|c_{k+1} - 1/q| \leqslant q^{-2}$, where $q = \lfloor 1/c_{k+1} \rfloor$. Furthermore, for $n \sim N$, we have $q \asymp N^{k+1-\delta}|h|^{-1}$, and the inequalities

$$q^{-1} < N^{-1} < M^{-1} \ll qM^{-k-1} \ll (|h|N^{\delta})^{(k-k_0)/(1+k_0)} < 1$$

hold. Thus, for $k = 4$ and $k = 5$ taking $k_0 = k + 1$ and applying Lemma 3, and for larger k applying Lemma 4, we derive that

$$\sum_{1 \leqslant m \leqslant M} e(P_{k_0}(m) + \gamma m) \ll M^{1+\varepsilon}(|h|N^{\delta})^{-\nu_0^{-1}}$$

uniformly for *any* $\gamma \in [0, 1)$, where $\nu_0 = 2^k(k + 2)$ for $k = 4, 5$, and

$$\nu_0 = \frac{1 + k_0}{k_0 - k}\sigma(k_0)^{-1}$$

for $k \geqslant 6$. Inserting this bound above then yields

$$\sum_{N < n \leqslant N'} e(f(n)) \ll M + N^{1+\varepsilon}(|h|N^{\delta})^{-\nu_0^{-1}} \ll N^{1+\varepsilon}(|h|N^{\delta})^{-\nu_0^{-1}},$$

since by definition of ν_0 the second term clearly dominates. This leads to the estimate

$$B(N) = N^{\delta-1} \sum_{1 \leqslant |h| \leqslant H_N} N^{1+\varepsilon}(|h|N^{\delta})^{-\nu_0^{-1}} \ll N^{\varepsilon+(1+\nu)(1-\nu_0^{-1})}.$$

It is not too hard to check that for $k \geqslant 6$, choosing

$$k_0 = k_0(k) = \begin{cases} 3k/2 & \text{if } 2 \mid k, k \geqslant 6, \\ (3k-1)/2 & \text{if } 2 \nmid k, k \geqslant 7, \end{cases}$$

gives the optimal value for ν_0 for any choice of $\sigma(k)$ in Remark 3. Choosing

$$\nu = \frac{\nu_0(\delta - 1) + 1}{2\nu_0 - 1}$$

yields under the assumption $c < \nu_0/(\nu_0 - 1)$ that $A(N) + B(N) \ll N^{\delta-\nu}$. Finally, summing over $N \leqslant P$, we obtain the desired result. □

§3. *Proof of Theorem 1.* Let $R_c(\mathcal{N})$ denote the number of representations of a positive integer $\mathcal{N}$ as in (1) for a fixed $k \geqslant 2$ and $c > 1$. Then

$$R_c(\mathcal{N}) = \int_{\mathcal{U}} S(\alpha)^s e(-\alpha\mathcal{N})\, d\alpha,$$

where $\mathcal{U}$ is any interval of unit length and

$$S(\alpha) = \sum_{n \in \mathcal{A}_c(P)} e(\alpha n^k) \quad \text{with } P = \lfloor \mathcal{N}^{1/k} \rfloor.$$

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3.1. *Major arcs.*

Definition 1. For fixed $\eta \in (0, 1)$, define

$$\mathfrak{M}_\eta(a, q) = \{\alpha \in \mathbb{R} : |\alpha - a/q| \leqslant q^{-1} P^{\eta-k}\}.$$

Let $\mathfrak{M}_\eta$ be the union of all $\mathfrak{M}_\eta(a, q)$ where a, q are coprime integers such that $1 \leqslant a \leqslant q \leqslant P^\eta$. Note that the sets $\mathfrak{M}_\eta(a, q)$ are pairwise disjoint and are contained in $\mathcal{U}_\eta = (P^{\eta-k}, 1 + P^{\eta-k}]$.

We recall the following familiar quantities from the classical Waring's problem

$$S(a, b; q) = \sum_{m=1}^{q} e\left(\frac{am^k + bm}{q}\right), \qquad S(a, q) = S(a, 0; q).$$

By [12, Lemmas 4.1, 4.2] the estimates

$$S(a, b, q) \ll q^{1/2+\varepsilon} \gcd(b, q) \quad \text{and} \quad S(a, q) \ll q^{1-1/k} \tag{14}$$

hold for $\gcd(a, q) = 1$.

LEMMA 6. *For $\alpha \in \mathfrak{M}_\eta(a, q)$ with $\gcd(a, q) = 1$ and $1 \leqslant a \leqslant q \leqslant P^\eta$,*

$$T_\delta(\alpha) = v(\alpha - a/q) + O(q^{1/2+2\varepsilon}),$$

where

$$T_\delta(\alpha) = \sum_{n \leqslant P} \delta n^{\delta-1} e(\alpha n^k), \qquad v(z) = q^{-1} S(a, q) I(z),$$

and

$$I(z) = \int_0^{N^{1/k}} \delta x^{\delta-1} e(zx^k)\, dx.$$

Proof. Let $\beta = \alpha - a/q$ and write

$$T_\delta(\alpha) = q^{-1} \sum_{-q/2 < b \leqslant q/2}^{q} S(a, b; q) F(b)$$

where

$$F(b) = \sum_{n \leqslant P} \delta n^{\delta-1} e(\beta n^k - bn/q).$$

Assume that $\beta \neq 0$. Then, for $b = 0$, we use [7, Lemma 8.8] to get

$$F(0) = \int_0^P \delta x^{\delta-1} e(\beta x^k)\, dx + O(1) = I(\beta) + O(1).$$

For $b \neq 0$, and sufficiently large P,

$$\theta \leqslant |k\beta x^{k-1} - b/q| \leqslant 1 - \theta,$$

where $\theta = (2|b| - 1)/(2q)$. Thus, it follows from [7, Corollary 8.11] that $F(b) \ll q/|b|$. The result then follows upon combining these estimates with those given by equation (14). □

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LEMMA 7. *Given $\nu \in (0,1)$, uniformly for $\alpha \in \mathfrak{M}_\eta(a,q)$ with $\gcd(a,q)=1$ and $1 \leqslant a \leqslant q \leqslant P^\eta$, we have*

$$S(\alpha) - v(\alpha - a/q) \ll P^{\delta-\nu} + P^{(1+\eta+3\nu)/2+\varepsilon},$$

provided that $\delta > \max(\eta, \nu)$.

Proof. It follows from the proof of Lemma 5 that

$$S(\alpha) - T_\delta(\alpha) \ll P^{\delta-\nu} + P^{(1+\nu)/2} + \sum_{N=2^l \leqslant P} B(N), \tag{15}$$

where $B(N)$ is given by (13). As we did in the proof of the previous lemma, we can write

$$\sum_{N<n\leqslant N'} e(hn^\delta + \alpha n^k) = q^{-1} \sum_{-q/2<b\leqslant q/2} S(a,b;q) \sum_{N<n\leqslant N'} e(g_b(n)),$$

where $g_b(n) = \beta n^k + hn^\delta - bn/q$. Since $\delta > \eta$, the inequality

$$k(k-1)P^{\eta-\delta}q^{-1} < \delta(1-\delta)/2$$

holds for sufficiently large P. Therefore, for $n \sim N$ and $\alpha \in \mathfrak{M}_\eta(a,q)$,

$$\frac{2^{\delta-2}}{2}\delta(1-\delta)|h|N^{\delta-2} \leqslant |g_b''(n)| \leqslant \frac{3}{2}\delta(1-\delta)|h|N^{\delta-2}.$$

Using Lemma 2 (with $q=0$ and $\lambda = |h|N^{\delta-2}$) then yields

$$\sum_{N<n\leqslant N'} e(hn^\delta + \alpha n^k) \ll q^{1/2+2\varepsilon}(N\lambda^{1/2} + \lambda^{-1/2}).$$

Thus, using the above estimate in (13) and recalling that $q \leqslant P^\eta$, we derive that

$$\begin{aligned}
\sum_N B(N) &\ll q^{1/2+2\varepsilon} \sum_N N^{\delta-1} \sum_h (N^{\delta/2}|h|^{1/2} + |h|^{-1/2}N^{1-\delta/2}) \\
&\ll q^{1/2+2\varepsilon} \sum_N N^{\delta-1}(N^{\delta/2+(3/2)(1-\delta+\nu)} + N^{1-\delta/2+(1/2)(1-\delta+\nu)}) \\
&\ll q^{1/2+2\varepsilon} P^{(1+3\nu)/2} \\
&\ll P^{(1+\eta+3\nu)/2+\varepsilon}.
\end{aligned}$$

Finally, inserting this estimate into (15) and using Lemma 6, the result follows. □

Before the next lemma we define

$$S_m(q) = \sum_{\substack{1\leqslant a\leqslant q \\ (a,q)=1}} (q^{-1}S(a,q))^s e(-ma/q) \quad (s \in \mathbb{N}, m \in \mathbb{Z})$$

and

$$\mathfrak{S}(m) = \sum_{q\geqslant 1} S_m(q).$$

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LEMMA 8. *Assume that* $s \geqslant \max(5, k+2)$. *If* c *satisfies*

$$1 < c < 1 + \min\left(\frac{s-k-1}{k}, \frac{1-\eta(1+10/s)}{1+\eta(1+10/s)}\right), \tag{16}$$

then there is a small positive $\rho = \rho(c, s, k)$ *such that, uniformly for* $1 \leqslant m \leqslant \mathcal{N}$,

$$\int_{\mathfrak{M}_\eta} S(\alpha)^s e(-\alpha m)\, d\alpha = \mathfrak{S}(m) m^{\delta s/k-1} \frac{\Gamma(1+\delta/k)^s}{\Gamma(s\delta/k)} + O(\mathcal{N}^{\delta s/k-1-\rho}).$$

Proof. Given c satisfying (16), there exists an $\epsilon = \epsilon(\delta) > 0$ such that the inequality

$$\delta > \frac{1+\eta(1+10/s)+7\epsilon}{2}$$

holds. Taking $\nu = 2\eta/s + \epsilon$, we see that the inequalities

$$\delta > \frac{1+\eta+5\nu}{2} + \epsilon > \eta > \nu$$

also hold. Thus, by Lemma 7,

$$S(\alpha) - v(\alpha - a/q) \ll P^{\delta-\nu},$$

uniformly for $\alpha \in \mathfrak{M}_\eta(a,q)$ with $(a,q) = 1$ and $1 \leqslant a \leqslant q \leqslant P^\eta$, so that

$$S(\alpha)^s - v(\alpha - a/q)^s \ll P^{s(\delta-\nu)} + P^{\delta-\nu}|v(\alpha - a/q)|^{s-1}.$$

Therefore, for any $m \in \mathbb{Z}$, the contribution from

$$\sum_{q \leqslant P^\eta} \sum_{\substack{a \leqslant q \\ (a,q)=1}} \int_{\mathfrak{M}_\eta(a,q)} (S(\alpha)^s - v(\alpha - a/q)^s) e(-\alpha m)\, d\alpha$$

is

$$\ll P^{\delta s-k-\epsilon} + P^{\delta-\nu} \sum_{q \leqslant P^\eta} \sum_{\substack{a \leqslant q \\ (a,q)=1}} |q^{-1} S(a,q)|^{s-1} \int_{|\beta| \leqslant P^{\eta-k}/q} |I(\beta)|^{s-1}\, d\beta.$$

The estimate $I(\beta) \ll \min(\mathcal{N}^{\delta/k}, |\beta|^{-\delta/k})$, together with [**12**, Lemma 4.9], implies that for $s \geqslant \max(5, k+2)$ and $\delta > k/(s-1)$, the last term is $\ll P^{\delta s-k-2\eta/s}$.

Substituting $\beta = \alpha - a/q$ into the integral in

$$\sum_{q \leqslant P^\eta} \sum_{\substack{a \leqslant q \\ (a,q)=1}} \int_{\mathfrak{M}_\eta(a,q)} v(\alpha - a/q)^s e(-\alpha m)\, d\alpha$$

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yields

$$\sum_{q\leqslant P^{\eta}} S_m(q)\int_{|\beta|\leqslant P^{\eta-k}/q} I(\beta)^s e(-\beta m)\,d\beta.$$

It follows from [12, Lemma 4.8] that extending the range of the last integral to $\mathbb{R}$ introduces an error

$$\begin{aligned}&\ll \sum_{q\leqslant P^{\eta}} |S_m(q)|\int_{\beta>P^{\eta-k}/q} \beta^{-\delta s/k}\,d\beta\\ &\ll |m|^{\varepsilon} P^{\delta s-k+\eta(\varepsilon-1/k+\max(0,1/k+1-\delta s/k))}.\end{aligned}$$

Furthermore, by the same lemma,

$$\int_{\mathbb{R}} I(\beta)^s e(-\beta m)\,d\beta\cdot\sum_{q>P^{\eta}} S_m(q)\ll |m|^{\varepsilon}P^{\delta s-k+\eta(\varepsilon-1/k)}.$$

We have shown so far that for some small $\rho=\rho(\delta,s,k)>0$ and $1\leqslant m\leqslant \mathcal{N}$,

$$\int_{\mathfrak{M}_{\eta}} S(\alpha)^s e(-\alpha m)\,d\alpha=\mathfrak{S}(m)\int_{\mathbb{R}} I(\beta)^s e(-\beta m)d\beta+O(\mathcal{N}^{\delta s/k-1-\rho}).$$

It remains to evaluate the integral above. By making the change of variable $y\mathcal{N}=x^k$ in $I(\beta)$, and then substituting $\gamma=\beta\mathcal{N}$, we obtain

$$\int_{\mathbb{R}} I(\beta)^s e(-m\beta)\,d\beta=(\delta k^{-1})^s\mathcal{N}^{\delta s/k-1}\int_{\mathbb{R}}\left(\int_0^1 x^{\delta/k-1}e(\gamma x)\,dx\right)^s e(-\gamma\theta)\,d\gamma,$$

where $\theta=m\mathcal{N}^{-1}\leqslant 1$. The integral on the right can be written as

$$\begin{aligned}&=\lim_{\lambda\to\infty}\int_{[0,1]^s}(x_1\cdots x_s)^{\delta/k-1}\left(\int_{-\lambda}^{\lambda} e(\gamma(x_1+\cdots+x_s-\theta))\,d\gamma\right)dx_1\cdots dx_s\\ &=\lim_{\lambda\to\infty}\int_{\mathbb{R}}\phi(u)\left(\int_{-\lambda}^{\lambda} e(\gamma(u-\theta))\,d\gamma\right)du,\end{aligned}$$

where

$$\phi(u)=\int\cdots\int_{\substack{x_1,\dots,x_{s-1}\in[0,1]\\ u-1\leqslant\sum x_i\leqslant u}}(x_1\cdots x_{s-1}(u-x_1-\cdots-x_{s-1}))^{\delta/k-1}\,dx_1\cdots dx_{s-1}.$$

By the Fourier integral theorem (cf. [15, §9.7])

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}}\phi(u)\left(\int_{-\lambda}^{\lambda} e(\gamma(u-\theta))\,d\gamma\right)du=\phi(\theta).$$

Upon substituting $x_i=\theta y_i$, $\phi(\theta)$ is given by

$$\theta^{\delta s/k-1}\int\cdots\int_{\substack{y_1,\dots,y_{s-1}\in[0,\theta^{-1}]\\ 0\leqslant\sum y_i\leqslant 1}}(y_1\cdots y_{s-1}(1-y_1-\cdots-y_{s-1}))^{\delta/k-1}\,dy_1\cdots dy_{s-1}.$$

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Thus, using Dirichlet's integral (see, for example, [15, §12.5]),

$$\begin{aligned}\phi(\theta) &= \theta^{\delta s/k-1}\frac{\Gamma(\delta/k)^{s-1}}{\Gamma((s-1)\delta/k)}\int_0^1 (1-x)^{\delta/k-1}x^{(s-1)\delta/k-1}\,dx\\ &= \theta^{\delta s/k-1}\frac{\Gamma(\delta/k)^{s-1}}{\Gamma((s-1)\delta/k)}\frac{\Gamma(\delta/k)\Gamma((s-1)\delta/k)}{\Gamma(s\delta/k)}\\ &= \theta^{\delta s/k-1}\frac{\Gamma(\delta/k)^{s}}{\Gamma(s\delta/k)}.\end{aligned}$$

Inserting this above and using $s\Gamma(s) = \Gamma(1+s)$, we have

$$\int_{\mathbb{R}} I(\beta)^s e(-m\beta)\,d\beta = m^{\delta s/k-1}\frac{\Gamma(1+\delta/k)^s}{\Gamma(s\delta/k)},$$

and the claimed result follows. □

3.2. *Minor arcs.* Put $\mathfrak{m}_\eta = \mathcal{U}_\eta \backslash \mathfrak{M}_\eta$. Recall that from the proof of Lemma 5,

$$S(\alpha) = T_\delta(\alpha) + \sum_{n\leqslant P} e(\alpha n^k)\Delta\psi(n) + O(\log P),$$

By partial summation,

$$\begin{aligned}\sum_{B<n\leqslant P}\delta n^{\delta-1}e(\alpha n^k) &\ll B^{\delta-1}\max_{B<P'\leqslant P}\left|\sum_{B<n\leqslant P'} e(\alpha n^k)\right|\\ &\ll B^{\delta-1}\sup_{\gamma\in[0,1]}\left|\sum_{1\leqslant n\leqslant P} e(\alpha n^k+\gamma n)\right|\log P.\end{aligned}$$

Given $\alpha\in\mathfrak{m}_\eta$, Dirichlet's approximation yields coprime integers a, q with

$$1\leqslant a\leqslant q\leqslant P^{k-\eta},\qquad |\alpha - a/q|\leqslant q^{-1}P^{\eta-k}.$$

Since $\alpha\in\mathfrak{m}_\eta$, by definition $q > P^\eta$. By Lemma 3 the estimate

$$\sum_{n\leqslant P} e(\alpha n^k+\gamma n) \ll P^{1-\eta 2^{1-k}+\varepsilon}$$

holds for $k\geqslant 2$, and uniformly for $\gamma\in[0,1)$. Choosing $B = P^{1-\eta 2^{1-k}}$ yields the bound

$$T_\delta(\alpha) \ll B^\delta + B^{\delta-1}P^{1+\varepsilon-\eta 2^{1-k}} \ll P^{\delta(1-\eta 2^{1-k})+\varepsilon}. \tag{17}$$

If, instead of Lemma 3, we use Lemma 4, then

$$T_\delta(\alpha) \ll P^{\delta(1-\eta\sigma(k))+\varepsilon} \tag{18}$$

uniformly for $\alpha\in\mathfrak{m}_\eta$, which will be used in the proof of Theorem 2 and 3.

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3.2.1. *Small k.* We start with an analog of Hua's inequality. For any arithmetic function f, we define

$$(\Delta_y f)(n) := f(n+y) - f(n),$$
$$(\Delta_{y_v,\dots,y_1} f)(n) := (\Delta_{y_v}\Delta_{y_{v-1},\dots,y_1} f)(n) \quad (v > 1),$$

and

$$S_v(y_1,\dots,y_v;\, f) = \sum_n{}' e(\alpha(\Delta_{y_v,\dots,y_1} f)(n)),$$

where Σ' indicates that the sum runs over n such that $n + \sum \xi_i y_i \in \mathcal{A}_c(P)$ for all 2^v v-tuples $(\xi_1,\dots,\xi_v)$ with $\xi_i = 0, 1$.

LEMMA 9. *For any $k \geqslant 1$, and $v \geqslant 1$,*

$$|S(\alpha)|^{2^v} \ll P^{2^v - v(1-\delta)-1} + P^{2^v-v-1} \operatorname{Re} \sum_{\substack{1\leqslant y_i\leqslant P\\ i=1,\dots,v}} \cdots \sum S_v(y_1,\dots,y_v;\, n^k).$$

Proof. For any arithmetic function f,

$$\left|\sum_{n\in\mathcal{A}_c(P)} e(\alpha f(n))\right|^2 = 2\operatorname{Re}\sum_{1\leqslant y\leqslant P} S_1(y;\, f) + O(P^\delta).$$

Thus, the result follows for $v = 1$ upon taking $f(n) = n^k$. Assuming that the result holds for a certain $v \geqslant 1$, we obtain, by the Cauchy–Schwarz inequality,

$$\begin{aligned}|S(\alpha)|^{2^{v+1}} &\ll P^{2(2^v - v(1-\delta)-1)} + P^{2(2^v-v-1)}\left|\sum_{y_1,\dots,y_v\leqslant P}\cdots\sum S_v(y_1,\dots,y_v;\, n^k)\right|^2\\ &\ll P^{2^{v+1}-2v(1-\delta)-2} + P^{2^{v+1}-(v+1)-1}\sum_{y_1,\dots,y_v\leqslant P}\cdots\sum |S_v(y_1,\dots,y_v;\, n^k)|^2.\end{aligned}$$

The result follows upon noting that

$$|S_v(y_1,\dots,y_v;\, n^k)|^2 = 2\operatorname{Re}\sum_{1\leqslant y_{v+1}\leqslant P} S_{v+1}(y_1,\dots,y_v,y_{v+1};\, n^k) + R,$$

where

$$R \ll \sum_{\substack{n, n+y_i\in\mathcal{A}_c(P)\\ i=1,\dots,v}} 1,$$

and hence

$$\sum_{y_1,\dots,y_v\leqslant P} R \ll (\#\mathcal{A}_c(2P))^{v+1} \ll P^{\delta(v+1)}. \qquad \square$$

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LEMMA 10. *For any fixed $\varepsilon > 0$ and $1 \leqslant v \leqslant k$,*

$$I_v := \int_0^1 |S(\alpha)|^{2^v}\, d\alpha \ll P^{2^v - v + (\delta-1)(v^2-v+2)/2+\varepsilon}.$$

Proof. The result holds for $v = 1$, as $I_1 = \#\mathcal{A}_c(P) \ll P^{\delta}$. Assuming that the result holds for a certain $1 \leqslant v < k$, we derive from Lemma 9 that

$$\begin{aligned} I_{v+1} \ll{}& P^{2^v - v - 1} \sum_{y_1 \leqslant P} \cdots \sum_{y_v \leqslant P} \int_0^1 S_v(y_1, \ldots, y_v; n^k) |S(\alpha)|^{2^v}\, d\alpha \\ &+ P^{2^v - v(1-\delta) - 1} I_v. \end{aligned}$$

The integral above, together with the sums over $y_1, \ldots, y_v$, counts the solutions of

$$(\Delta_{y_v,\ldots,y_1} x^k)(n) + n_1^k + \cdots + n_{2^{v-1}}^k \cdots - m_1^k - m_2^k \cdots - m_{2^{v-1}}^k = 0,$$

where $n_i, m_i \in \mathcal{A}_c(P)$. Since each y_i divides $(\Delta_{y_v,\ldots,y_1} x^k)(n)$ and since this function is strictly increasing, we see that for any given 2^v-tuples of n_i and m_i, there are at most P^{ε} choices for the y_i, and at most one choice of n such that the above equation holds. We thus conclude that

$$\begin{aligned} I_{v+1} &\ll P^{2^v - v(1-\delta) - 1} I_v + P^{2^v - v - 1 + \delta 2^v + \varepsilon} \\ &\ll P^{2^v - v(1-\delta) - 1 + 2^v - v + (\delta-1)(v^2-v+2)/2+\varepsilon} + P^{2^v - v - 1 + \delta 2^v + \varepsilon} \\ &\ll P^{2^{v+1} - (v+1) + (\delta-1)((v+1)^2 - (v+1) + 2)/2 + \varepsilon}, \end{aligned}$$

where the last estimate follows as $\delta < 1$. The bound for I_k is obtained by inserting $v = k - 1$ above. □

We are now ready to prove Theorem 1 in the case $2 \leqslant k \leqslant 4$. By Lemma 5 and equation (17),

$$\sup_{\alpha \in \mathfrak{m}_\eta} |S(\alpha)| \ll P^{\theta(k)} + P^{\delta(1 - \eta 2^{1-k}) + \varepsilon},$$

provided c satisfies the condition given by (11). Writing $\theta(k) = a_k \delta + b_k$ and applying Lemma 10, we derive that for $s > 2^k$,

$$\int_{\mathfrak{m}_\eta} |S(\alpha)|^s\, d\alpha \leqslant \sup_{\alpha \in \mathfrak{m}_\eta} |S(\alpha)|^{s-2^k} \int_0^1 |S(\alpha)|^{2^k}\, d\alpha \ll P^{\delta s - k - \rho} \tag{19}$$

for some small $\rho > 0$, provided c satisfies (11) and $1 < c < 1 + \Delta(s)$ with

$$\Delta(s) = (s - 2^k) \min\left(\frac{\eta 2^{1-k}}{2^k - \Lambda_k}, \frac{1 - a_k - b_k}{(s - 2^k) b_k + 2^k - \Lambda_k} \right)$$

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and $\Lambda_k = (k^2 - k + 2)/2$. Choosing

$$\eta = \frac{2^{k-1}(1 - a_k - b_k)(2^k - \Lambda_k)}{2^k - \Lambda_k + b_k(s - 2^k)}$$

balances the quantities in the definition of $\Delta(s)$ and yields the range given by (3). Furthermore, with this choice of η and assuming that $1 < c < 1 + \Delta(s)$, one can easily check for $2 \leqslant k \leqslant 4$ that both (11) and (16) hold. Therefore, Theorem 1 follows upon combining Lemma 8 and the minor arc estimate (19) in the case $2 \leqslant k \leqslant 4$.

3.2.2. *Large k.* Assume that $k \geqslant 5$. Recall that

$$S(\alpha) = T_\delta(\alpha) + \sum_{n \leqslant P} e(\alpha n^k)\Delta\psi(n) + O(\log P).$$

Combining Lemma 5, (17) and (18), we obtain, for $\alpha \in \mathfrak{m}_\eta$,

$$S(\alpha) \ll P^{\delta(1-\eta\lambda(k))+\varepsilon} + P^{\theta(k)},$$

where $\lambda(5) = 2^{-4}$ and $\lambda(k) = \sigma(k)$ for $k \geqslant 6$, provided that c satisfies (3). Choosing $\eta = 1/4$, we see that the second term dominates. By considering the underlying Diophantine equations, we note that

$$\int_0^1 |S(\alpha)|^{2t}\, d\alpha \leqslant \int_0^1 |T_1(\alpha)|^{2t}\, d\alpha.$$

Therefore, assuming that (2) holds for some t, we derive that

$$\int_{\mathfrak{m}_\eta} |S(\alpha)|^s \leqslant \sup_{\alpha \in \mathfrak{m}_\eta} |S(\alpha)|^{s-2t} \int_0^1 |T_1(\alpha)|^{2t}\, d\alpha \ll P^{\delta s - k - \rho}$$

for some $\rho > 0$, provided that (3) holds. Note that this condition on c implies the one in (3), which in turn implies (16) with our choice of η. Therefore, taking $m = \mathcal{N}$ in Lemma 8 completes the proof of Theorem 1.

§4. *Proof of Theorem 2.* Let $\mathcal{A}_R$ denote the set of R-smooth numbers,

$$\mathcal{A}_R = \{n \in \mathbb{N} : p \mid n \Rightarrow p \leqslant R\},$$

and set $\mathcal{A}_{c,R} = \mathcal{A}_R \cap \mathcal{A}_c$. Let $R_c(\mathcal{N})$ denote the number of representations of a positive integer $\mathcal{N}$ as

$$\mathcal{N} = n_1^k + \cdots + n_s^k + m_1^k + \cdots + m_{2t}^k,$$

where $n_1, \ldots, n_s \in \mathcal{A}_c$ and $m_1, \ldots, m_{2t} \in \mathcal{A}_{c,R}$, so that

$$R_c(\mathcal{N}) = \int_{\mathcal{U}} S(\alpha)^s U(\alpha)^{2t} e(-\alpha\mathcal{N})\, d\alpha,$$

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where $\mathcal{U}$ is any unit interval and

$$S(\alpha) = \sum_{n \in \mathcal{A}_c(P)} e(\alpha n^k), \qquad U(\alpha) = \sum_{n \in \mathcal{A}_{c,R}(P)} e(\alpha n^k).$$

Using Definition 1 (see §3.1), we define the major arcs $\mathfrak{M}_\eta$ with $\eta = 1/4$, fix the unit interval $\mathcal{U}_\eta$, set $\mathfrak{m}_\eta = \mathcal{U}_\eta \backslash \mathfrak{M}_\eta$ and define the corresponding integrals

$$R_{\mathfrak{m}_\eta}(\mathcal{N}) = \int_{\mathfrak{m}_\eta} S(\alpha)^s U(\alpha)^{2t} e(-\alpha\mathcal{N})\, d\alpha$$

and

$$R_{\mathfrak{M}_\eta}(\mathcal{N}) = \int_{\mathfrak{M}_\eta} S(\alpha)^s U(\alpha)^{2t} e(-\alpha\mathcal{N})\, d\alpha.$$

Here, the choice $1/4$ for η is not crucial.

LEMMA 11. *There is a positive integer k_0 such that whenever $k \geqslant k_0$, one can find an integer $t_0(k) > 0$ satisfying (7) and a $\kappa(k) > 0$ such that for $2 \leqslant R \leqslant P^\kappa$, and for any real number t with $t \geqslant t_0(k)$,*

$$\int_0^1 |T(\alpha)|^2 |V(\alpha)|^{2t}\, d\alpha \ll P^{2+2t-k},$$

where

$$T(\alpha) = T_1(\alpha) = \sum_{1 \leqslant n \leqslant P} e(\alpha n^k), \qquad V(\alpha) = \sum_{n \in \mathcal{A}_R(P)} e(\alpha n^k).$$

Proof. Let us denote the integral to be estimated by L. Let $\mathfrak{m}$ denote the set of real numbers α such that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$, and $|\alpha - a/q| \leqslant q^{-1} P^{1-k}$, one has $q > P$. Then, when $\kappa_1 = \kappa_1(\varepsilon, k)$ is a sufficiently small positive number and $2 \leqslant R \leqslant P^{\kappa_1}$, we have by [**16**, Theorem 1.1]

$$\sup_{\alpha \in \mathfrak{m}} |V(\alpha)| \ll_{\varepsilon,k} P^{1-\sigma_k+\varepsilon},$$

where, when k is large, $\sigma_k^{-1} = k(\log k + O(\log\log k))$. Thus, following the proof of [**16**, Theorem 5.1] and the remark that follows it, we conclude that

$$\int_{\mathfrak{m}} |T(\alpha)|^2 |V(\alpha)|^{2t}\, d\alpha \ll P^{2+2t-k-\rho}$$

for some small positive ρ and $R \leqslant P^{\kappa_2}$ for a sufficiently small κ_2, whenever $t \geqslant t_0 = v + \lceil \Delta_{v+1}/(2\sigma_k) \rceil$, where

$$v = \lfloor \tfrac{1}{2} k(\log k + \log\log k + 1) \rfloor \quad \text{and} \quad \Delta_{v+1} \leqslant 1/\log k.$$

This gives the desired upper bound for t_0.

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Using the classical major arc estimate

$$\int_{\mathfrak{M}} |T(\alpha)|^v \, d\alpha \ll P^{v-k} \quad (v \geqslant 2k+1)$$

together with Hölder's inequality, we obtain

$$\begin{aligned} L &\ll P^{2+2t-k} + L^{2t/(2+2t)} \left(\int_{\mathfrak{M}} |T(\alpha)|^{2+2t} \, d\alpha \right)^{2/(2+2t)} \\ &\ll P^{2+2t-k} + L^{2t/(2+2t)} (P^{2+2t-k})^{2/(2+2t)}, \end{aligned}$$

for sufficiently large k, from which one can derive the claimed upper bound for L. □

LEMMA 12. *Let t_0, k_0 and κ be as in Lemma 11. Assume that $k \geqslant k_0$, $s \geqslant 1$, and $t \geqslant t_0 + 1$. If c satisfies (8), then there is a small number ρ depending on c and k such that*

$$R_{\mathfrak{m}_\eta}(\mathcal{N}) \ll \mathcal{N}^{\delta(s+2t)/k-1-\rho}.$$

Proof. We have

$$R_{\mathfrak{m}_\eta}(\mathcal{N}) \leqslant \sup_{\alpha \in \mathfrak{m}_\eta} |S(\alpha)|^s \int_{\mathfrak{m}_\eta} |U(\alpha)|^{2t} \, d\alpha.$$

By replacing the integral above by one over $[0, 1]$ and interpreting the result in terms of the underlying Diophantine equation and then using Lemma 11, we see that

$$\int_{\mathfrak{m}_\eta} |U(\alpha)|^{2t} \, d\alpha \leqslant \int_0^1 |T(\alpha)|^2 |V(\alpha)|^{2t-2} \, d\alpha \ll P^{2t-k}$$

whenever $t \geqslant t_0 + 1$. For $\alpha \in \mathfrak{m}_\eta$, equation (18) and Lemma 5 yield

$$S(\alpha) \ll P^{\delta(1-\eta\sigma(k))+\varepsilon} + P^{(\delta+1)((\nu_0-1)/(2\nu_0-1))+\varepsilon}.$$

By our choice of η, we see that the second term dominates so that

$$R_{\mathfrak{m}_\eta}(\mathcal{N}) \ll P^{s(\delta+1)((\nu_0-1)/(2\nu_0-1))+\varepsilon+2t-k}.$$

This implies that $R_{\mathfrak{m}_\eta}(\mathcal{N}) \ll P^{(s+2t)\delta-k-\rho}$ for some $\rho > 0$ provided c satisfies (8), thereby establishing the claimed result. □

Next, we deal with $R_{\mathfrak{M}_\eta}(\mathcal{N})$ using the pruning method in [**13**, §5]. Let

$$\mathfrak{N}(a, q) = \{\alpha : |\alpha - a/q| \leqslant q^{-1} W P^{-k}\}.$$

We shall assume in what follows that W is a suitable power of $\log P$. Put $\mathfrak{N}$ for the union of the $\mathfrak{N}(a, q)$ for $1 \leqslant a \leqslant q \leqslant W$ with $(a, q) = 1$. Note that $\mathfrak{N}(a, q) \subset \mathfrak{M}_\eta(a, q)$ and $\mathfrak{N} \subset \mathfrak{M}_\eta$.

Write $R_{\mathfrak{M}_\eta} = R_{\mathfrak{N}} + R_{\mathfrak{M}_\eta \setminus \mathfrak{N}}$, where

$$R_{\mathfrak{M}_\eta \setminus \mathfrak{N}}(\mathcal{N}) = \int_{\mathfrak{M}_\eta \setminus \mathfrak{N}} S(\alpha)^s U(\alpha)^{2t} e(-\alpha\mathcal{N}) \, d\alpha.$$

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LEMMA 13. *Assume that $s \geqslant 2$ and $t \geqslant t_0+1$, where t_0 is given by Lemma 11. If c satisfies (8) and (16) (in which s is to be replaced by $2+2t$), then*

$$R_{\mathfrak{M}_\eta \backslash \mathfrak{N}}(\mathcal{N}) \ll P^{(2t+s)\delta-k} W^{-\lambda}$$

for some $\lambda > 0$.

Proof. For any c satisfying (16), there exists an $\epsilon = \epsilon(c) > 0$ such that for any $\alpha \in \mathfrak{M}_\eta(a, q)$,

$$|S(\alpha)|^{2+2t} \ll |v(\alpha - a/q)|^{2+2t} + P^{\delta(2+2t)-2\eta-\epsilon}.$$

This implies, upon using [**12**, Lemma 4.9], that

$$\int_M |S(\alpha)|^{2+2t}\, d\alpha \ll P^{(2t+2)\delta-k-\epsilon} + P^{(2+2t)\delta-k} W^{-\lambda} \ll P^{(2+2t)\delta-k} W^{-\lambda} \quad (20)$$

for some $\lambda > 0$ if $M = \mathfrak{M}_\eta \backslash \mathfrak{N}$, and for $\lambda = 0$ if $M = \mathfrak{M}_\eta$. Put

$$L = \int_0^1 |S(\alpha)|^2 |U(\alpha)|^{2t}\, d\alpha.$$

By Hölder's inequality and Lemma 12,

$$L \ll P^{(2+2t)\delta-k-\rho} + \left(\int_{\mathfrak{M}_\eta} |S(\alpha)|^{2+2t}\, d\alpha\right)^{2/(2+2t)} \left(\int_{\mathfrak{M}_\eta} |U(\alpha)|^{2+2t}\, d\alpha\right)^{2t/(2+2t)}.$$

Extending the range of the last integral to $[0, 1]$ and interpreting the result in terms of the underlying Diophantine equation, we see that it is bounded by L. We thus conclude that $L \ll P^{(2+2t)\delta-k}$. By the trivial estimate $S(\alpha) \ll P^\delta$, together with Hölder's inequality and (20), we derive that

$$\begin{aligned} R_{\mathfrak{M}_\eta \backslash \mathfrak{N}}(\mathcal{N}) &\ll P^{(s-2)\delta} \left(\int_{\mathfrak{M}_\eta \backslash \mathfrak{N}} |S(\alpha)|^{2+2t}\, d\alpha\right)^{2/(2+2t)} L^{2t/(2+2t)} \\ &\ll P^{(s+2t)\delta-k} W^{-\lambda'} \end{aligned}$$

for some *positive* λ', which proves the claim. □

LEMMA 14. *For any $c \in (1, 8/7)$, there is a $\kappa(c) > 0$ such that for each $\alpha \in \mathfrak{N}(a, q)$, where $1 \leqslant a \leqslant q \leqslant W$ with $(a, q) = 1$, and $R = P^{\kappa'}$ with $0 < \kappa' \leqslant \kappa(c)$, we have*

$$U(\alpha) = w(\alpha - a/q) + O\left(\frac{W P^\delta}{\log P}\right)$$

where $w(z) = q^{-1} S(a, q) J(z)$, with

$$J(z) = \int_R^{\mathcal{N}^{1/k}} \delta x^{\delta-1} e(zx^k) \rho\left(\frac{\log x}{\log R}\right) dx,$$

in which ρ is Dickman's function.

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Proof. Write

$$U(\alpha) = \sum_{n\in\mathcal{A}_R(P)} e(\alpha n^k)(\delta n^{\delta-1} + \Delta\psi(n) + O(n^{\delta-2})).$$

Put $\beta = \alpha - a/q$. Then

$$\sum_{n\in\mathcal{A}_R(P)} \delta n^{\delta-1} e(\alpha n^k) = \sum_{r=1}^{q} e(ar^k/q) \sum_{\substack{R<n\in\mathcal{A}_R(P)\\ n\equiv r \bmod q}} \delta n^{\delta-1} e(\beta n^k) + O(R^\delta).$$

For $R < x \leqslant P$,

$$\left| \sum_{\substack{n\in\mathcal{A}_R(x)\\ n\equiv r \bmod q}} 1 - \frac{1}{q}\#\mathcal{A}_R(x) \right| \leqslant E(x),$$

where $E(x)$ is the number of integers $1 \leqslant n \leqslant x$ that are coprime with primes not exceeding R. By [5, Theorem 2.2],

$$E(x) \ll x \prod_{p<R} (1 - 1/p) \ll \frac{x}{\log R}.$$

Also (cf. [12, Lemma 5.3])

$$\#\mathcal{A}_R(x) = x\rho\left(\frac{\log x}{\log R}\right) + O\left(\frac{x}{\log x}\right),$$

uniformly for $R \leqslant x \leqslant R^\lambda$ for any fixed $\lambda > 1$. Thus, applying partial integration,

$$\begin{aligned}\sum_{\substack{R<n\in\mathcal{A}_R(P)\\ n\equiv r \bmod q}} \delta n^{\delta-1} e(\beta n^k) &= \int_R^P \delta x^{\delta-1} e(\beta x^k)\, d(q^{-1}\#\mathcal{A}_R(x) + O(E(x)))\\ &= q^{-1}\int_R^P \delta x^{\delta-1} e(\beta x^k)\rho\left(\frac{\log x}{\log R}\right) dx + O\left(\frac{WP^\delta}{\log P}\right),\end{aligned}$$

which gives the stated main term.

As in the proof of Lemma 5,

$$\sum_{n\in\mathcal{A}_R(P)} e(\alpha n^k)\Delta\psi(n) \ll \sum_{qP^{1/3}<N=2^l\leqslant P} (A(N) + B(N)) + O(P^\delta/\log P),$$

where

$$\begin{aligned} A(N) &\ll NH_N^{-1} + H_N^{1/2}N^{\delta/2},\\ B(N) &= N^{\delta-1}\sum_{1\leqslant|h|\leqslant H_N} \max_{N<N'\leqslant 2N} \left| \sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R}} e(\alpha n^k + hn^\delta) \right|. \end{aligned}$$

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Put $\beta = \alpha - a/q$, and note that

$$\sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R\\ n\equiv r \bmod q}} e(\beta n^k + hn^\delta) \ll (1+|\beta|N^k) \max_{N<N'\leqslant 2N} \Bigg| \sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R\\ n\equiv r \bmod q}} e(hn^\delta)\Bigg|.$$

Suppose that $\gcd(r,q)=d$. Using Dirichlet characters modulo q/d,

$$\sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R\\ n\equiv r \bmod q}} e(hn^\delta) = \varphi(q/d)^{-1} \sum_{\chi \bmod q/d} \chi(dr^{-1}) \sum_{\substack{M<n\leqslant M'\\ n\in\mathcal{A}_R}} \chi(n) e(Dn^\delta),$$

where $D = hd^\delta$, $M = N/d$, and $M' = N'/d$. Suppose that $R \leqslant P^{1/3}$ and K is a number satisfying $R \leqslant K < M$. Note that this is possible since $P^{1/3} < M \leqslant P$. By [13, Lemma 10.1] it follows that

$$\sum_{\substack{M<n\leqslant M'\\ n\in\mathcal{A}_R}} \chi(n) e(Dn^\delta) = \sum_{p\leqslant R} \sum_{\substack{K/p<v\leqslant K\\ P^-(v)\geqslant p\\ v\in\mathcal{A}_R}} \sum_{\substack{u\\ M<uvp\leqslant M'\\ u\in\mathcal{A}_p}} \chi(puv) e(Dp^\delta u^\delta v^\delta),$$

where $P^-(n)$ denotes the smallest prime factor of n. The sums over u and v may be split into $\ll \log^2 M$ bilinear sums of the form

$$\mathcal{S}_p(X,Y) = \sum_{v\sim X} \sum_{u\sim Y} a_u b_v e(Dp^\delta u^\delta v^\delta) \quad (|a_u|, |b_v| \leqslant 1)$$

with

$$K/p \leqslant X \leqslant K, \qquad M/pK \leqslant Y \leqslant M'/K, \qquad XY \asymp M/p.$$

Using the method in the proof of [4, Lemma 4.13] with exponent pair $(1/2, 1/2)$, it follows that

$$\mathcal{S}_p(X,Y)\log^{-1} M \ll \frac{M^{1-\delta/2}}{pD^{1/2}} + \frac{M}{p^{1/2}K^{1/2}} + \frac{D^{1/6}M^{\delta/6+2/3}K^{1/6}}{p^{2/3}}.$$

We sum over X, Y, and p and then choose

$$K = M^{1/2-\delta/4}R^{1/4}D^{-1/4}, \quad R = P^{\kappa'},$$

where $\kappa' > 0$ is taken sufficiently small so as to have $R \leqslant K < M$. Recalling the definitions of M and D, we derive that for $1 \leqslant r \leqslant q$ with $(r,q) = d$,

$$\sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R\\ n\equiv r \bmod q}} e(hn^\delta) \ll (h^{-1/2}d^{-1}N^{1-\delta/2} + P^{3\kappa'/8}d^{-3/4}N^{3/4+\delta/8}h^{1/8})\log^4 N.$$

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Summing over such r with $(r,q)=d$ and then over the positive divisors d of q, we arrive at

$$\sum_{\substack{N<n\leqslant N'\\ n\in\mathcal{A}_R}} e(hn^\delta+\alpha n^k)\ll W(h^{-1/2}N^{1-\delta/2}+P^{3\kappa'/8}N^{3/4+\delta/8}h^{1/8})\log^4 N.$$

Therefore,

$$B(N)\ll W(H^{1/2}N^{\delta/2}+P^{3\kappa'/8}N^{-1/4+9\delta/8}H^{9/8})\log^4 N.$$

Choosing $H_N=N^{1-\delta}\log^2 N$ and summing over $N=2^l$ with $qP^{1/3}<N\leqslant P$, we conclude that for fixed $c\in(1,8/7)$, there is a sufficiently small $\kappa(c)>0$ such that for $\kappa'\leqslant\kappa(c)$,

$$\sum_{n\in\mathcal{A}_R(P)} e(\alpha n^k)\Delta\psi(n)\ll P^\delta/\log P,$$

which completes the proof. □

LEMMA 15. *For any $c\in(1,2)$, and any $\alpha\in\mathfrak{N}(a,q)$, $1\leqslant a\leqslant q\leqslant W$ with $(a,q)=1$,*

$$S(\alpha)=v(\alpha-a/q)+O\left(\frac{P^\delta}{\log P}\right).$$

Proof. This follows easily by modifying Lemma 7 and is therefore omitted. □

LEMMA 16. *Assume that $s+2t>2k$, $c\in(1,8/7)$, and $\kappa=\kappa(c)$ is given Lemma 14. Let $R=P^{\kappa'}$ with $0<\kappa'\leqslant\kappa$ such that $W\leqslant(\log P)^\lambda\leqslant R\leqslant P^\kappa$. There exist positive constants A, B such that for any $1\leqslant m\leqslant\mathcal{N}$,*

$$R_{\mathfrak{N}}(m)=\mathfrak{S}(m)\mathcal{I}(m)+O\left(P^{(s+2t)\delta-k}\left(\frac{W^A}{\log P}+W^{-B}\right)\right),$$

where

$$\mathcal{I}(m)=\int_{\mathbb{R}} I(\beta)^s J(\beta)^{2t}e(-\beta m)\,d\beta,\qquad \mathfrak{S}(m)=\sum_{q\geqslant 1} S_m(q).$$

Proof. Combining Lemmas 14 and 15 and using the trivial estimates $I,J\ll P^\delta$, $|S(a,q)|\leqslant q$, we obtain for any $\alpha\in\mathfrak{N}(a,q)$, where $1\leqslant a\leqslant q\leqslant W$ with $(a,q)=1$,

$$S(\alpha)^sU(\alpha)^{2t}=v(\alpha-a/q)^sw(\alpha-a/q)^{2t}+O\left(\frac{P^{(2t+s)\delta}W}{\log P}\right),$$

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which yields for any $m \in \mathbb{N}$,

$$R_{\mathfrak{N}}(m) = \sum_{q \leqslant W} S_m(q) \int_{|\beta| \leqslant WP^{-k}/q} I(\beta)^s J(\beta)^{2t} e(-\beta m)\, d\beta + O\left(\frac{P^{(2t+s)\delta - k} W^3}{\log P}\right),$$

where

$$S_m(q) = \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} (q^{-1} S(a,q))^{s+2t} e(-ma/q).$$

Using the bound (14) for $q^{-1}S(a,q)$ in completing the integral to $\mathbb{R}$ and then the sum over q to $\mathfrak{S}(m)$ produces the claimed error terms under the conditions stated in the lemma. Thus, the result follows. □

Taking $m = \mathcal{N}$ in Lemma 16, combining it with Lemma 13 and choosing W appropriately, we derive that

$$R_{\mathfrak{M}_\eta}(\mathcal{N}) = \mathfrak{S}(\mathcal{N})\mathcal{I}(\mathcal{N}) + O\left(\frac{P^{(s+2t)\delta - k}}{(\log P)^\lambda}\right) \tag{21}$$

for some positive number λ, whenever c satisfies (16), $s > 1$, and $t \geqslant t_0 + 1$ for $k \geqslant k_0$. We assume that k is large enough to have $s + 2t \geqslant 4k$, in which case $\mathfrak{S}(\mathcal{N}) \gg 1$.

Next, we shall prove that $\mathcal{I}(\mathcal{N}) \gg P^{(s+2t)\delta - k}$. Making the change of variable $y\mathcal{N} = x^k$ in $I(\beta)$ and $J(\beta)$, and then substituting $\gamma = \beta\mathcal{N}$ in $\mathcal{I}(m)$, we see that

$$\mathcal{I}(m) = (\delta/k)^{s+2t} \mathcal{N}^{(s+2t)\delta/k - 1} \lim_{\lambda \to \infty} \int_{\mathbb{R}} \phi(u) \left(\int_{-\lambda}^{\lambda} e(\gamma(u - m\mathcal{N}^{-1}))\, d\gamma \right) du,$$

where

$$\phi(u) = \int \cdots \int_{\substack{y_1, \ldots, y_{2t} \in [\theta, 1] \\ y_{2t+1}, \ldots, y_{2t+s-1} \in [0,1] \\ u - 1 \leqslant \sum y_i \leqslant u}} F(y_1, \ldots, y_{2t+s-1}, u - \Sigma_i y_i)\, dy_1 \cdots dy_{2t+s-1}$$

with

$$F(y_1, \ldots, y_{2t+s}) = \prod_{i=1}^{2t+s} y_i^{\delta/k - 1} \prod_{i=1}^{2t} \rho\left(\frac{\log(\mathcal{N} y_i)}{\log R}\right)$$

and $\theta = R^k/\mathcal{N}$. By the Fourier integral theorem, the limit above equals $\phi(m\mathcal{N}^{-1})$ so that

$$\mathcal{I}(m) = (\delta/k)^{s+2t} \mathcal{N}^{(s+2t)\delta/k - 1} \phi(m\mathcal{N}^{-1}). \tag{22}$$

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Since ρ is decreasing, positive and $R = P^{\kappa}$ for a suitable $\kappa > 0$,

$$\rho\left(\frac{\log(\mathcal{N}y)}{\log R}\right) \geqslant \rho(k/\kappa) > 0$$

for $y \in [\theta, 1]$. Therefore, taking $m = \mathcal{N}$, we conclude that

$$\mathcal{I}(\mathcal{N}) \gg \mathcal{N}^{(s+2t)\delta/k-1}.$$

We are now ready to prove Theorem 2. We first observe that the condition (8) on c implies the one in (16) for our choice of η. Thus, upon combining Lemma 12 with (21), we obtain under the above assumptions that

$$R_c(\mathcal{N}) \gg \mathcal{N}^{(s+2t)\delta/k-1}$$

for sufficiently large integers $\mathcal{N}$, as desired.

§5. *Proof of Theorem 3.* Given $\mathcal{N} \in \mathbb{N}$, consider

$$R_c(n) = \int_{\mathcal{U}} S(\alpha)^s U(\alpha)^{2t} e(-\alpha n)\, d\alpha \quad (1 \leqslant n \leqslant \mathcal{N})$$

where $S(\alpha)$ and $U(\alpha)$ are defined in §4. Using the same $\mathfrak{M}_\eta$, $\mathcal{U}_\eta$, $\mathfrak{m}_\eta$, and $\mathfrak{N}$ in §4, we write $R_c(n) = R_{\mathfrak{M}_\eta \setminus \mathfrak{N}} + R_{\mathfrak{m}_\eta}(n) + R_{\mathfrak{N}}(n)$. By Bessel's inequality it follows that

$$\begin{aligned}\sum_{n \leqslant \mathcal{N}} &|R_c(n) - R_{\mathfrak{N}}(n)|^2 \\ &\leqslant \int_{\mathfrak{m}_\eta} |S(\alpha)|^{2s}\, |U(\alpha)|^{4t}\, d\alpha + \int_{\mathfrak{M}_\eta \setminus \mathfrak{N}} |S(\alpha)|^{2s}\, |U(\alpha)|^{4t}\, d\alpha. \qquad (23)\end{aligned}$$

Assume that $t \geqslant \lceil (t_0+1)/2 \rceil$, where t_0 is defined in Lemma 11, $s \geqslant 1$, and c satisfies (8) and (16) (where s is to be replaced by $2 + 2t$). Following the proof of Lemma 13, we obtain

$$\int_{\mathfrak{M}_\eta \setminus \mathfrak{N}} |S(\alpha)|^{2s} |U(\alpha)|^{4t} \ll \mathcal{N}^{2\delta(s+2t)/k-1} (\log \mathcal{N})^{-2C} \qquad (24)$$

for some constant $C > 0$. Furthermore, arguing as in Lemma 12, we conclude that

$$\begin{aligned}\int_{\mathfrak{m}_\eta} |S(\alpha)|^{2s} |U(\alpha)|^{4t}\, d\alpha &\ll P^{2s(\delta+1)(\nu_0-1)/(2\nu_0-1)+4t-k+\varepsilon} \\ &\ll \mathcal{N}^{2\delta(s+2t)/k-1} (\log \mathcal{N})^{-2C}. \qquad (25)\end{aligned}$$

By Lemma 16, there exists a constant C_1 such that

$$R_{\mathfrak{N}}(n) = \mathfrak{S}(n)\mathcal{I}(n) + O\left(\frac{\mathcal{N}^{\delta(s+2t)/k-1}}{(\log \mathcal{N})^{C_1}}\right)$$

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for all $1 \leqslant n \leqslant \mathcal{N}$. Furthermore, for large $n \leqslant \mathcal{N}$, one can prove using (22) that

$$\mathcal{I}(n) \gg n^{\delta(s+2t)/k-1}.$$

Therefore, combining last two results, we conclude that there is a sufficiently small constant $C_2 > 0$ such that for $\mathcal{N}/(\log \mathcal{N})^{C_2} < n \leqslant \mathcal{N}$,

$$R_{\mathfrak{N}}(n) \gg \frac{\mathcal{N}^{\delta(s+2t)/k-1}}{(\log \mathcal{N})^{C/2}}. \tag{26}$$

Finally, let E be the exceptional set of integers that *cannot* be represented as a sum of $s + 2t$ positive kth powers of members of $\mathcal{A}_c$. Combining equations (23)–(26), we conclude that $\#E(\mathcal{N}) = o(\mathcal{N})$, thereby proving Theorem 3 under the conditions stated on the relevant parameters.

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