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Annihilation dynamics in the KPP-Fisher equation

M. RODRIGO[1]† and M. MIMURA[2]

[1]*Department of Mathematical Sciences, New Jersey Institute of Technology, 606 Cullimore Hall, University Heights, Newark, NJ 07102, USA*
e-mail: mrodrigo@math.sci.hiroshima-u.ac.jp
[2]*Institute for Nonlinear Sciences and Applied Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan*
e-mail: mimura@math.sci.hiroshima-u.ac.jp

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We study the annihilation dynamics arising in the KPP-Fisher equation, proposed by Fisher in 1936 to model the propagation of a mutant gene and subsequently studied rigorously in the seminal work of Kolmogorov, Petrovskii and Piskunov. The approach is via a comparison theorem, where the comparison functions satisfy equations which are linearizable to the heat equation. In some sense, we have obtained a 'linearization' of the KPP-Fisher equation.

1 Introduction

In this paper, we shall consider the following Cauchy problem for the KPP-Fisher equation:

$$\begin{aligned} &u_t = \epsilon^2 \triangle u + f(u), \quad \mathbf{x} \in \mathbb{R}^N \ (N \geqslant 1), \quad t > 0, \\ &u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $f \in C^1[0,1]$ satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f(u) > 0 \quad \text{for} \quad u \in (0,1), \tag{1.2}$$

and ϵ is any positive real number. This equation arises in several biological models for the propagation of genes and population dynamics (see elsewhere [1, 3, 4, 9], and the references therein).

In the one-dimensional case ($N = 1$), it is well-known that (1.1) admits a travelling wave front solution (unique up to translation) of the form $u(x,t) = \phi_c(x - c\,t)$ for every c satisfying $c \geqslant c^* > 0$. The constant c^* is called the minimal wave speed and ϕ_c is a monotonic decreasing function satisfying

$$\phi_c(-\infty) = 1, \quad \phi_c(+\infty) = 0.$$

The asymptotic behaviour of (1.1) has been well-studied, with special attention being given to finding appropriate initial conditions for which the solution converges to the travelling wave solution ϕ_{c^*} with minimal speed c^* [1, 2, 7, 8, 9]. In particular, when the initial function u_0 is a unit step function, Kolmogorov *et al.* [7] showed that the solution

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of (1.1) converges in some sense to ϕ_{c^*}. On the other hand, if the initial function has bounded support, then the solution converges to a pair of diverging travelling fronts [9].

Suppose that the initial condition is a pair of travelling fronts moving toward each other. Intuitively, one can expect that the fronts annihilate each other upon collision so that the solution tends asymptotically to $u \equiv 1$. The purpose of this paper is to show analytically that this is in fact what happens and, more importantly, to describe the annihilation dynamics of the solution as it evolves in time from the initial condition. We have in mind a general initial condition consisting of an arbitrary number of 'peaks' and 'valleys'.

In particular, when $0 < \epsilon \ll 1$, we can describe the annihilation dynamics quite accurately since (1.1) can be approximated by a *nonlinear* partial differential equation which is *linearizable* to the heat equation (see § 2). Our results are also applicable to higher-dimensional cases ($N \geqslant 2$). When $N = 2$, for example, we can consider an initial distribution consisting of an arbitrary number of 'spots'.

The method of proof is by a standard comparison theorem, where the comparison functions satisfy equations which are linearizable to the heat equation. In some sense, we have obtained a 'linearization' of the KPP-Fisher equation since we can describe, in principle, the evolution of the comparison functions for arbitrary initial conditions.

Some work related to ours was done by Hamel & Nadirashvili [5, 6]. They considered time-global solutions ($t \in \mathbb{R}$) of (1.1), and the mixing of any density of travelling fronts. Our method differs from theirs, and the results are obtained for more general initial conditions. In addition, we do not need to assume (as they did) that f is concave in $(0, 1)$.

In § 2, we construct upper and lower solutions of (1.1) which satisfy linearizable partial differential equations and then give our main result. In § 3, we apply this result to the Fisher case $f(u) = u(1-u)$. For various initial distributions, we give some numerical results showing how the comparison functions and the solution of Fisher's equation evolve in time. For some of these initial functions, it is possible to obtain explicit analytical forms for either the upper or lower solution. If $0 < \epsilon \ll 1$, then we can use these comparison functions to approximate the solution of Fisher's equation. Furthermore, we find a class of initial conditions such that if the solution of Fisher's equation should converge to a travelling front, then the speed would not be minimal. Finally, in § 4, we state some current work in progress which generalize our results.

2 Construction of upper and lower solutions of (1.1) and statement of main result

The derivation of our comparison functions will be done by using some explicit nonlinear transformations. More specifically, suppose that u can be expressed as

$$u = h(v), \tag{2.1}$$

where v satisfies the linear partial differential equation

$$v_t = \epsilon^2 \triangle v + \alpha v, \quad \alpha \neq 0. \tag{2.2}$$

We can then compute

$$\begin{aligned} N(u) &\equiv \epsilon^2 \triangle u - u_t + f(u) \\ &= \epsilon^2 (h_v \triangle v + h_{vv} |\nabla v|^2) - h_v v_t + f(h) \\ &= -\alpha v h_v + \epsilon^2 h_{vv} |\nabla v|^2 + f(h). \end{aligned}$$

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If we assume further that

$$h_v = \frac{f(h)}{\alpha v}, \tag{2.3}$$

then we get

$$N(u) = \epsilon^2 h_{vv} |\nabla v|^2 = \epsilon^2 \frac{h_{vv}}{h_v^2} |\nabla u|^2.$$

From (2.3), we can calculate that

$$\frac{h_{vv}}{h_v^2} = \frac{f'(u) - \alpha}{f(u)}.$$

Therefore, the function u satisfies the equation

$$u_t = \epsilon^2 \triangle u + f(u) - \epsilon^2 \frac{f'(u) - \alpha}{f(u)} |\nabla u|^2, \tag{2.4}$$

while v satisfies (2.2), and u and v are related by

$$\int_v^u \frac{ds}{f(s)} = \frac{1}{\alpha} \ln v, \quad v \in \mathbb{R}, \quad f(v) \neq 0. \tag{2.5}$$

If we can find a solution u of (2.4) satisfying $N(u) \leqslant 0$ (resp. $N(u) \geqslant 0$), then u is an upper (resp. lower) solution of (1.1).

A few remarks are in order here. First, we did not assume anything on the nonlinearity f except the fact that it should be differentiable. Second, the ansatz (2.1) did not involve the spatial variable $\mathbf{x}$ so we can also consider a more general initial-boundary value problem instead of (1.1). Finally, if we restrict ourselves to bounded solutions and let $0 < \epsilon \ll 1$, then the last term of (2.4) is very small so that in fact, (2.4) is a very good approximation to (1.1). This can be verified numerically (see § 3).

We now show that upper and lower solutions can be obtained straightforwardly if we assume that f satisfies (1.2). Letting $\beta \equiv \max_{u \in [0,1]} |f'(u)|$, it follows that

$$-\beta - \alpha \leqslant f'(u) - \alpha \leqslant \beta - \alpha$$

for every $u \in (0, 1)$.

For an upper solution u^+, we choose $\alpha = \alpha_1 \geqslant \beta$ so that u^+ satisfies the following:

$$\begin{aligned} u_t^+ &= \epsilon^2 \triangle u^+ + f(u^+) - \epsilon^2 \frac{f'(u^+) - \alpha_1}{f(u^+)} |\nabla u^+|^2, \\ \int_v^{u^+} \frac{ds}{f(s)} &= \frac{1}{\alpha_1} \ln v^+, \\ v_t^+ &= \epsilon^2 \triangle v^+ + \alpha_1 v^+. \end{aligned} \tag{2.6}$$

On the other hand, for a lower solution u^-, we choose $\alpha = \alpha_2 \leqslant -\beta$ so that u^- satisfies the following:

$$\begin{aligned} u_t^- &= \epsilon^2 \triangle u^- + f(u^-) - \epsilon^2 \frac{f'(u^-) - \alpha_2}{f(u^-)} |\nabla u^-|^2, \\ \int_v^{u^-} \frac{ds}{f(s)} &= \frac{1}{\alpha_2} \ln v^-, \\ v_t^- &= \epsilon^2 \triangle v^- + \alpha_2 v^-. \end{aligned} \tag{2.7}$$

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The corresponding initial functions for (2.6) and (2.7) will be denoted by u_0^+, v_0^+ and u_0^-, v_0^-, respectively.

We note that (2.2) can be mapped to the linear heat equation by the transformation $v = \exp(\alpha t)w$ to obtain

$$w_t = \epsilon^2 \triangle w, \quad w_0(\mathbf{x}) = v_0(\mathbf{x}) = \exp\left[\alpha \int_v^{u_0(\mathbf{x})} \frac{ds}{f(s)}\right]. \tag{2.8}$$

The general solution of (2.8) is given by

$$w(\mathbf{x}, t) = \frac{1}{(4\epsilon^2 \pi t)^{N/2}} \int_{\mathbb{R}^N} G(\mathbf{r}; \mathbf{x}, t) w_0(\mathbf{r})\, d\mathbf{r}, \tag{2.9}$$

where

$$G(\mathbf{r}; \mathbf{x}, t) = \exp\left(-\frac{|\mathbf{x} - \mathbf{r}|^2}{4\epsilon^2 t}\right), \ \mathbf{x} = (x_1, x_2, \ldots, x_N)^t, \ \mathbf{r} = (r_1, r_2, \ldots, r_N)^t.$$

Thus, the time-evolutions of $u^\pm$ with initial distributions $u_0^\pm$ are given by

$$\int_v^{u^\pm(\mathbf{x},t)} \frac{ds}{f(s)} =$$

$$\frac{1}{\alpha_{1,2}} \ln\left[\frac{\exp(\alpha_{1,2} t)}{(4\epsilon^2 \pi t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|\mathbf{x} - \mathbf{r}|^2}{4\epsilon^2 t} + \alpha_{1,2} \int_v^{u_0^\pm(\mathbf{r})} \frac{ds}{f(s)}\right) d\mathbf{r}\right]. \tag{2.10}$$

To see how (1.1) evolves, we choose the initial conditions such that $u_0^-(\mathbf{x}) \leqslant u_0(\mathbf{x}) \leqslant u_0^+(\mathbf{x})$ and substitute in (2.10).

Now, suppose that $u_0^\pm$ are both positive and continuous and satisfy

$$\inf_{\mathbf{x} \in \mathbb{R}^N} u_0^+(\mathbf{x}) > v, \quad \inf_{\mathbf{x} \in \mathbb{R}^N} u_0^-(\mathbf{x}) > v, \quad (0 < v < 1) \tag{2.11}$$

respectively. Then, it follows that $u_0^+(\mathbf{x}) > v$ for every $\mathbf{x} \in \mathbb{R}^N$ and

$$\alpha_1 \int_v^{u_0^+(\mathbf{r})} \frac{ds}{f(s)} > 0,$$

$$\alpha_1 \int_v^{u_0^+(\mathbf{r})} \frac{ds}{f(s)} - \frac{|\mathbf{x} - \mathbf{r}|^2}{4\epsilon^2 t} > -\frac{|\mathbf{x} - \mathbf{r}|^2}{4\epsilon^2 t},$$

$$\int_{\mathbb{R}^N} G(\mathbf{r}; \mathbf{x}, t) v_0^+(\mathbf{r})\, d\mathbf{r} > \int_{\mathbb{R}^N} G(\mathbf{r}; \mathbf{x}, t)\, d\mathbf{r} = (4\epsilon^2 \pi t)^{N/2},$$

$$v^+(\mathbf{x}, t) = \frac{\exp(\alpha_1 t)}{(4\epsilon^2 \pi t)^{N/2}} \int_{\mathbb{R}^N} G(\mathbf{r}; \mathbf{x}, t) v_0^+(\mathbf{r})\, d\mathbf{r} > \exp(\alpha_1 t).$$

Therefore, the above statement and the second statement in (2.6) imply that

$$\lim_{t \to +\infty} v^+(\mathbf{x}, t) = +\infty, \quad \lim_{t \to +\infty} u^+(\mathbf{x}, t) = 1.$$

In a similar manner, we obtain

$$v^-(\mathbf{x}, t) = \frac{\exp(\alpha_2 t)}{(4\epsilon^2 \pi t)^{N/2}} \int_{\mathbb{R}^N} G(\mathbf{r}; \mathbf{x}, t) v_0^-(\mathbf{r})\, d\mathbf{r} < \exp(\alpha_2 t).$$

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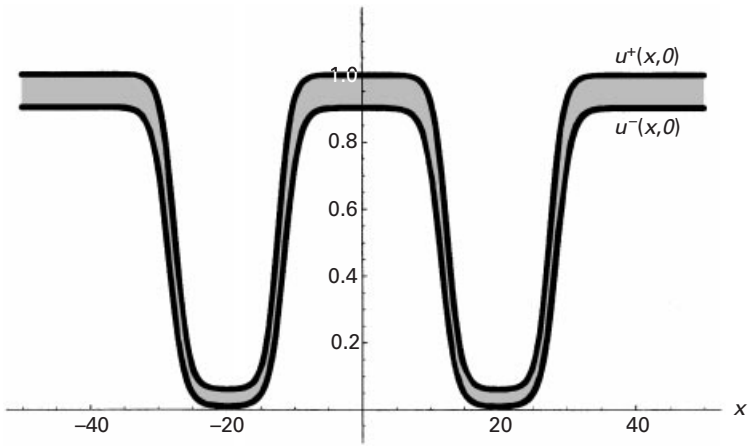


FIGURE 1. Profile of an initial condition of (1.1).

This statement and the second equation in (2.7) imply that

$$\lim_{t\to+\infty} v^-(\mathbf{x},t) = 0, \quad \lim_{t\to+\infty} u^-(\mathbf{x},t) = 1.$$

One of the referees remarked that with the hypotheses for the initial conditions, it immediately follows that $\lim_{t\to+\infty} u^-(\mathbf{x},t) = 1$, since one can use as a lower solution the function $u^- = u^-(t)$ given implicitly by $\int_v^{u^-(t)} ds/f(s) = t$. This is certainly true if one were interested only in the asymptotic behaviour. However, as we mentioned in the Introduction, we are interested not only in the asymptotic behaviour of the solution, but also in the dynamics of the annihilation process starting from the initial condition. With our choice of the lower solution, we can show both.

Based on the above results and invoking a comparison theorem for (1.1) [1], we can now state the following:

Main Result *Suppose that u_0^- and u_0^+ are nonconstant continuous functions in $\mathbb{R}^N$ satisfying $0 < v \leqslant u_0^-(\mathbf{x}) \leqslant u_0(\mathbf{x}) \leqslant u_0^+(\mathbf{x}) \leqslant 1$, where v is any arbitrary number satisfying $0 < v < 1$. Then,*

$$u^-(\mathbf{x},t) \leqslant u(\mathbf{x},t) \leqslant u^+(\mathbf{x},t)$$

for every $t \geqslant 0$, where the dynamics of $u^\pm$ are described by (2.10).

As a consequence of this result, we have

$$\lim_{t\to+\infty} u^-(\mathbf{x},t) = \lim_{t\to+\infty} u(\mathbf{x},t) = \lim_{t\to+\infty} u^+(\mathbf{x},t) = 1.$$

In the one-dimensional case, this result implies that if the initial function u_0 consists of an arbitrary number of 'peaks' and 'valleys' (see Figure 1, where u_0 is any continuous function in the shaded region), then they annihilate each other, and the solution eventually approaches the steady state $u \equiv 1$. What we would like to emphasize is that not only do we know the asymptotic behaviour of the solution, we can also describe the dynamics of the annihilation process from the dynamics of the upper and lower solutions as described by (2.10). A similar interpretation can be given for higher-dimensional cases as well.

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3 Numerical results and explicit approximate solutions of (1.1) when $f(u) = u(1-u)$

In this section, we consider Fisher's equation and specify $f(u) = u(1-u)$. From a direct computation, we get $\beta = 1$. Choosing $\alpha_1 = 1$, $\alpha_2 = -1$, and $v = 1/2$, we obtain the following upper and lower solutions:

$$\begin{aligned} u_t^+ &= \epsilon^2 \triangle u^+ + u^+(1-u^+) + \frac{2\epsilon^2}{1-u^+}|\nabla u^+|^2, \\ u^+ &= \frac{v^+}{1+v^+}, \quad v_t^+ = \epsilon^2 \triangle v^+ + v^+, \end{aligned} \tag{3.1}$$

$$\begin{aligned} u_t^- &= \epsilon^2 \triangle u^- + u^-(1-u^-) - \frac{2\epsilon^2}{u^-}|\nabla u^-|^2, \\ u^- &= \frac{1}{1+v^-}, \quad v_t^- = \epsilon^2 \triangle v^- - v^-. \end{aligned} \tag{3.2}$$

Alternatively, we also have

$$u^+ = \frac{v^+}{1+v^+}, \quad v^+(\mathbf{x},t) = \frac{\exp(t)}{(4\epsilon^2\pi t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|\mathbf{x}-\mathbf{r}|^2}{4\epsilon^2 t}\right) \frac{u_0^+(\mathbf{r})}{1-u_0^+(\mathbf{r})}\, d\mathbf{r}, \tag{3.3}$$

$$u^- = \frac{1}{1+v^-}, \quad v^-(\mathbf{x},t) = \frac{\exp(-t)}{(4\epsilon^2\pi t)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|\mathbf{x}-\mathbf{r}|^2}{4\epsilon^2 t}\right) \frac{1-u_0^-(\mathbf{r})}{u_0^-(\mathbf{r})}\, d\mathbf{r}. \tag{3.4}$$

For the numerical results, we use (3.1) and (3.2), while for the analytical results, we use (3.3) and (3.4). In all the numerical computations, we fix the mesh spacings to be $\Delta x = 0.01$, $\Delta t = 0.05$ and impose no-flux boundary conditions on a sufficiently large interval.

First, we study the one-dimensional case ($N = 1$). Let the initial conditions be

$$u_0(x) = u_0^+(x) = u_0^-(x) = \frac{1}{1+\exp(b\,x)}, \quad b > 0. \tag{3.5}$$

Making use of (3.3) and (3.4), we get

$$u^+(x,t) = \frac{1}{1+\exp(b\,x - (1+b^2\epsilon^2)\,t)}, \quad u^-(x,t) = \frac{1}{1+\exp(b\,x - (1-b^2\epsilon^2)\,t)}, \tag{3.6}$$

which are both travelling wave solutions. The wave speeds of u^+ and u^- are given by

$$c_{u^+} = \frac{1+b^2\epsilon^2}{b} \quad \text{and} \quad c_{u^-} = \frac{1-b^2\epsilon^2}{b}, \tag{3.7}$$

respectively. Note that if $0 < \epsilon \ll 1$, then

$$c_{u^+} \approx c_{u^-} \approx \frac{1}{b} \quad \text{and} \quad u^+ \approx u^- \approx \frac{1}{1+\exp(b\,x - t)}.$$

For Fisher's equation, it is known that the minimal wave speed is $c_F = 2\epsilon$. It is not difficult to see that $c_{u^+} \geqslant c_F$. On the other hand, if

$$\epsilon^2 b^2 + 2b\epsilon - 1 \leqslant 0,$$

then $c_{u^-} \geqslant c_F$. Assume that b is chosen such that the strict inequality is satisfied. It follows that if the solution of Fisher's equation converges to a travelling wave solution, then the speed will be *greater* than the minimal value c_F. This is in contrast to previous studies

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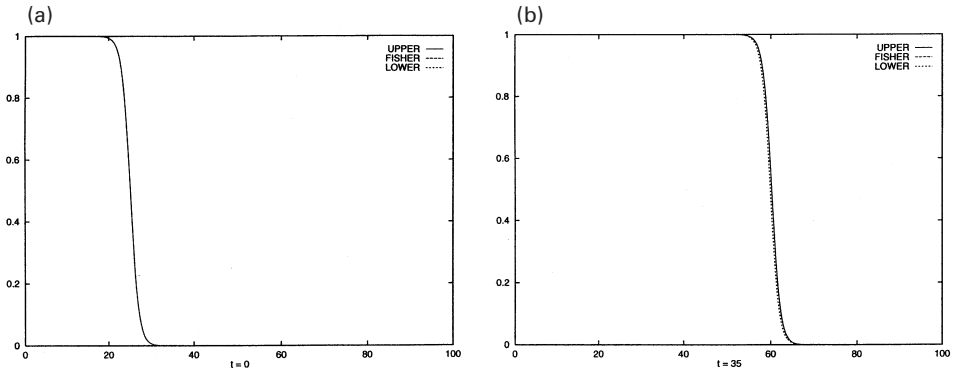


FIGURE 2. Profiles of (1.1), (3.1) and (3.2) with initial conditions (3.5) for $t = 0, 35$, $\epsilon^2 = 0.03$, and $b = 1$.

done on Fisher's equation, where initial conditions are sought for which the solution converges to the wave of minimal speed.

Solving (1.1), (3.1) and (3.2) numerically, we obtain the profiles of u, u^+ and u^-, shown in Figure 2, where $\epsilon^2 = 0.03$, $b = 1$, and $c_F = 0.34641016$. From (3.7), we get $c_{u^+} = 1.03$ and $c_{u^-} = 0.97$.

Next, we assume more complicated initial conditions such as

$$u_0(x) = u_0^+(x) = u_0^-(x) = \varphi(x - 30) + \varphi(x + 30) + \frac{1}{2}\varphi(x) - \frac{1}{4}, \tag{3.8}$$

where

$$\varphi(x) = \frac{1}{2} - \frac{1}{1 + m\cosh(b\,x)}, \quad b, m > 0.$$

In this case, it is not possible to obtain closed analytic forms from (3.3) and (3.4). However, we can integrate (1.1), (3.1) and (3.2) numerically, and compare the actual solution with the upper and lower solutions. The results are shown in Figure 3, where $\epsilon^2 = 0.1$, $b = 1$ and $m = 10^{-4}$.

For the remainder of this section, we consider other initial distributions for $N = 1, 2$. In all of the examples, it is only possible to obtain a closed analytical expression for either the upper or lower solution, but not both. In any case, we can still compare the actual solution with the upper and lower solutions by integrating (1.1), (3.1) and (3.2) numerically. Here, we will only show the numerical results for $N = 1$.

(1) Let

$$u_0(x) = u_0^+(x) = u_0^-(x) = 1 - \frac{1}{1 + m\cosh(b\,x)}, \quad b, m > 0. \tag{3.9}$$

Evaluating (3.3), we get

$$u^+(x,t) = \frac{m\exp(t + b^2\epsilon^2\,t)\,(1 + \exp(2\,b\,x))}{2\exp(b\,x) + m\exp(t + b^2\epsilon^2\,t)\,(1 + \exp(2\,b\,x))}. \tag{3.10}$$

Solving (1.1), (3.1) and (3.2) numerically, we obtain the profiles of u, u^+ and u^- shown in Figure 4, where $\epsilon^2 = 0.1$, $b = 0.9$ and $m = 10^{-7}$.

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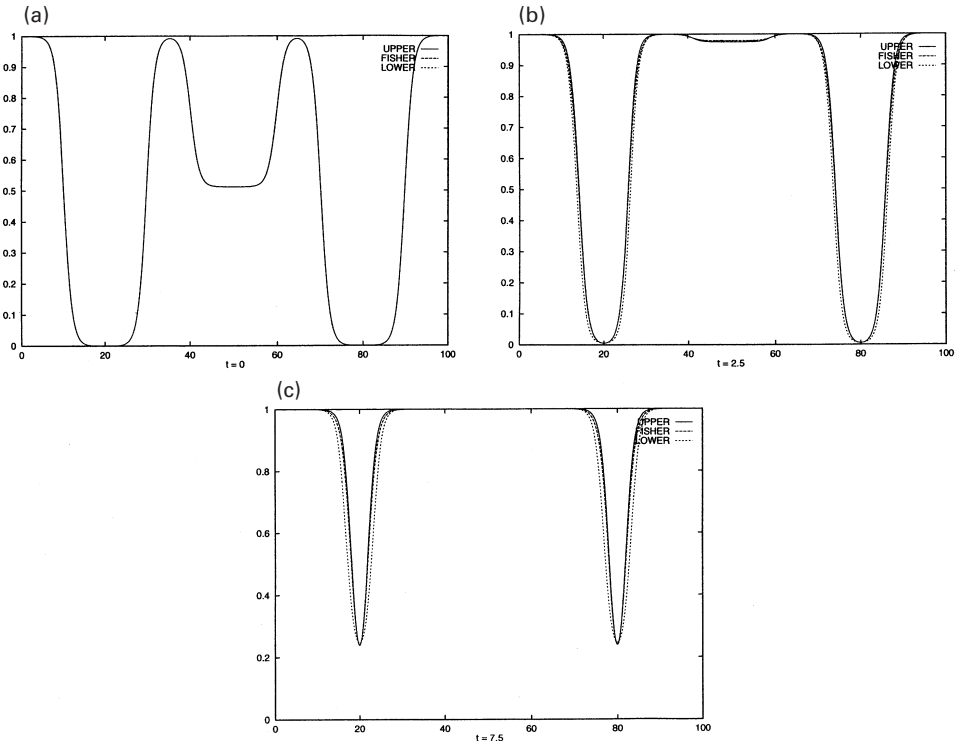


FIGURE 3. Profiles of (1.1), (3.1) and (3.2) with initial conditions (3.8) for $t = 0, 2.5, 7.5$, $\epsilon^2 = 0.1$, $b = 1$ and $m = 10^{-4}$.

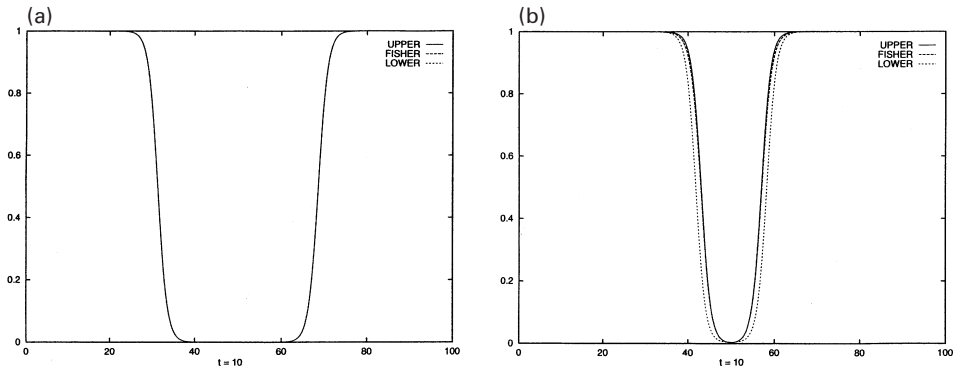


FIGURE 4. Profiles of (1.1), (3.1) and (3.2) with initial conditions (3.8) for $t = 0, 10$, $\epsilon^2 = 0.1$, $b = 0.9$ and $m = 10^{-7}$.

(2) Let

$$u_0(x) = u_0^+(x) = u_0^-(x) = \frac{1}{1 + m\cosh(b\,x)}, \quad b, m > 0. \tag{3.11}$$

Evaluating (3.4), we get

$$u^-(x,t) = \frac{2}{2 + m\exp(-b\,x - t + b^2\epsilon^2\,t)\,(1 + \exp(2\,b\,x))}. \tag{3.12}$$

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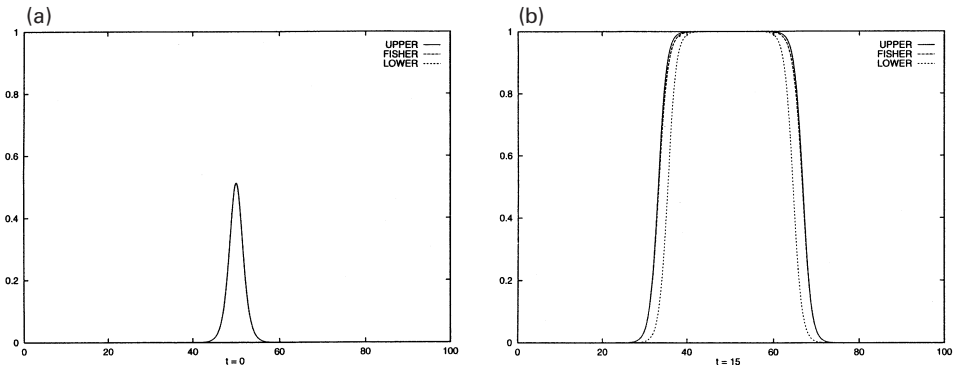


FIGURE 5. Profiles of (1.1), (3.1) and (3.2) with initial conditions (3.11) for $t = 0, 15$, $\epsilon^2 = 0.1$, $b = 1$ and $m = 1$.

Solving (1.1), (3.1) and (3.2) numerically, we obtain the profiles of u, u^+ and u^- shown in Figure 5, where $\epsilon^2 = 0.1$, $b = 1$, and $m = 1$.

(3) Let

$$u_0^-(x, y) = \frac{1}{1 + m \exp(-b(x^2 + y^2))}, \quad b, m > 0.$$

Evaluating (3.4), we get

$$u^-(x, y, t) = \frac{1 + 4b\epsilon^2 t}{1 + 4b\epsilon^2 t + m \exp\left(-t - \frac{b(x^2+y^2)}{1+4b\epsilon^2 t}\right)}. \tag{3.13}$$

This is a two-dimensional radially symmetric version of (3.10).

(4) Let

$$u_0^+(x, y) = \frac{m \exp(-b(x^2 + y^2))}{1 + m \exp(-b(x^2 + y^2))}, \quad b, m > 0.$$

Evaluating (3.3), we get

$$u^+(x, y, t) = \frac{m \exp(t)}{m \exp(t) + \exp\left(\frac{b(x^2+y^2)}{1+4b\epsilon^2 t}\right)(1 + 4b\epsilon^2 t)}. \tag{3.14}$$

This is a two-dimensional radially symmetric version of (3.12).

4 Conclusion

In this paper, we studied the annihilation dynamics for the KPP-Fisher equation. The method we used is to find 'nice' comparison functions which satisfy equations linearizable to the heat equation. Our method can be generalized to other problems, some of which are the following:

(1) dynamics of multi-dimensional fronts of (1.1) for different boundary conditions;

(2) dynamics of metastable layers in (1.1) (in bounded and unbounded intervals) where f is the derivative of a double-well potential;

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(3) extensions to systems of reaction-diffusion equations which satisfy a comparison principle, e.g. a Lotka-Volterra competition-diffusion system;

(4) free boundary problems for (1.1) which can be reduced to a two-phase Stefan problem;

(5) extensions to density-dependent nonlinear diffusion equations.

These problems are currently under investigation.

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