LAMINATED TUBES UNDER EXTENSION, INTERNAL AND EXTERNAL PRESSURE

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ABSTRACT

An exact analysis of deformation and stress field in a laminated elastic tube under extension and uniform pressuring is presented. The problem of finite laminated tube is considered with emphasis on the end effect. The problem is formulated on the basis of the state space formalism for axisymmetric deformation of transversely isotropic layer. The transfer matrix transmits the state vector in radial direction from inner surface to outer surface and takes into account the interfacial continuity and lateral boundary conditions in a rigorous manner. Upon delineating the symplectic orthogonality relations of the eigenvectors and by using eigenfunction expansion, an exact solution which satisfies the end conditions is determined. The results show that the end effect is significant but confined to a local region near the base where the displacement and stress distributions are remarkably different from those according to the simplified solutions that only satisfy lateral BC.

Keywords: State space approach, Laminated tube, Transfer matrix, End effects.

1. INTRODUCTION

Exact solutions of the deformation and stress in laminated tubes are of practical importance for designing riser tubes and conductors with layers. When the laminated tube with infinite length is subjected to extension and uniform pressuring, it is in the state of generalized plane strain [1]. Analysis of generalized plane strain problems is usually based on the Lekhnitskii stress function approach, but the displacements cannot be expressed by the stress functions in simple terms. Alternatively, the displacement approach derives the governing equations in terms of displacements, but the stress expressions may become complicated. Thus for laminates in which the interfacial continuity conditions require both the displacement and the traction be continuous, these formalisms are ineffective and may lead to intricate solutions. Tarn and Wang [2] developed the state space approach to treat the problem of laminated tube of infinite length. However, for the requirements to satisfying the end conditions, the problem of finite laminated tube is more complicated than that of infinite length.

In this paper we present an exact analysis of the displacement and stress fields in a laminated tube of finite length under extension and uniform pressuring. Since the loads are axially symmetric and the materials are transversely isotropic, the equations of elasticity for axisymmetric deformation are formulated into a state equation and an output equation of laminated layer. We show that the simplified solution alone cannot satisfy the end conditions. To determine an exact solution of the problem, it is necessary to find a complete solution of the state equation and make it satisfy the prescribed BC. The transfer matrix transmits the state vector in radial direction from inner surface to outer surface and takes into account the interfacial continuity and lateral boundary conditions in a rigorous manner. Regardless the number of layers, its determination requires only eigensolutions of matrices. We also delineate the symplectic orthogonality relations of the eigenvectors, and by means of eigenfunction expansion, an exact analysis of the displacement and stress fields in the laminated tube with the bottom plane perfectly bonded is carried out. Then it is examined by numerical examples on extension and uniform pressuring separately. The end effects are evaluated with reference to the simplified solutions.

2. STATE SPACE FORMULATION

Consider a transversely isotropic circular tube of laminated cross-section resting on a rigid base. Let the origin of the cylindrical coordinates (r, θ, z) be located at the center of the bottom plane, with the z axis pointing upward. When the tube is subjected to axisymmetric loadings, the deformation and stress fields are independent of θ. On the basis of the state space formalism for anisotropic elasticity [3], the state equation and output equation [4] of the k-th layer are as follows:

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$$\frac{\partial}{\partial z}\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \begin{bmatrix} 0 & -\partial_r & r^{-1}c_{44}^{-1} & 0 \\ l_{31} & 0 & 0 & r^{-1}c_{33}^{-1} \\ l_{41} & 0 & 0 & l_{46} \\ 0 & 0 & -\partial_r & 0 \end{bmatrix}_k \begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k, \quad (1)$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta} \end{bmatrix}_k = \begin{bmatrix} \tilde{c}_{11}(r\partial_r + 1) - 2c_{66} & \hat{c}_{13} \\ \tilde{c}_{12}(r\partial_r + 1) + 2c_{66} & \hat{c}_{13} \end{bmatrix}_k \begin{bmatrix} u_r \\ r\sigma_{zz} \end{bmatrix}_k, \quad (2)$$

where it is understood that the indices 1, 2, 3 stand for r, θ and z, respectively, c_{ij} are the elastic constants with reference to the cylindrical coordinates. With $c_{66} = (c_{11} - c_{12})/2$, there are five independent elastic constants for transversely isotropic materials. ∂_r, ∂_z denote partial differentiation with respect to r and z, respectively, and

$$l_{31} = -\hat{c}_{13}(\partial_r + r^{-1}),\ l_{41} = -\tilde{c}_{11}\left[\partial_r(r\partial_r) - r^{-1}\right],$$
$$l_{46} = -\hat{c}_{13}(\partial_r - r^{-1}),$$

$$\tilde{c}_{ij} = c_{ij} - c_{i3}c_{33}^{-1}c_{j3},\quad \hat{c}_{ij} = c_{ij}c_{33}^{-1}.$$

when the material is isotropic, $c_{33} = c_{11}$, $c_{13} = c_{12}$, $c_{44} = c_{66} = (c_{11} - c_{12})/2$.

The cylindrical surfaces at $r = a$ and $r = b$ are subjected to internal and external pressure:

$$[\sigma_{rr} \quad \sigma_{rz}]_{r=a} = [-R_a \ 0], \quad (3)$$

$$[\sigma_{rr} \quad \sigma_{rz}]_{r=b} = [-R_b \ 0]. \quad (4)$$

The bottom plane is assumed to be perfectly bonded with the rigid base (the fixed end), thus the end conditions at $z = 0$ are

$$[u_r \quad u_z]_{z=0} = [0 \ 0], \quad (5)$$

for the bottom plane perfectly bonded and the upper plane at $z = l$ are

$$[\sigma_{rz} \quad \sigma_{zz}]_{z=l} = [0 \quad p(r)]. \quad (6)$$

3. SIMPLIFIED SOLUTION OF INTERNAL AND EXTERNAL PRESSURE

In a laminated tube under uniform pressure on the inner and outer surfaces, the solution of Eq. (1) can be assumed as functions of r:

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \begin{bmatrix} \varphi_1(r) \\ \varphi_2(r) \\ \varphi_3(r) \\ \varphi_4(r) \end{bmatrix}_k. \quad (7)$$

Substituting Eq. (7) into Eq. (1), we obtain

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}_k = \begin{bmatrix} B_1^{(k)}r + B_1^{\prime(k)}r^{-1} \\ (c_{44}^{-1})^{(k)}B_3^{(k)}\ln r + B_2^{(k)} \\ B_3^{(k)} \\ 2(c_{13})^{(k)}B_1^{(k)}r \end{bmatrix}, \quad (8)$$

where $B_1^{(k)}$, $B_2^{(k)}$, $B_3^{(k)}$ and $B_1^{\prime(k)}$ are coefficients to be determined. Imposing the inner and outer surface BC, we have

$$\begin{bmatrix} B_3^{(1)} \\ 2(c_{11} - c_{66})_{(1)}B_1^{(1)}a - 2(c_{66})_{(1)}B_1^{\prime(1)}a^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ -aR_a \end{bmatrix},$$

$$\begin{bmatrix} B_3^{(m)} \\ 2(c_{11} - c_{66})_{(m)}B_1^{(m)}b - 2(c_{66})_{(m)}B_1^{\prime(m)}b^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ -bR_b \end{bmatrix},$$

i.e.,

$$B_3^{(1)} = 0,\quad B_3^{(m)} = 0, \quad (9)$$

$$B_1^{\prime(1)} = \frac{\left[2(c_{11} - c_{66})_{(1)}B_1^{(1)} + R_a\right]a^2}{2(c_{66})_{(1)}},$$
$$B_1^{\prime(m)} = \frac{\left[2(c_{11} - c_{66})_{(m)}B_1^{(m)} + R_b\right]b^2}{2(c_{66})_{(m)}}, \quad (10)$$

where m is the number of layers. The interfacial continuity conditions are

$$\begin{bmatrix} B_1^{(k)}c_k + B_1^{\prime(k)}c_k^{-1} \\ (c_{44}^{-1})_{(k)}B_3^{(k)}\ln c_k + B_2^{(k)} \\ B_3^{(k)} \\ 2(c_{11} - c_{66})_{(k)}B_1^{(k)}c_k - 2(c_{66})_{(k)}B_1^{\prime(k)}c_k^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} B_1^{(k+1)}c_k + B_1^{\prime(k+1)}c_k^{-1} \\ (c_{44}^{-1})_{(k+1)}B_3^{(k+1)}\ln c_k + B_2^{(k+1)} \\ B_3^{(k+1)} \\ 2(c_{11} - c_{66})_{(k+1)}B_1^{(k+1)}c_k - 2(c_{66})_{(k+1)}B_1^{\prime(k+1)}c_k^{-1} \end{bmatrix}, \quad (11)$$

where c_k is the interfacial radius. From Eqs. (9) and (11), we have for k-th and $(k + 1)$-th layers:

$$B_3^{(k)} = B_3^{(k+1)} = 0,\quad B_2^{(k)} = B_2^{(k+1)} = B_2, \quad (12)$$

$$\begin{bmatrix} B_1^{(k+1)} \\ B_1^{\prime(k+1)} \end{bmatrix} = \mathbf{P}_{0k}\begin{bmatrix} B_1^{(k)} \\ B_1^{\prime(k)} \end{bmatrix}, \quad (13)$$

where

$$\mathbf{P}_{0k} = \begin{bmatrix} c_k & c_k^{-1} \\ 2(c_{11} - c_{66})_{(k+1)}c_k & -2(c_{66})_{(k+1)}c_k^{-1} \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} c_k & c_k^{-1} \\ 2(c_{11} - c_{66})_{(k)}c_k & -2(c_{66})_{(k)}c_k^{-1} \end{bmatrix}.$$

Transferring the vector in Eq. (13) recursively, we arrive at

$$\begin{bmatrix} B_1^{(m)} \\ B_1^{\prime(m)} \end{bmatrix} = \left[\mathbf{P}_{0(m-1)}\ldots\mathbf{P}_{03}\mathbf{P}_{02}\mathbf{P}_{01}\right]\begin{bmatrix} B_1^{(1)} \\ B_1^{\prime(1)} \end{bmatrix}, \quad (14)$$

and $B_1^{(1)}$, $B_1^{(m)}$ can be solved by using Eq. (10).

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Thus the solution for k-th layer is

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \begin{bmatrix} B_1^{(k)} r + B_1^{'(k)} r^{-1} \\ 0 \\ 0 \\ 2(c_{13})_{(k)} B_1^{(k)} r \end{bmatrix}. \tag{15}$$

where

$$\begin{bmatrix} B_1^{(k)} \\ B_1^{'(k)} \end{bmatrix} = \left[\mathbf{P}_{0(k-1)} \ldots \mathbf{P}_{03} \mathbf{P}_{02} \mathbf{P}_{01} \right] \begin{bmatrix} B_1^{(1)} \\ B_1^{'(1)} \end{bmatrix}.$$

For a homogeneous tube, it can be verified that the solution in Eq. (15) reduces to

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix} = \begin{bmatrix} -\dfrac{R_b b^2 - R_a a^2}{b^2 - a^2} \dfrac{r}{c_{11} + c_{12}} + \dfrac{ab(R_a b^2 - R_b a^2)}{b^2 - a^2} \dfrac{r^{-1}}{c_{11} - c_{12}} \\ 0 \\ 0 \\ -\dfrac{2c_{13}}{c_{11} + c_{12}} \dfrac{R_b b^2 - R_a a^2}{b^2 - a^2} r \end{bmatrix}, \tag{16}$$

and

$$\sigma_{rr} = -\frac{R_b b^2 - R_a a^2}{b^2 - a^2} - \frac{ab(R_a b^2 - R_b a^2)}{b^2 - a^2} r^{-2}, \tag{17}$$

which is exactly the solution in a hollow cylinder submitted to uniform pressures [5].

Note that Eq. (15), which represents a plane stress state, satisfies the equations of elasticity of transversely isotropic materials, the BC on the inner and outer surfaces of the laminate, but not the end conditions at $z = 0$ and $z = l$. To account for the end conditions, the first thing to do is to determine a complete solution of Eq. (1), which consists of the eigensolution in addition to the simplified solution.

4. EIGENSOLUTION

We seek the homogeneous solution of Eq. (1) of the form

$$[u_r \;\; u_z \;\; r\sigma_{rz} \;\; r\sigma_{zz}]_k = [U_r \;\; U_z \;\; S_{rz} \;\; S_{zz}]_k e^{\mu z}, \tag{18}$$

where μ is a parameter to be determined; U_r, U_z, S_{rz}, S_{zz} are unknown functions of r, which reduces Eq. (1) to

$$\begin{bmatrix} 0 & -d_r & r^{-1}c_{44}^{-1} & 0 \\ -\hat{c}_{13}(d_r + r^{-1}) & 0 & 0 & r^{-1}c_{33}^{-1} \\ -\tilde{c}_{11}\left[d_r(rd_r) - r^{-1} \right] & 0 & 0 & -\hat{c}_{13}(d_r - r^{-1}) \\ 0 & 0 & -d_r & 0 \end{bmatrix}_k \cdot \begin{bmatrix} U_r \\ U_z \\ S_{rz} \\ S_{zz} \end{bmatrix}_k = \mu \begin{bmatrix} U_r \\ U_z \\ S_{rz} \\ S_{zz} \end{bmatrix}_k, \tag{19}$$

where d_r denotes ordinary differentiation with respect to r.

Since Eq. (19) can be reduced to a Bessel's equation, as shown in appendix *A*, it can be solved by letting

$$[U_r \;\; U_r \;\; S_{rz} \;\; S_{zz}]_k = \left[c_1 J_1(\lambda r) \;\; c_2 J_0(\lambda r) \;\; c_3 r J_1(\lambda r) \;\; c_4 r J_0(\lambda r) \right]_k, \tag{20}$$

where $J_0(\lambda r)$ and $J_1(\lambda r)$ are the Bessel functions of the first kind, of order 0 and 1, respectively; λ and c_i (i = 1, 2, 3, 4) are constants to be determined.

Substituting Eq. (20) into Eq. (19), making use of the derivative formulas [6]

$$\begin{aligned} \frac{d}{dr} J_0(\lambda r) &= -\lambda J_1(\lambda r), \\ \frac{d}{dr} J_1(\lambda r) &= \lambda J_0(\lambda r) - \frac{1}{r} J_1(\lambda r), \end{aligned} \tag{21}$$

we arrive at

$$\begin{bmatrix} 0 & \lambda & c_{44}^{-1} & 0 \\ -\hat{c}_{13}\lambda & 0 & 0 & c_{33}^{-1} \\ \tilde{c}_{11}\lambda^2 & 0 & 0 & \hat{c}_{13}\lambda \\ 0 & 0 & -\lambda & 0 \end{bmatrix}_k \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k = \mu \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k, \tag{22}$$

to which non-trivial solutions exist if and only if the determinant of the coefficient matrix equals to zero. This condition yields

$$\left\{ c_{11}c_{44}\kappa^4 - \left[c_{11}c_{33} - c_{13}(c_{13} + 2c_{44}) \right] \kappa^2 + c_{33}c_{44} \right\}_k = 0, \quad (\kappa_k = \lambda_k / \mu) \tag{23}$$

which has four roots, given by

$$(\kappa_1)_k = -(\kappa_3)_k = \left[\frac{c_{11}c_{33} - c_{13}(c_{13} + 2c_{44}) + \sqrt{(c_{11}c_{33} - c_{13}^2)\left[c_{11}c_{33} - (c_{13} + 2c_{44})^2 \right]}}{2c_{11}c_{44}} \right]_k^{1/2}, \tag{24}$$

$$(\kappa_2)_k = -(\kappa_4)_k = \left[\frac{c_{11}c_{33} - c_{13}(c_{13} + 2c_{44}) - \sqrt{(c_{11}c_{33} - c_{13}^2)\left[c_{11}c_{33} - (c_{13} + 2c_{44})^2 \right]}}{2c_{11}c_{44}} \right]_k^{1/2}. \tag{25}$$

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To each κ_{jk} there corresponds a solution of Eq. (22) determined within a constant A_{jk} as

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k = A_{jk} \begin{bmatrix} (\kappa_j^2 c_{44} - c_{33}) \\ \kappa_j (c_{44} + c_{13}) \\ -\mu c_{44} (c_{33} + \kappa_j^2 c_{44}) \\ \kappa_j \mu c_{44} (\kappa_j^2 c_{13} + c_{33}) \end{bmatrix}_k , \quad (j = 1,\ 2,\ 3,\ 4) \tag{26}$$

It can be shown that there are two linearly independent solutions derivable from Eqs. (20) ~ (26) since $J_0(-\kappa_j \mu r) = J_0(\kappa_j \mu r)$ and $J_1(-\kappa_j \mu r) = -J_1(\kappa_j \mu r)$. The solution of Eqs. (1) and (2) is

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \sum_{j=1}^{2} A_{jk} e^{\mu z} \begin{bmatrix} f_1(\kappa_j,\ \mu,\ r) \\ f_2(\kappa_j,\ \mu,\ r) \\ f_3(\kappa_j,\ \mu,\ r) \\ f_4(\kappa_j,\ \mu,\ r) \end{bmatrix}_k , \tag{27}$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta} \end{bmatrix}_k = \sum_{j=1}^{2} A_{jk} e^{\mu z} \begin{bmatrix} f_5(\kappa_j,\ \mu,\ r) \\ f_6(\kappa_j,\ \mu,\ r) \end{bmatrix}_k , \tag{28}$$

where

$$\begin{bmatrix} f_1(\kappa_j,\ \mu,\ r) \\ f_2(\kappa_j,\ \mu,\ r) \\ f_3(\kappa_j,\ \mu,\ r) \\ f_4(\kappa_j,\ \mu,\ r) \end{bmatrix}_k = \begin{bmatrix} (\kappa_j^2 c_{44} - c_{33}) J_1(\kappa_j \mu r) \\ \kappa_j (c_{44} + c_{13}) J_0(\kappa_j \mu r) \\ -\mu c_{44} (c_{33} + \kappa_j^2 c_{44}) r J_1(\kappa_j \mu r) \\ \kappa_j \mu c_{44} (\kappa_j^2 c_{13} + c_{33}) r J_0(\kappa_j \mu r) \end{bmatrix}_k ,$$

$$\begin{aligned} \left[f_5(\kappa_j,\ \mu,\ r) \right]_k &= \left\{ \left[c_{44} (\kappa_j^2 c_{11} + c_{13}) - \tilde{c}_{11} c_{33} \right] \kappa_j \mu r J_0(\kappa_j \mu r) \right. \\ &\quad \left. -2c_{66} (\kappa_j^2 c_{44} - c_{33}) J_1(\kappa_j \mu r) \right\}_k , \\ \left[f_6(\kappa_j,\ \mu,\ r) \right]_k &= \left\{ \left[c_{44} (\kappa_j^2 c_{12} + c_{13}) - \tilde{c}_{12} c_{33} \right] \kappa_j \mu r J_0(\kappa_j \mu r) \right. \\ &\quad \left. + 2c_{66} (\kappa_j^2 c_{44} - c_{33}) J_1(\kappa_j \mu r) \right\}_k . \end{aligned}$$

It can also be shown that there are another two linearly independent solutions derivable from

$$\begin{aligned} &[U_r \ \ U_z \ \ S_{rz} \ \ S_{zz}]_k \\ &= \left[c_1 Y_1(\lambda r) \ \ c_2 Y_0(\lambda r) \ \ c_3 r Y_1(\lambda r) \ \ c_4 r Y_0(\lambda r) \right]_k , \end{aligned} \tag{29}$$

and we have

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \sum_{j=1}^{2} B_{jk} e^{\mu z} \begin{bmatrix} g_1(\kappa_j,\ \mu,\ r) \\ g_2(\kappa_j,\ \mu,\ r) \\ g_3(\kappa_j,\ \mu,\ r) \\ g_4(\kappa_j,\ \mu,\ r) \end{bmatrix}_k , \tag{30}$$

$$\begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta} \end{bmatrix}_k = \sum_{j=1}^{2} B_{jk} e^{\mu z} \begin{bmatrix} g_5(\kappa_j,\ \mu,\ r) \\ g_6(\kappa_j,\ \mu,\ r) \end{bmatrix}_k , \tag{31}$$

where

$$\begin{bmatrix} g_1(\kappa_j,\ \mu,\ r) \\ g_2(\kappa_j,\ \mu,\ r) \\ g_3(\kappa_j,\ \mu,\ r) \\ g_4(\kappa_j,\ \mu,\ r) \end{bmatrix}_k = \begin{bmatrix} (\kappa_j^2 c_{44} - c_{33}) Y_1(\kappa_j \mu r) \\ \kappa_j (c_{44} + c_{13}) Y_0(\kappa_j \mu r) \\ -\mu c_{44} (c_{33} + \kappa_j^2 c_{44}) r Y_1(\kappa_j \mu r) \\ \kappa_j \mu c_{44} (\kappa_j^2 c_{13} + c_{33}) r Y_0(\kappa_j \mu r) \end{bmatrix}_k ,$$

$$\begin{aligned} \left[g_5(\kappa_j,\ \mu,\ r) \right]_k &= \left\{ \left[c_{44} (\kappa_j^2 c_{11} + c_{13}) - \tilde{c}_{11} c_{33} \right] \kappa_j \mu r Y_0(\kappa_j \mu r) \right. \\ &\quad \left. -2c_{66} (\kappa_j^2 c_{44} - c_{33}) Y_1(\kappa_j \mu r) \right\}_k , \\ \left[g_6(\kappa_j,\ \mu,\ r) \right]_k &= \left\{ \left[c_{44} (\kappa_j^2 c_{12} + c_{13}) - \tilde{c}_{12} c_{33} \right] \kappa_j \mu r Y_0(\kappa_j \mu r) \right. \\ &\quad \left. + 2c_{66} (\kappa_j^2 c_{44} - c_{33}) Y_1(\kappa_j \mu r) \right\}_k . \end{aligned}$$

5. DETERMINATION OF EIGENVALUES AND EIGENVECTORS

From Eqs. (27) ~ (28) and (30) ~ (31) the displacements and stresses of the k-th layer in radial direction can be written as

$$[u_r \ u_z \ r\sigma_{rr} \ r\sigma_{rz}]_k = \left[U_r(r) \ \ U_z(r) \ \ S_{rr}(r) \ \ S_{rz}(r) \right]_k e^{\mu z} , \tag{32}$$

where

$$\begin{bmatrix} U_r(r) \\ U_z(r) \\ S_{rr}(r) \\ S_{rz}(r) \end{bmatrix}_k = \begin{bmatrix} f_1(\kappa_1,\ \mu,\ r) & f_1(\kappa_2,\ \mu,\ r) & g_1(\kappa_1,\ \mu,\ r) & g_1(\kappa_2,\ \mu,\ r) \\ f_2(\kappa_1,\ \mu,\ r) & f_2(\kappa_2,\ \mu,\ r) & g_2(\kappa_1,\ \mu,\ r) & g_2(\kappa_2,\ \mu,\ r) \\ f_5(\kappa_1,\ \mu,\ r) & f_5(\kappa_2,\ \mu,\ r) & g_5(\kappa_1,\ \mu,\ r) & g_5(\kappa_2,\ \mu,\ r) \\ f_3(\kappa_1,\ \mu,\ r) & f_3(\kappa_2,\ \mu,\ r) & g_3(\kappa_1,\ \mu,\ r) & g_3(\kappa_2,\ \mu,\ r) \end{bmatrix}_k \cdot \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix}_k \tag{33}$$

Substituting the internal radii $r = c_{k-1}$ of the k-th layer to Eq. (33), we obtain

$$\begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix}_k = \begin{bmatrix} f_1(\kappa_1,\ \mu,\ c_{k-1}) & f_1(\kappa_2,\ \mu,\ c_{k-1}) & g_1(\kappa_1,\ \mu,\ c_{k-1}) & g_1(\kappa_2,\ \mu,\ c_{k-1}) \\ f_2(\kappa_1,\ \mu,\ c_{k-1}) & f_2(\kappa_2,\ \mu,\ c_{k-1}) & g_2(\kappa_1,\ \mu,\ c_{k-1}) & g_2(\kappa_2,\ \mu,\ c_{k-1}) \\ f_5(\kappa_1,\ \mu,\ c_{k-1}) & f_5(\kappa_2,\ \mu,\ c_{k-1}) & g_5(\kappa_1,\ \mu,\ c_{k-1}) & g_5(\kappa_2,\ \mu,\ c_{k-1}) \\ f_3(\kappa_1,\ \mu,\ c_{k-1}) & f_3(\kappa_2,\ \mu,\ c_{k-1}) & g_3(\kappa_1,\ \mu,\ c_{k-1}) & g_3(\kappa_2,\ \mu,\ c_{k-1}) \end{bmatrix}_k^{-1} \cdot \begin{bmatrix} U_r(c_{k-1}) \\ U_z(c_{k-1}) \\ S_{rr}(c_{k-1}) \\ S_{rz}(c_{k-1}) \end{bmatrix}_k \tag{34}$$

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Then substituting (34) back to (33), the solution can be rewritten in the form:

$$\mathbf{R}_k(r) = \mathbf{P}_k(r;\ c_{k-1})\mathbf{R}_k(c_{k-1}), \tag{35}$$

where the matrices and vectors are defined as

$$\mathbf{R}_k(r) = \left[U_r(r)\ \ U_z(r)\ \ S_{rr}(r)\ \ S_{rz}(r)\right]_k^T,$$

$$\mathbf{P}_k(r;\ c_{k-1}) = \mathbf{M}_k(r)\mathbf{M}_k^{-1}(c_{k-1}),$$

$$\mathbf{M}_k(r) = \left[\mathbf{f}_1(\kappa_1,\mu,r)\ \ \mathbf{f}_2(\kappa_2,\mu,r)\ \ \mathbf{g}_1(\kappa_3,\mu,r)\ \ \mathbf{g}_2(\kappa_4,\mu,r)\right]_k,$$

$$\mathbf{f}_j = \left[f_1(\kappa_j,\mu,r)\ \ f_2(\kappa_j,\mu,r)\ \ f_5(\kappa_j,\mu,r)\ \ f_3(\kappa_j,\mu,r)\right]^T,$$

$$\mathbf{g}_j = \left[g_1(\kappa_j,\ \mu,\ r)\ \ g_2(\kappa_j,\ \mu,\ r)\ \ g_5(\kappa_j,\ \mu,\ r)\ \ g_3(\kappa_j,\ \mu,\ r)\right]^T.$$

Specifically, the interfacial continuity conditions at external radii $r = c_k$ of k-th layer is

$$\mathbf{R}_{k+1}(c_k) = \mathbf{R}_k(c_k). \tag{36}$$

It follows that

$$\mathbf{R}_{k+1}(r) = \mathbf{P}_{k+1}(r;\ c_k)\mathbf{P}_k(c_k;\ c_{k-1})\mathbf{R}_k(c_{k-1}), \tag{37}$$

which transfers the state vector from the k-th layer to the $(k + 1)$-th layer.

Transferring the vector in Eq. (37) recursively from the inner radii $r = a$ outward yields

$$\mathbf{R}_k(r) = \mathbf{T}_k(r)\mathbf{R}_1(a), \tag{38}$$

where the transfer matrix:

$$\mathbf{T}_k(r) = \begin{cases} \mathbf{P}_1(r;a), & k=1; \\ \mathbf{P}_k(r;\ c_{k-1})\mathbf{T}_{k-1}(c_{k-1}), & k=2,\ 3,\ ...,\ m. \end{cases}$$

Following the same line by using the transfer matrix, the eigenvalue μ_j ($j = 1,\ 2,\ \cdots$) and the associated eigenvector are determined by satisfying the interfacial continuity conditions and the traction-free conditions on the inner and outer surfaces. Setting the outer radii $r = b$ in Eq. (38) gives

$$\mathbf{R}_m(b) = \mathbf{T}_m(b)\mathbf{R}_1(a), \tag{39}$$

which can be expressed as

$$\begin{bmatrix}\mathbf{U}_m(b)\\ \mathbf{S}_m(b)\end{bmatrix} = \begin{bmatrix}\mathbf{T}_{uu}(b) & \mathbf{T}_{us}(b)\\ \mathbf{T}_{su}(b) & \mathbf{T}_{ss}(b)\end{bmatrix}_m \begin{bmatrix}\mathbf{U}_1(a)\\ \mathbf{S}_1(a)\end{bmatrix}. \tag{40}$$

On $r = a$ and b either the traction or the displacement, or a mixed boundary conditions may be prescribed. Equation (40) consists of 4 equations in 4 unknowns. Among the 4 components of $\mathbf{U}_k$ and $\mathbf{S}_k$, 2 are prescribed and other 2 are unknown. When the tube is subjected to tractions, $\mathbf{S}_k$ are prescribed and $\mathbf{U}_k$ on $r = a$ and b are unknown. The traction-free BC on $r = a$ and b demand

$$\mathbf{S}_1(a) = 0,\ \ \mathbf{S}_m(b) = 0. \tag{41}$$

Note that when displacements are prescribed, $\mathbf{U}_k$ are prescribed and $\mathbf{S}_k$ on $r = a$ and b are unknown. Imposing them on Eq. (41) yields

$$\left|\left[\mathbf{T}_{su}(b)\right]_m\right| = 0, \tag{42}$$

from which the eigenvalues μ are determined.

By inspection on Eq. (35), if μ is a root of Eq. (42), so is $-\mu$. Moreover, $\mu = 0$ is a repeated root, which can be verified by replacing μ by $-\mu$ in Eq. (42) and using the fact that $\mu = 0$ is a zero of $J_1(\kappa\mu a)$ and $J_0(-\kappa_j\mu r) = J_0(\kappa_j\mu r)$, $J_1(-\kappa_j\mu r) = -J_1(\kappa_j\mu r)$. Since $\mu_0 = 0$ is a repeated root of Eq. (42), linearly independent eigenvectors associated with $\mu_0 = 0$ can be determined by considering the Jordan-chain solution given as

$$\begin{cases}\boldsymbol{\psi}_{01}^{(k)} = \boldsymbol{\psi}_0^{(0k)}, \\ \boldsymbol{\psi}_{02}^{(k)} = \boldsymbol{\psi}_0^{(1k)} + z\boldsymbol{\psi}_0^{(0k)}, \end{cases}, \tag{43}$$

where

$$\boldsymbol{\psi} = [U_r\ \ U_z\ \ S_{rz}\ \ S_{zz}]^T.$$

$\boldsymbol{\psi}_0^{(0k)}$ and $\boldsymbol{\psi}_0^{(1k)}$ are determined from the Jordan chain:

$$\begin{cases}\mathbf{H}_k\boldsymbol{\psi}_0^{(0k)} = \mathbf{0}, \\ \mathbf{H}_k\boldsymbol{\psi}_0^{(1k)} = \boldsymbol{\psi}_0^{(0k)}, \end{cases} \tag{44}$$

with the interfacial continuity conditions and the traction-free BC at $r = a$, b where

$$\mathbf{H}_k = \begin{bmatrix} 0 & -d_r & r^{-1}c_{44}^{-1} & 0 \\ -\hat{c}_{13}(d_r + r^{-1}) & 0 & 0 & r^{-1}c_{33}^{-1} \\ -\tilde{c}_{11}\left[d_r(rd_r) - r^{-1}\right] & 0 & 0 & -\hat{c}_{13}(d_r - r^{-1}) \\ 0 & 0 & -d_r & 0 \end{bmatrix}_k, \tag{45}$$

and we arrive at

$$\boldsymbol{\psi}_0^{(0k)} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix},\ \ \boldsymbol{\psi}_0^{(1k)} = \begin{bmatrix} D_1^{(k)}r + D_1^{'(k)}r^{-1} \\ 0 \\ 0 \\ 2(c_{13})_{(k)}D_1^{(k)}r + (c_{33})_{(k)}r \end{bmatrix}; \tag{46}$$

the constants $D_1^{(k)}$, $D_1^{'(k)}$ can be expressed by transfer matrices as

$$\begin{bmatrix}D_1^{(k)}\\ D_1^{'(k)}\end{bmatrix} = \left[\mathbf{P}_{0(k-1)}...\mathbf{P}_{02}\mathbf{P}_{01}\right]\begin{bmatrix}D_1^{(1)}\\ D_1^{'(1)}\end{bmatrix} + \sum_{j=1}^{k-1}\left[\mathbf{P}_{0(k-1)}...\mathbf{P}_{0(j+2)}\mathbf{P}_{0(j+1)}\right]\begin{bmatrix}X_j\\ Y_j\end{bmatrix}, \tag{47}$$

with

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$$\begin{bmatrix} X_j \\ Y_j \end{bmatrix} = \frac{\left[(c_{13})_{(j)} - (c_{13})_{(j+1)}\right] c_j}{2(c_{11})_{(j+1)}} \begin{bmatrix} c_j^{-1} \\ -c_j \end{bmatrix}; \tag{48}$$

$D_1^{(1)}$ can be determined from

$$\begin{bmatrix} 1 \\ \dfrac{2(c_{11}-c_{66})_{(m)}}{2(c_{66})_{(m)}} b^2 \end{bmatrix} D_1^{(m)} - \left[\mathbf{P}_{0(m-1)} \ldots \mathbf{P}_{03}\mathbf{P}_{02}\mathbf{P}_{01}\right] \begin{bmatrix} 1 \\ \dfrac{2(c_{11}-c_{66})_{(1)}}{2(c_{66})_{(1)}} a^2 \end{bmatrix} D_1^{(1)}$$
$$= \left[\mathbf{P}_{0(m-1)} \ldots \mathbf{P}_{03}\mathbf{P}_{02}\mathbf{P}_{01}\right] \begin{bmatrix} 0 \\ \dfrac{(c_{13})_{(1)}}{2(c_{66})_{(1)}} a^2 \end{bmatrix} - \begin{bmatrix} 0 \\ \dfrac{(c_{13})_{(m)}}{2(c_{66})_{(m)}} b^2 \end{bmatrix} + \sum_{j=1}^{m-1} \left[\mathbf{P}_{0(k-1)} \ldots \mathbf{P}_{0(j+2)}\mathbf{P}_{0(j+1)}\right] \begin{bmatrix} X_j \\ Y_j \end{bmatrix}, \tag{49}$$

and then

$$D_1^{(1)} = \frac{2(c_{11}-c_{66})_{(1)} D_1^{(1)} + (c_{13})_{(1)}}{2(c_{66})_{(1)}} a^2. \tag{50}$$

A linear combination of the eigensolutions and the simplified solution obtained above produces the complete solution of the k-th layer:

$$\begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \boldsymbol{\varphi}^{(k)} + (C_0 \boldsymbol{\psi}_{01}^{(k)} + C_0' \boldsymbol{\psi}_{02}^{(k)}) + \sum_{i=1}^{\infty} \left[C_i \boldsymbol{\psi}_i^{(k)} e^{\mu_i z} + C_{-i} \boldsymbol{\psi}_{-i}^{(k)} e^{-\mu_i z} \right], \tag{51}$$

where C_0, C_0', C_i and C_{-i} are constants of linear combination, and

$$\boldsymbol{\varphi}^{(k)} = \begin{bmatrix} B_1^{(k)} r + B_1^{\prime(k)} r^{-1} \\ 0 \\ 0 \\ 2(c_{13})_{(k)} B_1^{(k)} r \end{bmatrix}, \tag{52}$$

$$\boldsymbol{\psi}_i^{(k)} = \begin{bmatrix} \psi_{i1}^{(k)} \\ \psi_{i2}^{(k)} \\ \psi_{i3}^{(k)} \\ \psi_{i4}^{(k)} \end{bmatrix} = \sum_{j=1}^{2} \left(A_{jk} \begin{bmatrix} f_1(\kappa_j, \mu_i, r) \\ f_2(\kappa_j, \mu_i, r) \\ f_3(\kappa_j, \mu_i, r) \\ f_4(\kappa_j, \mu_i, r) \end{bmatrix}_k + B_{jk} \begin{bmatrix} g_1(\kappa_j, \mu_i, r) \\ g_2(\kappa_j, \mu_i, r) \\ g_3(\kappa_j, \mu_i, r) \\ g_4(\kappa_j, \mu_i, r) \end{bmatrix}_k \right), \tag{53}$$

$$\boldsymbol{\psi}_{-i}^{(k)} = \begin{bmatrix} \psi_{-i1}^{(k)} \\ \psi_{-i2}^{(k)} \\ \psi_{-i3}^{(k)} \\ \psi_{-i4}^{(k)} \end{bmatrix} = \sum_{j=1}^{2} \left(A_{jk} \begin{bmatrix} f_1(\kappa_j, -\mu_i, r) \\ f_2(\kappa_j, -\mu_i, r) \\ f_3(\kappa_j, -\mu_i, r) \\ f_4(\kappa_j, -\mu_i, r) \end{bmatrix}_k + B_{jk} \begin{bmatrix} g_1(\kappa_j, -\mu_i, r) \\ g_2(\kappa_j, -\mu_i, r) \\ g_3(\kappa_j, -\mu_i, r) \\ g_4(\kappa_j, -\mu_i, r) \end{bmatrix}_k \right). \tag{54}$$

6. SATISFACTION OF THE END CONDITIONS

The complete solution of k-th layer is required to satisfy the end conditions at $z = 0$ and $z = l$. To this end, imposing Eqs. (5) ~ (6) on Eq. (51), we obtain

$$\mathbf{F}_{0k}(r) = \boldsymbol{\varphi}^{(k)} + C_0 \boldsymbol{\psi}_0^{(0k)} + C_0' \boldsymbol{\psi}_0^{(1k)} + \sum_{i=1}^{\infty} \left[C_i \boldsymbol{\psi}_i^{(k)} + C_{-i} \boldsymbol{\psi}_{-i}^{(k)} \right], \tag{55}$$

$$\mathbf{F}_{lk}(r) = \boldsymbol{\varphi}^{(k)} + C_0 \boldsymbol{\psi}_0^{(0k)} + C_0' \left[\boldsymbol{\psi}_0^{(1k)} + l \boldsymbol{\psi}_0^{(0k)} \right] + \sum_{i=1}^{\infty} \left[C_i \boldsymbol{\psi}_i^{(k)} e^{\mu_i l} + C_{-i} \boldsymbol{\psi}_{-i}^{(k)} e^{-\mu_i l} \right], \tag{56}$$

where

$$\mathbf{F}_{0k}(r) = \begin{bmatrix} 0 \\ 0 \\ r\sigma_{rz}(r,\ 0) \\ r\sigma_{zz}(r,\ 0) \end{bmatrix}_k, \quad \mathbf{F}_{lk}(r) = \begin{bmatrix} u_r(r,\ l) \\ u_z(r,\ l) \\ 0 \\ rp(r) \end{bmatrix}_k,$$

in which C_0, C_0', C_i and C_{-i} are constants to be determined, $\mathbf{F}_{0k}(r)$ contains the unknown tractions $[\sigma_{rz}(r, 0)]_k$ and $[\sigma_{zz}(r, 0)]_k$ at the fixed end, $\mathbf{F}_{lk}(r)$ contains the unknown displacements $[u_r(r, l)]_k$ and $[u_z(r, l)]_k$ at the free end, which, in turn, can be expressed in terms of C_0, C_0', C_i and C_{-i} by using Eqs. (52) ~ (54) as

$$\left[r\sigma_{rz}(r,\ 0)\right]_k = \sum_{i=1}^{\infty} \left[C_i \psi_{i3}^{(k)} + C_{-i} \psi_{-i3}^{(k)} \right], \tag{57}$$

$$\left[r\sigma_{zz}(r,\ 0)\right]_k = 2(c_{13})_{(k)} B_1^{(k)} r + C_0' \left[2(c_{13})_{(k)} D_1^{(k)} r + (c_{33})_{(k)} r \right] + \sum_{i=1}^{\infty} \left[C_i \psi_{i4}^{(k)} + C_{-i} \psi_{-i4}^{(k)} \right], \tag{58}$$

$$\left[u_r(r,\ l)\right]_k = B_1^{(k)} r + B_1^{\prime(k)} r^{-1} + C_0' \left[D_1^{(k)} r + D_1^{\prime(k)} r^{-1} \right] + \sum_{i=1}^{\infty} \left[C_i \psi_{i1}^{(k)} e^{\mu_i l} + C_{-i} \psi_{-i1}^{(k)} e^{-\mu_i l} \right], \tag{59}$$

$$\left[u_z(r,\ l)\right]_k = z + C_0 + C_0' l + \sum_{i=1}^{\infty} \left[C_i \psi_{i2}^{(k)} e^{\mu_i l} + C_{-i} \psi_{-i2}^{(k)} e^{-\mu_i l} \right]. \tag{60}$$

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At this stage, it remains to determine the constants C_0, C_0', C_i and C_{-i} in Eqs. (55) ~ (56). For this purpose, it is necessary to derive the orthogonality relations among the eigenvectors so as to express a given vector function in terms of the eigenvectors by using eigenfunction expansion.

7. SYMPLECTIC ORTHOGONALITY

Symplectic orthogonality [7] in the Hamiltonian state space holds for the eigenvector derived from the state vector which is composed of the displacement vector and the associated stress vector. Let us express Eq. (19) with the interfacial continuity conditions at $r = c_{k-1}$ and $r = c_k$ as

$$\mathbf{H}_k \boldsymbol{\psi}_k = \mu \boldsymbol{\psi}_k \ , \quad \mathbf{B}_k(\boldsymbol{\psi}_k) = \mathbf{0} \ , \tag{61}$$

where

$$\mathbf{H}_k = \begin{bmatrix} \mathbf{L}_{11k} & \mathbf{L}_{12k} \\ \mathbf{L}_{21k} & -\mathbf{L}_{11k}^* \end{bmatrix}, \quad \boldsymbol{\psi}_k = \begin{bmatrix} \mathbf{U}_k \\ \mathbf{S}_k \end{bmatrix},$$

$$\mathbf{U}_k = \begin{bmatrix} \psi_{1k} \\ \psi_{2k} \end{bmatrix}, \quad \mathbf{S}_k = \begin{bmatrix} \psi_{3k} \\ \psi_{4k} \end{bmatrix},$$

$$\mathbf{B}_k(\boldsymbol{\psi}_k) = \begin{bmatrix} \mathbf{R}_k(c_k) - \mathbf{R}_{k+1}(c_k) \\ \mathbf{R}_k(c_{k-1}) - \mathbf{R}_{k-1}(c_{k-1}) \end{bmatrix},$$

$$\mathbf{L}_{11k} = \begin{bmatrix} 0 & -d_r \\ -\hat{c}_{13}(d_r + r^{-1}) & 0 \end{bmatrix}_k,$$

$$\mathbf{L}_{11k}^* = \begin{bmatrix} 0 & \hat{c}_{13}(d_r - r^{-1}) \\ d_r & 0 \end{bmatrix}_k,$$

$$\mathbf{L}_{12k} = \begin{bmatrix} r^{-1}c_{44}^{-1} & 0 \\ 0 & r^{-1}c_{33}^{-1} \end{bmatrix}_k,$$

$$\mathbf{L}_{21k} = \begin{bmatrix} -\tilde{c}_{11}\left[d_r(rd_r) - r^{-1}\right] & 0 \\ 0 & 0 \end{bmatrix}_k .$$

The matrix $\mathbf{H}_k$ in Eq. (61) exhibits the characteristics of a Hamiltonian matrix such that $\mathbf{L}_{12k}$ and $\mathbf{L}_{21k}$ are symmetric matrix and $\mathbf{L}_{11}$ and $\mathbf{L}_{11}^*$ are adjoint operators, which can be deduced as in appendix B. Such characteristics are of fundamental significance in the formulation. The eigenvector consists of two vector components, namely, the vector $\mathbf{U}_k$ and the vector $\mathbf{S}_k$ as defined in Eq. (20).

Consider two linearly independent eigenvectors $\boldsymbol{\psi}_{ki}$ and $\boldsymbol{\psi}_{kj}$ associated with the eigenvalues μ_i and μ_j, respectively,

$$\mathbf{H}_k \boldsymbol{\psi}_{ki} = \mu_i \boldsymbol{\psi}_{ki} \ , \quad \mathbf{B}_k(\boldsymbol{\psi}_{ki}) = \mathbf{0} \ , \tag{62}$$

$$\mathbf{H}_k \boldsymbol{\psi}_{kj} = \mu_j \boldsymbol{\psi}_{kj} \ , \quad \mathbf{B}_k(\boldsymbol{\psi}_{kj}) = \mathbf{0} \ . \tag{63}$$

Pre-multiplying Eq. (62) by $\boldsymbol{\psi}_{kj}^T \mathbf{J}$ and Eq. (63) by $\boldsymbol{\psi}_{ki}^T \mathbf{J}$, integrating the resulting equations with respect to r over (c_{k-1}, c_k) and then summating from 1-st layer to m-th layer, we obtain

$$\sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{kj}^T \mathbf{J}\mathbf{H}_k \boldsymbol{\psi}_{ki} \, dr = \mu_i \sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{kj}^T \mathbf{J} \boldsymbol{\psi}_{ki} dr \ , \tag{64}$$

$$\sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ki}^T \mathbf{J}\mathbf{H}_k \boldsymbol{\psi}_{kj} \, dr = \mu_j \sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ki}^T \mathbf{J} \boldsymbol{\psi}_{kj} \, dr \ , \tag{65}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}.$$

Substracting Eqs. (64) and (65) gives

$$(\mu_i + \mu_j) \sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ki}^T \mathbf{J} \boldsymbol{\psi}_{kj} \, dr = 0 \ . \tag{66}$$

Since $\mu_i + \mu_j \neq 0$ $(i \neq -j)$ according to the sequence of the eigenvectors defined before, there follows the orthogonal relation for the eigenvectors associated with μ_i, $\mu_j \neq 0$ $(i, j = 1, 2, \cdots)$:

$$\sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ki}^T \mathbf{J} \boldsymbol{\psi}_{kj} \, dr \begin{cases} = 0 & (i \neq -j) \\ \neq 0 & (i = -j) \end{cases} . \tag{67}$$

The case of the zero eigenvalue needs special consideration. It has been shown that $\mu_0 = 0$ is a repeated eigenvalue. The two linearly independent eigenvectors $\boldsymbol{\psi}_0^{(0k)}$ and $\boldsymbol{\psi}_0^{(1k)}$ can be determined from the Jordan chain associated with $\mu_0 = 0$ in Eq. (44), and the orthogonal relations of the eigenvectors associated with $\mu_0 = 0$ [4] are

$$\sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(\alpha k)}]^T \mathbf{J} \boldsymbol{\psi}_0^{(\beta k)} dr \begin{cases} = 0 \ , & (\beta = \alpha \ ; \ \alpha, \ \beta = 0, \ 1) \\ \neq 0, & (\beta \neq \alpha \ ; \ \alpha, \ \beta = 0, \ 1) \end{cases} . \tag{68}$$

Since $\mu_i + \mu_j \neq 0$ $(i \neq 0)$, there follows the orthogonal relation for the eigenvectors associated with μ_i $(i \neq 0)$, μ_0: $\sum_{k=1}^{m} \int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ki}^T \mathbf{J} \boldsymbol{\psi}_0^{(\alpha k)} dr = 0 \ , \quad (\alpha = 0, 1).$

8. DETERMINING THE UNKNOWN CONSTANTS

To determine C_0, C_0', C_i and C_{-i} in Eqs. (55) and (56) with the symplectic orthogonality given by Eqs. (67) and (68), it is possible to express an arbitrary integrable state vector in terms of a sequence of orthogonal eigenvectors in the state space by means of eigenfunction expansion.

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Multiplying both sides of Eqs. (55) and (56) by $\boldsymbol{\psi}_{-ik}^T\mathbf{J}$, $\boldsymbol{\psi}_{ik}^T\mathbf{J}$, $[\boldsymbol{\psi}_0^{(1k)}]^T\mathbf{J}$ and $[\boldsymbol{\psi}_0^{(0k)}]^T\mathbf{J}$, respectively, and integrating them over over (c_{k-1}, c_k) and then summating from 1-st layer to m-th layer, making use of the symplectic orthogonality, we obtain

$$C_i = \frac{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{-ik}^T(r)\mathbf{J}\left[\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{-ik}^T(r)\mathbf{J}\boldsymbol{\psi}_{ik}(r)\,dr},$$

$$C_{-i} = \frac{-\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ik}^T(r)\mathbf{J}\left[\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ik}^T(r)\mathbf{J}\boldsymbol{\psi}_{-ik}(r)\,dr}, \tag{69}$$

$$C_0 = \frac{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(1k)}(r)]^T\mathbf{J}\left[\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(1k)}]^T\mathbf{J}\boldsymbol{\psi}_0^{(0k)}dr},$$

$$C_0' = \frac{-\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(0k)}]^T\mathbf{J}\left[\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(0k)}]^T\mathbf{J}\boldsymbol{\psi}_0^{(1k)}dr}, \tag{70}$$

$$C_i = \frac{e^{-\mu_i l}\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{-ik}^T(r)\mathbf{J}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{-ik}^T(r)\mathbf{J}\boldsymbol{\psi}_{ik}(r)dr},$$

$$C_{-i} = \frac{-e^{\mu_i l}\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ik}^T(r)\mathbf{J}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ik}^T(r)\mathbf{J}\boldsymbol{\psi}_{-ik}(r)\,dr}, \tag{71}$$

$$C_0 + C_0' l = \frac{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \left[\boldsymbol{\psi}_0^{(1k)}(r)\right]^T\mathbf{J}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]\,dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \left[\boldsymbol{\psi}_0^{(1k)}\right]^T\mathbf{J}\boldsymbol{\psi}_0^{(0k)}dr},$$

$$C_0' = \frac{-\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \left[\boldsymbol{\psi}_0^{(0k)}(r)\right]^T\mathbf{J}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]dr}{\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(0k)}]^T\mathbf{J}\boldsymbol{\psi}_0^{(1k)}dr}. \tag{72}$$

In Eqs. (69) ~ (72) each of the vectors $\mathbf{F}_{0k}(r)$ and $\mathbf{F}_{lk}(r)$ contains four components, of which the tractions $[\sigma_{rz}(r, 0)]_k$ and $[\sigma_{zz}(r, 0)]_k$ are unknown at the fixed end $z = 0$, the displacements $[u_r(r, l)]_k$ and $[u_z(r, l)]_k$ are unknown at the free end $z = l$. Substituting Eqs. (69) ~ (70) in Eqs. (71) ~ (72) gives

$$\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{-ik}^T(r)\mathbf{J}\left\{\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}-e^{-\mu_i l}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]\right\}\,dr = 0, \tag{73}$$

$$\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \boldsymbol{\psi}_{ik}^T(r)\mathbf{J}\left\{\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}-e^{\mu_i l}\left[\mathbf{F}_{lk}(r)-\boldsymbol{\varphi}^{(k)}\right]\right\}\,dr = 0, \tag{74}$$

$$\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} [\boldsymbol{\psi}_0^{(0k)}]^T\mathbf{J}\left[\mathbf{F}_{0k}(r)-\mathbf{F}_{lk}(r)\right]dr = 0, \tag{75}$$

$$\sum_{k=1}^{m}\int_{c_{k-1}}^{c_k} \left\{\left[\boldsymbol{\psi}_0^{(1k)}(r)\right]^T\mathbf{J}\left[\mathbf{F}_{0k}(r)-\mathbf{F}_{lk}(r)\right] - l\left[\boldsymbol{\psi}_0^{(0k)}\right]^T\mathbf{J}\left[\mathbf{F}_{0k}(r)-\boldsymbol{\varphi}^{(k)}\right]\right\}dr = 0, \tag{76}$$

in which the four unknowns in $\mathbf{F}_{0k}(r)$ and $\mathbf{F}_{lk}(r)$ are related to the constants C_0, C_0', C_i and C_{-i} by Eqs. (57) ~ (60); the terms on the right-hand sides are completely known.

Equation (73) ~ (76) are, in fact, a system of four integral equations for the four unknown functions of r (2 stress components at the fixed end: $[r\sigma_{rz}(r, 0)]_k$ and $[r\sigma_{zz}(r, 0)]_k$; 2 displacement components at the free end: $[u_r(r, l)]_k$ and $[u_z(r, l)]_k$, which in turn are given by Eqs. (57) ~ (60) in series forms). Thus, substituting Eqs. (57) ~ (60) in Eqs. (73) ~ (76), taking n terms of the series for computation, we obtain a system of $2n + 2$ algebraic equations ($2n$ equations from Eqs. (73) and (74) for $i = 1, 2, \cdots, n$; 2 equations from Eqs. (75) and (76) for $i = 0$) for the $2n + 2$ unknowns: C_0, C_0', C_i and C_{-i} ($i = 1, 2, \cdots, n$), which can be solved by using a standard method for simultaneous linear algebraic equations.

9. RESULTS AND DISCUSSIONS

For numerical calculations using MATLAB, a laminated tube with multilayerd section is examined as shown in Fig. 1. The dimensionless length of the tube is taken to be $l/b = 10$ and the inner radii is $a/b = 0.25$, where b is the outer radii. The number of layers is 3, and each layer has the same thickness, thus at the interfaces between layers: $c_1/b = 0.5$ and $c_2/b = 0.75$. In the outer layer, the following dimensionless elastic constants for transversely isotropic material are taken: $(c_{33} / c_{11})_{(3)} = 5$; $(c_{12} / c_{11})_{(3)} = (c_{13} / c_{11})_{(3)} = 1/3$; $(c_{44} / c_{11})_{(3)} = (c_{66} / c_{11})_{(3)}$. In the middle layer and in the inner layer, the elastics constants are 2 and 3 times that of the outer layer, respectively. The internal pressure is $R_a / (c_{11})_{(3)} = 1$, and the external pressure is $R_b / (c_{11})_{(3)} = 2$.

Table 1 shows the first 20 dimensionless quantities $\mu_i b$ associated with the eigenvalues for the material parameters considered. Note that the eigenvalues in the present analysis are independent of the prescribed end conditions. The first eigenvalue in all cases is zero and it is a repeated eigenvalue, with which associated linearly z-dependent displacements and z-independent stresses. The non-zero eigenvalues are related to the stress disturbance by the end conditions. Variations of the displacements and stresses in the laminated tube are evaluated by taking 40 terms in the eigenfunction expansion. This number is chosen to

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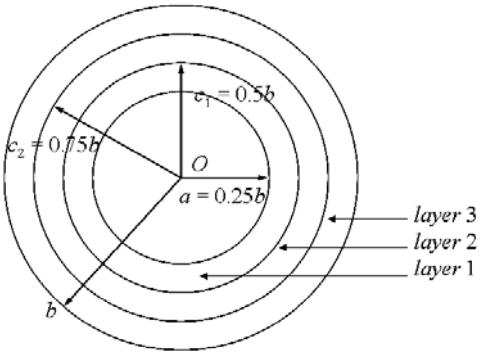


Fig. 1 A laminated tube composed of 3 layers of transversely isotropic materials

Table 1 The dimensionless eigenvalues (μb) for transversely isotropic materials

mode	μb
1	0
2	1.270
3	2.242
4	3.365+0.461i
5	3.365–0.461i
6	4.746
7	5.556
8	7.403
9	8.091+0.441i
10	8.091–0.441i
11	8.336
12	10.308
13	11.408
14	12.704
15	13.741+0.534i
16	13.741–0.534i
17	14.353
18	15.868
19	16.866
20	18.334

achieve a required accuracy. To evaluate the end effects on the deformation and stress distribution in the laminated tube, the tube with a fixed end are considered and the results are compared with the corresponding ones according to the simplified solution that satisfies only the lateral BC.

When the laminated tube is only subjected to extension at the end $z = l$, the cylindrical surfaces at $r = a$ and $r = b$ are traction-free. The only simplified solution is $C_0\boldsymbol{\psi}_{01}^{(k)} + C_0'\boldsymbol{\psi}_{02}^{(k)}$ in Eq. (51) in which $\boldsymbol{\psi}_{01}^{(k)}$ and $\boldsymbol{\psi}_{02}^{(k)}$ are the solutions for repeated $\mu_0 = 0$ and satisfy the lateral BC. Figure 2 depict variations of displacements and stresses in the radial direction at the fixed end $z = 0$. The dot lines in the figure are the results given by the simplified solution. The lateral BC $\sigma_{rr} = \sigma_{rz} = 0$ at $r = a$ and at $r = b$ are satisfied as required. The stresses σ_{rr}, $\sigma_{\theta\theta}$ and σ_{rz} are small but nonzero, whereas the simplified solution gives $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0$. The points at $r = a$ and at $r = b$ are around the rim of the cylinder where results significantly different from the simplified ones are to be expected. Remarkable differences in the axial stress σ_{zz} from the one given by the simplified solution exist around the rim of the cylinder. This is expected because the fixed end conditions $u_r = u_z = 0$ at $z = 0$ are not satisfied by the simplified solution. Comparing with the stresses σ_{zz} and σ_{rz} according to the simplified solution, the stresses of complete solution overshoot near the boundary points, indicating stress concentration appears at the rims on the base plane at the fixed end due to geometric discontinuity. It is expected that the end effect of the stresses is more significant in the layer with stronger anisotropy. Therefore, as c_{33} / c_{11} increases as the inner layer, the curves of the rigorous results should deviate from the one based on the simplified solution.

Figures 3 and 4 show the variations of displacements and stresses in the radial direction at $z = 0.1b$ and at $z = 1b$ near the fixed end. Remarkable differences of u_r, σ_{zz} and $\sigma_{\theta\theta}$ from those of the simplified solution exist in the region within a distance of outer diameter of the laminated tube. The fixed end effects diminish rapidly. At a distance of a diameter from the base plane, the simplified solution produces essentially the same results as those obtained from the present analysis. This implies that the end effect is confined to a local region near the fixed end.

When the laminated tube is only subjected to uniform pressure at the cylindrical surfaces $r = a$ and $r = b$, the only simplified solution is $\boldsymbol{\phi}^{(k)}$ in Eq. (51) which satisfies the lateral BC. To show the effect of uniform pressure, Fig. 5 depicts the displacements and stresses

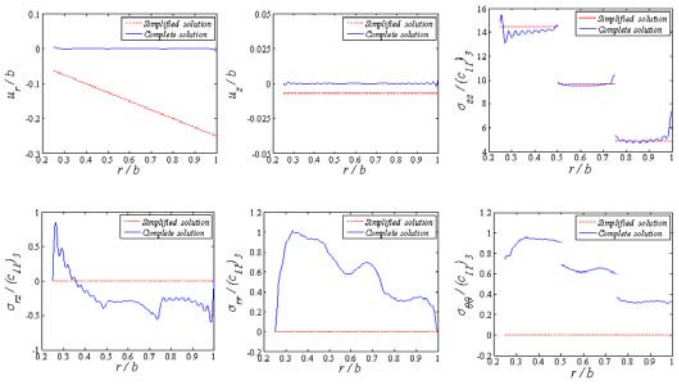


Fig. 2 Variations of displacements and stresses of the laminated tube subjected to extension in radial direction at fixed end $z = 0$

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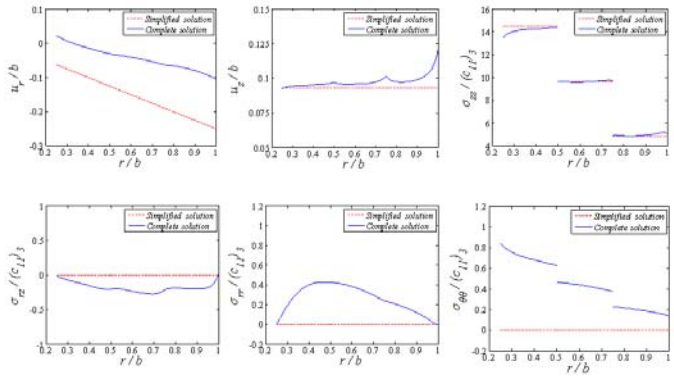


Fig. 3 Variations of displacements and stresses of the laminated tube subjected to extension in radial direction at $z = 0.1b$

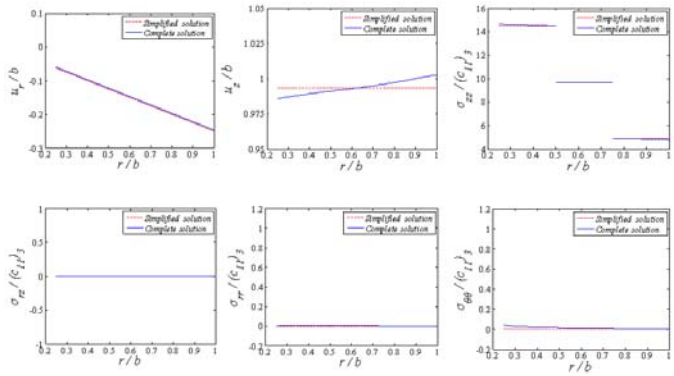


Fig. 4 Variations of displacements and stresses of the laminated tube subjected to extension in radial direction at $z = 1b$

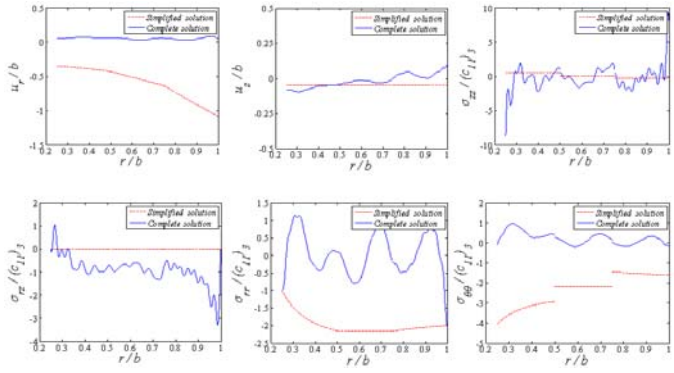


Fig. 5 Variations of displacements and stresses of the laminated tube subjected to uniform pressuring in radial direction at fixed end $z = 0$

in the radial direction at the fixed end $z = 0$. The lateral BC at $r = a$ and at $r = b$ are satisfied as required. Remarkable differences in the stresses σ_{zz} and σ_{rz} from the one given by the simplified solution exist around the rims of the laminated tube. This is expected because the fixed end conditions $u_r = 0$ at $z = 0$ are not satisfied by the simplified solution. The stresses of complete solution overshoot near the boundary points, comparing with the stresses σ_{zz} and σ_{rz} according to the simplified solution. In Fig. 6 the fixed end effects diminish rapidly at a distance of a outer diameter from the base plane. The present analysis produces essentially the same results as those obtained from the simplified solution. The end effect is confined to a local region near the fixed end.

In Figs. 2 and 5, some fluctuations of the curves are observed. The stresses at the fixed end are oscillating with respect to mean values. It's a consequence of the series solution by eigenfunction expansion and using small scales to illustrate the variations. At a distance $z = 0.1a$ away from the fixed end, the oscillations reduce rapidly, as shown in Fig. 3.

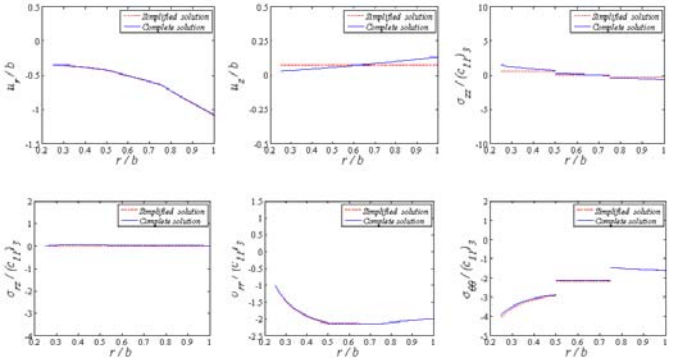


Fig. 6 Variations of displacements and stresses of the laminated tube subjected to uniform pressuring in radial direction at $z = 1b$

10. CONCLUSIONS

In this paper we have formulated the problems of generalized plane strain of laminated composite tubes in a state space setting and developed a systematic method for stress analysis of the tube subjected to extension and uniform pressuring and the end effect is considered. On the basis of Hamiltonian state space formulation, an exact analysis of the deformations and stress distributions in the tube of transversely isotropic elastic materials is conducted.

The state space approach can treat multilayered systems in an elegant and effective way. By means of the transfer matrix we have derived Eq. (40) which transmits the state vector in radial direction from the inner surface to the outer surface. On $r = a$ and b either the traction or the displacement, or a mixed boundary conditions may be prescribed. Equation (40) consists of 4 equations in 4 unknowns. Among the 4 components of $\mathbf{U}_k$ and $\mathbf{S}_k$, 2 are prescribed and other 2 are unknown. For a well-posed problem, we can always find a unique solution for the unknowns following the same solution procedure. Once $\mathbf{U}_k$ and $\mathbf{S}_k$ on $r = a$ are found, the displacement and stress in the tube are determined via Eq. (38).

The results show that the displacement and stress fields in the region near the base plane of the tube are significantly different from those according to the simplified solutions that only satisfy the lateral BC. The end effect is pronounced in the tube within an outer diameter from the bottom plane being perfectly bonded.

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APPENDIX

A. Bessel'S Equation Reduced from Equation (19)

From Eq. (19), we can get two sets of equations:

$$\begin{bmatrix} S_{rz} \\ S_{zz} \end{bmatrix}_k = r \begin{bmatrix} c_{44}\mu & c_{44}d_r \\ c_{33}\hat{c}_{13}(d_r + r^{-1}) & c_{33}\mu \end{bmatrix}_k \begin{bmatrix} U_r \\ U_z \end{bmatrix}_k, \tag{A1}$$

$$\begin{bmatrix} -\tilde{c}_{11}\left[d_r(rd_r) - r^{-1} \right] & 0 \\ 0 & 0 \end{bmatrix}_k \begin{bmatrix} U_r \\ U_z \end{bmatrix}_k - \begin{bmatrix} \mu & \hat{c}_{13}(d_r - r^{-1}) \\ d_r & \mu \end{bmatrix}_k \begin{bmatrix} S_{rz} \\ S_{zz} \end{bmatrix}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A2}$$

Substituting Eq. (A1) in Eq. (A2), we have

$$-\begin{bmatrix} c_{11}(rd_{rr} + d_r - r^{-1}) + c_{44}\mu^2 r & (c_{44} + c_{13})\mu r d_r \\ (c_{44} + c_{13})\mu(rd_r + 1) & c_{44}(rd_{rr} + d_r) + c_{33}\mu^2 r \end{bmatrix}_k \cdot \begin{bmatrix} U_r \\ U_z \end{bmatrix}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A3}$$

From Eq. $(A3)_1$ we can get

$$(c_{44} + c_{13})_k \mu d_r U_{zk} = -\left[c_{11}(d_{rr} + r^{-1}d_r - r^{-2}) + c_{44}\mu^2 \right]_k U_{rk}. \tag{A4}$$

Substituting Eq. (A4) in Eq. $(A3)_2$, we have

$$\left[c_{11}c_{44}(d_{rr} + r^{-1}d_r - r^{-2})^2 + (c_{33}\tilde{c}_{11} - 2c_{44}c_{13})(d_{rr} + r^{-1}d_r - r^{-2})\mu^2 + c_{33}c_{44}\mu^4 \right]_k U_{rk} = 0, \tag{A5}$$

i.e., Bessel's equation:

$$\begin{aligned} &\left[r^2 d_{rr} + r d_r + (\kappa_1\mu)^2 r^2 - 1 \right]_k \\ &\left[r^2 d_{rr} + r d_r + (\kappa_2\mu)^2 r^2 - 1 \right]_k U_{rk} = 0, \end{aligned} \tag{A6}$$

where

$$(\kappa_1)_k = \left[\frac{c_{11}c_{33} - c_{13}(c_{13} + 2c_{44}) + \sqrt{(c_{11}c_{33} - c_{13}^2)\left[c_{11}c_{33} - (c_{13} + 2c_{44})^2 \right]}}{2c_{11}c_{44}} \right]_k^{1/2},$$

$$(\kappa_2)_k = \left[\frac{c_{11}c_{33} - c_{13}(c_{13} + 2c_{44}) - \sqrt{(c_{11}c_{33} - c_{13}^2)\left[c_{11}c_{33} - (c_{13} + 2c_{44})^2 \right]}}{2c_{11}c_{44}} \right]_k^{1/2}.$$

B. Adjoint Operator of $\mathbf{L}_{11}$

For simplicity, consider the case of homogeneous tube in which the operator

$$\mathbf{L}_{11} = \begin{bmatrix} 0 & -d_r \\ -\hat{c}_{13}(d_r + r^{-1}) & 0 \end{bmatrix}. \tag{B1}$$

Then the adjoint operator of $\mathbf{L}_{11}$ can be deduced from

$$\begin{aligned} &\int_a^b \mathbf{S}_j^T \mathbf{L}_{11} \mathbf{U}_i \, dr = \int_a^b \begin{bmatrix} S_{rz} \\ S_{zz} \end{bmatrix}_j^T \begin{bmatrix} 0 & -d_r \\ -\hat{c}_{13}(d_r + r^{-1}) & 0 \end{bmatrix} \begin{bmatrix} U_r \\ U_z \end{bmatrix}_i dr = \int_a^b \left[-(S_{rz})_j d_r (U_z)_i - (S_{zz})_j \hat{c}_{13}(d_r + r^{-1})(U_r)_i \right] dr \\ &= -\left[(S_{rz})_j (U_z)_i + (S_{zz})_j \hat{c}_{13} (U_r)_i \right]_a^b + \int_a^b \left[(U_z)_i d_r (S_{rz})_j + (U_r)_i \hat{c}_{13}(d_r - r^{-1})(S_{zz})_j \right] dr \\ &= -\left[(S_{rz})_j (U_z)_i + (S_{zz})_j \hat{c}_{13} (U_r)_i \right]_a^b + \int_a^b \begin{bmatrix} U_r \\ U_z \end{bmatrix}_i^T \begin{bmatrix} 0 & \hat{c}_{13}(d_r - r^{-1}) \\ d_r & 0 \end{bmatrix} \begin{bmatrix} S_{rz} \\ S_{zz} \end{bmatrix}_j dr \\ &= -\left[(S_{rz})_j (U_z)_i + \hat{c}_{13} (S_{zz})_j (U_r)_i \right]_a^b + \int_a^b \mathbf{U}_i^T \mathbf{L}_{11}^* \mathbf{S}_j \, dr, \end{aligned} \tag{B2}$$

which contains a part of the BC at $r = a, b$. Thus the adjoint operator of $\mathbf{L}_{11}$ is

$$\mathbf{L}_{11}^* = \begin{bmatrix} 0 & \hat{c}_{13}(d_r - r^{-1}) \\ d_r & 0 \end{bmatrix}. \tag{B3}$$

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