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On free actions on Λ-trees

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Abstract

In this paper we show that if a group generated by two elements acts freely and without phantom inversions on a Λ-tree, then it is either free or free abelian.

0. *Introduction*

In [**4**] Fine, Rohl and Rosenberger ask if a 2-generator group acting freely on a non-archimedean Λ-tree T must be free or free abelian. In this paper we show that this is indeed the case for both archimedean and non-archimedean Λ-trees. Our main result is as follows:

THEOREM. *Let G be a group acting freely by isometries and without phantom inversions on a Λ-tree T, where Λ is an arbitrary ordered abelian group. Let $g, h \in G \setminus \{1\}$. Then the subgroup $\langle g, h \rangle$ of G generated by g and h is either free or free abelian.*

In Section 1 we establish the result for archimedean Λ-trees. In this case, the fact that $\langle g, h \rangle$ is either free or free abelian was proved by Harrison in [**5**] in the context of abstract length functions. Our proof is short and generalizes to the non-archimedean setting. In Section 2 we establish some technical results concerning ordered abelian groups. Finally, in Section 3 we use the results of Section 2 to extend the methods of Section 1 to the general case.

Let Λ be a (linearly) ordered abelian group. Λ is said to be archimedean if it is order-isomorphic to an additive subgroup of $\mathbb{R}$. Otherwise Λ is said to be non-archimedean.

Following [**1**] and [**6**], a Λ-tree is a Λ-metric space (T, d), i.e. a set T together with a distance function $d: T \times T \to \Lambda$ satisfying the usual metric axioms, subject to the following three conditions:

(1) T is uniquely connected by segments. (By a segment we mean a subset of T isometric to a closed interval in Λ. The unique segment between two points x and y in T is denoted $[x, y]$.)

(2) If $x, y, z \in T$, then $[x, y] \cap [x, z] = [x, w]$ for some $w \in T$.

(3) If $x, y, z \in T$ and $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$.

In the case when $\Lambda = \mathbb{R}$, this is equivalent to saying that T is a real metric space in which any two points are joined by a unique topological arc isometric to a closed interval in $\mathbb{R}$: see [**6**].

Suppose that G is a group acting by isometries on a Λ-tree T. If $g \in G$ is not a phantom inversion, meaning that if g leaves some closed segment A in T invariant,

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then either g has a fixed point in A or g acts on A by a translation, then the length of g is defined by

$$|g| = \inf_{x \in T} \{d(x, g(x))\}.$$

In [1] and [6] it is shown that $|g|$ always exists and that $|g|$ is equal to 0 if and only if g has a fixed point. In the case when $|g| > 0$, the axis of g, denoted by C_g and defined by

$$C_g = \{x \in T \mid d(x, g(x)) = |g|\},$$

is a subtree of T isometric to a subtree of Λ and the action of g restricted to C_g is by translation $|g|$. In the case when $\Lambda = \mathbb{R}$, C_g is isometric to $\mathbb{R}$. The action of g on C_g induces a natural orientation on C_g so that $x < g(x)$. Therefore g^{-1} induces the opposite orientation.

1. *The Archimedean case*

In this section we establish our main result in the case when Λ is an archimedean ordered abelian group.

Let G be a group admitting a free action without phantom inversions on an $\mathbb{R}$-tree T. We begin with a result of Culler and Morgan. (See lemma 2·6 in [2].)

LEMMA 1·1. *If $g, h \in G \backslash \{1\}$ and $|C_g \cap C_h| < \min\{|g|, |h|\}$, then $\langle g, h \rangle$ is a free group of rank two.*

LEMMA 1·2. *If $g, h \in G \backslash \{1\}$ and $|C_g \cap C_h| \geq |g| + |h|$, then $\langle g, h \rangle$ is free abelian.*

Proof. We can assume without loss of generality that the orientations induced by g and h agree on the overlap $C_g \cap C_h$; otherwise we merely replace g by g^{-1}. By hypothesis, there exists an element $x \in C_g \cap C_h$ such that both $gh(x)$ and $hg(x)$ are in $C_g \cap C_h$. It follows that $gh(x) = hg(x)$ and therefore that $gh = hg$. This implies that $\langle g, h \rangle$ is abelian and that $C_g = C_h$. Since C_g is isometric to $\mathbb{R}$ it follows that the abelian group $\langle g, h \rangle$ acts freely by isometries on $\mathbb{R}$ and hence must be isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. ∎

PROPOSITION 1·3. *If $g, h \in G \backslash \{1\}$ and $\min\{|g|, |h|\} \leq |C_g \cap C_h| < |h| + |g|$, then $\langle g, h \rangle$ is a free group of rank two. In fact, there exists a pair of generators for $\langle g, h \rangle$ whose axes are disjoint.*

Proof. Note that the hypotheses of Proposition 1·3 imply that g and h do not commute, for otherwise $C_g = C_h$. As in the case of Lemma 2, we can assume that the orientations induced by g and h agree on the overlap $C_g \cap C_h$. Let Δ denote $C_g \cap C_h$ and $|\Delta|$ denote its length. Let a and b denote the two endpoints of Δ so that $\Delta = [a, b] \subset T$. We can assume without loss of generality that $|h| = \min\{|g|, |h|\}$. It follows therefore that $|h| < |g|$, for otherwise $\langle g, h \rangle$ would be abelian, contrary to our hypothesis. The proof of Proposition 1·3 depends on the following key lemma.

LEMMA 1·4. *Let $n = n(g, h)$ be the largest positive integer such that $h^n(a) \leq \min\{b, g(a)\}$. Then $|h^{-n}g| \leq |g| - n|h|$ and $C_h \cap C_{h^{-n}g} \subset [a, h^{-n}(b)]$. Moreover, $|h^{-n}g| = |g| - n|h|$ if and only if $C_h \cap C_{h^{-n}g} \neq \emptyset$, in which case $C_h \cap C_{h^{-n}g} = [a, h^{-n}(b)]$.*

Proof. Let $x \in C_h$. We estimate the distance $d(x, h^{-n}g(x))$. Note that $h^{-n}g \neq 1$, for otherwise $C_g = C_h$, contrary to assumption.

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Case 1. $x < a$. Then $d(x, h^{-n}g(x)) \geqslant 2d(x,a) + |g| - n|h| > |g| - n|h|$.

Case 2. $a \leqslant x \leqslant h^{-n}(b)$. Then $d(x, h^{-n}g(x)) = |g| - n|h|$.

Case 3. $h^{-n}(b) < x \leqslant b$. Then $d(x, h^{-n}g(x)) = |g| + n|h| - 2d(x,b) > |g| + n|h| - 2n|h| = |g| - n|h|$.

Case 4. $x \geqslant b$. Then $d(x, h^{-n}g(x)) \geqslant |g| + n|h| > |g| - n|h|$.

Thus $|h^{-n}g| \leqslant |g| - n|h|$, and $C_h \cap C_{h^{-n}g} \subset [a, h^{-n}(b)]$. The rest of the lemma follows immediately from Cases 1–4 and the definition of the length of an element of G. ∎

We now return to the proof of Proposition 1·3. Consider the new set of generators $\{h, h^{-n}g\}$ for $\langle g, h\rangle$. Let Δ' denote $C_h \cap C_{h^{-n}g}$. If $\Delta' = \varnothing$, then it follows from Lemma 1·1 that $\langle g, h\rangle$ is a free group of rank two. On the other hand if $\Delta' \neq \varnothing$, then by Lemma 1·4, $\Delta' = [a, h^{-n}(b)]$ and

$$|h^{-n}g| + |h| - |\Delta'| = |g| + |h| - |\Delta| > 0.$$

Thus, if $\Delta' \neq \varnothing$, we can apply Lemma 1·4 to the new set of generators $\{h, h^{-n}g\}$ to obtain still a new set of generators. If the corresponding axes are disjoint then we are done by Lemma 1·1. Otherwise, we can repeatedly apply the construction of Lemma 1·4. At some point we will obtain a pair of generators for $\langle g, h\rangle$ whose axes are disjoint, for otherwise we would obtain an infinite sequence $\{h_k, g_k\}$ of pairs of generators satisfying $h_{k+1} = g_k$, $|g_{k+1}| = |h_k| - n_k|g_k|$, for some positive integer n_k, and

$$|h_k| + |g_k| - |C_{h_k} \cap C_{g_k}| = |h| + |g| - |\Delta| > 0$$

with $|C_{h_k} \cap C_{g_k}| \geqslant 0$. But this is impossible since $\lim_{k\to\infty}(|h_k| + |g_k|) = 0$. In fact, since the $|g_k|$'s form a decreasing sequence of positive real numbers, the $\lim_{k\to\infty}|g_k|$ exists. Moreover as $h_{k+1} = g_k$, it follows that $\lim_{k\to\infty}|h_k| = \lim_{k\to\infty}|g_k|$. Finally, the relation $|g_{k+1}| = |h_k| - n_k|g_k|$ implies that both limits are equal to zero. ∎

THEOREM 1·5. *Let Λ be an archimedean ordered abelian group, and let G be a group acting freely by isometries on a Λ-tree T. Let $g, h \in G\backslash\{1\}$. Then $\langle g, h\rangle$ is either a free group of rank two or is free abelian.*

Proof. In [**6**] Morgan and Shalen show that the free action of G on T can be extended to a free action of G on an $\mathbb{R}$-tree $T_{\mathbb{R}}$. This is because Λ is a subgroup of $\mathbb{R}$. In fact, if Λ is discrete, then $T_{\mathbb{R}}$ is obtained as the geometric realization of a simplicial complex, while if Λ is dense in $\mathbb{R}$, $T_{\mathbb{R}}$ is the metric completion of T. Finally the result follows from Lemma 1·1, Lemma 1·2 and Proposition 1·3. ∎

2. *A division algorithm in ordered abelian groups*

In Section 3 we generalize the methods of Section 1 to prove the main result for general ordered abelian groups Λ. In particular we need to modify the proof of Proposition 1·3 to the non-archimedean case. The first difficulty which arises is that the integer $n(g,h)$ in Lemma 1·4 may fail to exist, and the second is that $\lim_{k\to\infty}(|h_k| + |g_k|)$ may not exist in Λ. In this section we establish some technical results concerning arbitrary ordered abelian groups. The main result of this section is Proposition 2·5.

Definition 2·1. Let Λ be an ordered abelian group. A pair of positive elements

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$\{a_0, a_1\}$ in Λ is called *strongly archimedean* if there exists a sequence of positive elements $\{a_k\}_{k\geqslant 0}$ in Λ with $a_0 > a_1 > a_2 > \ldots$ where $a_{k+1} = a_{k-1} - n_k a_k$ for $k \geqslant 1$, and where n_k's are positive integers chosen so that $a_{k-1} - (n_k+1)\, a_k < 0$.

Thus, associated to a strong archimedean pair is the sequence $\{a_k\}$ in Λ as well as the sequence of positive integers $\{n_k\}$.

LEMMA 2·2. *Suppose that $\{a_0, a_1\}$ is strongly archimedean in an ordered abelian group Λ. Then, given $ma_0 - na_1 \in \Lambda$ with $m, n \in \mathbb{Z}^+$, there exists an effective procedure, depending only on the induced sequence of positive integers $\{n_k\}$, to check the following inequality: $ma_0 - na_1 > 0$.*

Proof. Since $mn_1 a_1 < ma_0$ while $(mn_1 + m)\, a_1 > ma_0$, to check whether $na_1 < ma_0$ it suffices to check whether

$$(mn_1 + i_1)\, a_1 > ma_0$$

for all $1 \leqslant i_1 \leqslant (m-1)$, or, equivalently, whether

$$ma_2 < i_1 a_1 \qquad (*)$$

for all $1 \leqslant i_1 \leqslant (m-1)$. Since $i_1 n_2 a_2 < i_1 a_1$ while $(i_1 n_2 + i_1)\, a_2 > i_1 a_1$, to check (*) for each i_1 it suffices to check whether

$$(i_1 n_2 + i_2)\, a_2 > i_1 a_1$$

for all $1 \leqslant i_2 \leqslant (i_1 - 1)$, or, equivalently, whether

$$i_1 a_3 < i_2 a_2$$

for all $1 \leqslant i_2 \leqslant (i_1 - 1)$. Similarly, to check this last inequality for each such i_2 it suffices to check whether

$$i_2 a_4 < i_3 a_3$$

for all $1 \leqslant i_3 \leqslant (i_2 - 1)$.

Eventually this process terminates with having to check whether

$$i_{k-1} a_{k+1} < i_k a_k$$

where $i_k = 1$, that is, whether $i_{k-1} a_{k+1} < a_k$. But this last inequality is readily checked by comparing i_{k-1} with n_{k+1}; thus the inequality holds if and only if $i_{k-1} \leqslant n_{k+1}$. ∎

As a consequence of the above Lemma we have

COROLLARY 2·3. *Let $\{a_0, a_1\}$ and $\{a_0', a_1'\}$ be strongly archimedean in Λ and Λ' respectively, where Λ and Λ' are arbitrary ordered abelian groups. Then the mapping $a_0 \mapsto a_0'$, $a_1 \mapsto a_1'$ induces an order-preserving group isomorphism from the subgroup of Λ generated by a_0 and a_1 to the subgroup of Λ' generated by a_0' and a_1' if and only if the two pairs induce the same sequence of positive integers $\{n_k\}$.*

The following lemma is a well known fact from the theory of continued fractions. (For example, see Davenport [3].)

LEMMA 2·4. *Let $\{n_k\}_{k\geqslant 1}$ be an arbitrary sequence of positive integers. Then there exist positive real numbers r_0 and r_1 such that the pair $\{r_0, r_1\}$ is strongly archimedean in $\mathbb{R}$ and induces the sequence of positive integers $\{n_k\}_{k\geqslant 1}$.*

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PROPOSITION 2·5. *Let a_0 and a_1 be positive elements in an ordered abelian group Λ, with $a_0 > a_1$. Then $\{a_0, a_1\}$ is strongly archimedean in Λ if and only if the subgroup of Λ generated by the pair $\{a_0, a_1\}$ is an archimedean ordered abelian group of rank two.*

Proof. Suppose that $\{a_0, a_1\}$ generate an archimedean ordered abelian group of rank two. Then we can define a sequence of non-negative elements $\{a_k\}$ as in Definition 2·1. The fact that the group is archimedean ensures the existence of the n_k's while the condition on the rank ensures that a_ks are all different from zero. As for the converse, suppose that $\{a_0, a_1\}$ is strongly archimedean in Λ. Then by Lemma 2·4 there exist real numbers r_0 and r_1 such that the pair $\{r_0, r_1\}$ is strongly archimedean in $\mathbb{R}$ and induces the same sequence of positive integers $\{n_k\}_{k \geqslant 1}$ as that induced by $\{a_0, a_1\}$. Using Corollary 2·3 it follows that the mapping which sends a_0 to r_0 and a_1 to r_1 induces an order-preserving group isomorphism from the group generated by $\{a_0, a_1\}$ to the additive subgroup of $\mathbb{R}$ generated by $\{r_0, r_1\}$. Moreover, the latter group is clearly of $\mathbb{Q}$-rank two. ∎

3. *The general case*

If a group G acts freely on an $\mathbb{R}$-tree T, then the intersection of two distinct axes is either empty or a closed segment (possibly degenerate) of finite length. We begin by showing that this holds for all Λ-trees.

Throughout this section we will assume that G is a group admitting a free action without phantom inversions on a Λ-tree T.

LEMMA 3·1. *Let $g, h \in G \backslash \{1\}$. If $\langle g, h \rangle$ is not abelian, then $C_g \cap C_h$ is either empty or a closed segment of the form $[a, b]$ for some $a, b \in T$.*

Proof. Suppose that $C_g \cap C_h \neq \varnothing$. Let Δ denote $C_g \cap C_h$. As usual we assume that the orientations induced by g and h agree on Δ. We will show that there exists a point $x \in C_g$ such that $x \geqslant y$ for all $y \in \Delta$. Suppose to the contrary. Then, for any $x \in \Delta$, we have $g(x) \in \Delta$. Clearly $h(x)$ and $hg(x)$ cannot simultaneously be in Δ for otherwise $gh(x)$ and $hg(x)$ would both lie in Δ, implying that g and h commute. Hence there exists some point $z_0 \in C_h$ which is strictly greater than each point $x \in \Delta$. Now fix a point $x \in \Delta$ and let y denote the second endpoint of $[x, z_0] \cap [x, g(z_0)]$. (See the definition of Λ-trees in Section 0.) Then $z_0 \geqslant y > z$ for all $z \in \Delta$. Moreover, since

$$y \in [x, g(z_0)] = [x, g(x)] \cup [g(x), g(z_0)],$$

it follows that $y \in [g(x), g(z_0)] = g([x, z_0])$. Thus there exists a point $z \in C_h \backslash \Delta$ with $y = g(z)$. Regarding C_h as a subtree of Λ we can write $z - x = g(z) - g(x)$ or $g(z) - z = g(x) - x = |g|$, contradicting that $z \notin C_g$. This establishes that there exists a point $x \in C_g$ with $x \geqslant y$ for all $y \in \Delta$.

Similarly there exists a point x' in C_h which is greater than or equal to all points in Δ. Let y be any point in Δ and let b be the second endpoint of $[y, x] \cap [y, x']$. Then b is independent of the choice of point $y \in \Delta$ and points x, x' having the above properties. Clearly $b \in \Delta$ and represents the 'right'-hand endpoint of Δ. By reversing all of the inequalities above we can show that there exists a point $a \in \Delta$ representing the 'left'-hand endpoint of Δ. ∎

Thus, if $g, h \in G \backslash \{1\}$ do not commute, then $\Delta = C_g \cap C_h$ is either empty or a closed segment of the form $[a, b]$ and of length $|\Delta| = d(a, b)$.

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LEMMA 3·2. *Let $g, h \in G\backslash\{1\}$. Suppose that $|\Delta| < \min\{|g|, |h|\}$. Then $\langle g, h\rangle$ is a free group of rank two.*

Proof. Note that this is merely the extension of Lemma 1·1 (which is lemma 2·6 in [**2**]) to the general setting. Once it is established that Δ is either empty or of the form $[a, b]$, Culler and Morgan's proof goes through word for word in the non-archimedean case. Alternatively, a straightforward induction argument on the length $|w|$ of a reduced word w in g and h shows that if $x \in \Delta$ then $w(x) \notin \Delta$. (See Proposition 3·5.) ∎

LEMMA 3·3. *If $g, h \in G\backslash\{1\}$ and $gh \neq hg$, then $|\Delta| < |g| + |h|$.*

Proof. The proof of Lemma 1·2 goes through word for word to show that if $|\Delta| \geqslant |g| + |h|$ then g and h commute. ∎

THEOREM 3·4. *Let G be a group acting freely by isometries and without phantom inversions on a Λ-tree T where Λ is an ordered abelian group. Let $g, h \in G\backslash\{1\}$. Then $\langle g, h\rangle$ is either free or free abelian.*

Proof. In view of Lemma 3·2 and Lemma 3·3, it remains to consider the case in which

$$\min\{|g|, |h|\} \leqslant |C_g \cap C_h| < |h| + |g|.$$

Thus, henceforth we shall assume that $g, h \in G\backslash\{1\}$ with $gh \neq hg$, $\Delta = C_g \cap C_h = [a, b]$, the orientations induced by g and h agree on Δ, and

$$|h| = \min\{|g|, |h|\} \leqslant |C_g \cap C_h| < |h| + |g|.$$

We begin by introducing some notation which will occur in the next Proposition.

Let $g, h \in G\backslash\{1\}$, $\Delta = C_g \cap C_h = [a, b] \subset T$. Let W_g be the set of all reduced words in g and h which end (on the left) with g. (For example, $g^3h^{-5} \in W_g$.) The set W_h is defined analogously. We next define the set $W_g(a)$ to be the set of points $w(a)$ in T where $w \in W_g$ and such that for each initial subword u of w (that is, a word u such that we can write $w = vu$ with v possibly the empty word), $u(a) \in [a, b]$. Similarly we define the sets $W_g(b)$, $W_h(a)$, and $W_h(b)$. Next we put

$$W_g(a, b) = W_g(a) \cup W_g(b) \cup \{a, b\}.$$

$W_h(a, b)$ is defined analogously with g replaced by h.

PROPOSITION 3·5. *Suppose that for each non-zero integer n*

$$h^n(W_g(a, b)) \cap \{a, b\} = g^n(W_h(a, b)) \cap \{a, b\} = \varnothing.$$

Then $\langle g, h\rangle$ is a free group of rank two.

Proof. We begin by establishing a few technical Lemmas.

LEMMA 3·6. *Under the hypotheses of Proposition 3·5, if $[x, y] \subset T$ is a segment with $y \in W_h(a, b)$ and $[x, y] \cap C_g = \{y\}$, then for every integer $k \neq 0$ there exists a point $z \in W_g(a, b)$ with $[g^k(x), z] \cap C_h = \{z\}$ and $d(g^k(x), z) \geqslant d(x, y)$. In particular if $x \neq y$, then $g^k(x) \notin [a, b]$.*

Proof. If $g^k(y) \in [a, b]$, then $z = g^k(y) \in W_g(a, b)$ has all the desired properties; in fact one readily verifies in this case that $d(g^k(x), z) = d(x, y)$ and that $[g^k(x), z] \cap C_h =$

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$g^k([x,y]) \cap C_h = \{g^k(y)\}$. If $g^k(y) \notin [a,b]$, then take $z \in \{a,b\}$ to be the other endpoint of the geodesic from $g^k(y)$ to the closed segment $[a,b]$. By definition, $z \in \{a,b\} \subset W_g(a,b)$, and

$$d(g^k(x),z) = d(g^k(x),g^k(y)) + d(g^k(y),z) = d(x,y) + d(g^k(y),z) > d(x,y).$$

Finally one readily verifies that $[g^k(x),z] \cap C_h = \{z\}$. ∎

By reversing the role of g and h we obtain

LEMMA 3·7. *Under the hypotheses of Proposition* 3·5, *if* $[x,y] \subset T$ *is a segment with* $y \in W_g(a,b)$ *and* $[x,y] \cap C_h = \{y\}$, *then for every integer* $k \neq 0$ *there exists a point* $z \in W_h(a,b)$ *with* $[h^k(x),z] \cap C_g = \{z\}$ *and* $d(h^k(x),z) \geqslant d(x,y)$. *In particular if* $x \neq y$, *then* $h^k(x) \notin [a,b]$.

We now return to the proof of Proposition 3·5. Let $w \in W_g \cup W_h$. It is enough to show that $w(a) \neq a$. If $w(a) \in W_g(a) \cup W_h(a)$, then this follows immediately from the hypotheses. Otherwise let u be the shortest initial subword of w such that $u(a) \notin [a,b]$ and such that if $w = vu$, then the first letter of v is distinct from the last letter of u. Without loss of generality, we can suppose that $u \in W_h$. Then $u(a) \in C_h \backslash [a,b]$, and again without loss of generality, we can suppose that $[a,b] \subset [a,u(a)] \subset C_h$. Thus $[u(a),b] \cap C_g = \{b\}$ and $d(u(a),b) > 0$. Applying now Lemma 3·6 and Lemma 3·7 to all the remaining powers of g and h in w, we conclude that $w(a) \notin [a,b]$. ∎

COROLLARY 3·8. *If* $|h| \leqslant |C_g \cap C_h| < |g|$, *then* $\langle g,h \rangle$ *is a free group of rank two.*

Proof. If for each positive integer n, we have $h^n(a) \neq b$, then the result follows immediately from Proposition 3·5. Otherwise, if $h^n(a) = b$ for some positive integer n, then by Lemma 1·4 we deduce that $C_h \cap C_{h^{-n}g}$ either is empty or consists of a single point. We should remark at this point that the proof of Lemma 1·4 holds word for word in this general setting. The result now follows by Lemma 3·2. ∎

COROLLARY 3·9. *Suppose that for each positive integer* n, *we have* $h^n(a) < \min\{b, g(a)\}$. *Then* $\langle g,h \rangle$ *is a free group of rank two.*

Proof. If $g(a) \neq b$, then the result follows immediately from Proposition 3·5. If $g(a) = b$, then one verifies that $|h^{-1}g| = |C_{h^{-1}g} \cap C_g| = |g| - |h| < |g|$. The result now follows from Corollary 3·8. ∎

We now return to the Proof of Theorem 3·4. In view of Corollary 3·8 and Corollary 3·9, it suffices to consider the case in which there exists a largest positive integer $n = n(g,h)$ such that $h^n(a) \leqslant \min\{b, g(a)\} = g(a)$. In this case we can apply Lemma 1·4 to obtain a new pair of generators $\{h, h^{-n}g\}$ for $\langle g,h \rangle$. If $\Delta' = C_h \cap C_{h^{-n}g} = \varnothing$, then we are done by Lemma 3·2. Otherwise, by Lemma 1·4, $\Delta' = [a, h^{-n}(b)]$, and

$$|h^{-n}g| + |h| - |\Delta'| = |g| + |h| - |\Delta| > 0.$$

We can now try to iterate this procedure indefinitely to obtain a sequence of elements $g_0, g_1, g_2, \ldots$ in G with $g_0 = g$, $g_1 = h$, $\langle g_k, g_{k+1} \rangle = \langle g,h \rangle$, $|g_0| > |g_1| > |g_2| > \ldots$, $|g_{k+1}| = |g_{k-1}| - n_k|g_k|$, where n_k is the largest positive integer such that

$$|g_{k-1}| - n_k|g_k| > 0.$$

By Lemma 1·4 this process can be iterated indefinitely as long as at each stage the n_k's exist and $\Delta_k = C_{g_k} \cap C_{g_{k+1}} \neq \varnothing$. By Lemma 3·2 and Corollary 3·9 we can assume

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that the n'_k exist and that the Δ'_k are non-empty. This implies that the pair $\{|g_0|, |g_1|\}$ is strongly archimedean in Λ. (See Definition 2·1).

By Lemma 1·4 we have

$$|g_k| + |g_{k+1}| - |\Delta_k| = |g| + |h| - |\Delta| > 0$$

and $|\Delta_k| \geqslant 0$. It follows from Proposition 2·5 that $\lim_{k\to\infty}(|g_k|) = 0$ in $\langle |g|, |h| \rangle$. By this we mean that for each $\epsilon > 0$ in $\langle |g|, |h| \rangle$, for k sufficiently large $0 < |g_k| < \epsilon$. Therefore $|\Delta| \notin \langle |g|, |h| \rangle$. This together with the fact that $\langle |g|, |h| \rangle$ is free abelian of rank two (see Proposition 2.5) imply that for each non-zero integer n we have

$$h^n(W_g(a,b)) \cap \{a,b\} = g^n(W_h(a,b)) \cap \{a,b\} = \varnothing.$$

By Proposition 3·5, this completes the proof of Theorem 3·4. ∎

COROLLARY 3·10. *Under the hypotheses of Theorem* 3·4, *G either is abelian or contains a free group of rank two.*

Proof. Although this is an immediate consequence of Theorem 3·4, we exhibit an alternative proof to illustrate that the statement of Corollary 3·10 is in fact much weaker than that of Theorem 3·4. Let g and h be arbitrary elements of G. If G does not contain a free group of rank two, it follows from Lemma 3·2 that

$$|gC_h \cap C_h| = |C_{ghg^{-1}} \cap C_h| \geqslant \min\{|ghg^{-1}|, |h|\} = |h|,$$

and consequently that $|C_g \cap C_h| \geqslant |g| + |h|$. The result now follows from Lemma 3·3. ∎

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