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LIMIT THEOREMS OF OCCUPATION TIMES OF NORMALIZED BINARY CONTACT PATH PROCESSES ON LATTICES

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Abstract

The binary contact path process (BCPP) introduced in Griffeath (1983) describes the spread of an epidemic on a graph and is an auxiliary model in the study of improving upper bounds of the critical value of the contact process. In this paper, we are concerned with limit theorems of the occupation time of a normalized version of the BCPP (NBCPP) on a lattice. We first show that the law of large numbers of the occupation time process is driven by the identity function when the dimension of the lattice is at least 3 and the infection rate of the model is sufficiently large conditioned on the initial state of the NBCPP being distributed with a particular invariant distribution. Then we show that the centered occupation time process of the NBCPP converges in finite-dimensional distributions to a Brownian motion when the dimension of the lattice and the infection rate of the model are sufficiently large and the initial state of the NBCPP is distributed with the aforementioned invariant distribution.

Keywords: Binary contact path process; occupation time; central limit theorem

2020 Mathematics Subject Classification: Primary 60F05
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1. Introduction

In this paper we are concerned with the normalized binary contact path process (NBCPP). For later use, we first introduce some notation. For $d \geq 1$, the d-dimensional lattice is denoted by $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when they are neighbors. The origin of $\mathbb{Z}^d$ is denoted by O. Now we recall the definition of the binary contact path process (BCPP) introduced in [6]. The binary contact path process $\{\phi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$ is a continuous-time Markov process with state space $\mathbb{X} = \{0, 1, 2, \ldots\}^{\mathbb{Z}^d}$ and evolves as follows. For any $x \in \mathbb{Z}^d$, $x \sim y$, and $t \geq 0$,

$$\phi_t(x) \to \begin{cases} 0 & \text{at rate } \dfrac{1}{1+2\lambda d}, \\ \phi_t(x) + \phi_t(y) & \text{at rate } \dfrac{\lambda}{1+2\lambda d}, \end{cases}$$

where λ is a positive constant. As a result, the generator Ω of $\{\phi_t\}_{t\geq 0}$ is given by

$$\Omega f(\phi) = \frac{1}{1+2\lambda d} \sum_{x\in\mathbb{Z}^d} (f(\phi^{x,-}) - f(\phi)) + \frac{\lambda}{1+2\lambda d} \sum_{x\in\mathbb{Z}^d} \sum_{y\sim x} (f(\phi^{x,y}) - f(\phi))$$

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for any f from $\mathbb{X}$ to $\mathbb{R}$ depending on finitely many coordinates and $\phi \in \mathbb{X}$, where, for all $x, y, z \in \mathbb{Z}^d$,

$$\phi^{x,-}(z) = \begin{cases} \phi(z) & \text{if } z \neq x, \\ 0 & \text{if } z = x; \end{cases} \qquad \phi^{x,y}(z) = \begin{cases} \phi(z) & \text{if } z \neq x, \\ \phi(x) + \phi(y) & \text{if } z = x. \end{cases}$$

Intuitively, $\{\phi_t\}_{t\geq 0}$ describes the spread of an epidemic on $\mathbb{Z}^d$. The integer value a vertex takes is the seriousness of the illness on this vertex. A vertex taking value 0 is healthy and one taking a positive value is infected. An infected vertex becomes healthy at rate $1/(1 + 2\lambda d)$. A vertex x is infected by a given neighbor y at rate $\lambda/(1 + 2\lambda d)$. When the infection occurs, the seriousness of the illness on x is added with that on y.

The BCPP $\{\phi_t\}_{t\geq 0}$ was introduced in [6] to improve upper bounds of critical values of contact processes (CPs) according to the fact that the CP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$ can be equivalently defined as

$$\xi_t(x) = \begin{cases} 0 & \text{if } \phi_{(1+2\lambda d)t}(x) = 0, \\ 1 & \text{if } \phi_{(1+2\lambda d)t}(x) > 0 \end{cases}$$

for any $x \in \mathbb{Z}^d$. For a detailed survey of CPs, see [11, Chapter 6] and [12, Part II]. To emphasize the dependence on λ, we also write ξ_t as ξ_t^λ. The critical value λ_c of the CP is defined as

$$\lambda_c = \sup\left\{\lambda > 0 \colon \mathbb{P}\left(\sum_{x\in\mathbb{Z}^d} \xi_t^\lambda(x) = 0 \text{ for some } t > 0 \mid \sum_x \xi_0^\lambda(x) = 1\right) = 1\right\}.$$

Applying the above coupling relationship between BCPP and CP, it is shown in [6] that $\lambda_c(d) \leq 1/2d(2\gamma_d - 1)$ for $d \geq 3$, where γ_d is the probability that the simple random walk on $\mathbb{Z}^d$ starting at O never returns to O again. In particular, $\lambda_c(3) \leq 0.523$ as a corollary of the above upper bound. In [17], a modified version of BCPP is introduced and then a further improved upper bound of $\lambda_c(d)$ is given for $d \geq 3$. It is shown in [17] that $\lambda_c(d) \leq (2 - \gamma_d)/(2d\gamma_d)$ and consequently $\lambda_c(3) \leq 0.340$.

The BCPP belongs to a family of continuous-time Markov processes called linear systems defined in [11, Chapter 9], since there are a series of linear transformations $\{\mathcal{A}_k \colon k \geq 1\}$ on $\mathbb{X}$ such that $\phi_t = \mathcal{A}_k\phi_{t-}$ for some $k \geq 1$ at each jump moment t. As a result, for each $m \geq 1$, Kolmogorov–Chapman equations for

$$\left\{\mathbb{E}\left(\prod_{i=1}^m \phi_t(x_i)\right) \colon x_1, \ldots, x_m \in \mathbb{Z}^d\right\}$$

are given by a series of linear ordinary differential equations, where $\mathbb{E}$ is the expectation operator. For the mathematical details, see [11, Theorems 9.1.27 and 9.3.1].

For technical reasons, it is convenient to investigate a rescaled time change $\{\eta_t\}_{t\geq 0}$ of the BCPP defined by

$$\eta_t = \exp\left\{\frac{1 - 2d\lambda}{2d\lambda}t\right\}\phi_{(1+2d\lambda)/2d\lambda t}.$$

The process $\{\eta_t\}_{t\geq 0}$ is introduced in [11, Chapter 9] and is called the normalized binary contact path process (NBCPP) since $\mathbb{E}\eta_t(x) = 1$ for all $x \in \mathbb{Z}^d$ and $t > 0$ conditioned on $\mathbb{E}\eta_0(x) = 1$ for all $x \in \mathbb{Z}^d$ (see Proposition 2.1).

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Here we recall some basic properties of the NBCPP given in [11, Section 9.6]. The state space of the NBCPP $\{\eta_t\}_{t\geq 0}$ is $\mathbb{Y}=[0,+\infty)^{\mathbb{Z}^d}$ and the generator $\mathcal{L}$ of $\{\eta_t\}_{t\geq 0}$ is given by

$$\mathcal{L}f(\eta)=\frac{1}{2\lambda d}\sum_{x}(f(\eta^{x,-})-f(\eta))+\frac{1}{2d}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}(f(\eta^{x,y})-f(\eta))+\left(\frac{1}{2\lambda d}-1\right)\sum_{x\in\mathbb{Z}^d}f_x'(\eta)\eta(x)$$

for any $\eta\in\mathbb{Y}$, where f_x' is the partial derivative of f with respect to the coordinate $\eta(x)$ and, for all $x, y, z\in\mathbb{Z}^d$,

$$\eta^{x,-}(z)=\begin{cases}\eta(z) & \text{if } z\neq x,\\ 0 & \text{if } z=x;\end{cases}\qquad \eta^{x,y}(z)=\begin{cases}\eta(z) & \text{if } z\neq x,\\ \eta(x)+\eta(y) & \text{if } z=x.\end{cases}$$

Applying [11, Theorem 9.1.27], for any $x\in\mathbb{Z}^d$, $t\geq 0$, and $\eta\in\mathbb{Y}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta\eta_t(x)=-\mathbb{E}_\eta\eta_t(x)+\frac{1}{2d}\sum_{y\sim x}\mathbb{E}_\eta\eta_t(y), \tag{1.1}$$

where $\mathbb{E}_\eta$ is the expectation operator with respect to the NBCPP with initial state $\eta_0=\eta$. For any $x, y\in\mathbb{Z}^d$, $t\geq 0$, and $\eta\in\mathbb{Y}$, let $F_{\eta,t}(x,y)=\mathbb{E}_\eta(\eta_t(x)\eta_t(y))$. Consider $F_{\eta,t}$ as a column vector from $\mathbb{Z}^d\times\mathbb{Z}^d$ to $[0,+\infty)$; applying [11, Theorem 9.3.1], there exists a $(\mathbb{Z}^d)^2\times(\mathbb{Z}^d)^2$ matrix $H=\{H((x,y),(v,w))\}_{x,y,v,w\in\mathbb{Z}^d}$, independent of η, t, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\eta,t}=HF_{\eta,t} \tag{1.2}$$

for any $\eta\in\mathbb{Y}$ and $t\geq 0$. For a precise expression for H, see Section 3, where proofs of main results of this paper are given. Roughly speaking, both $\mathbb{E}_\eta\eta_t(x)$ and $\mathbb{E}_\eta(\eta_t(x)\eta_t(y))$ are driven by linear ordinary differential equations, which is a basic property of general linear systems as we recalled above. For more properties of the NBCPP as consequences of (1.1) and (1.2), see Section 2.

In this paper we study the law of large numbers and the central limit theorem (CLT) of the occupation time process $\left\{\int_0^t\eta_u(O)\,\mathrm{d}u\right\}_{t\geq 0}$ of an NBCPP as $t\to+\infty$. The occupation time is the simplest example of the so-called additive functions of interacting particle systems, whose limit theorems have been popular research topics since the 1970s (see the references cited in [7]). In detail, let $\{X_t\}_{t\geq 0}$ be an interacting particle system with state space $S^{\mathbb{Z}^d}$, where $S\subseteq\mathbb{R}$, then $\int_0^t f(X_s)\,\mathrm{d}s$ is called an additive function for any $f\colon S^{\mathbb{Z}^d}\to\mathbb{R}$. If $\{X_t\}_{t\geq 0}$ starts from a stationary probability measure π and $f\in L^1(\pi)$, then the law of large numbers of $\int_0^t f(X_s)\,\mathrm{d}s$ is a direct application of the Birkhoff ergodic theorem such that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s=\mathbb{E}_\pi(f\mid\mathcal{I})\text{ almost surely (a.s.) and in }L^1,$$

where $\mathcal{I}$ is the set of all invariant events. Furthermore, if the stationary process $\{X_t\}_{t\geq 0}$ starting from π is ergodic, i.e. $\mathcal{I}$ is trivial, then $\mathbb{E}_\pi(f\mid\mathcal{I})=\mathbb{E}_\pi f=\int_{S^{\mathbb{Z}^d}}f(\eta)\,\pi(\mathrm{d}\eta)$ almost surely. In the ergodic case, it is natural to further investigate whether a CLT corresponding to the above Birkhoff ergodic theorem holds, i.e. whether

$$\lim_{t\to+\infty}\frac{1}{\sqrt{t}}\int_0^t(f(X_s)-\mathbb{E}_\pi f)\,\mathrm{d}s=N(0,\sigma^2)$$

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in distribution for some $\sigma^2 = \sigma^2(f) > 0$ as $t \to +\infty$. It is shown in [7] that if the generator of the ergodic process $\{X_t\}_{t\geq 0}$ satisfies a so-called sector condition, then the CLT above holds. As an application of this result, [7, Part 2] shows that mean-zero asymmetric exclusion processes starting from Bernoulli product measures are ergodic and satisfy the sector condition. Consequently, corresponding CLTs of additive functions are given. For exclusion processes, particles perform random walks on $\mathbb{Z}^d$ but jumps to occupied sites are suppressed. We refer the reader to [11, Chapter 8] for the basic properties of exclusion processes.

CLTs of additive functions are also discussed for the supercritical contact process $\xi_t^{\lambda,d}$, where $\lambda > \lambda_c$. It is shown in [11, Chapter 6] that the supercritical contact process has a nontrivial ergodic stationary distribution $\overline{\nu}$, which is called the upper invariant distribution. CLTs of additive functions of supercritical contact processes starting from $\overline{\nu}$ are given in [16]. Since marginal distributions of $\overline{\nu}$ do not have clear expressions as product measures, the proof of the main theorem in [16] follows a different strategy than that given in [7]. It is shown in [16] that additive functions of the supercritical contact process satisfy a so-called FKG condition, and then corresponding CLTs follow from [15, Theorem 3].

For a general interacting particle system starting from a stationary distribution π which is not ergodic (or we do not know whether π is ergodic), we should first study which type of f makes $\mathbb{E}_\pi(f \mid \mathcal{I}) = \mathbb{E}_\pi f$ almost surely and then the investigation of a corresponding CLT is meaningful. It is natural to first consider f with the simplest form $f(\eta) = \eta(x)$ for some $x \in \mathbb{Z}^d$, then the corresponding additive function reduces to the occupation time. CLTs of occupation times of interacting particle systems are discussed for critical branching processes and voter models on $\mathbb{Z}^d$ in [1, 3] respectively. For voter models, each $x \in \mathbb{Z}^d$ has an opinion in $\{0, 1\}$ and x adopts each neighbor's opinion at rate 1. For critical branching random walks, there are several particles on each $x \in \mathbb{Z}^d$, the number of which is considered as the state of x. All particles perform independent random walks on $\mathbb{Z}^d$ and each particle dies with probability $\frac{1}{2}$ or splits into two new particles with probability $\frac{1}{2}$ at rate 1. As shown in [1, 3], CLTs of occupation times of both models perform dimension-dependent phase transition phenomena. For critical branching random walks, the CLTs above have three different forms in the respective cases where $d = 3$, $d = 4$, and $d \geq 5$. For voter models, the CLTs above have five different forms in the respective cases where $d = 1$, $d = 2$, $d = 3$, $d = 4$, and $d \geq 5$.

The investigation of the CLT of the occupation time of an NBCPP in this paper is greatly inspired by [1, 3]. Several basic properties of NBCPPs are similar to those of the voter model and the critical branching random walk. For example, in all these three models, the expectations of states of sites on $\mathbb{Z}^d$ are all driven by linear ordinary differential equations. Furthermore, when $d \geq 3$ (and λ is sufficiently large for an NBCPP), each model has a class of nontrivial extreme stationary distributions distinguished by a parameter representing the mean particle number on each site, the probability of a site taking opinion 1, and the mean seriousness of the illness on each site in critical branching random walks, voter models, and NBCPPs respectively. In addition, the voter model and NBCPP are both linear systems. In conclusion, according to all these similarities, it is natural to ask whether CLTs similar to those of voter models and critical branching random walks hold for the NBCPP.

On the other hand, the evolution of NBCPPs has different mechanisms than those of voter models and critical branching random walks. For the voter model, the state of each site is bounded by 1. For the critical branching random walk, although the number of particles on a site is not bounded from above, each jump, birth, and death of a particle can only change one or two sites' states by one. The above properties make it easy to bound high-order moments of states of sites for voter models and critical branching random walks. For NBCPPs, the

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seriousness of the illness on a site is unbounded and the increment of this seriousness when being further infected is also unbounded, which makes the estimation of $\mathbb{E}((\eta_t(x))^m)$ difficult for large m. Hence, it is also natural to discuss the influence of the above differences between NBCPPs and the other two models on the techniques utilized in the proof of the CLT of NBCPPs.

According to a calculation of the variance (see Remark 2.3), it is natural to guess that the CLT of the occupation time of an NBCPP should take three different forms in the respective cases where $d \geq 5$, $d = 4$, and $d = 3$, as in [3, Theorem 1] and [1, Theorem 1.1]. However, in this paper we only give a rigorous result for part of the first case, i.e. the CLT of the occupation time for sufficiently large d, since we cannot bound the fourth moment of $\eta_t(O)$ uniformly for $t \geq 0$ when the dimension d is low. For our main result and the mathematical details of the proof, see Sections 2 and 3, respectively.

Since the 1980s, the large deviation principle (LDP) of the occupation time has also been a popular research topic for models such as exclusion processes, voter models, critical branching random walks, and critical branching α-stable processes [2, 4, 8–10, 13]. It is natural to ask whether an LDP holds for the occupation time of an NBCPP. We think the core difficulty in the study of this problem is the estimation of $\mathbb{E}\exp\left\{\alpha \int_0^t \eta_s(O)\,\mathrm{d}s\right\}$ for $\alpha \neq 0$. We will work on this problem as a further investigation.

We are also inspired by investigations of CLTs of empirical density fields of NBCPPs, which are also called fluctuations. For sufficiently large d and λ, [18] gives the fluctuation of the NBCPP $\{\eta_t^{\lambda,d}\}_{t\geq 0}$ starting from the configuration where all sites are in state 1; [14] gives another type of fluctuation of a class of linear systems starting from a configuration with finite total mass, including NBCPP as a special case. The boundedness of $\sup_{t\geq 0}\mathbb{E}\left(\left(\eta_t^{\lambda,d}(O)\right)^4\right)$ given in [18] for sufficiently large d and λ is a crucial property for the proofs of the main results in this paper. For the mathematical details, see Section 2.

2. Main results

In this section we give our main result. For later use, we first introduce some notation and definitions. We denote by $\{S_n\}_{n\geq 0}$ the discrete-time simple random walk on $\mathbb{Z}^d$, i.e. $\mathbb{P}(S_{n+1} = y \mid S_n = x) = 1/2d$ for any $n \geq 0$, $x \in \mathbb{Z}^d$, and $y \sim x$. We denote by $\{Y_t\}_{t\geq 0}$ the continuous-time simple random walk on $\mathbb{Z}^d$ with generator L given by $Lh(x) = (1/2d)\sum_{y\sim x}(h(y) - h(x))$ for any bounded h from $\mathbb{Z}^d$ to $\mathbb{R}$ and $x \in \mathbb{Z}^d$. As defined in Section 1, we use γ_d to denote the probability that $\{S_n\}_{n\geq 1}$ never returns to O again conditioned on $S_0 = O$, i.e.

$$\gamma_d = \mathbb{P}(S_n \neq O \text{ for all } n \geq 1 \mid S_0 = O) = \mathbb{P}(Y_t \neq O \text{ for any } t \geq 0 \mid Y_0 = x)$$

for any $x \sim O$. For any $t \geq 0$, we use $p_t(\cdot\,,\cdot\,)$ to denote the transition probabilities of Y_t, i.e. $p_t(x, y) = \mathbb{P}(Y_t = y \mid Y_0 = x)$ for any $x, y \in \mathbb{Z}^d$. For any $x \in \mathbb{Z}^d$, we define

$$\Phi(x) = \mathbb{P}(S_n = x \text{ for some } n \geq 0 \mid S_0 = O). \tag{2.1}$$

We use $\vec{1}$ to denote the configuration in $\mathbb{Y}$ where all vertices take the value 1. For any $\eta \in \mathbb{Y}$, as defined in Section 1, we denote by $\mathbb{E}_\eta$ the expectation operator of $\{\eta_t\}_{t\geq 0}$ conditioned on $\eta_0 = \eta$. Furthermore, for any probability measure μ on $\mathbb{Y}$, we use $\mathbb{E}_\mu$ to denote the expectation operator of $\{\eta_t\}_{t\geq 0}$ conditioned on η_0 being distributed with μ.

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Now we recall more properties of the NBCPP $\{\eta_t\}_{t\geq 0}$ proved in [11, Chapter 9] and [18, Section 2].

Proposition 2.1. (Liggett, [11].) *For any $\eta \in \mathbb{Y}$, $t \geq 0$, and $x \in \mathbb{Z}^d$,*

$$\mathbb{E}_\eta \eta_t(x) = \sum_{y\in\mathbb{Z}^d} p_t(x, y)\eta(y).$$

Furthermore, $\mathbb{E}_{\vec{1}}\, \eta_t(x) = 1$ for any $x \in \mathbb{Z}^d$ and $t \geq 0$.

Proposition 2.1 follows from (1.1) directly. In detail, since the generator L of the simple random walk on $\mathbb{Z}^d$ satisfies

$$L(x, y) = \begin{cases} -1 & \text{if } x = y, \\ \dfrac{1}{2d} & \text{if } x \sim y, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\mathbb{E}_\eta \eta_t(x) = \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{tL}(x, y)\mathbb{E}_\eta \eta_t(y)$ by (1.1), where $\mathrm{e}^{tL} = \sum_{n=0}^{+\infty} t^n L^n/n! = p_t$.

Proposition 2.2. (Liggett, [11].) *There exist a series of functions $\{q_t\}_{t\geq 0}$ from $(\mathbb{Z}^d)^2$ to $\mathbb{R}$ and a series of functions $\{\hat{q}_t\}_{t\geq 0}$ from $(\mathbb{Z}^d)^4$ to $\mathbb{R}$ such that*

$$\mathbb{E}_{\vec{1}}\, (\eta_t(O)\eta_t(x)) = \mathbb{E}_{\vec{1}}\, (\eta_t(y)\eta_t(x + y)) = \sum_{z\in\mathbb{Z}^d} q_t(x, z), \tag{2.2}$$

$$\mathbb{E}_\eta(\eta_t(x)\eta_t(y)) = \sum_{z,w\in\mathbb{Z}^d} \hat{q}_t((x, y), (z, w))\eta(z)\eta(w) \tag{2.3}$$

for any $t \geq 0$, $x, y \in \mathbb{Z}^d$. and $\eta \in \mathbb{Y}$. Furthermore,

$$q_t(O, w) = \sum_{(y,z):\, y-z=w} \hat{q}_t((x, x), (y, z)) \tag{2.4}$$

for any $x, w \in \mathbb{Z}^d$.

Proposition 2.2 follows from (1.2). In detail, (2.3) holds by taking $\hat{q}_t = \mathrm{e}^{tH}$. When $\eta_0 = \vec{1}$, applying the spatial homogeneity of η_t, we have

$$F_{\vec{1},t}(O, x) = F_{\vec{1},t}(O, -x) = F_{\vec{1},t}(y, y + x) = F_{\vec{1},t}(y + x, y). \tag{2.5}$$

Denote by $A_t(x)$ the expression $F_{\vec{1},t}(O, x)$, applying (1.2) and (2.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A_t = QA_t$$

for any $t \geq 0$ and a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix $Q = \{q(x, y)\}_{x,y\in\mathbb{Z}^d}$ independent of t. Consequently, (2.2) holds by taking $q_t = \mathrm{e}^{tQ}$. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_t(x) &= \frac{\mathrm{d}}{\mathrm{d}t}F_{\vec{1},t}(z + x, z) \\ &= \sum_u \sum_v H((z + x, z), (u, v))F_{\vec{1},t}(u, v) = \sum_{y\in\mathbb{Z}^d} \sum_{(u,v):u-v=y} H((z + x, z), (u, v))A_t(y),\end{aligned}$$

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the $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix Q can be chosen such that $q(x, y) = \sum_{(u,v):u-v=y} H((z+x, z), (u, v))$ for any $x, y, z \in \mathbb{Z}^d$. Applying this equation repeatedly, we have

$$q^n(x, y) = \sum_{(u,v):u-v=y} H^n((z+x, z), (u, v))$$

for all $n \geq 1$ by induction, and then

$$q_t(x, y) = \sum_{(u,v):u-v=y} \hat{q}_t((z+x, z), (u, v)). \tag{2.6}$$

As a result, (2.4) holds.

Proposition 2.3. (Liggett, [11].) *When $d \geq 3$ and $\lambda > 1/(2d(2\gamma_d - 1))$, we have the following properties:*

(i) For any $x, y, z, w \in \mathbb{Z}^d$, $\lim_{t\to+\infty} q_t^{\lambda,d}(x, y) = \lim_{t\to+\infty} \hat{q}_t^{\lambda,d}(x, y, z, w) = 0$.

(ii) Conditioned on $\eta_0 = \vec{1}$, η_t converges weakly as $t \to +\infty$ to a probability measure $\nu_{\lambda,d}$ on $\mathbb{Y}$.

(iii) For any $x, y \in \mathbb{Z}^d$,

$$\lim_{t\to+\infty} \mathbb{E}_{\vec{1}}((\eta_t(x))^2) = \sup_{t\geq 0} \mathbb{E}_{\vec{1}}((\eta_t(x))^2) = \mathbb{E}_{\nu_{\lambda,d}}((\eta_0(O))^2) = 1 + \frac{1}{h_{\lambda,d}},$$

$$\lim_{t\to+\infty} \mathrm{Cov}_{\vec{1}}(\eta_t(x), \eta_t(y)) = \mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(x), \eta_0(y)) = \frac{\Phi(x-y)}{h_{\lambda,d}},$$

where

$$h_{\lambda,d} = \frac{2\lambda d(2\gamma_d - 1) - 1}{1 + 2d\lambda}$$

and Φ is defined as in (2.1).

Proposition 2.3 follows from [11, Theorem 2.8.13, Corollary 2.8.20, Theorem 9.3.17]. In detail, let $\{\overline{p}(x.y)\}_{x,y\in\mathbb{Z}^d}$ be the one-step transition probabilities of the discrete-time simple random walk on $\mathbb{Z}^d$, and $\{\overline{q}(x, y)\}_{x,y\in\mathbb{Z}^d}$ be defined as

$$\overline{q}(x, y) = \begin{cases} \frac{1}{2}q(x, y) & \text{if } x \neq y, \\ \frac{1}{2}q(x, y) + 1 & \text{if } x = y. \end{cases}$$

Furthermore, let $\overline{h}\colon \mathbb{Z}^d \to (0, +\infty)$ be defined as

$$\overline{h}(x) = \frac{\Phi(x) + h_{\lambda,d}}{h_{\lambda,d}}.$$

Then, direct calculations show that $\overline{p}$, $\overline{q}$, and $\overline{h}$ satisfy the assumption of [11, Theorem 2.8.13]. Consequently, applying [11, Theorem 2.8.13], $\lim_{t\to+\infty} \overline{q}_t(x, y) = 0$ for all $x, y \in \mathbb{Z}^d$, where $\overline{q}_t = \mathrm{e}^{-t}\mathrm{e}^{t\overline{q}}$. It is easy to check that $q_t = \overline{q}_{2t}$ and hence the first limit in (i) is zero. By (2.6), $\hat{q}_t(x, y, z, w) \leq q_t((x-y), (z-w))$ and hence the second limit in (i) is zero. Therefore, (i) holds. It is not difficult to show that $\overline{p}$, $\overline{q}$, and $\overline{h}$ also satisfy the assumption of [11, Corollary

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2.8.20]. Applying this corollary, we have $\lim_{t\to+\infty}\sum_y q_t(x,y)=\overline{h}(x)$. Equation (2.2) shows that $\sum_y q_t(x,y)=\mathbb{E}(\eta_t(z)\eta_t(z+x))$, and hence

$$\lim_{t\to+\infty}\mathbb{E}_{\vec{1}}\left((\eta_t(x))^2\right)=\overline{h}(0)=1+\frac{1}{h_{\lambda,d}},$$
$$\lim_{t\to+\infty}\mathrm{Cov}_{\vec{1}}\left(\eta_t(x),\eta_t(y)\right)=\overline{h}(x,y)-1=\frac{\Phi(x-y)}{h_{\lambda,d}}.$$

Note that the second limit also applies Proposition 2.1. These last two limits along with (i) ensure that $\{q_t\}_{t\ge0}$ and $\vec{1}$ satisfy the assumption of [11, Theorem 9.3.17]. By this theorem, (ii) and (iii) hold.

Proposition 2.4 ([18].) *There exist an integer $d_0\ge5$ and a real number $\lambda_0>0$ satisfying the following properties:*

(i) If $d\ge d_0$ and $\lambda\ge\lambda_0$, then $\lambda>1/2d(2\gamma_d-1)$.

(ii) For any $d\ge d_0$, $\lambda\ge\lambda_0$, and $x\in\mathbb{Z}^d$,

$$\mathbb{E}_{\nu_{\lambda,d}}((\eta_0(O)^4))\le\liminf_{t\to+\infty}\mathbb{E}^{\lambda,d}_{\vec{1}}((\eta_t(x))^4)<+\infty.$$

(iii) For any $d\ge d_0$ and $\lambda\ge\lambda_0$,

$$\lim_{M\to+\infty}\sup_{\substack{(x,y,z)\in(\mathbb{Z}^d)^3:\\ \|x-y\|_1\wedge\|x-z\|_1\ge M}}\mathrm{Cov}_{\nu_{\lambda,d}}((\eta_0(x))^2,\eta_0(y)\eta_0(z))=0,$$

where $\|\cdot\|_1$ is the l_1-norm on $\mathbb{R}^d$ and $a\wedge b=\min\{a,b\}$ for $a,b\in\mathbb{R}$.

Here we briefly recall how to obtain Proposition 2.4 from the main results given in [18]. Note that, in [18], limit behaviors are discussed for η_{tN^2} as $N\to+\infty$ and t is fixed. All these behaviors can be equivalently given for η_t as $t\to+\infty$. In [18], a discrete-time symmetric random walk $\{S^{\lambda,d}_n\}_{n\ge0}$ on $(\mathbb{Z}^d)^4$ and function $\overline{H}^{\lambda,d}\colon(\mathbb{Z}^d)^4\times(\mathbb{Z}^d)^4\to[1,+\infty)$ are introduced such that

$$\mathbb{E}^{\lambda,d}_{\vec{1}}\left(\prod_{i=1}^4\eta_t(x_i)\right)\le\mathbb{E}_{\vec{x}}\left(\prod_{n=0}^{+\infty}\overline{H}(S_n,S_{n+1})\right)$$

for any $\vec{x}=(x_1,x_2,x_3,x_4)\in(\mathbb{Z}^d)^4$ and $t\ge0$. We refer the reader to [18] for precise expressions of $\overline{H}$ and the transition probabilities of $\{S_n\}_{n\ge0}$. It is shown in [18] that there exist $\hat{d}_0$ and $\hat{\lambda}_0$ such that

$$\sup_{\vec{x}\in(\mathbb{Z}^d)^4}\mathbb{E}_{\vec{x}}\left(\left(\prod_{n=0}^{+\infty}\overline{H}(S_n,S_{n+1})\right)^{1+\varepsilon}\right)<+\infty \tag{2.7}$$

for some $\varepsilon=\varepsilon(\lambda,d)>0$ when $d\ge\hat{d}_0$ and $\lambda\ge\hat{\lambda}_0$. Taking

$$d_0=\max\{5,\hat{d}_0\},\qquad \lambda_0=\max\left\{\hat{\lambda}_0,\,1+\frac{1}{2d_0(2\gamma_{d_0}-1)}\right\},$$

(i) holds. By Proposition 2.3(ii) and Fatou's lemma, we have

$$\mathbb{E}_{\nu_{\lambda,d}}((\eta_0(O)^4))\le\liminf_{t\to+\infty}\mathbb{E}^{\lambda,d}_{\vec{1}}((\eta_t(x))^4).$$

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Then, Proposition 2.4(ii) follows from (2.7). It is further shown in [18] that

$$|\mathrm{Cov}_{\vec{1}}(\eta_t(x)\eta_t(v), \eta_t(y)\eta_t(z))| \le \mathbb{E}_{(x,v,y,z)}\left(\prod_{n=0}^{+\infty} \overline{H}(S_n, S_{n+1}), \sigma < +\infty\right)$$

for any $x, v, y, z \in \mathbb{Z}^d$ and $t \ge 0$, where $\sigma = \inf\{n\colon \{S_n(1), S_n(2)\} \bigcap \{S_n(3), S_n(4)\} \ne \emptyset\}$ and $S_n(j)$ is the jth component of S_n. Applying Proposition 2.4(ii) and (iii), the left-hand side in the last inequality can be replaced by $|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(x)\eta_0(v), \eta_0(y)\eta_0(z))|$. By (2.7) and the Hölder inequality, (iii) holds by checking that

$$\lim_{M\to+\infty} \sup_{\substack{(x,y,z)\in(\mathbb{Z}^d)^3:\\ \|x-y\|_1 \wedge \|x-z\|_1 \ge M}} \mathbb{P}_{(x,x,y,z)}(\sigma < +\infty) = 0. \tag{2.8}$$

By the transition probabilities of $\{S_n\}_{n\ge0}$, $\{S_n(i) - S_n(j)\}_{n\ge0}$ is a lazy version of the simple random walk on $\mathbb{Z}^d$ for any $1 \le i \ne j \le 4$. Hence, (2.8) follows from the transience of the simple random walk on $\mathbb{Z}^d$ for $d \ge 3$, and then (iii) holds.

Now we give our main results. The first one is about the law of large numbers of the occupation time process $\left\{\int_0^t \eta_u(O)\,\mathrm{d}u\right\}_{t\ge0}$.

Theorem 2.1. *Assuming that $d \ge 3$ and $\lambda > 1/2d(2\gamma_d - 1)$.*

(i) If the NBCPP on $\mathbb{Z}^d$ starts from $\nu_{\lambda,d}$, then

$$\lim_{N\to+\infty} \frac{1}{N}\int_0^{tN} \eta_u(O)\,\mathrm{d}u = t$$

a.s., and is in L^2 for any $t > 0$.

(ii) If the NBCPP on $\mathbb{Z}^d$ starts from $\vec{1}$, then

$$\lim_{N\to+\infty} \frac{1}{N}\int_0^{tN} \eta_u(O)\,\mathrm{d}u = t$$

is in L^2 for any $t > 0$.

Remark 2.1. By Proposition 2.3, an NBCPP starting from $\nu_{\lambda,d}$ is stationary. Hence, Theorem 2.1(i) will hold immediately when we can check the ergodicity. However, as far as we know, it is still an open question as to whether an NBCPP starting from $\nu_{\lambda,d}$ is ergodic. As far as we understand, the currently known properties of NBCPPs do not provide convincing evidence for either a positive or negative answer to this problem. We will work on this problem as a further investigation.

It is natural to ask what happens for the occupation time for small λ and $d = 1, 2$. Currently, we have the following result.

Theorem 2.2. *Assuming that $d \ge 1$ and $\lambda < 1/2d$, if the NBCPP on $\mathbb{Z}^d$ starts from $\vec{1}$, then*

$$\lim_{N\to+\infty} \mathbb{E}_{\vec{1}}\left(\left(\frac{1}{N}\int_0^{tN} \eta_u(O)\,\mathrm{d}u\right)^2\right) = +\infty$$

for any $t > 0$.

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Remark 2.2. Theorem 2.2 shows that there is no convergence in L^2 for the occupation time when $\lambda < 1/2d$ and the process starts from $\vec{1}$. Combined with Theorem 2.1, the occupation time for the NBCPP on $\mathbb{Z}^d$ for $d \geq 3$ performs a phase transition as λ grows from small to large. More problems arise from Theorem 2.2. What occurs in cases where $d = 1, 2$ and $\lambda \geq 1/2d$, or $d \geq 3$ and $1/2d \leq \lambda \leq 1/(2d(2\gamma_d - 1))$? Furthermore, does the occupation time converge to 0 in distribution when λ is sufficiently small? Our current strategy cannot solve these questions, which we will work on as further investigations.

Our third main result is about the CLT of the occupation time process. To state our result, we first introduce some notation and definitions. For any $t \geq 0$ and $N \geq 1$, we define

$$X_t^N = \frac{1}{\sqrt{N}} \int_0^{tN} (\eta_u(O) - 1)\, \mathrm{d}u.$$

We write X_t^N as $X_{t,\lambda,d}^N$ when we need to distinguish d and λ.

Here we recall the definition of 'weak convergence'. Let S be a topological space. Assuming that $\{Y_n\}_{n\geq 1}$ is a sequence of S-valued random variables and Y is an S-valued random variable, then we say that Y_n converges weakly to Y when and only when $\lim_{n\to+\infty} \mathbb{E}f(Y_n) = \mathbb{E}f(Y)$ for any bounded and continuous f from S to $\mathbb{R}$.

Now we give our central limit theorem.

Theorem 2.3. *Let d_0 and λ_0 be defined as in Proposition 2.4. Assume that $d \geq d_0$, $\lambda > \lambda_0$, and the NBCPP on $\mathbb{Z}^d$ starts from $\nu_{\lambda,d}$. Then, for any integer $m \geq 1$ and $t_1, t_2, \ldots, t_m \geq 0$, $\left(X_{t_1,\lambda,d}^N, X_{t_2,\lambda,d}^N, \ldots, X_{t_m,\lambda,d}^N\right)$ converges weakly, with respect to the Euclidean topology of $\mathbb{R}^m$, to $\sqrt{C_1(\lambda, d)}(B_{t_1}, B_{t_2}, \ldots, B_{t_m})$ as $N \to +\infty$, where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion and*

$$C_1(\lambda, d) = \frac{2\int_0^{+\infty}\int_0^{+\infty} p_{r+\theta}(O, O)\, \mathrm{d}r\, \mathrm{d}\theta}{h_{\lambda,d}\int_0^{+\infty} p_\theta(O, O)\, \mathrm{d}\theta}.$$

Remark 2.3. Theorem 2.3 is consistent with a calculation of variance. Applying Propositions 2.1–2.3 and the Markov property of $\{\eta_t\}_{t\geq 0}$,

$$\lim_{N\to+\infty} \mathrm{Var}_{\nu_{\lambda,d}}(X_t^N) = \frac{2t}{h_{\lambda,d}} \int_0^{+\infty} \sum_x \Phi(x) p_r(O, x)\, \mathrm{d}r.$$

Applying the strong Markov property of the simple random walk,

$$\int_0^{+\infty} p_\theta(x, O)\, \mathrm{d}\theta = \Phi(x) \int_0^{+\infty} p_\theta(O, O)\, \mathrm{d}\theta$$

and hence

$$\begin{aligned}\int_0^{+\infty} \sum_x \Phi(x) p_\theta(O, x)\, \mathrm{d}\theta &= \frac{\int_0^{+\infty}\int_0^{+\infty} \sum_x p_r(O, x) p_\theta(x, O)\, \mathrm{d}r\, \mathrm{d}\theta}{\int_0^{+\infty} p_\theta(O, O)\, \mathrm{d}\theta} \\ &= \frac{\int_0^{+\infty}\int_0^{+\infty} p_{r+\theta}(O, O)\, \mathrm{d}r\, \mathrm{d}\theta}{\int_0^{+\infty} p_\theta(O, O)\, \mathrm{d}\theta}.\end{aligned}$$

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Note that $p_t(O, O) = O(t^{-d/2})$. Hence,

$$\frac{\int_0^{+\infty}\int_0^{+\infty} p_{r+\theta}(O, O)\,\mathrm{d}r\,\mathrm{d}\theta}{\int_0^{+\infty} p_\theta(O, O)\,\mathrm{d}\theta} \in (0, +\infty)$$

when and only when $d \geq 5$. That is why we guess that Theorem 2.3 holds for all $d \geq 5$ and sufficiently large λ. Note that

$$\gamma_d = \frac{1}{\int_0^{+\infty} p_\theta(O, O)\,\mathrm{d}\theta}$$

by the strong Markov property, and hence

$$C_1(\lambda, d) = \frac{1}{h_{\lambda,d}} C(x)\big|_{x=O},$$

where $\{C(x)\}_{x\in\mathbb{Z}^d}$ are constants given in [3, Theorem 1] for cases where $d \geq 5$. Explicit calculations of the constants occurring in the main theorems and their dependency on d are given in [1, Section 1] and [3, Sections 0 and 2].

Remark 2.4. Under assumptions of Theorem 2.3, it is natural to ask, for any given $T > 0$, whether $\{\{X^N_{t,\lambda,d}\}\}_{0\leq t\leq T}$ converges in distribution to $\{\sqrt{C_1(\lambda, d)}B_t\}_{0\leq t\leq T}$ with respect to the Skorokhod topology. We think the answer is positive but we cannot prove this claim according to our current approach. Roughly speaking, in the proof of Theorem 2.3 we decompose the centralized occupation time as a martingale plus an error function. We can show that the error function converges to 0 in distribution at each moment $t > 0$ but we cannot check the tightness of this error function to show that it converges weakly to the zero function with respect to the Skorokhod topology. We will work on this problem as a further investigation.

Theorem 2.3 shows that the central limit theorem of the NBCPP is an analogue of that of voter models and branching random walks given in [1, 3] when the dimension d is sufficiently large. Using Proposition 2.3 and the fact that $p_t(O, O) = O(t^{-d/2})$, calculations of variances similar to that in Remark 2.3 show that

$$\lim_{N\to+\infty} \frac{1}{N\log N}\mathrm{Cov}_{\nu_{\lambda,d}}\left(\int_0^{tN}(\eta_u(O) - 1)\,\mathrm{d}u, \int_0^{sN}(\eta_u(O) - 1)\,\mathrm{d}u\right) = K_4 \min\{t, s\}$$

when $d = 4$ and

$$\lim_{N\to+\infty} \frac{1}{N^{3/2}}\mathrm{Cov}_{\nu_{\lambda,d}}\left(\int_0^{tN}(\eta_u(O) - 1)\,\mathrm{d}u, \int_0^{sN}(\eta_u(O) - 1)\,\mathrm{d}u\right) = K_3\left(t^{3/2} + s^{3/2} - |t - s|^{3/2}\right)$$

when $d = 3$ for any $t, s > 0$ and some constants $K_3, K_4 \in (0, +\infty)$. These two limits provide evidence for us to guess that, as N grows to infinity,

$$\left\{\frac{1}{N^{3/4}}\int_0^{tN}(\eta_u(O) - 1)\,\mathrm{d}u\right\}_{t\geq 0}$$

converges weakly to $\sqrt{K_3}$ times the fractional Brownian motion with Hurst parameter $\frac{3}{4}$ when $d = 3$, and

$$\left\{\frac{1}{\sqrt{N\log N}}\int_0^{tN}(\eta_u(O) - 1)\,\mathrm{d}u\right\}_{t\geq 0}$$

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converges weakly to $\{\sqrt{K_4}B_t\}_{t\geq 0}$ when $d=4$, where $\{B_t\}_{t\geq 0}$ is the standard Brownian motion. Furthermore, Remark 2.3 provides evidence for us to guess that Theorem 2.3 holds for all $d\geq 5$. That is to say, central limit theorems of occupation times of NBCPPs on $\mathbb{Z}^3$, $\mathbb{Z}^4$, and $\mathbb{Z}^d$ for $d\geq 5$ are guessed to be analogues of those of voter models and branching random walks in each case. However, our current proof of Theorem 2.3 relies heavily on the fact that $\sup_{t\geq 0}\mathbb{E}_{\vec{1}}^{\lambda,d}((\eta_t(O))^4)<+\infty$ when d and λ are sufficiently large, which we have not managed to prove yet for small d. That is why we currently only discuss the NBCPP on $\mathbb{Z}^d$ with d sufficiently large.

Another possible way to extend Theorem 2.3 to cases with small d is to check the ergodicity of the NBCPP, as mentioned in Remark 2.1. If the NBCPP is ergodic, then we can apply Birkhoff's theorem for f with the form $f(\eta)=\eta^2(O)$ and then an upper bound of fourth moments of $\{\eta_t(O)\}_{t\geq 0}$ is not required. We will work on this way as a further investigation.

Proofs of Theorems 2.1, 2.2, and 2.3 are given in Section 3. The proof of Theorem 2.1 is relatively easy, where it is shown that

$$\lim_{N\to+\infty}\mathbb{E}_{\nu_{\lambda,d}}\left(\frac{1}{N}\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right)=t,\qquad \lim_{N\to+\infty}\mathrm{Var}_{\nu_{\lambda,d}}\left(\frac{1}{N}\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right)=0$$

according to Propositions 2.1 and 2.3. The proof of Theorem 2.2 utilizes the coupling relationship between the NBCPP and the contact process to show that

$$\mathbb{E}_{\vec{1}}\left((\eta_t(O))^2\right)\geq \exp\left\{\frac{(1-2\lambda d)t}{(1+2\lambda d)}\right\}.$$

The proof of Theorem 2.3 is inspired by [1], which gives CLTs of occupation times of critical branching random walks. The core idea of the proof is to decompose $\int_0^t(\eta_u(O)-1)\,\mathrm{d}u$ as a martingale M_t plus a remainder R_t such that the quadratic variation process $(1/N)\langle M\rangle_{tN}$ of $(1/\sqrt{N})M_{tN}$ converges to C_1t in L^2 and $(1/\sqrt{N})R_{tN}$ converges to 0 in probability as $N\to+\infty$. To give the above decomposition, a resolvent function of the simple random walk $\{Y_t\}_{t\geq 0}$ on $\mathbb{Z}^d$ is utilized. There are two main technical difficulties in the execution of the above strategy for the NBCPP. The first one is to prove that $\mathrm{Var}((1/N)\langle M\rangle_{tN})$ converges to 0 as $N\to+\infty$, which ensures that the limit of $(1/N)\langle M\rangle_{tN}$ is deterministic. The second one is to show that $\mathbb{E}((1/N)\sup_{0\leq s\leq tN}(M_s-M_{s-})^2)$ converges to 0 as $N\to+\infty$, which ensures that the limit process of $\{(1/\sqrt{N})M_{tN}\}_{t\geq 0}$ is continuous. To overcome the first difficulty, we relate the calculation of $\mathrm{Var}(\langle M\rangle_{tN})$ to a random walk on $\mathbb{Z}^{2d}$ according to the Kolmogorov–Chapman equation of linear systems introduced in [11]. To overcome the second difficulty, we bound $\mathbb{P}((1/N)\sup_{0\leq s\leq tN}(M_s-M_{s-})^2>\varepsilon)$ from above by $O\big((1/N^2)\int_0^{tN}\mathbb{E}(\eta_s^4(O))\,\mathrm{d}s\big)$ according to the strong Markov property of our process. For the mathematical details, see Section 3.

3. Theorem proofs

In this section we prove Theorems 2.1, 2.2, and 2.3. We first give the proof of Theorem 2.1.

Proof of Theorem 2.1. Throughout this proof we assume that $d\geq 3$ and $\lambda>1/(2d(2\gamma_d-1))$. We only prove (i), since (ii) follows from an analysis similar to that leading to the L^2 convergence in (i). So in this proof we further assume that η_0 is distributed with $\nu_{\lambda,d}$. By

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Proposition 2.3, the NBCPP starting from $\nu_{\lambda,d}$ is stationary. Hence, applying Birkhoff's ergodic theorem,

$$\lim_{N\to+\infty}\frac{1}{N}\int_0^{tN}\eta_u(O)\,\mathrm{d}u = t\mathbb{E}_{\nu_{\lambda,d}}(\eta(O)\mid\mathcal{I}) \quad \text{a.s.},$$

where $\mathcal{I}$ is the set of all invariant events. As a result, we only need to show that the convergence in Theorem 2.1 is in L^2.

Applying Proposition 2.3, $\{\eta_u(O)\}_{u\geq 0}$ are uniformly integrable and hence

$$\mathbb{E}_{\nu_{\lambda,d}}\eta(O) = \lim_{t\to+\infty}\mathbb{E}_{\vec{1}}\eta_t(O) = 1.$$

Therefore,

$$\mathbb{E}_{\nu_{\lambda,d}}\left(\frac{1}{N}\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right) = t$$

for any $N\geq 1$ and $t>0$. Consequently,

$$\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{N}\int_0^{tN}\eta_u(O)\,\mathrm{d}u - t\right)^2\right) = \frac{1}{N^2}\mathrm{Var}_{\nu_{\lambda,d}}\left(\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right).$$

Hence, to prove the L^2 convergence in Theorem 2.1 we only need to show that

$$\lim_{N\to+\infty}\frac{1}{N^2}\mathrm{Var}_{\nu_{\lambda,d}}\left(\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right) = 0 \tag{3.1}$$

for any $t\geq 0$. Since $\nu_{\lambda,d}$ is an invariant measure of our process,

$$\begin{aligned}\mathrm{Var}_{\nu_{\lambda,d}}\left(\int_0^{tN}\eta_u(O)\,\mathrm{d}u\right) &= 2\int_0^{tN}\left(\int_0^{\theta}\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_r(O),\,\eta_\theta(O))\,\mathrm{d}r\right)\mathrm{d}\theta\\ &= 2\int_0^{tN}\left(\int_0^{\theta}\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O),\,\eta_{\theta-r}(O))\,\mathrm{d}r\right)\mathrm{d}\theta.\end{aligned} \tag{3.2}$$

Applying Propositions 2.1 and 2.3,

$$\begin{aligned}\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O),\,\eta_t(O)) &= \mathbb{E}_{\nu_{\lambda,d}}(\eta_0(O)\eta_t(O)) - 1\\ &= \mathbb{E}_{\nu_{\lambda,d}}\left(\eta_0(O)\left(\sum_{y\in\mathbb{Z}^d}p_t(O,y)\eta_0(y)\right)\right) - 1\\ &= \sum_{y\in\mathbb{Z}^d}p_t(O,y)(\mathbb{E}_{\nu_{\lambda,d}}(\eta_0(O)\eta_0(y)) - 1)\\ &= \sum_{y\in\mathbb{Z}^d}p_t(O,y)\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O),\,\eta_0(y))\\ &= \frac{1}{h_{\lambda,d}}\sum_{y\in\mathbb{Z}^d}p_t(O,y)\Phi(y).\end{aligned} \tag{3.3}$$

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Therefore, by (3.3), for any $t \geq 0$,

$$-1 \leq \mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O), \eta_t(O)) \leq \frac{1}{h_{\lambda,d}} \sum_{y \in \mathbb{Z}^d} p_t(O, y) = \frac{1}{h_{\lambda,d}}, \tag{3.4}$$

and hence $\sup_{t \geq 0} |\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O), \eta_t(O))| < +\infty$. By (3.2) and (3.4), to prove (3.1) we only need to show that

$$\lim_{t \to +\infty} \mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O), \eta_t(O)) = 0. \tag{3.5}$$

Since $d \geq 3$, for any $\varepsilon > 0$ there exists $M > 0$ such that $\Phi(y) < \varepsilon$ when $\|y\|_1 \geq M$. As a result, by (3.3),

$$\begin{aligned}\limsup_{t \to +\infty} \mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0(O), \eta_t(O)) &\leq \limsup_{t \to +\infty} \frac{\varepsilon}{h_{\lambda,d}} \sum_{y:\|y\|_1 \geq M} p_t(O, y) + \frac{1}{h_{\lambda,d}} \sum_{y:\|y\|_1 \leq M} \Big(\lim_{t \to +\infty} p_t(O, y) \Big) \\ &= \limsup_{t \to +\infty} \frac{\varepsilon}{h_{\lambda,d}} \sum_{y:\|y\|_1 \geq M} p_t(O, y) \leq \frac{\varepsilon}{h_{\lambda,d}}.\end{aligned}$$

Since ε is arbitrary, let $\varepsilon \to 0$ and then (3.5) holds. As we have explained, Theorem 2.1 follows from (3.5), and the proof is complete. □

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Throughout this proof we assume that $\lambda < 1/2d$. According to a calculation similar to that leading to (3.3),

$$\begin{aligned}\mathbb{E}_{\vec{1}}\left(\left(\frac{1}{N} \int_0^{tN} \eta_u(O)\, \mathrm{d}u \right)^2 \right) &= \frac{2}{N^2} \int_0^{tN} \int_0^s \sum_x p_{s-r}(O, x) \mathbb{E}_{\vec{1}}\, (\eta_r(O)\eta_r(x))\, \mathrm{d}r\, \mathrm{d}s \\ &\geq \frac{2}{N^2} \int_0^{tN} \int_0^s p_{s-r}(O, O) \mathbb{E}_{\vec{1}}\, (\eta_r^2(O))\, \mathrm{d}r\, \mathrm{d}s. \end{aligned} \tag{3.6}$$

Let ξ_t be the contact process defined as in Section 1, then a well-known property of ξ_t is that

$$\mathbb{P}_{\vec{1}}\, (\xi_t(O) = 1) \leq \exp\left\{ \frac{(2\lambda d - 1)t}{1 + 2d\lambda} \right\}.$$

This property can be proved by coupling the contact process with a branching process, the detail of which we omit here. According to the coupling relationship of the NBCPP and the contact process given in Section 1, utilizing the Hölder inequality and Proposition 2.1, we have

$$\mathbb{P}_{\vec{1}}\, (\xi_t(O) = 1) = \mathbb{P}_{\vec{1}}\, (\eta_t(O) > 0) \geq \frac{(\mathbb{E}_{\vec{1}}\, \eta_t(O))^2}{\mathbb{E}_{\vec{1}}\, (\eta_t^2(O))} = \frac{1}{\mathbb{E}_{\vec{1}}\, (\eta_t^2(O))}.$$

Hence,

$$\mathbb{E}_{\vec{1}}\, (\eta_t^2(O)) \geq \exp\left\{ \frac{(1 - 2\lambda d)t}{1 + 2d\lambda} \right\}. \tag{3.7}$$

Since $p_t(O, O) = \Theta(t^{-d/2})$ as $t \to +\infty$, Theorem 2.2 follows directly from (3.6) and (3.7). □

From now on we assume that $d \geq d_0$ and $\lambda \geq \lambda_0$, where d_0 and λ_0 are defined as in Proposition 2.4. As mentioned at the end of Section 2, our proof follows the strategy introduced in [1] where X_t^N is decomposed as a martingale plus a remainder term. As $N \to +\infty$,

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the martingale converges weakly to a Brownian motion and the remainder converges to 0 in probability. In detail, for any $\theta > 0$, we define

$$G_\theta(\eta) = \sum_{x\in\mathbb{Z}^d} g_\theta(x)(\eta(x) - 1),$$

where g_θ is the resolvent function of the simple random walk given by

$$g_\theta(x) = \int_0^{+\infty} \mathrm{e}^{-\theta u} p_u(O, x)\,\mathrm{d}u.$$

According to the Markov property of the simple random walk,

$$\sum_x g_0^2(x) = \int_0^{+\infty}\int_0^{+\infty} p_{u+\theta}(O, O)\,\mathrm{d}u\,\mathrm{d}\theta.$$

Then, applying $p_t(O, O) = \Theta(t^{-d/2})$ as $t \to +\infty$, $\sum_x g_0^2(x)$ is finite when and only when $d \geq 5$. Hence, $\sum_x g_0^k(x) < +\infty$ when $k \geq 2$ and $d \geq 5$. Let

$$M_t^\theta = G_\theta(\eta_t) - G_\theta(\eta_0) - \int_0^t \mathcal{L}G_\theta(\eta_s)\,\mathrm{d}s,$$

then by Dynkin's martingale formula, $\{M_t^\theta\}_{t\geq 0}$ is a martingale with quadratic variation process $\{\langle M^\theta\rangle_t\}_{t\geq 0}$ given by

$$\langle M^\theta\rangle_t = \int_0^t (\mathcal{L}((G_\theta(\eta_s))^2) - 2G_\theta(\eta_s)\mathcal{L}G_\theta(\eta_s))\,\mathrm{d}u.$$

Applying the definition of $\mathcal{L}$,

$$\langle M^\theta\rangle_t = \frac{1}{2\lambda d}\int_0^t \sum_{x\in\mathbb{Z}^d} \eta_s^2(x)\left(g_\theta^2(x) + \lambda\sum_{y\sim x} g_\theta^2(y)\right)\mathrm{d}s. \tag{3.8}$$

Applying $\theta g_\theta(\cdot) - Lg_\theta(\cdot) = \mathbf{1}_O(\cdot)$,

$$\mathcal{L}G_\theta(\eta) = \theta G_\theta(\eta) - \left(\eta(O) - \theta\sum_x g_\theta(x)\right) = \theta G_\theta(\eta) - (\eta(O) - 1).$$

As a result,

$$X_t^N = \frac{1}{\sqrt{N}} M_{tN}^{1/N} + \frac{1}{\sqrt{N}} R_{tN}^{1/N}, \tag{3.9}$$

where $R_t^\theta = -G_\theta(\eta_t) + G_\theta(\eta_0) + \int_0^t \theta G_\theta(\eta_s)\,\mathrm{d}s$. To prove Theorem 2.3, we need the following lemmas.

Lemma 3.1. *For any $t \geq 0$, $(1/N)\langle M^{1/N}\rangle_{tN}$ converges to $C_1(\lambda, d)t$ in L^2 as $N \to +\infty$.*

Lemma 3.2. *For any $t \geq 0$, $(1/\sqrt{N})R_{tN}^{1/N}$ converges to 0 in L^2 as $N \to +\infty$.*

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Lemma 3.3. *For any $t \geq 0$,*

$$\lim_{N\to+\infty} \mathbb{E}\Bigg(\sup_{0\leq s\leq tN} \frac{1}{N}\big(M_s^{1/N} - M_{s-}^{1/N}\big)^2 \Bigg) = 0,$$

where $M_{s-}^{1/N} = \lim_{u<s,u\to s} M_u^{1/N}$, i.e. the state of $M_\cdot^{1/N}$ at the moment just before s.

We first utilize Lemmas 3.1, 3.2, and 3.3 to prove Theorem 2.3.

Proof of Theorem 2.3. We denote by $\mathcal{D}[0, +\infty)$ the set of càdlàg functions from $[0, +\infty)$ to $\mathbb{R}$ endowed with the Skorokhod topology. According to Lemmas 3.1 and 3.3,

$$\left(\left\{ \frac{1}{\sqrt{N}} M_{tN}^{1/N} \right\}_{t\geq 0}, \left\{ \frac{1}{N} \langle M^{1/N} \rangle_{tN} \right\}_{t\geq 0} \right)$$

satisfies condition (b) of [5, Theorem 1.4, Chapter 7]. Hence, using that theorem, $\big\{(1/\sqrt{N})M_{tN}^{1/N} : t \geq 0\big\}_{N\geq 1}$ converges weakly, with respect to the Skorokhod topology of $\mathcal{D}[0, +\infty)$, to a continuous martingale $\{W_t\}_{t\geq 0}$ as $N \to +\infty$, where $\{W_t\}_{t\geq 0}$ satisfies that $\{W_t^2 - C_1(\lambda, d)t\}_{t\geq 0}$ is also a martingale. Hence, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion times $\sqrt{C_1(\lambda, d)}$. Then, for given $0 < t_1 < t_2 < \cdots < t_m$, since $\{\eta_t\}_{t\geq 0}$ is continuous at each t_i with probability 1, we have that $(1/\sqrt{N})\big(M_{t_1N}^{1/N}, \ldots, M_{t_mN}^{1/N}\big)$ converges weakly to $\sqrt{C_1(\lambda, d)}(B_{t_1}, \ldots, B_{t_m})$ as $N \to +\infty$. Consequently, Theorem 2.3 follows from (3.9) and Lemma 3.2. □

Now we only need to prove Lemmas 3.1, 3.2, and 3.3. We first give the proof of Lemma 3.1.

Proof of Lemma 3.1. Conditioned on $Y_0 = O$, the number of times $\{Y_t\}_{t\geq 0}$ visits O follows a geometric distribution with parameter γ_d, and at each time Y_t stays at O for an exponential time with mean 1. Therefore,

$$\int_0^{+\infty} p_s(O, O)\, \mathrm{d}s = \mathbb{E}_O \int_0^{+\infty} \mathbf{1}_{\{X_s=O\}}\, \mathrm{d}s = 1 \times \frac{1}{\gamma_d} = \frac{1}{\gamma_d}.$$

Then, by (3.8) and Proposition 2.3,

$$\begin{aligned}\lim_{N\to+\infty} \mathbb{E}_{\nu_{\lambda,d}}\left(\frac{1}{N}\langle M^{1/N}\rangle_{tN}\right) &= \frac{t(1+2\lambda d)}{2\lambda d}\left(1 + \frac{1}{h_{\lambda,d}}\right)\sum_x g_0^2(x) \\ &= \frac{2\gamma_d t}{h_{\lambda,d}} \int_0^{+\infty}\int_0^{+\infty} p_{\theta+r}(O, O)\, \mathrm{d}\theta\, \mathrm{d}r = C_1(\lambda, d)t.\end{aligned}$$

Hence, to complete the proof we only need to show that

$$\lim_{N\to+\infty} \frac{1}{N^2} \mathrm{Var}_{\nu_{\lambda,d}}(\langle M^{1/N}\rangle_{tN}) = 0. \tag{3.10}$$

By Proposition 2.2 and the Markov property of $\{\eta_t\}_{t\geq 0}$, for $s < u$ and $x, w \in \mathbb{Z}^d$,

$$\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_s^2(x), \eta_u^2(w)) = \sum_{y,z} \hat{q}_{u-s}((w, w), (y, z))\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x), \eta_0(y)\eta_0(z)).$$

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Therefore, to prove (3.10) we only need to show that

$$\lim_{r\to+\infty}\sum_{x,w,y,z} V_0(x)V_0(w)\hat{q}_r((w,w),(y,z))|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))| = 0, \tag{3.11}$$

where $V_\theta(x) = g_\theta^2(x) + \sum_{y\sim x} g_\theta^2(y)$, which is decreasing in θ.

According to Propositions 2.2–2.4 and the Cauchy–Schwarz inequality,

$$\sum_{y,z}\hat{q}_r((w,w),(y,z)) = \sum_y q_r(O,y) = \mathbb{E}_{\bar{1}}((\eta_r(O))^2) \le 1 + \frac{1}{h_{\lambda,d}},$$
$$\sup_{w,y,z}|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(w),\eta_0(y)\eta_0(z))| < 2\mathbb{E}_{\nu_{\lambda,d}}((\eta_0(O))^4) < +\infty.$$

Then, using the fact that $\sum_x V_0(x) < +\infty$ and the dominated convergence theorem, to prove (3.11) we only need to show that

$$\lim_{r\to+\infty}\sum_{y,z}\hat{q}_r((w,w),(y,z))|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))| = 0 \tag{3.12}$$

for any $x, w \in \mathbb{Z}^d$. By Proposition 2.4, for any $\varepsilon > 0$, there exists $M > 0$ such that $|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))| < \varepsilon$ when $\|x-y\|_1 \wedge \|x-z\|_1 \ge M$, and hence

$$\sum_{(y,z):\|x-y\|_1\wedge\|x-z\|_1\ge M}\hat{q}_r((w,w),(y,z))|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))|$$
$$\le \varepsilon\sum_{(y,z):\|x-y\|_1\wedge\|x-z\|_1\ge M}\hat{q}_r((w,w),(y,z)) \le \varepsilon\left(1+\frac{1}{h_{\lambda,d}}\right).$$

We claim that

$$\lim_{t\to+\infty}\sum_v \hat{q}_t((w,w),(y,v)) = \lim_{t\to+\infty}\sum_v \hat{q}_t((w,w),(v,z)) = 0 \tag{3.13}$$

for any $w, y, z \in \mathbb{Z}^d$. The proof of (3.13) is given later. By (3.13), Proposition 2.4, and the Cauchy–Schwarz inequality,

$$\lim_{r\to+\infty}\sum_{\substack{(y,z):\|x-y\|_1\le M \text{ or}\\ \|x-z\|_1\le M}}\hat{q}_r((w,w),(y,z))|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))| = 0$$

and hence

$$\limsup_{r\to+\infty}\sum_{y,z}\hat{q}_r((w,w),(y,z))|\mathrm{Cov}_{\nu_{\lambda,d}}(\eta_0^2(x),\eta_0(y)\eta_0(z))| \le \varepsilon\left(1+\frac{1}{h_{\lambda,d}}\right).$$

Since ε is arbitrary, let $\varepsilon \to 0$ and then (3.12) holds. □

Now we prove (3.13) to complete the proof of Lemma 3.1.

Proof of (3.13). By [11, Theorem 9.3.1] and the definition of $\mathcal{L}$,

$$\hat{q}_t((w,w),(y,v)) = \mathrm{e}^{-2t}\sum_{n=0}^{+\infty}\frac{t^n}{n!}H^n((w,w),(y,v)),$$

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where H is a $(\mathbb{Z}^d)^2 \times (\mathbb{Z}^d)^2$ matrix given by

$$H((x,y),(u,v)) = \begin{cases} 1/2d & \text{if } x \neq y,\ u \sim x,\ \text{and } v = y, \\ 1/2d & \text{if } x \neq y,\ u = x,\ \text{and } v \sim y, \\ 1/2d\lambda & \text{if } x = y \text{ and } (u,v) = (x,x), \\ 1/2d & \text{if } x = y,\ u \sim x,\ \text{and } v = u, \\ 1/2d & \text{if } x = y,\ u = x,\ \text{and } v \sim x, \\ 1/2d & \text{if } x = y,\ u \sim x. \text{ and } v = x, \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^n((w,w),(y,v)) = \sum_{\substack{\{(u_i,v_i)\}_{0\le i\le n}: \\ (u_0,v_0)=(w,w),(u_n,v_n)=(y,v)}} \prod_{i=0}^{n-1} H((u_i,v_i),(u_{i+1},v_{i+1})).$$

As a result,

$$\sum_{v\in\mathbb{Z}^d} \hat{q}_t((w,w),(y,v)) = \mathrm{e}^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} \mathbb{E}_{(w,w)} \left(\prod_{i=0}^{n-1} \Theta(\beta_i,\beta_{i+1}) \mathbf{1}_{\{\beta_n(1)=y\}} \right),$$

where $\{\beta_n = (\beta_n(1), \beta_n(2)) : n \ge 0\}$ is a random walk on $(\mathbb{Z}^d)^2$ such that

$$\mathbb{P}(\beta_{n+1} = (u,v) \mid \beta_n = (x,y)) = \begin{cases} 1/4d & \text{if } x \neq y,\ u \sim x. \text{ and } v = y, \\ 1/4d & \text{if } x \neq y,\ u = x,\ \text{and } v \sim y, \\ 1(6d+1) & \text{if } x = y \text{ and } (u,v) = (x,x), \\ 1/(6d+1) & \text{if } x = y,\ u \sim x. \text{ and } v = u, \\ 1/(6d+1) & \text{if } x = y,\ u \sim x,\ \text{and } v = x, \\ 1/6d+1 & \text{if } x = y,\ u = x, \textit{and } v \sim x, \\ 0 & \text{otherwise} \end{cases}$$

and Θ is a function from $(\mathbb{Z}^d)^2$ to $\mathbb{R}$ such that $\Theta((x,y),(u,v)) = \frac{1}{2}H((x,y),(u,v))\deg(x,y)$, where

$$\deg(x,y) = \begin{cases} 4d & \text{if } x \neq y, \\ 6d+1 & \text{if } x = y. \end{cases}$$

Note that $\deg(\cdot)$ is the degree function of the graph generated by adding edges on $\mathbb{Z}^{2d}$ to connect (x,x) with itself and (y,y) for all $x \in \mathbb{Z}^d$ and all y such that $y \sim x$ on $\mathbb{Z}^d$.

According to the definition of Θ, for each $i \ge 0$, $\Theta(\beta_i,\beta_{i+1}) \ge 1$ and $\Theta(\beta_i,\beta_{i+1}) = 1$ when and only when $\beta_i(1) = \beta_i(2)$. Hence, by Hölder's inequality,

$$\begin{aligned} &\sum_{v\in\mathbb{Z}^d} \hat{q}_t((w,w),(y,v)) \\ &\quad \le \left(\mathbb{E}_{(w,w)} \left(\prod_{i=0}^{+\infty} \Theta^{1+\varepsilon}(\beta_i,\beta_{i+1}) \right) \right)^{1/(1+\varepsilon)} \mathrm{e}^{-2t} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} (\mathbb{P}(\beta_n(1) = y))^{\varepsilon/(1+\varepsilon)} \end{aligned} \tag{3.14}$$

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for any $\varepsilon > 0$. Since $\{\beta_n(1) - \beta_n(2)\}_{n\geq 0}$ is a lazy version of the simple random walk on $\mathbb{Z}^d$ and $H(\beta_i, \beta_{i+1}) > 1$ only when $\beta_i(1) = \beta_i(2)$, by the strong Markov property of $\{\beta_n\}_{n\geq 0}$,

$$\mathbb{E}_{(w,w)}\left(\prod_{i=0}^{+\infty}\Theta^{1+\varepsilon}(\beta_i, \beta_{i+1})\right) = \frac{4d}{6d+1}\left(\frac{6d+1}{4d}\right)^{1+\varepsilon}\gamma_d + \sum_{k=1}^{+\infty}(\alpha_\varepsilon(d, \lambda))^k\gamma_d,$$

where

$$\alpha_\varepsilon(d, \lambda) = \frac{1}{6d+1}\left(\frac{6d+1}{2\cdot 2d\lambda}\right)^{1+\varepsilon} + \frac{2d}{6d+1}\left(\frac{6d+1}{2\cdot 2d}\right)^{1+\varepsilon} + \frac{4d}{6d+1}\left(\frac{6d+1}{2\cdot 2d}\right)^{1+\varepsilon}(1-\gamma_d).$$

Since $\lambda > 1/2d(2\gamma_d - 1)$, $\lim_{\varepsilon\to 0}\alpha_\varepsilon(d, \lambda) = 1/4d\lambda + \frac{1}{2} + 1 - \gamma_d < 1$ and hence there exists $\varepsilon_0 > 0$ such that $\mathbb{E}_{(w,w)}\big(\prod_{i=0}^{+\infty}\Theta^{1+\varepsilon_0}(\beta_i, \beta_{i+1})\big) < +\infty$. By (3.14), to complete the proof we only need to show that

$$\lim_{t\to+\infty} \mathrm{e}^{-2t}\sum_{n=0}^{+\infty}\frac{(2t)^n}{n!}(\mathbb{P}(\beta_n(1) = y))^{\varepsilon_0/(1+\varepsilon_0)} = 0. \tag{3.15}$$

Since $d \geq 3$ and $\{\beta_n(1)\}_{n\geq 1}$ is a lazy version of the simple random walk on $\mathbb{Z}^d$,

$$\lim_{n\to+\infty}\mathbb{P}(\beta_n(1) = y) = 0.$$

Hence, for any $\varepsilon > 0$, there exists $M \geq 1$ such that $(\mathbb{P}(\beta_n(1) = y))^{\varepsilon_0/(1+\varepsilon_0)} \leq \varepsilon$ when $n \geq M$, and then

$$\mathrm{e}^{-2t}\sum_{n=M}^{+\infty}\frac{(2t)^n}{n!}(\mathbb{P}(\beta_n(1) = y))^{\varepsilon_0/(1+\varepsilon_0)} \leq \varepsilon\mathrm{e}^{-2t}\sum_{n=0}^{+\infty}\frac{(2t)^n}{n!} = \varepsilon.$$

Since $\lim_{t\to+\infty}\sum_{n=0}^{M-1}\mathrm{e}^{-2t}(2t)^n/n! = 0$, we have

$$\limsup_{t\to+\infty}\mathrm{e}^{-2t}\sum_{n=0}^{+\infty}\frac{(2t)^n}{n!}(\mathbb{P}(\beta_n(1) = y))^{\varepsilon_0/(1+\varepsilon_0)} \leq \varepsilon.$$

Since ε is arbitrary, let $\varepsilon \to 0$ and then (3.15) holds. □

Now we prove Lemma 3.2.

Proof of Lemma 3.2. Applying Propositions 2.1 and 2.3, for any moment $c \geq 0$,

$$\begin{aligned}\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_c)\right)^2\right) &= \mathrm{Var}_{\nu_{\lambda,d}}\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_c)\right)\\ &= \mathrm{Var}_{\nu_{\lambda,d}}\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_0)\right) = \frac{1}{N}\sum_x\sum_y g_{1/N}(x)g_{1/N}(y)\frac{\Phi(y-x)}{h_{\lambda,d}}.\end{aligned}$$

As shown in Remark 2.3,

$$\Phi(y-x) = \frac{\int_0^{+\infty}p_s(y-x, O)\,\mathrm{d}s}{\int_0^{+\infty}p_s(O, O)\,\mathrm{d}s} = \int_0^{+\infty}p_s(x, y)\,\mathrm{d}s\gamma_d.$$

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Hence,

$$\begin{aligned}
&\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_c)\right)^2\right)\\
&\quad=\frac{\gamma_d}{Nh_{\lambda,d}}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-(r+s)/N}\sum_x\sum_y p_r(0,x)p_s(x,y)p_u(y,0)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}u\\
&\quad=\frac{\gamma_d}{Nh_{\lambda,d}}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-(r+s)/N}p_{r+s+u}(O,O)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}u\\
&\quad=\frac{\gamma_d}{Nh_{\lambda,d}}\int_0^{+\infty}p_\theta(O,O)\left(\int_0^{\theta}\mathrm{e}^{-v/N}\left(\int_0^{v}1\,\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}\theta\\
&\quad=\frac{N\gamma_d}{h_{\lambda,d}}\int_0^{+\infty}p_\theta(O,O)\left(1-\mathrm{e}^{-\theta/N}\left(1+\frac{\theta}{N}\right)\right)\mathrm{d}\theta. \qquad (3.16)
\end{aligned}$$

Since $p_\theta(O,O)=O(\theta^{-d/2})$ as $\theta\to+\infty$ and $\lim_{x\to+\infty}1-\mathrm{e}^{-x}(1+x)=1$, there exist $0<M_1,\widehat{C}_4<+\infty$ such that $p_\theta(O,O)\le\widehat{C}_4\theta^{-d/2}$ and $1-\mathrm{e}^{-x}(1+x)\le\widehat{C}_4$ when $\theta,x\ge M_1$. Since $\lim_{x\to0}(1-\mathrm{e}^{-x}(1+x))/x^2=\frac{1}{2}$, we have

$$\widetilde{C}_4:=\sup_{0<x\le M_1}\frac{1-\mathrm{e}^{-x}(1+x)}{x^2}<+\infty.$$

We denote by C_4 the maximum of $\widehat{C}_4$ and $\widetilde{C}_4$. Then,

$$p_\theta(O,O)\left(1-\mathrm{e}^{-\theta/N}\left(1+\frac{\theta}{N}\right)\right)\le\begin{cases}\theta^2C_4/N^2 & \text{when } 0\le\theta\le M_1,\\ C_4^2\theta^{2-(d/2)}/N^2 & \text{when } M_1\le\theta\le NM_1,\\ C_4^2\theta^{-d/2} & \text{when } \theta>NM_1.\end{cases}$$

Hence, for $d\ge5$,

$$\begin{aligned}
&\frac{N\gamma_d}{h_{\lambda,d}}\int_0^{M_1}p_\theta(O,O)\left(1-\mathrm{e}^{-\theta/N}\left(1+\frac{\theta}{N}\right)\right)\mathrm{d}\theta\le\frac{C_4\gamma_d}{Nh_{\lambda,d}}\int_0^{M_1}\theta^2\,\mathrm{d}\theta=O(N^{-1}),\\
&\frac{N\gamma_d}{h_{\lambda,d}}\int_{M_1}^{NM_1}p_\theta(O,O)\left(1-\mathrm{e}^{-\theta/N}\left(1+\frac{\theta}{N}\right)\right)\mathrm{d}\theta\le\frac{C_4^2\gamma_d}{Nh_{\lambda,d}}\int_{M_1}^{NM_1}\theta^{2-(d/2)}\,\mathrm{d}\theta\\
&\qquad\le\frac{C_4^2\gamma_d}{Nh_{\lambda,d}}\int_{M_1}^{NM_1}\theta^{2-5/2}\,\mathrm{d}\theta=O(N^{-1/2}),\\
&\frac{N\gamma_d}{h_{\lambda,d}}\int_{NM_1}^{+\infty}p_\theta(O,O)\left(1-\mathrm{e}^{-\theta/N}\left(1+\frac{\theta}{N}\right)\right)\mathrm{d}\theta\le\frac{C_4^2N\gamma_d}{h_{\lambda,d}}\int_{NM_1}^{+\infty}\theta^{-d/2}\,\mathrm{d}\theta=O(N^{2-(d/2)}).
\end{aligned}$$

Consequently, applying (3.16),

$$\lim_{N\to+\infty}\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_0)\right)^2\right)=\lim_{N\to+\infty}\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}G_{1/N}(\eta_{tN^2})\right)^2\right)=0. \qquad (3.17)$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}\int_0^{tN}\frac{1}{N}G_{1/N}(\eta_s)\,\mathrm{d}s\right)^2\right)&\le\frac{tN}{N^3}\int_0^{tN}(\mathbb{E}_{\nu_{\lambda,d}}((G_{1/N}(\eta_s))^2))\,\mathrm{d}s\\
&=\frac{t^2N^2}{N^3}\mathbb{E}_{\nu_{\lambda,d}}((G_{1/N}(\eta_0))^2).
\end{aligned}$$

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Hence, applying (3.17),

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_{\lambda,d}}\left(\left(\frac{1}{\sqrt{N}}\int_0^{tN}\frac{1}{N}G_{1/N}(\eta_s)\,\mathrm{d}s\right)^2\right)=0. \tag{3.18}$$

Lemma 3.2 follows from (3.17) and (3.18). □

Finally, we prove Lemma 3.3.

Proof of Lemma 3.3. By (3.9) and the fact that $\{X_t^N\}_{t\geq 0}$ is continuous in t,

$$\sup_{0\leq s\leq tN}\frac{1}{N}\big(M_s^{1/N}-M_{s-}^{1/N}\big)^2\leq \sup_{0\leq s\leq tN}\frac{1}{N}\sup_x\max_{y\sim x}\{g_{1/N}^2(x)\eta_s^2(x),\, g_{1/N}^2(x)\eta_s^2(y)\}. \tag{3.19}$$

For any $M>0$, let

$$\tau_M=\inf\left\{s>0\colon \frac{1}{N}\sup_x\max_{y\sim x}\{g_{1/N}^2(x)\eta_s^2(x),\, g_{1/N}^2(x)\eta_s^2(y)\}>M\right\};$$

then

$$\left\{\sup_{0\leq s\leq tN}\frac{1}{N}\sup_x\max_{y\sim x}\{g_{1/N}^2(x)\eta_s^2(x),\, g_{1/N}^2(x)\eta_s^2(y)\}>M\right\}=\{\tau_M\leq tN\}.$$

Conditioned on $\{\tau_M\leq tN\}$, there exists $x_0\in\mathbb{Z}^d$ such that

$$\max_{y\sim x_0}\{g_{1/N}^2(x_0)\eta_{\tau_M}^2(x_0),\, g_{1/N}^2(x_0)\eta_{\tau_M}^2(y)\}>NM.$$

If $\{\eta_s(y)\}_{y\sim x_0}$ and $\eta_s(x_0)$ do not jump to 0 during $s\in[\tau_M,\,\tau_M+1]$, then

$$\max_{y\sim x_0}\{g_{1/N}^2(x)\eta_s^2(x_0),\, g_{1/N}^2(x_0)\eta_s^2(y)\}\geq NM\exp\left\{\min\left\{\frac{1}{2\lambda d}-1,\,0\right\}\right\}$$

for $s\in[\tau_M,\,\tau_M+1]$ and hence

$$\int_0^{tN+1}\sum_x g_{1/N}^4(x)\left(\eta_s^4(x)+\sum_{y\sim x}\eta_s^4(y)\right)\mathrm{d}s\geq N^2M^2\exp\left\{2\min\left\{\frac{1}{2\lambda d}-1,\,0\right\}\right\}.$$

Since the state of a vertex jumps to 0 at rate $1/(2\lambda d)$, applying the strong Markov property of $\{\eta_t\}_{t\geq 0}$,

$$\begin{aligned}\mathbb{P}\left(\int_0^{tN+1}\sum_x g_{1/N}^4(x)\left(\eta_s^4(x)+\sum_{y\sim x}\eta_s^4(y)\right)\mathrm{d}s\geq N^2M^2\exp\left\{2\min\left\{\frac{1}{2\lambda d}-1,\,0\right\}\right\}\right)\\ \geq\mathbb{P}(\tau_M\leq tN)\exp\left\{-\frac{2d+1}{2\lambda d}\right\}.\end{aligned}$$

Then, by Markov's inequality, Proposition 2.4, and the fact that

$$\sum_x g_{1/N}^4(x)\leq\sum_x g_0^4(x)<+\infty,$$

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we have

$$\mathbb{P}\bigg(\sup_{0\le s\le tN}\frac{1}{N}\sup_{x}\max_{y\sim x}\{g_{1/N}^2(x)\eta_s^2(x),\, g_{1/N}^2(x)\eta_s^2(y)\}>M\bigg)$$
$$\le\frac{\exp\{(2d+1)/2\lambda d\}}{N^2M^2\exp 2\min\{1/(2\lambda d)-1,0\}}\int_0^{tN+1}\sum_x g_0^4(x)(\mathbb{E}_{\lambda,d}(\eta^4(O))(2d+1))\,\mathrm{d}s\le\frac{C_3}{NM^2},\tag{3.20}$$

where $C_3<+\infty$ is independent of N and M. Applying the Fubini theorem, for any positive random variable V and any $c>0$,

$$\mathbb{E}\big(V\mathbf{1}_{\{V>c\}}\big)=c\mathbb{P}(V>c)+\int_c^{+\infty}\mathbb{P}(V\ge u)\,\mathrm{d}u.$$

Hence, for any $\varepsilon>0$,

$$\mathbb{E}\bigg(\sup_{0\le s\le tN}\frac{1}{N}(M_s^{1/N}-M_{s-}^{1/N})^2\bigg)\le\varepsilon+\varepsilon\frac{C_3}{N\varepsilon^2}+\int_\varepsilon^{+\infty}\frac{C_3}{Nu^2}\,\mathrm{d}u=\varepsilon+\frac{2C_3}{N\varepsilon}$$

according to (3.19) and (3.20). As a result,

$$\limsup_{N\to+\infty}\mathbb{E}\bigg(\sup_{0\le s\le tN}\frac{1}{N}(M_s^{1/N}-M_{s-}^{1/N})^2\bigg)\le\varepsilon.$$

Since ε is arbitrary, let $\varepsilon\to 0$ and the proof is complete. □

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