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SAMPLE PATH MODERATE DEVIATIONS FOR NON-STATIONARY POWER-LAW SHOT NOISE PROCESSES

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Abstract

We establish a sample path moderate deviation principle for the integrated shot noise process with Poisson arrivals and non-stationary noises. As in Pang and Taqqu (2019), we assume that the noise is conditionally independent given the arrival times, and the distribution of each noise depends on its arrival time. As applications, we derive moderate deviation principles for the workload process and the running maximum process for a stochastic fluid queue with the integrated shot noise process as the input; we also show that a steady-state distribution exists and derive the exact tail asymptotics.

Keywords: Moderate deviations; Poisson shot noise; generalized fractional Brownian motion

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1. Introduction

Power-law shot noise models are a key class of physical models for long-range dependence and self-similarity; see, e.g., [31, Chapter 9] and [39, Section 3.4]. They frequently arise in applications such as wireless networks [47], finance [26, 46], insurance [8, 16, 27], and queueing theory [7, 15, 36, 50]. A shot noise process captures the cumulative effect of arriving noise (or 'shots') whose influence persists according to certain response functions. When the noise is independent and identically distributed with a finite second moment, the integrated shot noise process converges weakly to a fractional Brownian motion (FBM); see, e.g., [26] and [39, Chapter 3.4]. When the noise has an infinite second moment, such as belonging to the domain of attraction of a symmetric α-stable distribution, the scaling limit of the integrated shot noise process may be a fractionally integrated α-stable Lévy process; see, e.g., [22, Chapter 3.3] and [28].

Non-stationary shot noise processes are widely used to model many stochastic systems [3, 36–38]. Recently, [35] studied a class of integrated non-stationary shot noise processes where the noise is assumed to be conditionally independent given the arrival times, with the

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distribution of each noise depending on its arrival time. In [35, Theorem 3.1], it is shown that the scaled integrated shot noise process converges in distribution to a generalized FBM [11]. This weak convergence result intuitively suggests that a sample path moderate deviation principle (MDP) for the scaled integrated shot noise process can be formulated, based on the limiting generalized FBM. The objective of this paper is to establish the MDP rigorously.

Large and moderate deviation principles for Poisson shot processes and related models have been extensively studied [6, 9–11, 13, 14, 18, 23, 29, 34, 43, 51]. However, most of these studies focus on situations where the increments are either independent or stationary. In those cases, Ottaviani's inequality and the maximal inequalities for martingales [41] play important roles in the proofs. When the noise is neither independent nor identically distributed, as in the framework of [35], no such maximal inequality is available, to the best of our knowledge. In this paper, we address this difficulty by applying Schied's exponential tightness criterion on the path space [44, 45] to replace the maximal inequality.

Recently, [1, Theorem 3.1] established the large deviation principle (LDP) for the generalized Riemann–Liouville FBM – see (17) – by using the weak convergence approach. Building on this LDP result, the LDP for a stochastic fluid queue with a generalized Riemann–Liouville FBM as the input process was obtained in [1, Theorem 4.1], the existence of a steady-state distribution was derived in [1, Corollary 5.1], and the exact logarithmic tail asymptotics for the steady-state distribution were obtained in [1, Theorem 5.4]. These results extend those of [10] regarding the stochastic fluid queue with a Riemann–Liouville FBM as the input process.

As pointed out in [2, Introduction], two asymptotic scaling regimes for shot noise processes have been studied in the literature. The first is the conventional time-space scaling regime, where time is sped up and space is scaled down. The second is known as the high-intensity regime, in which the arrival rate is scaled up while time remains unchanged in the shot response function, and space is scaled down. The sample path MDP for shot noise processes in the high-intensity regime was established in [2, Theorem 2.1]. The approach was based on the Dawson–Gärtner theorem [12, Theorem 4.6.1], the Gärtner–Ellis theorem [12, Theorem 2.3.6], and exponential tightness in appropriate functional spaces. However, shot noise processes with non-Markovian and non-stationary characteristics, as considered in [2], introduce new challenges. They applied a maximal inequality for the sum of independent random variables (which are not necessarily identically distributed), as well as Lusin's theorem for exponential tightness. In comparison, they also established the sample-path MDP result in the conventional time–space scaling regime (see [2, Theorem 2.2]). Although both scaling regimes yield MDPs with the same speed function, the rate functions differ significantly. For more details, see [2, Remarks 2.8, 2.9].

The MDP result in this paper (see Theorem 2) also applies to the conventional time–space scaling regime. However, we remove [2, Assumption 2.1 (A.4)] for the integrated shot noise process (3). This introduces difficulties in proving an MDP for finite-dimensional distributions (see Lemma 3) and exponential tightness (see Lemma 5). Surprisingly, by applying Schied's exponential tightness criterion on the path space from [44, 45] and utilizing Hölder's continuity of the limiting process X (see (7)), we are able to prove Lemmas 3 and 5 using the same strategy. We believe this method will also be useful for studying the sample path MDP for shot noise processes in the high-intensity regime.

1.1. Organization of the paper

In Section 2, we first describe the model of non-stationary power-law shot noise processes and state the assumptions on the scaled processes. We then present a functional central limit

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theorem in the space of continuous functions, followed by the sample path MDP result for non-stationary power-law shot noise processes and its applications to fluid queues with integrated shot noise process input. The proofs of the central limit theorem and MDP are provided in Sections 3 and 4, respectively. The proofs for the associated fluid queues are given in Section 5. A proof of the MDP without the fourth-moment condition (Condition 5) is provided in Section 6. Finally, a short proof of Schied's exponential tightness criterion [44] is included in the Appendix.

1.2. Notation

Throughout the paper, let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and natural numbers respectively, and let $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$, $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$. We denote by $\mathcal{D}[0, T]$ the space of all càdlàg functions on the interval $[0,T]$, equipped with the Skorokhod J_1-norm. Similarly, we denote by $\mathcal{C}[0, T]$ the space of all continuous functions on $[0,T]$, equipped with the uniform norm $\|z\|_T = \sup_{t\in[0,T]}|z(t)|$. For more details on these spaces and their properties, see, e.g., [4].

2. The models, main results, and applications

2.1. Non-stationary power-law shot noise processes

As in [35], we consider the shot noise process $S^* = \{S^*(y)\}_{y\in\mathbb{R}}$ defined by

$$S^*(y) := \sum_{j\in\mathbb{Z}_*} g^*(y - S_j)R_j(S_j), \quad y \in \mathbb{R}, \tag{1}$$

where $\{S_j\}_{j\in\mathbb{Z}_*}$ is a sequence of Poisson arrival times with rate 1, satisfying $\cdots < S_{-1} < 0 < S_1 < \cdots$, and R_j denotes the noise caused by the shot at the time S_j. The non-negative shot shape function g^* accounts for the lingering effect of the noise shocks.

Assume that the random variables $\{R_j\}_{j\in\mathbb{Z}_*}$ are conditionally independent given $\{S_j\}_{j\in\mathbb{Z}_*}$, and that, for each $j \in \mathbb{Z}_*$, the distribution of R_j depends on the time S_j. Specifically, for any $u, r \in \mathbb{R}$, we have the conditional distribution $\mathbb{P}(R_j \le r \mid S_j = u) = F_u(r)$. We write $R_j(S_j)$ to indicate explicitly that R_j depends on S_j.

Condition 1. *Given the arrival times $\{S_j\}_{j\in\mathbb{Z}_*}$, we assume that the conditional mean of R_j is zero, and the conditional variance of R_j is finite. Specifically, for each $u \in \mathbb{R} \setminus \{0\}$,*

$$K_1(u) := \int_{\mathbb{R}} r \, \mathrm{d}F_u(r) = 0, \qquad K_2(u) := \int_{\mathbb{R}} r^2 \, \mathrm{d}F_u(r) \in (0, \infty). \tag{2}$$

Furthermore, we assume that $K_2(u) = |u|^{-\gamma}\tilde{L}(u)\mathbf{1}_{\{u\neq 0\}}$, where $\gamma \in (0, 1)$ is a constant, $\tilde{L}(u)$ is a positive slowly varying function at $\pm\infty$, and $\tilde{L}(u)$ is bounded away from 0 and ∞ on every compact subset of $\mathbb{R}$.

Consider the integrated shot noise process $S := \{S(t)\}_{t\in\mathbb{R}_+}$ defined by

$$S(t) := \sum_{j\in\mathbb{Z}_*} \big(g(t - S_j) - g(-S_j)\big)R_j(S_j), \quad t \in \mathbb{R}_+, \tag{3}$$

where $S(0) = 0$, and the shot shape function $g(t)$ is given by

$$g(t) := \int_0^t g^*(y) \, \mathrm{d}y, \quad t \in \mathbb{R}_+. \tag{4}$$

We assume that the function g^* satisfies the following condition.

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Condition 2. *There exists a constant* $\beta \in \left(\frac{1}{2} - \gamma/2, 1\right)$ *such that*

$$g^*(y) := y^{-\beta} L^*(y) \mathbf{1}_{\{y>0\}}, \tag{5}$$

where L^* *is a positive slowly varying function at* $+\infty$ *that is bounded away from 0 and* ∞ *on every bounded subset of* $[0, \infty)$.

Under Condition 2, by using [5, Proposition 1.5.8] we know that the function g in (4) satisfies $g(t) = t^{1-\beta} L(t) \mathbf{1}_{\{t\geq 0\}}$, where L is a positive slowly varying function bounded away from 0 and ∞ on every bounded subset of $[0, \infty)$, and $L(t) \sim L^*(t)/(1-\beta)$ as $t \to \infty$. Let N be a Poisson random measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ with intensity $\mathrm{d}F_u(r)\,\mathrm{d}u$, that is,

(i) N is independently scattered on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, i.e. for any disjoint sets $G_1, \ldots, G_n$, $N(G_1), \ldots, N(G_n)$ are independent;

(ii) for any bounded Borel set $G \in \mathcal{B}(\mathbb{R}^2)$, $N(G)$ is a Poisson random variable with mean $\int_G \mathrm{d}F_u(r)\,\mathrm{d}u$.

Let $\widetilde{N}$ be the compensated Poisson measure defined by $\widetilde{N}(G) = N(G) - \int_G \mathrm{d}F_u(r)\,\mathrm{d}u$ for any bounded Borel set $G \in \mathcal{B}(\mathbb{R}^2)$. Since the conditional mean of R_j in (2) is zero, the integrated shot noise process $\{S(t)\}_{t\in\mathbb{R}_+}$ can be written as

$$S(t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big) r \widetilde{N}(\mathrm{d}u, \mathrm{d}r), \quad t \geq 0.$$

The integral on the right-hand-side is well defined in the sense of [42], since

$$\begin{aligned} \mathbb{E}[S(t)^2] &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} \big(g(t-u) - g(-u)\big)^2 r^2\,\mathrm{d}F_u(r)\,\mathrm{d}u \\ &= \int_{-\infty}^{t} \big(g(t-u) - g(-u)\big)^2 K_2(u)\,\mathrm{d}u, \end{aligned} \tag{6}$$

which is finite under Conditions 1 and 2. We also refer to [39, Appendix B.1.4] for more information on integration with respect to Poisson measures.

2.2. Weak convergence of the scaled integrated shot noise process

Condition 3. *Let* $b\colon \mathbb{N}_* \to \mathbb{R}_+$ *be a function that satisfies*

$$\lim_{n\to\infty} \frac{b(n)}{n^{3/2-\beta-\gamma/2} L(n) \tilde{L}(n)^{1/2}} = 1.$$

When $\beta \in \left(\frac{1}{2}, 1\right)$ and $\gamma \in (0, 1)$, under Conditions 1–3 and the fourth moment condition of the distribution F_u (see (9)), [35, Theorem 3.1 and Remark 3.3] proved that, as $n \to \infty$, $\{S(nt)/b(n)\}_{t\in[0,T]}$ converges weakly to a generalized fractional Brownian motion (GFBM) X in $(\mathcal{D}[0, T], J_1)$, where $X := \{X(t)\}_{t\in[0,T]}$ is defined by the integral representation

$$X(t) = \int_{\mathbb{R}} \left((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\right) |u|^{-\gamma/2}\, B(\mathrm{d}u),$$

with $u_+ = \max\{0, u\}$ for all $u \in \mathbb{R}$, and $B(\mathrm{d}u)$ a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure $\mathrm{d}u$. The process X is a continuous mean-zero Gaussian process with

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$X(0) = 0$, and it has the self-similarity property with Hurst parameter $H = \frac{3}{2} - \beta - \gamma/2$. When $\gamma = 0$, X becomes a standard FBM with Hurst parameter H; when $\gamma \in (0, 1)$, X is a GFBM as defined in [35]. A GFBM preserves the self-similarity property, while the factor $|u|^{-\gamma/2}$ introduces the non-stationarity of increments, making it suitable for modeling non-stationary physical systems. For example, [21] studied the semimartingale properties of GFBMs, and applied them to model volatility processes in finance.

It follows from [20, Theorem 3.1] that for any $\gamma \in (0, 1)$, $\beta \in \left(\frac{1}{2} - \gamma/2, 1\right)$, there exists a constant $c_{2,1}(\gamma, \beta) \in (0, \infty)$ such that, for any $t, s \geq 0$,

$$\mathbb{E}[|X(t) - X(s)|^2] = \int_{\mathbb{R}} \left((t-u)_+^{1-\beta} - (s-u)_+^{1-\beta}\right)^2 |u|^{-\gamma} \mathrm{d}u \leq c_{2,1}(\gamma, \beta)|t-s|^{3-2\beta-\gamma}. \quad (7)$$

The following theorem proves the convergence of $\{S(nt)/b(n)\colon t \in [0, T]\}_{n\in\mathbb{N}_*}$ in the space $\left(\mathcal{C}[0, T], \|\cdot\|_T\right)$.

Theorem 1. *Assume that Conditions 1–3 hold with $\gamma \in (0, 1)$ and $\beta \in \left(\frac{1}{2} - \gamma/2, 1 - \gamma/2\right)$. Then, as $n \to \infty$, $\{S(nt)/b(n)\}_{t\in[0,T]}$ converges weakly to GFBM X in $\left(\mathcal{C}[0, T], \|\cdot\|_T\right)$.*

Remark 1. Compared with [35, Theorem 3.1], Theorem 1 does not assume the fourth moment condition on F_u, and the convergence of the processes occurs in the function space $\left(\mathcal{C}[0, T], \|\cdot\|_T\right)$, replacing the Skorokhod space $(\mathcal{D}[0, T], J_1)$ used in [35]. If $\gamma = 0$, then the range of β in Theorem 1 is $\left(\frac{1}{2}, 1\right)$, which coincides with the result for the standard FBM (see, e.g., [39, Theorem 3.3.8]).

2.3. Sample path MDP

To establish the sample path MDP for the integrated shot noise processes with Poisson arrivals and non-stationary noise, we impose the following conditions.

Condition 4. *There are constants $u_0 > 0$ and $\theta_0 > 0$ such that $K_2(u) = |u|^{-\gamma}\tilde{L}(u)\mathbf{1}_{\{|u|\geq u_0\}}$, where $\tilde{L}(u)$ is a positive slowly varying function at $\pm\infty$ that it is bounded away from 0 and ∞ on every compact subset of $\mathbb{R}$, and*

$$M_{\theta_0} := \sup_{u\in\mathbb{R}} \int_{\mathbb{R}} \exp\left(\theta_0|r|\right) \mathrm{d}F_u(r) < +\infty. \quad (8)$$

Condition 5. *There exists a positive constant $\kappa < \infty$ such that, for all $|u| \geq u_0$,*

$$\frac{K_4(u)}{K_2(u)^2} \leq \kappa, \quad (9)$$

where $K_4(u) = \int_{\mathbb{R}} r^4 \, \mathrm{d}F_u(r)$.

Example 1. Suppose that F_u is Gaussian with mean zero and variance $K_2(u) = |u|^{-\gamma}\mathbf{1}_{\{|u|\geq u_0\}}$ for $\gamma \in (0, 1)$. Then Conditions 4 and 5 are satisfied.

Condition 6. *Assume that the function $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:*

$$\lim_{n\to\infty} \pi(n) = \infty, \qquad \lim_{n\to\infty} \frac{\pi(n)}{n^{1/2-\gamma/2}\tilde{L}(n)^{1/2}} = 0.$$

For any $n > 0$ and $t \geq 0$, we define

$$S_n(t) := \frac{S(nt)}{b(n)\pi(n)}. \quad (10)$$

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Theorem 2. *Assume Conditions 1–6 hold with* $\gamma \in (0, 1)$ *and* $\beta \in \left(\frac{1}{2} - \gamma/2, 1\right)$. *Then, as* $n \to \infty$, $\{S_n(t)\colon t \in [0, T]\}_{n\in\mathbb{N}_*}$ *satisfies the large deviation principle on* $\left(\mathcal{C}[0, T], \|\cdot\|_T\right)$ *with the speed function* $\pi(n)^2$ *and the good rate function* $I_{\beta,\gamma}$ *given by*

$$I_{\beta,\gamma}(\xi) := \inf_{\mathcal{S}_\xi} \frac{1}{2} \int_{-\infty}^{T} \psi(u)^2 \, \mathrm{d}u, \tag{11}$$

where, for $\xi \in \mathcal{C}[0, T]$, $\mathcal{S}_\xi$ *is the collection of all* $\psi \in L^2((-\infty, T], \mathbb{R})$ *such that*

$$\xi(t) = \int_{-\infty}^{t} \left((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\right)|u|^{-\gamma/2}\psi(u) \, \mathrm{d}u \quad \text{for } t \in [0, T].$$

To explain why the distribution F_u is assumed to be degenerate when u is in a neighborhood of the origin, let us recall that, when applying the Gärtner–Ellis theorem [12, Section 2.3], it is usually required that, for every θ in a neighborhood of the origin,

$$\Lambda(\theta) := \lim_{n\to\infty} \frac{1}{\pi(n)^2} \log \mathbb{E}[\exp(\theta\pi(n)^2 S_n(t))] \text{ exists as a real number.} \tag{12}$$

For the MDP of the empirical mean of independent and identically distributed random variables, Condition (12) reduces to the well-known Cramér condition; see, e.g., [12, Theorem 3.7.1] or [17, Theorem 1].

Let us continue with Example 1. If F_u is Gaussian with mean zero and variance $K_2(u) = |u|^{-\gamma}\mathbf{1}_{\{|u|\neq 0\}}$ for $\gamma \in (0, 1)$ then, for every $\theta > 0$, $\mathbb{E}[\exp(\theta\pi(n)^2 S_n(t))] = \infty$, which contradicts (12). Thus, to study the MDP, it is necessary to assume that F_u is degenerate when $|u| < u_0$ for some $u_0 > 0$, which is equivalent to saying that there is no noise when the shot time $|S_j| < u_0$ for every $j \in \mathbb{Z}_*$. Under this condition, the integrated shot noise process $\{S(t)\}_{t\in[0,T]}$ defined by (3) becomes the following truncated model:

$$\tilde{S}(t) := \sum_{j\in\mathbb{Z}_*} \left(g(t - S_j) - g(-S_j)\right)R_j(S_j)\mathbf{1}_{\{|S_j|\geq u_0\}}, \quad t \in \mathbb{R}_+. \tag{13}$$

Since $S_j \to +\infty$ as $|j| \to \infty$, the truncation can only affect a finite number of terms in (13). Therefore, $\{\tilde{S}(nt)/b(n)\}_{t\in[0,T]}$ shares many properties of $\{S(nt)/b(n)\}_{t\in[0,T]}$. In particular, Theorem 1 also holds for $\{\tilde{S}(nt)/b(n)\colon t \in [0, T]\}_{n\in\mathbb{N}_*}$ under Conditions 1–3.

2.4. Applications

Shot noise processes and integrated shot noise processes have been applied to model the claim amount processes in insurance risk theory or input processes in queueing theory [8, 16, 27, 28, 35, 50]. Those models are also applied to study the dynamics in storage systems or dams [40].

The process $\{S(t)\}_{t\geq 0}$ in the context of insurance can be interpreted as follows. Consider an insurance company with initial capital $M_0 > 0$, premium rate $\nu_1 > 0$, and insurance rate $\nu_2 \in (0, \nu_1)$. Suppose that claims occur according to a renewal process $\{S_j\}_{j\in\mathbb{Z}_*}$, and that the insurance company pays claims occurring at time $\{S_j\}_{j\in\mathbb{Z}_*}$, at the rate $g^*(\cdot - S_j)R_j(S_j) + \nu_2$, with $g^*(\cdot)$ given in (1). Then, the total claim paid in the time interval (0,t] is the integrated shot noise process $\{S(t) + \nu_2 t\}_{t\geq 0}$, with $S(t)$ defined in (3). See [16] for further details.

Write $\nu = \nu_1 - \nu_2$. For any $t \geq 0$, let

$$M(t) := \sup_{0\leq s\leq t}(S(s) - \nu s); \qquad V(t) := \sup_{0\leq s\leq t}(S(t) - S(s) - \nu(t - s)). \tag{14}$$

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Then, $M(t) > 0$ represents the event of the insurance company's ruin in the time interval $[0,t]$, and $V(t)$ is the total amount of residual claims, which are paid from the claims occurring up to time t. In the language of queueing models, $\{S(t) + \nu_2 t\}_{t\geq 0}$ is called an input process, $\{M(t)\}_{t\geq 0}$ is called a running maximum process, and $\{V(t)\}_{t\geq 0}$ is called a workload process [40, 50].

A systematic method for generating an input process that exhibits long-range dependence was developed in [10, 11],] which showed that $V^n = \{V(nt)/t\colon t \geq 0\}_{n\in\mathbb{N}_*}$ and $M^n = \{M(nt)/t\colon t \geq 0\}_{n\in\mathbb{N}_*}$ satisfy the MDP [10, Theorem 4.1, Lemma 5.2] and that the limit $M(\infty)$ has a sub-exponential tail distribution [10, Theorem 5.1]. On the other hand, [1] studied the LDP for the stochastic fluid queue with the generalized Riemann–Liouville FBM as the input process, examining the long-time behavior of the processes $\{M(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$, as defined by (14), with S replaced by the generalized Riemann–Liouville FBM. The methods there rely on large deviation theory and some special properties of the generalized Riemann-Liouville FBM.

We are interested in the asymptotic probabilities of the following events associated with the integrated shot noise process, as $t \to \infty$:

(i) the ruin probability of the insurance company in the time interval $[0, t]$, i.e.

$$\mathbb{P}\big(\sup_{0\leq s\leq t}(S(s) - \nu s) > M_0\big);$$

(ii) the probability that the total amount of residual claims paid from the time of the claim to time t exceeds a given value M_1, i.e.

$$\mathbb{P}\big(\sup_{0\leq s\leq t}(S(t) - S(s) - \nu(t-s)) > M_1\big).$$

Next, we give estimates for these probabilities using the MDP in Theorem 2. Let $c_{\beta,\gamma} = \int_{\mathbb{R}} \big((1-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)^2 |u|^{-\gamma}\,\mathrm{d}u$.

Proposition 1. *Assume that Conditions 1–5 hold with* $\gamma \in (0, 1)$ *and* $\beta \in \big(\frac{1}{2} - \gamma/2, 1 - \gamma\big)$.

(i) *For all* $\lambda > 0$,

$$\lim_{n\to\infty} \frac{b(n)^2}{n^2} \log \mathbb{P}\left(\frac{M(n)}{n} > \lambda\right) = -\mathcal{I}(\lambda), \tag{15}$$

where

$$\mathcal{I}(\lambda) = \begin{cases} \dfrac{(\lambda+\nu)^2}{2c_{\beta,\gamma}} & \text{if } \lambda \geq \dfrac{\nu(1-H)}{H}, \\ \dfrac{\nu^{2H}}{2c_{\beta,\gamma}H^{2H}(1-H)^{2(1-H)}}\lambda^{2(1-H)} & \text{if } \lambda < \dfrac{\nu(1-H)}{H}. \end{cases}$$

(ii) *For any* $\lambda > 0$,

$$\lim_{n\to\infty} \frac{b(n)^2}{n^2} \log \mathbb{P}(V(n) > \lambda) = -\inf_{0\leq s\leq 1} \frac{(\nu(1-s)+\lambda)^2}{v_1(s) + v_2(s)}, \tag{16}$$

where, for any $s \in (-\infty, 1]$, $v_1(s) = \int_{-\infty}^{s} \big[(1-u)^{1-\beta} - (s-u)^{1-\beta}\big]^2 |u|^{-\gamma}\,\mathrm{d}u$ *and* $v_2(s) = \int_s^1 (1-u)^{2(1-\beta)}|u|^{-\gamma}\,\mathrm{d}u$.

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Proposition 2. *Under the conditions of Proposition 1, $M(t)$ converges almost surely to a finite random variable $M(\infty)$, which has a sub-exponential tail distribution. Specifically,*

$$\lim_{n\to\infty} \frac{b(n)^2}{n^2} \log \mathbb{P}(M(\infty) > n) = -\frac{\nu^{2H}}{2c_{\beta,\gamma}H^{2H}(1-H)^{2(1-H)}}.$$

Remark 2.

(i) In this subsection, to study the long-time behavior of the running maximum process $M(n)$ and workload process $V(n)$, we assume a stronger condition: $\beta \in \left(\frac{1}{2} - \gamma/2, 1 - \gamma\right)$, in which case the Hurst parameter of GFBM X is $H = \frac{3}{2} - \beta - \gamma/2 \in \left(\frac{1}{2}, 1\right)$. This is the long-range dependence case. If $H < \frac{1}{2}$, the limiting behavior may be different. This condition is also necessary for the study of the long-term behaviors of these two processes for the stochastic fluid queue with input a (generalized Riemann–Louisville) FBM; see [10, Theorem 3.1] and [1, Remark 5.1].

(ii) In queueing theory, the steady-state limit

$$V(\infty) := \lim_{t\to\infty} \sup_{0\le s\le t}(Y(t) - Y(s) - \nu(t-s))$$

is obtained through the limit of $M(t) = \sup_{0\le s\le t}(Y(s) - \nu s)$. The key linkage between these two processes is the stationary increment property of the input process Y, which ensures that $V(\infty)$ and $M(\infty)$ are equal in distribution. If the input process Y has non-stationary increments, there is no simple way to ensure the existence of $V(\infty)$. By using the self-similarity and the modulus of continuity-type estimates of the input process Y, [1, Corollary 5.1, Theorem 5.4] established that the law of $V(t)$ has a weak limit as $t \to \infty$, and further derived the exact tail asymptotics for the steady state of V. See [1, p. 64] for more details. However, those two approaches do not apply to the queueing process with the integrated shot noise process $S(t)$ input (see (3)), since it lacks both properties of stationary increments and self-similarity.

2.5. The shot noise process with positive Poisson arrival times of shots

Consider the integrated shot noise process $\widetilde{S} := \{\widetilde{S}(t)\}_{t\in\mathbb{R}_+}$ defined by

$$\widetilde{S}(t) := \sum_{j\in\mathbb{N}_*} \big(g(t - S_j) - g(-S_j)\big)R_j(S_j), \quad t \in \mathbb{R}_+.$$

Here, the processes g, $\{S_j\}_{j\in\mathbb{N}_*}$, and $\{R_j\}_{j\in\mathbb{N}_*}$ satisfy Conditions 1 and 2 with $\gamma \in (0, 1)$ and $\beta \in \left(\frac{1}{2} - \gamma/2, 1\right)$.

When $b(n)$ satisfies Condition 3, by following similar steps to the proof of Theorem 1, we can show that $\{\widetilde{S}(nt)/b(n)\}_{t\in[0,T]}$ converges weakly to $\{Z(t)\}_{t\in[0,T]}$ in $\big(\mathcal{C}[0, T], \|\cdot\|_T\big)$ as $n \to \infty$. Here, $\{Z(t)\}_{t\ge0}$ is the generalized Riemann–Liouville FBM defined by

$$Z(t) := \int_0^t (t-u)^{1-\beta}u^{-\gamma/2}B(\mathrm{d}u), \quad t \ge 0, \tag{17}$$

where $B(\mathrm{d}u)$ is a Gaussian measure. This process was introduced in [35, Remark 5.1] and further studied in [20, Section 2.2] and [49, Section 3].

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For every $n \in \mathbb{N}_*$ and $t \in [0, T]$, let $\widetilde{S}_n(t)$ be defined as in (10) with $S(nt)$ replaced by $\widetilde{S}(nt)$. By following the same reasoning as in the proof of Theorem 2 we obtain that, under the same conditions as Theorem 2, $\{\widetilde{S}_n(t)\colon t \in [0, T]\}_{n\in\mathbb{N}_*}$ satisfies the LDP on $\big(\mathcal{C}[0, T], \|\cdot\|_T\big)$ as $n \to \infty$, with speed function $\pi(n)^2$ and good rate function $\widetilde{I}_{\beta,\gamma}$ given by

$$\widetilde{I}_{\beta,\gamma}(\xi) := \inf_{\psi\in\widetilde{\mathcal{S}}_\xi} \frac{1}{2}\int_0^T \psi(u)^2\,\mathrm{d}u, \tag{18}$$

where, for $\xi \in \mathcal{C}[0, T]$, $\widetilde{\mathcal{S}}_\xi$ is the collection of all $\psi \in L^2[0, T]$ such that

$$\xi(t) = \int_0^t \big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)|u|^{-\gamma/2}\psi(u)\,\mathrm{d}u \quad \text{for } t \in [0, T].$$

Notice that the rate function $\widetilde{I}_{\beta,\gamma}(\xi)$ in (18) is exactly the rate function for LDP of the generalized Riemann–Liouville FBM; see [1, Theorem 3.1].

As an application of the LDP for $\{\widetilde{S}_n\}_{n\geq 1}$, following the steps in Section 2.4, we can also obtain similar results for $\widetilde{M}$ and $\widetilde{V}$, where $\widetilde{M}(t) := \sup_{0\leq s\leq t}(\widetilde{S}(s) - \nu s)$ and $\widetilde{V}(t) := \sup_{0\leq s\leq t}(\widetilde{S}(t) - \widetilde{S}(s) - \nu(t-s))$.

2.6. Results without Condition 5

When Condition 5 is absent, by using a strategy similar to the one in the previous subsections we can prove the same conclusion as Theorem 2, provided that the function $\pi(n)$ has a slower rate as n tends to infinity. More precisely, we assume $\pi(n)$ satisfies the following condition.

Condition 7. *Assume that there exists some constant $\eta > 0$ satisfying $\lim_{n\to\infty}\pi(n) = \infty$ and*

$$\lim_{n\to\infty}\frac{\pi(n)}{n^{1/2-\gamma/2-\eta}} = 0.$$

Theorem 3. *Assume that Conditions 1–4 and 7 hold with $\gamma \in (0, 1)$ and $\beta \in \big(\frac{1}{2} - \gamma/2, 1\big)$. Then the conclusion of Theorem 2 holds.*

3. Proof of Theorem 1

For any $n \in \mathbb{N}_*$, let $\hat{S}_n(t) = S(nt)/b(n)$ for all $t \geq 0$.

3.1. Two lemmas

First, we provide a uniform moment estimate for the increments of $\hat{S}_n$.

Lemma 1. *Assume that Conditions 1–3 hold with $\gamma \in (0, 1)$ and $\beta \in \big(\frac{1}{2} - \gamma/2, 1 - \gamma/2\big)$. Then, for any $0 < \eta < \min\{2\beta + \gamma - 1, 2 - 2\beta - \gamma\}$ there exist positive constants $c_{3,1}$ and $c_{3,2}$ such that, for any $n > c_{3,2}$,*

$$\mathbb{E}\big[|\hat{S}_n(t) - \hat{S}_n(s)|^2\big] \leq c_{3,1}|t-s|^{3-2\beta-\gamma-\eta} \quad \text{for all } s, t \in [0, T]. \tag{19}$$

Consequently, by Kolmogorov's continuity theorem (see, e.g., [24, Theorem C.6]), $\hat{S}_n$ has a modification, still denoted by $\hat{S}_n$, that is almost surely continuous on [0, T].

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Proof. For any $n > 0$, $0 \le s < t \le T$, we write

$$\hat{S}_n(t) - \hat{S}_n(s) = \frac{1}{b(n)} \int_{-\infty}^{ns} \int_{-\infty}^{\infty} \big(g(nt-u) - g(ns-u)\big) r\, N(\mathrm{d}u, \mathrm{d}r) + \frac{1}{b(n)} \int_{ns}^{nt} \int_{-\infty}^{\infty} g(nt-u) r\, N(\mathrm{d}u, \mathrm{d}r).$$

Since the two terms on the right-hand side are independent and have expectation zero, we have

$$\begin{aligned}\mathbb{E}\big[(\hat{S}_n(t) - \hat{S}_n(s))^2\big] &= \frac{1}{b(n)^2} \int_{-\infty}^{ns} \big(g(nt-u) - g(ns-u)\big)^2 K_2(u)\, \mathrm{d}u \\ &\quad + \frac{1}{b(n)^2} \int_{ns}^{nt} g(nt-u)^2 K_2(u)\, \mathrm{d}u \\ &= \frac{n}{b(n)^2} \int_{-\infty}^{s} \big(g(nt-nu) - g(ns-nu)\big)^2 K_2(nu)\, \mathrm{d}u \\ &\quad + \frac{n}{b(n)^2} \int_{s}^{t} g(nt-nu)^2 K_2(nu)\, \mathrm{d}u, \end{aligned} \tag{20}$$

where a change of variables is used in the last step.

Since the function L^* is slowly varying at ∞, it follows from Potter's theorem [5, Theorem 1.5.6] that, for any $\delta_1 > 0$, there exists a constant $c_{3,3}(\delta_1) > 1$ such that the function L^* in (5) satisfies

$$\left|\frac{L^*(u)}{L^*(v)}\right| \le c_{3,3}(\delta_1) \max\left\{\left(\frac{u}{v}\right)^{\delta_1}, \left(\frac{u}{v}\right)^{-\delta_1}\right\} \quad \text{for all } u, v > 0. \tag{21}$$

By (4), (5), and (21), for any $\delta_1 \in (0, 1-\beta)$,

$$\begin{aligned}&\frac{|g(nt-nu) - g(ns-nu)|}{n^{1-\beta} L^*(n)} \\ &= \int_{(s-u)_+}^{(t-u)_+} y^{-\beta} \frac{L^*(ny)}{L^*(n)}\, \mathrm{d}y \\ &\le c_{3,3}(\delta_1) \max\left\{\int_{(s-u)_+}^{(t-u)_+} y^{-\beta+\delta_1}\, \mathrm{d}y, \int_{(s-u)_+}^{(t-u)_+} y^{-\beta-\delta_1}\, \mathrm{d}y\right\} \\ &= c_{3,3}(\delta_1) \max\left\{\frac{(t-u)_+^{1-\beta+\delta_1} - (s-u)_+^{1-\beta+\delta_1}}{1-\beta+\delta_1}, \frac{(t-u)_+^{1-\beta-\delta_1} - (s-u)_+^{1-\beta-\delta_1}}{1-\beta-\delta_1}\right\}. \end{aligned} \tag{22}$$

Applying Potter's theorem to the function $\tilde{L}$ given in Condition 1, we obtain that for any $\delta_2 > 0$ there exists a constant $c_{3,4}(\delta_2) > 0$ such that $\tilde{L}(nu)/\tilde{L}(n) \le c_{3,4}(\delta_2) \max\{|u|^{\delta_2}, |u|^{-\delta_2}\}$ for any $u \ne 0$. Hence, from this, Condition 3, and (22), we derive that for any $\delta_1 \in (0, 1-\beta)$ and $\delta_2 > 0$ there exist constants $c_{3,5} > 0$ and $c_{3,6} = c_{3,6}(\delta_1, \delta_2, \beta) > 0$ such that, for any $n \ge c_{3,5}$,

$$\begin{aligned}&\frac{n}{b(n)^2} \big(g(nt-nu) - g(ns-nu)\big)^2 K_2(nu) \\ &= \frac{n^{3-2\beta} L^*(n)^2 \tilde{L}(n)}{b(n)^2} \left(\frac{g(nt-nu) - g(ns-nu)}{n^{(1-\beta)} L^*(n)}\right)^2 \left(|nu|^{-\gamma} \frac{\tilde{L}(nu)}{\tilde{L}(n)}\right) \\ &\le c_{3,6} \max\left\{\big((t-u)_+^{1-\beta+\delta_1} - (s-u)_+^{1-\beta+\delta_1}\big)^2, \big((t-u)_+^{1-\beta-\delta_1} - (s-u)_+^{1-\beta-\delta_1}\big)^2\right\} \\ &\quad \times \max\{|u|^{\delta_2-\gamma}, |u|^{-\delta_2-\gamma}\}. \end{aligned} \tag{23}$$

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By (7) and (23), we obtain that for any $s, t \in [0, T]$, $n \geq c_{3,5}$, and positive constants $\delta_1 < 1-\beta$, $\delta_2 < \min\{\gamma, 1-\gamma\}$ with $\delta_1 + \delta_2/2 \leq \min\left\{\beta + \gamma/2 - \frac{1}{2}, 1-\beta-\gamma/2\right\}$,

$$\begin{aligned}
&\frac{n}{b(n)^2}\int_{-\infty}^{s}\big(g(nt-nu)-g(ns-nu)\big)^2K_2(nu)\,\mathrm{d}u\\
&\quad\leq c_{3,6}(\delta_1,\delta_2,\beta)\bigg[\int_{-\infty}^{s}\Big[(t-u)_+^{1-\beta+\delta_1}-(s-u)_+^{1-\beta+\delta_1}\Big]^2(|u|^{\delta_2-\gamma}+|u|^{-\delta_2-\gamma})\,\mathrm{d}u\\
&\qquad\qquad+\int_{-\infty}^{s}\Big[(t-u)_+^{1-\beta-\delta_1}-(s-u)_+^{1-\beta-\delta_1}\Big]^2(|u|^{\delta_2-\gamma}+|u|^{-\delta_2-\gamma})\,\mathrm{d}u\bigg]\\
&\quad\leq c_{3,7}(\delta_1,\delta_2,\beta,\gamma)\bigg[|t-s|^{3-2\beta-\gamma+2\delta_1+\delta_2}+|t-s|^{3-2\beta-\gamma+2\delta_1-\delta_2}\\
&\qquad\qquad+|t-s|^{3-2\beta-\gamma-2\delta_1+\delta_2}+|t-s|^{3-2\beta-\gamma-2\delta_1-\delta_2}\bigg]\\
&\quad\leq c_{3,7}(\delta_1,\delta_2,\beta,\gamma)T^{4\delta_1+2\delta_2}|t-s|^{3-2\beta-\gamma-2\delta_1-\delta_2},
\end{aligned}\tag{24}$$

where $c_{3,7}(\delta_1, \delta_2, \beta, \gamma)$ is a finite constant.

Similarly to the proof of (24), there exists a constant $c_{3,8} > 0$ such that, for all $n \geq c_{3,8}$, $s, t \in [0, T]$, and positive constants $\delta_1 < 1-\beta$, $\delta_2 < \min\{\gamma, 1-\gamma\}$ with $\delta_1 + \delta_2/2 \leq \min\left\{\beta + \gamma/2 - \frac{1}{2}, 1-\beta-\gamma/2\right\}$, we have

$$\frac{1}{b(n)^2}\int_{s}^{t} g(nt-nu)^2K_2(nu)\,\mathrm{d}u \leq c_{3,9}(\delta_1,\delta_2,\beta,\gamma)T^{4\delta_1+2\delta_2}|t-s|^{3-2\beta-\gamma-2\delta_1-\delta_2}.\tag{25}$$

Here, $c_{3,9}(\delta_1, \delta_2, \beta, \gamma)$ is a finite constant.

Putting (20), (24), and (25) together, we obtain (19) with $\eta = 2\delta_1 + \delta_2$. The proof is complete. □

3.2. Proof of Theorem 1

In order to verify the (exponential) tightness in the proofs of Theorems 1 and 2, we will make use of the following Hölder norm. For any $\tau \in (0, 1)$,

$$\|\xi\|_{T,\tau} := \sup\left\{\frac{|\xi(s)-\xi(t)|}{|s-t|^{\tau}} : s, t \in [0, T], s \neq t\right\}.$$

Let $\mathcal{C}^{\tau}[0, T]$ be the space of all τ-Hölder continuous functions, equipped with the $\|\cdot\|_{T,\tau}$ norm.

Proof of Theorem 1. The convergence of finite-dimensional distributions was proved in [35, Theorem 3.1]. Specifically, for any integer $k \geq 1$ and $t_i \geq 0$, $i = 1, \ldots, k$, $(\hat{S}_n(t_1), \ldots, \hat{S}_n(t_k))$ converges weakly to $(X(t_1), \ldots, X(t_k))$ in $\mathbb{R}^k$ as $n \to \infty$. To prove Theorem 1, it remains to establish the tightness of $\{\hat{S}_n(t) : t \in [0, T]\}_{n\in\mathbb{N}_*}$ in the space $\big(\mathcal{C}[0, T], \|\cdot\|_T\big)$.

Using Lemma 1 and Kolmogorov's continuity theorem (see, e.g., [24, Theorem C.6]), there exist constants $\tau > 0$, $c_{3,10} > 0$, and $c_{3,11} > 0$ such that, for all $n \geq c_{3,10}$, $\mathbb{E}\big(\|\hat{S}_n\|_{T,\tau}^2\big) \leq c_{3,11}$. It follows from Chebyshev's inequality that, for any $M > 0$, $\mathbb{P}\big(\|\hat{S}_n\|_{T,\tau} \geq M\big) \leq c_{3,11}/M^2$, which implies the tightness of $\{\hat{S}_n\}_{n\in\mathbb{N}_*}$ in $\big(\mathcal{C}[0, T], \|\cdot\|_T\big)$, since the Hölder space $\mathcal{C}^{\tau}[0, T]$ is compactly embedded in $\mathcal{C}[0, T]$. The proof is complete. □

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4. Proof of Theorem 2

First, we apply the Gärtner–Ellis theorem [12, Theorem 2.3.6] to prove the MDP for the finite-dimensional distributions of $\{S_n\}_{n\in\mathbb{N}_*}$; see Lemma 3. Then, we use the Dawson–Gärtner theorem [12, Section 4.6] to conclude sample-path MDP of $\{S_n\}_{n\in\mathbb{N}_*}$ in the topology of pointwise convergence; see Lemma 4. Finally, we strengthen the MDP to the topology of uniform convergence by establishing the exponential tightness of $\{S_n\}_{n\in\mathbb{N}_*}$ in $\big(\mathcal{C}[0,T],\|\cdot\|\big)$; see Lemma 5. By combining all these results together, we obtain the sample-path MDP for $\{S_n\}_{n\in\mathbb{N}_*}$ in the space $\mathcal{C}[0,T]$ endowed with the topology of uniform convergence.

4.1. MDP of the finite-dimensional distributions

We first give the following formula for the exponential moment of $S(t)$.

Lemma 2. *Under the conditions of Theorem 2, for any $T>0$ there exists a constant $\theta_1>0$ such that, for any $|\theta|\le\theta_1$ and $t\in[0,T]$,*

$$\mathbb{E}\big[\mathrm{e}^{\theta S(t)}\big]=\exp\bigg(\int_{-\infty}^{t}\int_{-\infty}^{\infty}\Big[\exp\big(\theta\big(g(t-u)-g(-u)\big)r\big)-1\Big]\,\mathrm{d}F_u(r)\,\mathrm{d}u\bigg)<\infty. \tag{26}$$

Proof. Let $G(t,u,r):=\big(g(t-u)-g(-u)\big)r$. For any $M>0$, let

$$G_{(M)}(t,u,r):=G(t,u,r)\cdot\mathbf{1}_{\{u\in[-M,t],|r|\le M\}},$$

and let $S_{(M)}(t):=\int_{-\infty}^{t}\int_{-\infty}^{\infty}G_{(M)}(t,u,r)\,N(\mathrm{d}u,\mathrm{d}r)$, $t\ge0$. By Condition 1 and (6), $S_{(M)}(t)\to S(t)$ in $L^2(\Omega,\mathcal{F},\mathbb{P})$ and $G_{(M)}(t,\cdot,\cdot)\in L^1(\mathbb{R}^2,\mathrm{d}F_u(r)\,\mathrm{d}u)$. Consequently, by Campbell's theorem [25, p. 28],

$$\log\mathbb{E}\big[\exp\big(\theta S_{(M)}(t)\big)\big]=\int_{-\infty}^{t}\int_{-\infty}^{\infty}\big[\exp(\theta G_{(M)}(t,u,r))-1\big]\,\mathrm{d}F_u(r)\,\mathrm{d}u \tag{27}$$

for any $\theta\in\mathbb{R}$ for which the integral on the right-hand side of (27) converges.

We claim that there exists a constant $\theta_1>0$ such that, for any $|\theta|\le\theta_1$,

$$\int_{-\infty}^{t}\int_{-\infty}^{\infty}[\exp(\theta G(t,u,r))-1]\,\mathrm{d}F_u(r)\,\mathrm{d}u<\infty. \tag{28}$$

We postpone the proof of (28). Let us first prove (26).

Notice that $G_{(M)}(t,u,r)\to G(t,u,r)$ and $|G_{(M)}(t,u,r)|\le|G(t,u,r)|$ for any $t\ge0$ and $(u,r)\in\mathbb{R}^2$. Thus, by (27), (28), and the dominated convergence theorem, we obtain (26).

Now we prove (28). For any $t\in[0,T]$ and $u\in\mathbb{R}$, by (2),

$$\begin{aligned}&\int_{-\infty}^{\infty}\big[\exp\big(\theta\big(g(t-u)-g(-u)\big)r\big)-1\big]\,\mathrm{d}F_u(r)\\&\qquad=\int_{-\infty}^{\infty}\big[\exp\big(\theta\big(g(t-u)-g(-u)\big)r\big)-1-\theta\big(g(t-u)-g(-u)\big)r\big]\,\mathrm{d}F_u(r).\end{aligned} \tag{29}$$

By using the elementary inequality

$$\big|\mathrm{e}^x-1-x\big|\le x^2\mathrm{e}^{|x|}\quad\text{for all } x\in\mathbb{R}, \tag{30}$$

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and by using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\exp\big(\theta\big(g(t-u)-g(-u)\big)r\big) - 1 - \theta\big(g(t-u)-g(-u)\big)r \right] \mathrm{d}F_u(r) \\
&\quad \le \theta^2\big(g(t-u)-g(-u)\big)^2 \int_{-\infty}^{\infty} |r|^2 \exp\big(\theta\big|\big(g(t-u)-g(-u)\big)r\big|\big)\, \mathrm{d}F_u(r) \\
&\quad \le \theta^2\big(g(t-u)-g(-u)\big)^2 \bigg(\int_{-\infty}^{\infty} |r|^4\, \mathrm{d}F_u(r) \bigg)^{1/2} \\
&\qquad \times \bigg(\int_{-\infty}^{\infty} \exp\big(2\theta\big|\big(g(t-u)-g(-u)\big)r\big|\big)\, \mathrm{d}F_u(r) \bigg)^{1/2}. \qquad (31)
\end{aligned}$$

By (4), we know that $\sup_{t\in[0,T],u\in\mathbb{R}}|g(t-u)-g(-u)| \le \sup_{t\in[0,T]} g(t) =: c_{g,T} < \infty$. By using Condition 4, for any $\theta \le \theta_0/(2c_{g,T})$ and $u \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \exp\big(2\theta|\big(g(t-u)-g(-u)\big)r|\big)\, \mathrm{d}F_u(r) \le \int_{-\infty}^{\infty} \exp(\theta_0|r|)\, \mathrm{d}F_u(r) \le M_{\theta_0}. \qquad (32)$$

By (9), we obtain

$$\int_{-\infty}^{t} \big(g(t-u)-g(-u)\big)^2 \bigg(\int_{-\infty}^{\infty} |r|^4\, \mathrm{d}F_u(r) \bigg)^{1/2} \mathrm{d}u \le \kappa^{1/2} \int_{-\infty}^{t} \big(g(t-u)-g(-u)\big)^2 K_2(u)\, \mathrm{d}u, \qquad (33)$$

which is finite by using Conditions 1 and 2, with $\gamma \in (0,1)$ and $\beta \in \big(\frac{1}{2}-\gamma/2, 1\big)$.

Putting (29)–(33) together, we obtain (28). The proof is complete. □

For any $t \ge 0$ and $u \in \mathbb{R}\setminus\{0\}$, let $g_{\beta,\gamma}(t,u) := \big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)|u|^{-\gamma/2}$. For any $\boldsymbol{t} = (t_1,\dots,t_m)^\top \in \mathbb{R}_+^m$, let $M_{\beta,\gamma}(\boldsymbol{t}) := \big(M_{\beta,\gamma}(t_i,t_j)\big)_{i,j\in\{1,\dots,m\}}$ be the $m\times m$ matrix with entries $M_{\beta,\gamma}(t_i,t_j) := \int_{\mathbb{R}} g_{\beta,\gamma}(t_i,u) g_{\beta,\gamma}(t_j,u)\, \mathrm{d}u$.

Lemma 3. *Under the conditions of Theorem 2, for any $m \in \mathbb{N}$, $\boldsymbol{\theta} = (\theta_1,\dots,\theta_m)^\top \in \mathbb{R}^m$ and $\boldsymbol{t} = (t_1,\dots,t_m)^\top \in \mathbb{R}_+^m$, the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{\pi(n)^2} \log \mathbb{E}\bigg[\exp\bigg(\pi(n)^2 \sum_{i=1}^{m} \theta_i S_n(t_i) \bigg) \bigg] = \frac{1}{2} \big\langle \boldsymbol{\theta}, M_{\beta,\gamma}(\boldsymbol{t})\boldsymbol{\theta} \big\rangle. \qquad (34)$$

Consequently, as $n\to\infty$, the sequence $\{(S_n(t_1),\dots,S_n(t_m))\}_{n\in\mathbb{N}_*}$ satisfies a large deviation principle on $\mathbb{R}^m$ with speed function $\pi(n)^2$ and rate function $J^{M_{\beta,\gamma}(\boldsymbol{t})}$ defined for $\boldsymbol{x}\in\mathbb{R}^m$ by

$$J^{M_{\beta,\gamma}(\boldsymbol{t})}(\boldsymbol{x}) := \sup_{\boldsymbol{y}\in\mathbb{R}^m} \bigg\{ \boldsymbol{x}^\top \boldsymbol{y} - \frac{1}{2}\boldsymbol{y}^\top M_{\beta,\gamma}(\boldsymbol{t})\boldsymbol{y} \bigg\}.$$

Proof. First, let us state the following two facts, which will be used frequently:

(i) Under Conditions 2, 3, and 6, it follows from (22) that

$$\lim_{n\to\infty} \sup_{-\infty<u\le t\le T} \frac{\pi(n)}{b(n)} |g(nt-nu)-g(-nu)| = 0. \qquad (35)$$

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(ii) Under Conditions 2, 3, and 4, we derive, as in [35, (52)], that

$$\lim_{n\to\infty}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)=\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}. \tag{36}$$

Furthermore, it follows from the dominated convergence theorem, (23), and (36) that

$$\lim_{n\to\infty}\int_{-\infty}^{t}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)\,\mathrm{d}u = \int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u. \tag{37}$$

We first prove (34) in the case $m=1$, that is,

$$\lim_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}[\exp(\theta\pi(n)^2S_n(t))]=\frac{\theta^2}{2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u. \tag{38}$$

The proof of (38) is divided into two parts: the upper bound and the lower bound.

Step 1: Upper bound. For any fixed $\theta\in\mathbb{R}$, $t\ge 0$, by (2), (26), and (35), we have

$$\begin{aligned}
&\frac{1}{\pi(n)^2}\log\mathbb{E}[\exp(\theta\pi(n)^2S_n(t))]\\
&\quad=\frac{1}{\pi^2(n)}\int_{-\infty}^{at}\int_{\mathbb{R}}\left[\exp\left(\theta\frac{\pi(n)}{b(n)}\big(g(nt-u)-g(-u)\big)r\right)-1\right]\mathrm{d}F_u(r)\,\mathrm{d}u\\
&\quad=\frac{1}{\pi^2(n)}\int_{-\infty}^{t}\int_{\mathbb{R}}\left[\exp\left(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right)-1\right.\\
&\qquad\left.-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right]\mathrm{d}F_{nu}(r)\,n\mathrm{d}u.
\end{aligned} \tag{39}$$

For any $\varepsilon>0$, there exists a constant $c_{4,1}(\varepsilon)>0$ such that

$$\left|\mathrm{e}^x-1-x-\frac{x^2}{2}\right|\le\frac{\varepsilon}{2}x^2\quad\text{for all } |x|\le c_{4,1}(\varepsilon). \tag{40}$$

It follows from (35) and (40) that, for any $\varepsilon>0$ and $r_0>0$,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\pi^2(n)}\left(\int_{-\infty}^{t}\int_{|r|\le r_0}\left[\exp\left(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right)-1\right.\right.\\
&\qquad\left.\left.-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right]\mathrm{d}F_{nu}(r)\,n\mathrm{d}u\right)\\
&\quad\le\frac{(1+\varepsilon)\theta^2}{2}\limsup_{n\to\infty}\int_{-\infty}^{t}\int_{|r|\le r_0}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2r^2\,\mathrm{d}F_{nu}(r)\,\mathrm{d}u\\
&\quad\le\frac{(1+\varepsilon)\theta^2}{2}\lim_{n\to\infty}\int_{-\infty}^{t}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)\,\mathrm{d}u\\
&\quad=\frac{(1+\varepsilon)\theta^2}{2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u,
\end{aligned} \tag{41}$$

where (37) was used in the last step.

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It follows from (30) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\int_{|r|>r_0}\left[\exp\left(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right)-1-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right]\mathrm{d}F_{nu}(r)\\
&\le\theta^2\frac{\pi(n)^2}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2\int_{|r|>r_0}|r|^2\exp\left(\theta\frac{\pi(n)}{b(n)}\big|\big(g(nt-nu)-g(-nu)\big)r\big|\right)\mathrm{d}F_{nu}(r)\\
&\le\theta^2\frac{\pi(n)^2}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2\left(\int_{\mathbb{R}}|r|^4\,\mathrm{d}F_{nu}(r)\right)^{1/2}\\
&\quad\times\left(\int_{|r|>r_0}\exp\left(2\theta\frac{\pi(n)}{b(n)}\big|\big(g(nt-nu)-g(-nu)\big)r\big|\right)\mathrm{d}F_{nu}(r)\right)^{1/2}. \qquad (42)
\end{aligned}$$

By (35) and Chebyshev's inequality we have, for sufficiently large n,

$$\begin{aligned}
&\int_{|r|>r_0}\exp\left(2\theta\frac{\pi(n)}{b(n)}\big|\big(g(nt-nu)-g(-nu)\big)r\big|\right)\mathrm{d}F_{nu}(r)\\
&\quad\le\int_{|r|>r_0}\exp\left(\frac{\theta_0}{2}|r|\right)\mathrm{d}F_{nu}(r)\\
&\quad\le\exp\left(-\frac{\theta_0 r_0}{2}\right)\int_{|r|>r_0}\exp(\theta_0|r|)\,\mathrm{d}F_{nu}(r)\le\exp\left(-\frac{\theta_0 r_0}{2}\right)M_{\theta_0}, \qquad (43)
\end{aligned}$$

where (8) in Condition 4 was used in the last step.

By using Condition 5 and (37), we derive

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(-nu)\big)^2\left(\int_{\mathbb{R}}|r|^4\,\mathrm{d}F_{nu}(r)\right)^{1/2}\mathrm{d}u\\
&\quad=\limsup_{n\to\infty}\frac{n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(-nu)\big)^2K_4(nu)^{1/2}\,\mathrm{d}u\\
&\quad\le\kappa^{1/2}\limsup_{n\to\infty}\frac{n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)\,\mathrm{d}u\\
&\quad=\kappa^{1/2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u. \qquad (44)
\end{aligned}$$

Putting (39), (41), (42), (43), and (44) together, and using the arbitrariness of r_0, we obtain that, for any $\theta\in\mathbb{R}$, $t\ge 0$, and $\varepsilon\in(0,1)$,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}[\exp(\theta\pi(n)^2S_n(t))]\\
&\qquad\le\frac{\theta^2(1+\varepsilon)}{2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u. \qquad (45)
\end{aligned}$$

Step 2: Lower bound. Since $\mathrm{e}^x-1-x\ge 0$ for all $x\in\mathbb{R}$, it follows from (35) and (40) that, for any $\varepsilon\in(0,1)$ and $r_0>0$,

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$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{\pi^2(n)}\Bigg(\int_{-\infty}^{t}\int_{\mathbb{R}}\Bigg[\exp\Bigg(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\Bigg)-1\\
&\qquad\qquad-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\Bigg]\,\mathrm{d}F_{nu}(r)\,n\mathrm{d}u\Bigg)\\
&\geq\liminf_{n\to\infty}\frac{1}{\pi^2(n)}\Bigg(\int_{-\infty}^{t}\int_{|r|\leq r_0}\Bigg[\exp\Bigg(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\Bigg)-1\\
&\qquad\qquad-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\Bigg]\,\mathrm{d}F_{nu}(r)\,n\mathrm{d}u\Bigg)\\
&\geq\frac{(1-\varepsilon)\theta^2}{2}\liminf_{n\to\infty}\int_{-\infty}^{t}\int_{|r|\leq r_0}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2r^2\,\mathrm{d}F_{nu}(r)\,\mathrm{d}u\\
&\geq\frac{(1-\varepsilon)\theta^2}{2}\liminf_{n\to\infty}\int_{-\infty}^{t}\int_{\mathbb{R}}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)\,\mathrm{d}u\\
&\quad-\frac{(1-\varepsilon)\theta^2}{2}\limsup_{n\to\infty}\int_{-\infty}^{t}\int_{|r|>r_0}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2r^2\,\mathrm{d}F_{nu}(r)\,\mathrm{d}u\\
&=\frac{(1-\varepsilon)\theta^2}{2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u\\
&\quad-\frac{(1-\varepsilon)\theta^2}{2}\limsup_{n\to\infty}\int_{-\infty}^{t}\int_{|r|>r_0}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2r^2\,\mathrm{d}F_{nu}(r)\,\mathrm{d}u,
\end{aligned}\tag{46}$$

where (37) was used in the last step. Next, we prove that the last term in (46) can be ignored as $r_0\to\infty$.

By using the Cauchy–Schwarz inequality and Condition 5, we have

$$\begin{aligned}
&\int_{-\infty}^{t}\int_{|r|>r_0}\frac{n}{b(n)^2}\big(g(nt-nu)-g(-nu)\big)^2r^2\,\mathrm{d}F_{nu}(r)\,\mathrm{d}u\\
&\quad\leq\frac{n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(-nu)\big)^2\Bigg(\int_{\mathbb{R}}|r|^4\,\mathrm{d}F_{nu}(r)\Bigg)^{1/2}\Bigg(\int_{|r|>r_0}\mathrm{d}F_{nu}(r)\Bigg)^{1/2}\mathrm{d}u\\
&\quad\leq\frac{\kappa^{1/2}n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(-nu)\big)^2K_2(nu)\,\mathrm{d}u\cdot\sup_{u\in\mathbb{R}}\Bigg(\int_{|r|>r_0}\mathrm{d}F_{nu}(r)\Bigg)^{1/2}.
\end{aligned}\tag{47}$$

By Chebyshev's inequality and Condition 4, we have

$$\begin{aligned}
\sup_{u\in\mathbb{R}}\int_{|r|>r_0}\mathrm{d}F_{nu}(r)&\leq\exp(-\theta_0r_0)\cdot\sup_{u\in\mathbb{R}}\int_{|r|>r_0}\exp(\theta_0|r|)\,\mathrm{d}F_{nu}(r)\\
&\leq\exp(-\theta_0r_0)M_{\theta_0}.
\end{aligned}\tag{48}$$

Putting (39), (46), (47), and (48) together, and using the arbitrariness of r_0, we obtain that, for any $\theta\in\mathbb{R}$, $t\geq0$, and $\varepsilon\in(0,1)$,

$$\liminf_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}[\exp(\theta\pi(n)^2S_n(t))]\geq\frac{\theta^2(1-\varepsilon)}{2}\int_{-\infty}^{t}\big((t-u)_+^{1-\beta}-(-u)_+^{1-\beta}\big)^2|u|^{-\gamma}\,\mathrm{d}u.$$

This, together with (45), implies (38).

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Next, we prove (34) for $m \geq 2$. Notice that, for any $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_m)^\top \in \mathbb{R}^m$ and $\boldsymbol{t} := (t_1, \ldots, t_m)^\top \in \mathbb{R}_+^m$, we can write

$$\mathbb{E}\left[\exp\left(\frac{\pi(n)}{b(n)} \sum_{i=1}^m \theta_i S(nt_i) \right) \right] = \mathbb{E}\left[\exp\left(\frac{\pi(n)}{b(n)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{j=1}^k \theta_j \big(g(nt_j - u) - g(-u) \big) \right) r\, N(\mathrm{d}u, \mathrm{d}r) \right) \right].$$

The proof of (34) is similar to that of (38). More specifically, we replace $\theta\big(g(nt-u)-g(-u)\big)$ in (39) and (46) with $\sum_{j=1}^m \theta_j \big(g(nt_j-u)-g(-u)\big)$ and follow the same computation to obtain

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\pi(n)^2} \log \mathbb{E}\left[\exp\left(\pi(n)^2 \sum_{i=1}^m \theta_i S_n(t_i) \right) \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left(\sum_{j=1}^m \theta_j \big((t_j - u)_+^{1-\beta} - (-u)_+^{1-\beta} \big) \right)^2 |u|^{-\gamma}\, \mathrm{d}u \\ &= \frac{1}{2} \langle \boldsymbol{\theta}, M_{\beta,\gamma}(\boldsymbol{t}) \boldsymbol{\theta} \rangle. \end{aligned}$$

Thus, (34) holds. By the Gärtner–Ellis theorem (see, e.g., [12, Theorem 2.3.6]), we obtain the desired result. □

4.2. MDP in the topology of pointwise convergence

Lemma 4. *Under the conditions of Theorem 2, $\{S_n(t)\colon t \in [0,T]\}_{n\in\mathbb{N}_*}$ satisfies a large deviation principle on $\mathcal{C}[0,T]$ in the topology of pointwise convergence as $n \to \infty$, with speed function $\pi(n)^2$ and the rate function $I_{\beta,\gamma}$ given by (11).*

Proof. It follows from the Dawson–Gärtner theorem [12, Theorem 4.6.1] and Lemma 3 that the family $\{S_n(t)\colon t \in [0,T]\}_{n\in\mathbb{N}_*}$ satisfies a large deviation principle on $\mathcal{C}[0,T]$ in the topology of pointwise convergence as $n \to \infty$, with the speed function $\pi(n)^2$ and the rate function $I'_{\beta,\gamma}$ given by

$$I'_{\beta,\gamma}(\xi) := \sup_{0\leq t_1<\cdots<t_m\leq T, m\in\mathbb{N}} J^{M_{\beta,\gamma}(t_1,\ldots,t_m)}((\xi(t_1), \ldots, \xi_{t_m})^\top). \tag{49}$$

Following exactly the same steps as in the proof of [33, Lemma 11], we conclude that the rate functions $I'_{\beta,\gamma}$ in (49) and $I_{\beta,\gamma}$ in (11) agree on $\mathcal{C}[0,T]$. The proof is complete. □

4.3. Exponential tightness in $\big(\mathcal{C}[0,T], \|\cdot\|_T\big)$

To strengthen the MDP to the topology of uniform convergence, we establish the associated exponential tightness of $\{S_n(t)\colon t \in [0,T]\}_{n\in\mathbb{N}_*}$.

Lemma 5. *Under the conditions of Theorem 2, for any $M > 0$ there exists a compact set $\mathcal{K}_M \subset \big(\mathcal{C}[0,T], \|\cdot\|_T\big)$ such that $\limsup_{n\to\infty} (1/\pi(n)^2) \log \mathbb{P}(S_n \notin \mathcal{K}_M) \leq -M$.*

Proof. The result follows from Lemma 6 and Theorem 4. □

Lemma 6. *Under the conditions of Theorem 2, for any $T > 0$ there exist some positive constants $c_{4,2}$, $c_{4,3}$, and $c_{4,4}$ such that, for any $n \geq c_{4,2}$, and $0 \leq s, t \leq T$, with $0 < \eta < \frac{3}{2} - \beta - \gamma/2$, the following inequality holds:*

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$$\mathbb{E}\left[\exp\left(\pi(n)^2\frac{|S_n(t)-S_n(s)|}{|t-s|^\eta}\right)\right]\le c_{4,3}\exp(c_{4,4}\pi(n)^2). \tag{50}$$

Proof. By using the same argument as in the proof of (38) we obtain, for any $\theta\in\mathbb{R}$ and $0\le s<t\le T$,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}[\exp(\theta\pi(n)^2(S_n(t)-S_n(s))] &= \frac{\theta^2}{2}\int_{-\infty}^{t}\left((t-u)_+^{1-\beta}-(s-u)_+^{1-\beta}\right)^2|u|^{-\gamma}\,\mathrm{d}u\\ &\le c_{2,1}(\gamma,\beta)\frac{\theta^2}{2}|t-s|^{3-2\beta-\gamma},\end{aligned} \tag{51}$$

where (7) was used in the last step. Taking $\theta=\pm1/|t-s|^\eta$ in (51), we have

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}\left[\exp\left(\pi(n)^2\frac{|S_n(t)-S_n(s)|}{|t-s|^\eta}\right)\right]\\ &\quad=\max\left\{\lim_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}\left[\exp\left(\pi(n)^2\frac{S_n(t)-S_n(s)}{|t-s|^\eta}\right)\right],\right.\\ &\qquad\left.\lim_{n\to\infty}\frac{1}{\pi(n)^2}\log\mathbb{E}\left[\exp\left(-\pi(n)^2\frac{S_n(t)-S_n(s)}{|t-s|^\eta}\right)\right]\right\}\\ &\quad\le c_{2,1}(\gamma,\beta)|t-s|^{3-2\beta-\gamma-2\eta}.\end{aligned}$$

This implies (50). The proof is complete. □

5. Proofs of Propositions 1 and 2

Proof of Proposition 1. The proof for (15) follows the same lines as the proof of [10, Lemma 5.2] or [1, Lemma 4.1], and the proof for (16) follows that of [10, Theorem 4.1] or [1, Theorem 4.1]. We adopt the same approach for our process. To save space, we provide the details only for the proof of (15), while (16) can be proved using a similar argument.

Let $M_n(t):=M(nt)/n$. Take $\pi(n)=n/b(n)$. Then Condition 6 holds by the choice of $\beta\in\left(\frac{1}{2}-\gamma/2,\,1-\gamma\right)$. Clearly,

$$M_n(t)=\frac{\sup_{0\le s\le nt}(S(s)-\nu s)}{n}=\sup_{0\le s\le t}(S_n(s)-\nu s).$$

Since the mapping $\{S_n(t)\}_{t\in[0,1]}\in\mathcal{C}[0,1]\mapsto\{M_n(t)\}_{t\in[0,1]}\in\mathcal{C}[0,1]$ is continuous, it follows from the contraction principle that $\{M_n(t)\colon t\in[0,1]\}_{n\in\mathbb{N}_*}$ satisfies the LDP with speed $n^2/b(n)^2$ and a good rate function $I_M(\psi)\colon\mathcal{C}[0,1]\to\mathbb{R}$ given by

$$\begin{aligned}I_M(\psi):=\inf\Big\{&I_{\beta,\gamma}(\phi)\colon\text{ for all } t\in[0,1],\\ &\sup_{0\le s\le t}\left[\int_{-\infty}^{s}\left((s-u)_+^{1-\beta}-(-u)_+^{1-\beta}\right)|u|^{-\gamma/2}\phi(u)\,\mathrm{d}u-\nu s\right]=\psi(t)\Big\},\end{aligned}$$

where $I_{\beta,\gamma}$ is given by (11). In addition, from the contraction principle and $M_n(1)=M(n)/n$, we know that $\{M(n)/n\}_{n\in\mathbb{N}_*}$ satisfies the LDP with speed $n^2/b(n)^2$ and a good rate function $\Lambda_M^*(\lambda)=\inf_{\psi(1)=\lambda}I_M(\psi)$. In particular,

$$\limsup_{n\to\infty}\frac{b(n)^2}{n^2}\log\mathbb{P}\left(\frac{M(n)}{n}>\lambda\right)\le-\inf_{x>\lambda}\Lambda_M^*(x)=-\inf_{\phi\in\mathcal{G}_1}\int_{-\infty}^{1}\frac{1}{2}\phi(u)^2\,\mathrm{d}u,$$

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where

$$\mathcal{G}_1 = \left\{\phi \in L^2((-\infty, 1]) \colon \sup_{0\le s\le 1}\left[\int_{-\infty}^{s} \left((s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\right)|u|^{-\gamma/2}\phi(u)\,\mathrm{d}u - \nu s\right] > \lambda\right\}.$$

Now, define

$$\mathcal{G}_s = \left\{\phi \in L^2((-\infty, s]) \colon \left[\int_{-\infty}^{s} \left((s-u)_+^{1-\beta} - (-u)_+^{1-\beta}\right)|u|^{-\gamma/2}\phi(u)\,\mathrm{d}u - \nu s\right] > \lambda\right\}.$$

Then, $\mathcal{G}_1 = \cup_{0\le s\le 1}\mathcal{G}_s$. By Lemma 8(i),

$$\inf_{\phi\in\mathcal{G}_s}\int_{-\infty}^{s}\phi(u)^2\,\mathrm{d}u = \frac{(\nu s+\lambda)^2}{c_{\beta,\gamma}s^{2H}}.$$

Note that $\inf_{\phi\in\mathcal{G}_s}\int_{-\infty}^{1}\phi(u)^2\,\mathrm{d}u = \inf_{\phi\in\mathcal{G}_s}\int_{-\infty}^{s}\phi(u)^2\,\mathrm{d}u$, since we can simply set $\phi(u)=0$ for $s<u\le 1$. Hence,

$$\inf_{\phi\in\mathcal{G}_1}\int_{-\infty}^{1}\phi(u)^2\,\mathrm{d}u = \inf_{0\le s\le 1}\inf_{\phi\in\mathcal{G}_s}\int_{-\infty}^{1}\phi(u)^2\,\mathrm{d}u = \inf_{0\le s\le 1}\frac{(\nu s+\lambda)^2}{c_{\beta,\gamma}s^{2H}}.$$

To solve the above minimization problem, it suffices to minimize the function $f(s) := (\nu s+\lambda)/s^H$. It is easy to verify that $f'(s)$ has a single zero at $s^* = H\lambda/\nu(1-H)$, and that $f(s)$ is decreasing for $s<s^*$ and increasing for $s>s^*$. Hence, when $\lambda < \nu(1-H)/H$ the optimal solution is $s=s^*<1$, and when $\lambda\ge\nu(1-H)/H$ the optimal solution is $s=1$.

The desired optimal value $\mathcal{I}(\lambda)$ follows. Therefore, we have proved that

$$\limsup_{n\to\infty}\frac{b(n)^2}{n^2}\log\mathbb{P}\left(\frac{M(n)}{n}>\lambda\right)\le -\mathcal{I}(\lambda).$$

Analogously, we can prove that

$$\liminf_{n\to\infty}\frac{b(n)^2}{n^2}\log\mathbb{P}\left(\frac{M(n)}{n}>\lambda\right)\ge -\mathcal{I}(\lambda).$$

Thus, the proof is complete. □

Proof of Proposition 2. This proposition can be proved along exactly the same lines as the proof of [10, Theorem 5.1]. To save space, the details are omitted. □

6. Proof of Theorem 3

Proof of Theorem 3. The absence of Condition 5 makes the situation slightly different. For the reader's convenience, we provide a self-contained proof of (38) under Conditions 1–4 and 7. The proofs for the remaining results in Lemmas 2, 3, and 5, and Theorem 2, follow similarly to those in Section 4, and are therefore omitted.

Step 1: Upper bound. Under Conditions 3 and 7, by using Potter's theorem (see (21)) and the procedure as in (22), there exist constants $c_{6,1}, c_{6,2}\in(0,\infty)$ such that, for every $n\ge c_{6,1}$,

$$\frac{\pi(n)}{b(n)}|g(nt-nu)-g(-nu)|\le c_{6,2}n^{-\eta/2}\quad\text{for all } -\infty<u\le t\le T. \tag{52}$$

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Recall the constant $c_{4,1}(\varepsilon)$ from (40) for any fixed $\varepsilon > 0$. By (36), (40), and (52), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\pi^2(n)} \Bigg(\int_{-\infty}^{t} \int_{|r| \le \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \Bigg[\exp\Bigg(\theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu) \big) r \Bigg) - 1 \\
&\qquad\qquad - \theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu) \big) r \Bigg] \mathrm{d}F_{nu}(r)\, n \mathrm{d}u \Bigg) \\
&\le \frac{(1+\varepsilon)\theta^2 n}{2b(n)^2} \limsup_{n\to\infty} \int_{-\infty}^{t} \int_{|r| \le \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \big(g(nt-nu) - g(-nu) \big)^2 r^2 \, \mathrm{d}F_{nu}(r)\, \mathrm{d}u \\
&\le \frac{(1+\varepsilon)\theta^2 n}{2b(n)^2} \lim_{n\to\infty} \int_{-\infty}^{t} \big(g(nt-nu) - g(-nu) \big)^2 K_2(nu)\, \mathrm{d}u \\
&= \frac{(1+\varepsilon)\theta^2}{2} \int_{-\infty}^{t} \big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta} \big)^2 |u|^{-\gamma}\, \mathrm{d}u. \qquad (53)
\end{aligned}$$

By (30) and Hölder's inequality, we obtain that, for any $p \in (1, \gamma/(1-2\beta)_+)$,

$$\begin{aligned}
&\int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \Bigg[\exp\Bigg(\theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu) \big) r \Bigg) - 1 \\
&\qquad\qquad - \theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu) \big) r \Bigg] \mathrm{d}F_{nu}(r) \\
&\le \theta^2 \frac{\pi(n)^2}{b(n)^2} \big(g(nt-nu) - g(-nu) \big)^2 \\
&\quad \times \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} |r|^2 \exp\Bigg(\theta \frac{\pi(n)}{b(n)} \big| \big(g(nt-nu) - g(-nu) \big) r \big| \Bigg) \mathrm{d}F_{nu}(r) \\
&\le \theta^2 \frac{\pi(n)^2}{b(n)^2} \big(g(nt-nu) - g(-nu) \big)^2 \Bigg(\int_{\mathbb{R}} |r|^2 \, \mathrm{d}F_{nu}(r) \Bigg)^{1/p} \\
&\quad \times \Bigg(\int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} |r|^2 \exp\Bigg(\frac{p\theta}{p-1} \cdot \frac{\pi(n)}{b(n)} \big| \big(g(nt-nu) - g(-nu) \big) r \big| \Bigg) \mathrm{d}F_{nu}(r) \Bigg)^{(p-1)/p}.
\end{aligned}$$

For every fixed $\theta \in \mathbb{R}$ and $p \in (1, \gamma/(1-2\beta)_+)$, by (52), there exists a constant $c_{6,3} > 0$ such that, for every $n \ge c_{6,3}$,

$$\frac{p|\theta|}{p-1} \cdot \frac{\pi(n)}{b(n)} |(g(nt-nu) - g(-nu)| \le \frac{\theta_0}{4} \quad \text{for all } -\infty < u \le t \le T.$$

This, together with the elementary inequality

$$x^2 \le 2c^{-2} \mathrm{e}^{c|x|} \quad \text{for every } c \ge 0,\ x \in \mathbb{R}, \qquad (54)$$

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implies that

$$\begin{aligned}
&\int_{|r|>\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}} |r|^2 \exp\left(\frac{p\theta}{p-1}\cdot\frac{\pi(n)}{b(n)}\left|\big(g(nt-nu)-g(-nu)\big)r\right|\right) \mathrm{d}F_{nu}(r)\\
&\quad\le \int_{|r|>\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}} |r|^2 \exp\left(\frac{\theta_0}{4}|r|\right) \mathrm{d}F_{nu}(r)\\
&\quad\le 32\theta_0^{-2} \int_{|r|>\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}} \exp\left(\frac{\theta_0}{2}|r|\right) \mathrm{d}F_{nu}(r)\\
&\quad\le 32\theta_0^{-2} \exp\left(-\frac{\theta_0}{2}\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}\right)\left(\int_{|r|>\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}} \exp(\theta_0|r|)\,\mathrm{d}F_{nu}(r)\right)\\
&\quad\le 32\theta_0^{-2} \exp\left(-\frac{\theta_0}{2}\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}\right)M_{\theta_0}, \qquad (55)
\end{aligned}$$

where Chebyshev's inequality and (8) were used in the last two steps, respectively.

By using a similar argument to (23), for any $\delta_1 \in (0,\, 1-\beta)$ and $\delta_2 > 0$, we have

$$\begin{aligned}
&\frac{n}{b(n)^2}\big(g(nt-nu)-g(ns-nu)\big)^2 K_2(nu)^{1/p}\\
&\quad= \frac{n^{3-2\beta}L^*(n)^2\tilde{L}(n)}{b(n)^2}\left(\frac{g(nt-nu)-g(ns-nu)}{n^{(1-\beta)}L^*(n)}\right)^2\left(|nu|^{-\gamma}\frac{\tilde{L}(nu)}{\tilde{L}(n)}\right)^{1/p}\\
&\quad\le c_{6,4}(\delta_1,\,\delta_2,\,\beta,\,p)n^{\gamma(1-1/p)}\tilde{L}(n)^{1-1/p}\\
&\qquad\times \max\left\{\big((t-u)_+^{1-\beta+\delta_1}-(s-u)_+^{1-\beta+\delta_1}\big)^2,\,\big((t-u)_+^{1-\beta-\delta_1}-(s-u)_+^{1-\beta-\delta_1}\big)^2\right\}\\
&\qquad\times \max\left\{|u|^{\delta_2/p-\gamma/p},\,|u|^{-\delta_2/p-\gamma/p}\right\}. \qquad (56)
\end{aligned}$$

By (7) and (56), there exist positive constants $c_{6,5} > 0$ and $c_{6,6} = c_{6,6}(\delta_1,\,\delta_2,\,\beta,\,\gamma,\,p,\,T)$ such that, for any $n \ge c_{6,5}$ and $s,\,t \in [0,\,T]$, and for $\delta_1 < 1-\beta$, $\delta_2 < \min\{\gamma,\,p-\gamma\}$ with

$$\delta_1 + \frac{\delta_2}{2p} \le \min\left\{\beta+\frac{\gamma}{2p}-\frac{1}{2},\,1-\beta-\frac{\gamma}{2p}\right\},$$

the following estimate holds:

$$\frac{n}{b(n)^2}\int_{-\infty}^{t}\big(g(nt-nu)-g(ns-nu)\big)^2 K_2(nu)^{1/p}\,\mathrm{d}u \le c_{6,6}(\delta_1,\,\delta_2,\,\beta,\,\gamma,\,p,\,T)n^{\gamma(1-1/p)}. \qquad (57)$$

By (55) and (57), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\pi^2(n)}\int_{-\infty}^{t}\int_{|r|>\frac{c_{4,1}(\varepsilon)n^{\eta/2}}{\theta c_{6,2}}}\left[\exp\left(\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right)-1\right.\\
&\qquad\qquad\left.-\theta\frac{\pi(n)}{b(n)}\big(g(nt-nu)-g(-nu)\big)r\right]\mathrm{d}F_{nu}(r)\,n\mathrm{d}u\\
&\le c_{6,6}(\delta_1,\,\delta_2,\,\beta,\,\gamma,\,p,\,T)\big(32\theta_0^{-2}M_{\theta_0}\big)^{(p-1)/p}\limsup_{n\to\infty}\exp\left(-\frac{c_{4,1}(\varepsilon)n^{\eta/2}\theta_0}{2c_{6,2}\theta}\cdot\frac{p-1}{p}\right)n^{\gamma(1-1/p)}\\
&= 0. \qquad (58)
\end{aligned}$$

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Putting (39), (53), and (58) together, we obtain that, for any $\theta \in \mathbb{R}$, $t \geq 0$, and $\varepsilon \in (0, 1)$,

$$\limsup_{n\to\infty} \frac{1}{\pi(n)^2} \log \mathbb{E}[\exp(\theta\pi(n)^2 S_n(t))] \leq \frac{\theta^2(1+\varepsilon)}{2} \int_{-\infty}^{t} \big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)^2 |u|^{-\gamma}\, \mathrm{d}u. \tag{59}$$

Step 2: Lower bound. By (40), we have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{\pi^2(n)} \int_{-\infty}^{t} \int_{\mathbb{R}} \left[\exp\left(\theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu)\big) r\right) - 1 \right. \\
&\qquad \left. - \theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu)\big) r \right] \mathrm{d}F_{nu}(r)\, n\, \mathrm{d}u \\
&\geq \liminf_{n\to\infty} \frac{1}{\pi^2(n)} \int_{-\infty}^{t} \int_{|r| \leq \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \left[\exp\left(\theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu)\big) r\right) - 1 \right. \\
&\qquad \left. - \theta \frac{\pi(n)}{b(n)} \big(g(nt-nu) - g(-nu)\big) r \right] \mathrm{d}F_{nu}(r)\, n \mathrm{d}u \\
&\geq \frac{(1-\varepsilon)\theta^2 n}{2b(n)^2} \liminf_{n\to\infty} \int_{-\infty}^{t} \int_{|r| \leq \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \big(g(nt-nu) - g(-nu)\big)^2 r^2\, \mathrm{d}F_{nu}(r)\, \mathrm{d}u \\
&\geq \frac{(1-\varepsilon)\theta^2}{2} \liminf_{n\to\infty} \int_{-\infty}^{t} \frac{n}{b(n)^2} \big(g(nt-nu) - g(-nu)\big)^2 K_2(nu)\, \mathrm{d}u \\
&\quad - \frac{(1-\varepsilon)\theta^2 n}{2b(n)^2} \limsup_{n\to\infty} \int_{-\infty}^{t} \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \big(g(nt-nu) - g(-nu)\big)^2 r^2\, \mathrm{d}F_{nu}(r)\, \mathrm{d}u.
\end{aligned} \tag{60}$$

By using Hölder's inequality we have, for $p \in (1, \gamma/(1-2\beta)_+)$,

$$\begin{aligned}
&\int_{-\infty}^{t} \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \frac{n}{b(n)^2} \big(g(nt-nu) - g(-nu)\big)^2 r^2\, \mathrm{d}F_{nu}(r)\, \mathrm{d}u \\
&\quad \leq \frac{n}{b(n)^2} \int_{-\infty}^{t} \mathrm{d}u \big(g(nt-nu) - g(-nu)\big)^2 \left(\int_{\mathbb{R}} |r|^2\, \mathrm{d}F_{nu}(r) \right)^{1/p} \\
&\qquad \times \left(\int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} |r|^2\, \mathrm{d}F_{nu}(r) \right)^{1-1/p}.
\end{aligned} \tag{61}$$

By using (54), Chebyshev's inequality, and Condition 4, we have

$$\begin{aligned}
\sup_{u\in\mathbb{R}} \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} |r|^2\, \mathrm{d}F_{nu}(r) &\leq 8\theta_0^{-2} \cdot \sup_{u\in\mathbb{R}} \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \exp\left(\frac{\theta_0 |r|}{2}\right) \mathrm{d}F_{nu}(r) \\
&\leq 8\theta_0^{-2} \cdot \exp\left(-\frac{c_{4,1}(\varepsilon) n^{\eta/2} \theta_0}{2c_{6,2}\theta}\right) M_{\theta_0}.
\end{aligned}$$

This, together with (57) and (61), implies that

$$\limsup_{n\to\infty} \int_{-\infty}^{t} \int_{|r| > \frac{c_{4,1}(\varepsilon) n^{\eta/2}}{\theta c_{6,2}}} \frac{n}{b(n)^2} \big(g(nt-nu) - g(-nu)\big)^2 r^2\, \mathrm{d}F_{nu}(r)\, \mathrm{d}u = 0.$$

By (37) and (60), we have

$$\liminf_{n\to\infty} \frac{1}{\pi(n)^2} \log \mathbb{E}[\exp(\theta\pi(n)^2 S_n(t))] \geq \frac{\theta^2(1-\varepsilon)}{2} \int_{-\infty}^{t} \big((t-u)_+^{1-\beta} - (-u)_+^{1-\beta}\big)^2 |u|^{-\gamma}\, \mathrm{d}u.$$

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By using the arbitrariness of $\varepsilon \in (0, 1)$ in both this and (59), we obtain (38). The proof is complete. □

Appendix A. Schied's exponential tightness criterion

Let (E, d) be a metric space equipped with its Borel σ-field $\mathcal{E}$. A function $I\colon E \to [0, \infty]$ is called a good rate function if it is inf-compact, meaning that for any $L \geq 0$, the level set $\{x\colon I(x) \leq L\}$ is compact.

Let $\lambda(n)$ be a sequence of positive numbers tending to ∞ as $n \to \infty$.

Definition 1. A sequence $(\mu_n)_{n\in\mathbb{N}_*}$ of probability measures on $(E, \mathcal{E})$ satisfies the large deviation principle with the speed function $\lambda(n)$ and good rate function I if the following conditions hold:

(i) for any open subset G of E, $\liminf_{n\to\infty} (1/\lambda(n)) \log \mu_n(G) \geq -\inf_{x\in G} I(x)$;

(ii) for any closed subset F of E, $\limsup_{n\to\infty} (1/\lambda(n)) \log \mu_n(F) \leq -\inf_{x\in F} I(x)$.

Definition 2. A sequence $(\mu_n)_{n\in\mathbb{N}_*}$ of probability measures on $(E, \mathcal{E})$ is said to be exponentially tight in $(E, \mathcal{E})$ if, for every $M > 0$, there exists a compact set $K_M \subset E$ such that

$$\limsup_{n\to\infty} \frac{1}{\lambda(n)} \log \mu_n(K_M^{\mathrm{c}}) \leq -M.$$

Theorem 4 ([44, Theorem 3].) *A family of random variables $\{Z_n\}_{n\in\mathbb{N}_*}$ is exponentially tight in $\big(\mathcal{C}[0, T], \|\cdot\|_T\big)$ with respect to the speed function $\lambda(n)$ if the following two conditions hold:*

(i) *Condition for increments: there exist positive constants n_0, α, θ, κ, and c such that, for all $n > n_0$ and all $t, s \in [0, T]$ with $t \neq s$,*

$$\mathbb{E}\left[\exp\left(\theta\lambda(n)\frac{|Z_n(t) - Z_n(s)|}{|t - s|^{\alpha}}\right)\right] \leq c\kappa^{\lambda(n)}. \tag{62}$$

(ii) *Condition at $t = 0$:*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{1}{\lambda(n)} \log \mathbb{P}\big(|Z_n(0)| > R\big) = -\infty. \tag{63}$$

This theorem is due to Schied [44, Theorem 3]; see also [45, Corollary 7.1] or [32, Theorem 1.1]. Here, we provide a proof using the Garsia–Rodemich–Rumsey inequality, as [44] may not be readily accessible to all readers.

First, we recall the Garsia–Rodemich–Rumsey inequality (see, e.g., [19, Lemma 1.1] or [48, Theorem 1.1]). Let $\Psi\colon [0, \infty) \to [0, \infty)$ be a non-decreasing function with $\lim_{x\to\infty} \Psi(x) = \infty$, and let $p\colon [0, 1] \to [0, 1]$ be continuous and non-decreasing with $p(0) = 0$. For $u \geq \Psi(0)$, define $\Psi^{-1}(u) = \sup\{v\colon \Psi(v) \leq u\}$.

Lemma 7 ([19, Lemma 1.1].) *Suppose that $f(s)$ is a continuous function on [0,1], and that*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(s) - f(t)}{p(s - t)}\right) \mathrm{d}s\, \mathrm{d}t \leq B < \infty.$$

Then, for any $s, t \in [0, 1]$,

$$|f(s) - f(t)| \leq 8 \int_0^{|s-t|} \Psi^{-1}\left(\frac{4B}{u^2}\right) \mathrm{d}p(u).$$

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Proof of Theorem 4. Without loss of generality, we assume $T = 1$. For any $n \geq 1$ and $r \geq 0$, define $\Psi(r) := \exp(\theta\lambda(n)r)$, $f(r) := Z_n(r)$, and $p(r) := r^\alpha$, and let

$$B := \int_0^1 \int_0^1 \exp\left(\frac{\theta\lambda(n)|Z_n(t) - Z_n(s)|}{|t-s|^\alpha}\right) \mathrm{d}s\,\mathrm{d}t.$$

By Fubini's theorem and (62), we have

$$\mathbb{E}[B] = \int_0^1 \int_0^1 \mathbb{E}\left[\exp\left(\theta\lambda(n)\frac{|Z_n(t) - Z_n(s)|}{|t-s|^\alpha}\right)\right] \mathrm{d}s\,\mathrm{d}t \leq c\kappa^{\lambda(n)}. \tag{64}$$

Using Lemma 7 we obtain, for any $\epsilon \in (0, \alpha)$,

$$\begin{aligned}
|Z_n(t) - Z_n(s)| &\leq \frac{8}{\theta\lambda(n)} \int_0^{|s-t|} [\log(4B) - 2\log u]\,\mathrm{d}u^\alpha \\
&= \frac{8}{\theta\lambda(n)}\left(\log(4B)\cdot|s-t|^\alpha - 2\alpha\int_0^{|s-t|} \log u \cdot u^\epsilon \cdot u^{\alpha-1-\epsilon}\,\mathrm{d}u\right) \\
&\leq \frac{8}{\theta\lambda(n)}\left(\log(4B)\cdot|s-t|^\alpha + \frac{2\alpha}{\mathrm{e}\epsilon}\int_0^{|s-t|} u^{\alpha-1-\epsilon}\,\mathrm{d}u\right) \\
&= \frac{8}{\theta\lambda(n)}\left(\log(4B)\cdot|s-t|^\alpha + \frac{2\alpha}{\mathrm{e}\epsilon(\alpha-\epsilon)}|s-t|^{\alpha-\epsilon}\right),
\end{aligned}$$

where in the second-last step we used the inequality $u^\epsilon \log u \geq -1/(\mathrm{e}\epsilon)$ for any $u \in [0, 1]$, $\epsilon \in (0, \alpha)$. Hence,

$$\|Z_n\|_{\alpha-\epsilon} := \sup_{s,t\in[0,1],s\neq t} \frac{|Z_n(t) - Z_n(s)|}{|t-s|^{\alpha-\epsilon}} \leq \frac{8}{\theta\lambda(n)}\left(\log(4B) + \frac{2\alpha}{\mathrm{e}\epsilon(\alpha-\epsilon)}\right).$$

From this and (64) we obtain

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{\theta\lambda(n)\|Z_n\|_{\alpha-\epsilon}}{8}\right)\right] &\leq \mathbb{E}\left[\exp\left(\log(4B) + \frac{2\alpha}{\mathrm{e}\epsilon(\alpha-\epsilon)}\right)\right] \\
&= 4\exp\left(\frac{2\alpha}{\mathrm{e}\epsilon(\alpha-\epsilon)}\right)\mathbb{E}[B] \leq 4c\exp\left(\frac{2\alpha}{\mathrm{e}\epsilon(\alpha-\epsilon)}\right)\kappa^{\lambda(n)}.
\end{aligned} \tag{65}$$

For any $R > 0$, define $\Gamma_R := \{\xi \in \mathcal{C}[0, 1]\colon |\xi(0)| \leq R \text{ and } \|\xi\|_{\alpha-\epsilon} \leq R\}$. Then, by the Arzelà–Ascoli theorem, Γ_R is compact in $(\mathcal{C}[0, T], \|\cdot\|_T)$. Using (63), (65), and Chebyshev's inequality, we obtain

$$\lim_{R\to\infty}\lim_{n\to\infty}\frac{1}{\lambda(n)}\log\mathbb{P}(Z_n \notin \Gamma_R) = -\infty,$$

which completes the proof. □

The following lemma is proved in the same manner as [10, Lemma 3.3].

Lemma 8.

(i) *For any $g \in L^2((-\infty, 1])$ such that $G^2 = \int_a^b g^2(t)\,\mathrm{d}t > 0$ for any $-\infty \leq a < b \leq 1$, the unique optimal solution to the minimization problem*

$$\min_{f\in L^2((-\infty,1])} \int_a^b f^2(t)\,\mathrm{d}t \quad \text{subject to} \quad \int_a^b g(t)f(t)\,\mathrm{d}t > c$$

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for some $c \geq 0$ *is* $f^*(t) = (c/G^2)g(t)$ *for* $t \in (-\infty, 1]$*, and the corresponding minimum is* c^2/G^2.

(ii) *For any* $g_1, g_2 \in L^2((-\infty, 1])$ *such that* $G_1^2 := \int_a^{b_1} g_1^2(t)\,\mathrm{d}t > 0$ *and* $G_2^2 := \int_{b_1}^{b} g_2^2(t)\,\mathrm{d}t > 0$ *for some* $-\infty \leq a < b_1 < b \leq 1$*, the unique optimal solution to the minimization problem*

$$\min_{f \in L^2((-\infty,1])} \int_a^b f^2(t)\,\mathrm{d}t \quad \text{subject to } \int_a^{b_1} g_1(t)f(t)\,\mathrm{d}t + \int_{b_1}^{b} g_2(t)f(t)\,\mathrm{d}t > c$$

for some $c \geq 0$ *is given by*

$$f^*(t) = \begin{cases} \dfrac{c}{G_1^2 + G_2^2} g_1(t) & \text{if } t \in (a, b_1], \\ \dfrac{c}{G_1^2 + G_2^2} g_2(t) & \text{if } t \in (b_1, b], \end{cases}$$

and the corresponding minimum is $c^2/(G_1^2 + G_2^2)$.

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Competing interests

There were no competing interests to declare during the preparation or publication of this article.

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