*Mathematika* 63 (2017) 250–259 © 2016 University College London
doi:10.1112/S0025579316000243

# INTRINSIC DIOPHANTINE APPROXIMATION ON GENERAL POLYNOMIAL SURFACES

MORTEN HEIN TILJESET

*Abstract.* We study the Hausdorff measure and dimension of the set of intrinsically simultaneously $\psi$-approximable points on a curve, surface, etc, given as a graph of integer polynomials. We obtain complete answers to these questions for algebraically "nice" manifolds. This generalizes earlier work done in the case of curves.

§1. *Introduction.* Let $\mathbf{x} \in \mathbb{R}^n$ be a real vector. Let $\mathbb{R}^+$ denote the positive real numbers. Here and throughout, let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function, which we will refer to as an *approximation function*. We say that $\mathbf{x}$ is *(simultaneously) $\psi$-approximable* if there exist infinitely many rational points $\mathbf{p}/q$ with $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ such that

$$\|\mathbf{x} - \mathbf{p}/q\|_\infty \leqslant \psi(q).$$

We denote the set of $\psi$-approximable vectors by $\mathcal{S}_\psi$, and for the particular approximation functions $\psi_\tau(r) = r^{-\tau}$ we use the notation $\mathcal{S}_\tau = \mathcal{S}_{\psi_\tau}$.

In this notation, the classical theorem of Dirichlet states that $\mathcal{S}_{1+1/n} = \mathbb{R}^n$. We say that $\mathbf{x} \in \mathbb{R}^n$ is *very well approximable (VWA)* if $\mathbf{x} \in \mathcal{S}_\tau$ for some $\tau > 1+1/n$. Otherwise, we say that $\mathbf{x}$ is not very well approximable or *extremal*.

The theory of metric Diophantine approximation seeks to quantify, in terms of measure, the size of $\mathcal{S}_\psi$. The starting point is the following theorem due to Khintchine.

THEOREM 1 (Khintchine). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing approximation function. Let $\lambda_n$ denote the $n$-dimensional Lebesgue measure. Then*

$$\lambda_n(\mathcal{S}_\psi) = \begin{cases} 0 & \text{if } \displaystyle\sum_{r=1}^{\infty} r^n \psi(r)^n < \infty, \\ \infty & \text{if } \displaystyle\sum_{r=1}^{\infty} r^n \psi(r)^n = \infty. \end{cases}$$

In particular, this theorem shows that almost all points are extremal. Furthermore, it gives metric answers not only for functions of the form $\psi_\tau(r) = r^{-\tau}$ but for general approximation functions. In what follows, it will be useful to understand where this theorem comes from. The starting point is to

Received 25 January 2016, published online 2 November 2016.
MSC (2010): 11J83 (primary), 11J54 (secondary).
The research was partially supported by the Danish Research Council for Independent Research.

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

realize that we can write the set of $\psi$-approximable vectors as a limsup set:

$$\mathcal{S}_\psi = \bigcap_{N=1}^{\infty} \bigcup_{q>N} \bigcup_{\mathbf{p}\in\mathbb{Z}^n} B\left(\frac{\mathbf{p}}{q}, \psi(q)\right),$$

where $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\|_\infty \leqslant r\}$ is the ball in the sup-norm.

The convergence part of the theorem now follows by restricting to a countable cover given by sets of the form $\mathcal{S}_\psi \cap I^n$, where $I \subset \mathbb{R}$ is a bounded interval, and applying the first Borel–Cantelli lemma. The hard part of the theorem is then to prove the divergence case by establishing that the sets do not overlap too much, so that the conclusion of the second Borel–Cantelli lemma may be recovered.

A more refined viewpoint is given by replacing the Lebesgue measure with the Hausdorff measure and studying in more detail the size of the null sets. Before proceeding, we recall the definition of the Hausdorff measure and dimension. A complete account is available in [7].

*Definition* 2. Let $E \subseteq \mathbb{R}^n$ be some set and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function such that $f(r) \to 0$ as $r \to 0$, which we refer to as a dimension function. For any $\delta > 0$, we define

$$\mathcal{H}^f_\delta(E) = \inf\left\{\sum f(|U_i|) : \{U_i\} \text{ is a cover of } E \text{ with } |U_i| < \delta\right\},$$

where $|U_i| = \sup\{|x - y| : x, y \in U_i\}$ is the diameter of $U_i$. We now define the *(outer) Hausdorff $f$-measure* on $E$ by

$$\mathcal{H}^f(E) = \lim_{\delta\to 0} H^f_\delta(E).$$

As a special case, for any $s \geqslant 0$, the *Hausdorff $s$-measure* is the Hausdorff $f$-measure given by the dimension function $f(r) = r^s$ and we denote it by $\mathcal{H}^s$. It turns out that for any subset $E \subseteq \mathbb{R}^n$ there is some number $s$ such that $\mathcal{H}^t(E) = \infty$ for any $0 \leqslant t < s$ (which is an empty set when $s = 0$) and $\mathcal{H}^t(E) = 0$ for any $t > s$. We call this number the *Hausdorff dimension* of $E$.

There is an analogue of Khintchine's theorem for Hausdorff measures, known as Jarník's theorem. A modern version of the theorem is the following (see [2, Theorem DV]).

THEOREM 3 (Jarník; Dickinson, Velani). *Let $f$ be a dimension function such that $r^{-n} f(r) \to \infty$ as $r \to 0$ and $r \mapsto r^{-n} f(r)$ is decreasing. Then*

$$\mathcal{H}^f(\mathcal{S}_\psi) = \begin{cases} 0 & \text{if } \displaystyle\sum_{r=1}^{\infty} f(\psi(r))r^n < \infty, \\ \infty & \text{if } \displaystyle\sum_{r=1}^{\infty} f(\psi(r))r^n = \infty. \end{cases}$$

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

When $f(r) = r^n$, the conclusion of the theorem is the same as that of Khintchine's theorem, but Jarník's theorem does not imply Khintchine's theorem as $f$ does not satisfy the growth condition. However, it is a rather surprising fact that Khintchine's theorem implies Jarník's theorem by the mass transference principle of Beresnevich and Velani [3]. The essence of this principle is that when we are rescaling the measure, we also have to rescale the balls in the limsup set.

Let $M \subset \mathbb{R}^n$ be a manifold. Our problem is to study the metric nature of the set

$$\mathcal{S}_\psi(M) = \mathcal{S}_\psi \cap M.$$

This problem originates in a problem of Mahler, who conjectured that almost all points (with respect to the induced Lebesgue measure) on the Veronese curve

$$\mathcal{V} = \{(x, x^2, \dots, x^n)\}$$

are extremal. A manifold with this property is called extremal. The question of Mahler was answered in the affirmative by Sprindžuk, and the result has later been generalized to a large class of non-degenerate manifolds by Kleinbock and Margulis using dynamical methods and the Dani–Margulis correspondence [12].

Having established the correct exponent for approximation on $M$, two natural problems emerge.

(i) To replace functions of the form $\psi_\tau(r) = r^{-\tau}$ by more general approximation functions, by obtaining a Khintchine-type theorem for manifolds.

(ii) To further study the size of the null sets, by obtaining the Hausdorff measure and dimension of the set $\mathcal{S}_\psi(M)$.

Some general theory has been established for the first problem. For a survey, see the monograph of Bernik and Dodson [4] as well as the more recent paper of Beresnevich [1].

It is very tempting to think that the second problem would follow from the first by the mass transference principle as in the classical case. However, this is not quite so as we are approximating by points *outside* the manifold and we are now considering the intersection of a limsup set with the manifold. In fact, it turns out that the second problem is quite different from the first and depends on the subtle arithmetic nature of the manifold. The reason for this is the following: for many manifolds, when we have a sufficiently good approximation, the approximating points must eventually lie on the manifold itself. To the author's knowledge, this was first observed in [6] and the argument there easily generalizes to any variety of the form $x^n + y^n = r$, where $r, n \in \mathbb{N}$ are fixed natural numbers. In the case of a circle of radius 1, we have an abundance of rational points; however, for a circle of radius 3 or indeed the Fermat curve, we have only finitely many rational points and Diophantine approximation is not possible at all.

This gives rise to another type of Diophantine approximation on manifolds, which has gained interest in recent years: the question of *intrinsic Diophantine approximation*. In contrast, we will refer to the previous form of approximation

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

as *ambient approximation*. We introduce the notation

$$\mathcal{I}_\psi(M) = \{\mathbf{x} \in M : \|\mathbf{x} - \mathbf{p}/q\|_\infty \leqslant \psi(q) \text{ for infinitely many } \mathbf{p}/q \in \mathbb{Q}^n \cap M\}$$

for the set of intrinsically $\psi$-approximable points on $M$ and for $\psi_\tau(r) = r^{-\tau}$ we define $\mathcal{I}_\tau = \mathcal{I}_{\psi_\tau}$.

As there is no known method for determining whether a given variety has infinitely many rational points, a general theory for intrinsic Diophantine approximation is far away. However, some results have been obtained in special cases: the case of the circle is well understood [6], as is the case of certain polynomial curves [5]. More recently, results for spheres [13] and more general quadratic surfaces [8] as well as homogenous varieties [10] have been obtained by dynamic methods.

Throughout we will use the Vinogradov notation, that is, for $a, b > 0$, $a \ll b$ will mean that there exists a constant $c > 0$ such that $a \leqslant cb$.

§2. *Statement and proof of main results.* Let $P_1, \dots, P_m \in \mathbb{Z}[x_1, \dots, x_n]$ be integer polynomials in $n$ variables, and consider a variety of the form

$$\Gamma = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : y_1 = P_1(\mathbf{x}), \dots, y_m = P_m(\mathbf{x})\}.$$

Put $d_j = \deg P_j$ and let $d = \max_j d_j$ be the maximum degree. In this paper, we aim to establish a Jarník-type zero–infinity law for the Hausdorff measure of $\mathcal{I}_\psi(\Gamma)$. In the case where $n = 1$, this has been studied previously by Budarina *et al* in [5]. The special case of Veronese manifolds is covered in [9, §2]. Furthermore, in the case where the defining polynomials depend only on one variable this has been studied by Schleischitz [14]. Our main result is the following theorem.

THEOREM 4. *Let $\psi$ be an approximation function and let $f$ be a dimension function such that for any $\delta > 0$ we have $f(\psi(\delta r)) \ll f(\psi(r))$ when $r$ is sufficiently large, and for any $C > 0$ we have $f(Cx) \ll f(x)$ when $x$ is sufficiently small. Suppose that $r^{-n} f(r) \to \infty$ as $r \to 0$ and $r \mapsto r^{-n} f(r)$ is decreasing. Finally, write $P_i = P_{i,0} + \cdots + P_{i,d}$, where $P_{i,k}$ are homogenous polynomials of degree $k$, and suppose that the only common point of vanishing for $\{P_{i,d}\}_{i=1}^m$ over $\overline{\mathbb{Q}}$ is 0. The Hausdorff $f$ measure of $\mathcal{I}_\psi(\Gamma)$ satisfies*

$$\mathcal{H}^f(\mathcal{I}_\psi(\Gamma)) = \begin{cases} 0 & \text{if } \displaystyle\sum_{r=1}^{\infty} r^n f(\psi(r^d)) < \infty, \\ \infty & \text{if } \displaystyle\sum_{r=1}^{\infty} r^n f(\psi(r^d)) = \infty. \end{cases}$$

*Furthermore, if $r^d \psi(r) \to 0$ as $r \to \infty$, we have $\mathcal{I}_\psi(\Gamma) = \mathcal{S}_\psi(\Gamma)$.*

As a corollary, we derive the Hausdorff dimension of $\mathcal{I}_\tau(\Gamma)$.

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

COROLLARY 5. *Suppose that the polynomials defining* $\Gamma$ *satisfy the condition of the theorem. For* $\tau > (n+1)/nd$, *the Hausdorff dimension of* $\mathcal{I}_\tau(\Gamma)$ *is given by*

$$\dim \mathcal{I}_\tau(\Gamma) = \frac{1+n}{d\tau}.$$

*Furthermore, if* $\tau > d$, *we also have* $\mathcal{S}_\tau(\Gamma) = \mathcal{I}_\tau(\Gamma)$.

The proof will be split into three key lemmas, which establish the cases of divergence, convergence, and the equality of ambient and intrinsic approximation separately. Our approach mimics that of the proof of the main theorem in [5], where the novelty in our argument is the use of algebraic geometry to obtain an upper bound in the case of convergence.

Before proceeding, we make some reductions in order to write $\mathcal{I}_\psi(\Gamma)$ as a manageable limsup set. As we are aiming for a zero–infinity law, it suffices to show that the Hausdorff measure of the $\psi$-approximable points are either full or null for sets of the form

$$\Gamma_I = \{(\mathbf{x}, P_1(\mathbf{x}), \dots, P_m(\mathbf{x})) \in I^n \times \mathbb{R}^m\},$$

where $I \subset \mathbb{R}$ is some arbitrary bounded interval. For notational simplicity we take $I = [0, 1]$; however, the argument does not use this in any essential way.

Define the function $F : \mathbb{R}^n \to \Gamma$ by $F(\mathbf{x}) = (\mathbf{x}, P_1(\mathbf{x}), \dots, P_m(\mathbf{x}))$. Now, by the mean value theorem, we can find a constant $K \geqslant 1$ such that for any $\mathbf{x_1}, \mathbf{x_2} \in I^n$, we have

$$\|\mathbf{x_1} - \mathbf{x_2}\|_\infty \leqslant \|F(\mathbf{x_1}) - F(\mathbf{x_2})\|_\infty \leqslant K\|\mathbf{x_1} - \mathbf{x_2}\|_\infty,$$

so $F$ is a bi-Lipschitz function on $I$. Since $f(Kx) \ll f(x)$ when $x$ is sufficiently small, the Hausdorff measure is changed by at most a constant under a bi-Lipschitz mapping. It thus suffices to show that the measure is full or null for the set

$$V_\psi(\Gamma_I) = \{\mathbf{x} \in I^n : F(\mathbf{x}) \in \mathcal{I}_\psi(\Gamma)\}.$$

*Definition* 6. For a rational vector $\mathbf{x}$ in $\mathbb{R}^k$, we define the *affine height* of $\mathbf{x}$ to be the least natural number $D$ such that

$$\mathbf{x} = (r_1/D, \dots, r_k/D) \quad \text{and} \quad \gcd(r_1, \dots, r_k, D) = 1$$

for some $r_1, \dots, r_k \in \mathbb{Z}$. We also define the height function $H : \mathbb{Q}^n \to \mathbb{N}$ by $H(\mathbf{x}) = D$.

We are now in a position to write $V_\psi(\Gamma_I)$ as a limsup set. Recall that $V_\psi(\Gamma_I)$ consists of the set of $\mathbf{x} \in I^n$ such that $\|F(\mathbf{x}) - \mathbf{r}\|_\infty \leqslant \psi(H(\mathbf{r}))$ for infinitely many $\mathbf{r} \in \Gamma_I \cap \mathbb{Q}^{n+m}$. Such rationals are necessarily of the form $\mathbf{r} = F(\mathbf{p}/q)$ for

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

some rational $\mathbf{p}/q \in \mathbb{Q}^n$. We thus have

$$\begin{aligned} &\bigcap_{N=1}^{\infty} \bigcup_{q>N} \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n \\ 0\leqslant p_1,\dots,p_n\leqslant q \\ \gcd(p_1,\dots,p_n,q)=1}} B\left(\frac{\mathbf{p}}{q}, \frac{\psi(H(F(\mathbf{p}/q)))}{K}\right) \subseteq V_\psi(\Gamma_I), \\ &V_\psi(\Gamma_I) \subseteq \bigcap_{N=1}^{\infty} \bigcup_{q>N} \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n \\ 0\leqslant p_1,\dots,p_n\leqslant q \\ \gcd(p_1,\dots,p_n,q)=1}} B\left(\frac{\mathbf{p}}{q}, \psi(H(F(\mathbf{p}/q)))\right). \end{aligned} \tag{1}$$

LEMMA 7 (Divergence case). *Let $\psi$ be an approximation function. Let $f$ be a dimension function such that for any $C > 0$ we have $f(Cx) \ll f(x)$ when $x$ is sufficiently small. Suppose that $r^{-n} f(r) \to \infty$ as $r \to 0$ and $r \mapsto r^{-n} f(r)$ is decreasing. If $\sum_{r=1}^{\infty} r^n f(\psi(r^d)) = \infty$, then $\mathcal{H}^f(\mathcal{I}_\psi(\Gamma)) = \infty$.*

*Proof of Lemma 7.* Let $\mathbf{p}/q \in I^n$ be some rational vector. It is clear that $q^d$ is a common multiple of the denominators in $F(\mathbf{p}/q)$, so $H(F(\mathbf{p}/q)) \leqslant q^d$. As $\psi$ is decreasing, we have

$$\frac{\psi(H(F(\mathbf{p}/q)))}{K} \geqslant \frac{\psi(q^d)}{K}.$$

This implies that

$$\bigcap_{N=1}^{\infty} \bigcup_{q>N} \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n \\ 0\leqslant p_1,\dots,p_n\leqslant q \\ \gcd(p_1,\dots,p_n,q)=1}} B\left(\frac{\mathbf{p}}{q}, \frac{\psi(q^d)}{K}\right) \subseteq V_\psi(\Gamma_I).$$

The set on the left is simply the set of $\phi(q) := \psi(q^d)/K$-approximable points on $I^n$. By Theorem 3, this set is full if

$$\sum_{r=1}^{\infty} f(\phi(r)) r^n = \infty.$$

If $\psi(r^d)$ does not tend to 0 as $r \to \infty$, this is trivial. Otherwise, we can apply the estimate $f(\psi(r^d)/K) \gg f(\psi(r^d))$ when $r$ is large to obtain

$$\sum_{r=1}^{\infty} f(\psi(r^d)/K) \gg \sum_{r=1}^{\infty} f(\psi(r^d)) = \infty. \qquad \square$$

LEMMA 8 (Convergence case). *Let $\psi$ be an approximation function and let $f$ be a dimension function such that for any $\delta > 0$ we have $f(\psi(\delta r)) \ll f(\psi(r))$ when $r$ is sufficiently large, and for any $C > 0$ we have $f(Cx) \ll f(x)$ when $x$ is sufficiently small. Write $P_i = P_{i,0} + \cdots + P_{i,d}$, where $P_{i,k}$ are homogenous polynomials of degree $k$, and suppose that the only common point of vanishing for $\{P_{i,d}\}_{i=1}^m$ over $\overline{\mathbb{Q}}$ is 0. If $\sum_{r=1}^{\infty} r^n f(\psi(r^d)) < \infty$, then $\mathcal{H}^f(\mathcal{I}_\psi(\Gamma)) = 0$.*

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

The key ingredient in the proof of the convergence case is to get a lower bound on the height of $F(\mathbf{p}/q)$. Our approach to doing this uses projective methods from algebraic geometry, which we briefly introduce.

Recall that *projective n-space* over a field $k$ is the set $\mathbb{P}^n(k) = (k^{n+1}\setminus\{0\})/\sim$, where $P \sim Q$ when $P = \lambda Q$ for some $\lambda \in k^*$. We can go from $k^n$ to $\mathbb{P}^n(k)$ by the embedding $(x_1, \ldots, x_n) \hookrightarrow (1 : x_1 : \cdots : x_n)$. As we cannot tell the difference between numerators and denominators in projective space, we define a new height on these spaces.

*Definition* 9. Let $P \in \mathbb{P}^n(\mathbb{Q})$. By clearing denominators, we can write $P$ uniquely (up to a sign) as

$$P = (x_0, \ldots, x_n),$$

where $x_0, \ldots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. Define the *projective height* of $P$ as

$$H_{\text{proj}}(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

We also want to define functions between these spaces. For this we introduce the rational maps and the morphisms. We should remark that our definition is more restrictive than the one usually found in the literature, but it is sufficient for our purposes.

*Definition* 10. Let $k$ be a field. A map $\phi : \mathbb{P}^n(k) \to \mathbb{P}^m(k)$ is called a *rational map* of degree $d$ if

$$\phi(P) = (f_1(P), \ldots, f_m(P)),$$

where $f_1, \ldots, f_m$ are homogenous polynomials of the same degree $d$. Note that $\phi$ is defined only at $P \in \mathbb{P}^n(k)$ if the polynomials $f_1, \ldots, f_m$ do not all vanish at $P$. If $\phi$ is defined everywhere, we say that $\phi$ is a *morphism over* $k$.

The way we control the height is given by the following theorem, which is a special case of [**11**, Theorem B.2.5].

THEOREM 11. *Let $\phi : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{P}^m(\overline{\mathbb{Q}})$ be a morphism of degree $d$ over the algebraic closure of $\mathbb{Q}$. For all $P \in \mathbb{P}^n(\mathbb{Q})$, we have*

$$H_{\text{proj}}(P)^d \ll H_{\text{proj}}(\phi(P)) \ll H_{\text{proj}}(P)^d,$$

*where the implied constants depend on $\phi$ but not on $P$.*

*Proof of Lemma 8.* In order to apply Theorem 11, we need to extend $F$ to $\mathbb{P}^n$. For each of the defining polynomials $P_i \in \mathbb{Z}[X_1, \ldots, X_n]$, write $P_i = P_{i,0} + \cdots + P_{i,d}$, where $P_{i,k}$ is a homogenous polynomial of degree $k$. Define the degree-$d$ homogenization of $P_i$ by

$$P_i^* = X_0^d P_{i,0} + X_0^{d-1} P_{i,1} + \cdots + P_{i,d}.$$

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

Note that $P_i^* \in \mathbb{Z}[X_0, \ldots, X_n]$ is a homogenous polynomial of degree $d$. We now define the (rational) map $F^* : \mathbb{P}^n \to \mathbb{P}^{n+m}$ by

$$F^*(X_0, \ldots, X_n) = (X_0^d, X_0^{d-1}X_1, \ldots, X_0^{d-1}X_n, P_1^*(X_0, \ldots, X_n), \ldots, P_m^*(X_0, \ldots, X_n)).$$

In the affine patch $X_0 = 1$, this corresponds to the map $F$ from above. We further claim that this map is a morphism. For $X_0 \neq 0$, we clearly have $X_0^d \neq 0$, so it is well defined. If $X_0 = 0$, the defining polynomials vanish if and only if $P_{1,d}, \ldots, P_{n,d}$ have a common point of vanishing away from 0 over the algebraic closure of $\mathbb{Q}$, but this does not happen by our assumption. Thus, $F^*$ is a morphism over the algebraic numbers.

Now let $\mathbf{p}/q \in I^n$ be some rational vector. Since $I$ is a bounded interval, the ratio between the projective and affine heights is at most a multiplicative constant. We now have

$$\begin{aligned} q^d \leqslant H_{\text{proj}}((1, p_1/q, \ldots, p_n/q))^d &\ll H_{\text{proj}}(F^*(1, p_1/q, \ldots, p_n/q)) \\ &\ll H(F(\mathbf{p}/q)). \end{aligned}$$

Here the implied constants depend only on the variety $\Gamma$ and the interval $I$. Let $\delta > 0$ be the constant such that $H(F(\mathbf{p}/q)) \geqslant \delta q^d$. Now, by the inclusion (1) and the estimate $f(\psi(\delta q^d)) \ll f(\psi(q^d))$, we have, for any $N \in \mathbb{N}$,

$$\begin{aligned} \mathcal{H}^f(V_\psi(\Gamma_I)) &\ll \sum_{q>N} \sum_{\substack{\mathbf{p}\in\mathbb{Z}^n \\ 0\leqslant p_1,\ldots,p_n\leqslant q \\ \gcd(p_1,\ldots,p_n,q)=1}} f(\psi(H(F(\mathbf{p}/q)))) \\ &\ll \sum_{q>N} q^n f(\psi(q^d)) < \infty. \end{aligned}$$

So, the Hausdorff measure is bounded by the tail of a convergent series, and we conclude that $\mathcal{H}^f(V_\psi(\Gamma_I)) = 0$. □

Finally, the equivalence of intrinsic and ambient approximation is given by the following lemma. This was already shown in full generality in [**5**, Lemma 1] and we hence omit the proof.

LEMMA 12. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximation function satisfying the growth condition $r^d\psi(r) \to 0$ as $r \to \infty$. Let $\mathbf{x} \in \mathcal{S}_\psi(\Gamma)$. If*

$$\|\mathbf{x} - \mathbf{r}\|_\infty \leqslant \psi(H(\mathbf{r}))$$

*for $\mathbf{r} \in \mathbb{Q}^{n+m}$ with $H(\mathbf{r})$ sufficiently large, then $\mathbf{r} \in \Gamma$.*

*Proof of Theorem 4.* This is an immediate consequence of Lemmas 7, 8, and 12. □

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

*Proof of Corollary 5.* Put $\psi(r) = r^{-\tau}$ and $f(r) = r^s$, where $s = (1+n)/d\tau$. We should verify that $f$ satisfies the conditions of the theorem. We have

$$r^{-n} f(r) = r^{s-n} \to \infty \quad \text{as } r \to 0$$

precisely when

$$s - n = \frac{1+n}{d\tau} - n < 0,$$

but this is satisfied as $\tau > (1+n)/dn$.

By the strict inequality, the theorem is satisfied for dimension functions $f(r) = r^t$, where $t$ is in some small interval around $s$. It follows that $\mathcal{H}^t(\mathcal{I}_\tau(\Gamma)) = \infty$ when $t \leqslant s$ and $\mathcal{H}^t(\mathcal{I}_\tau(\Gamma)) = 0$ when $t > s$. □

§3. *End notes and examples.* The most restrictive condition in the theorem is the requirement that the degree-$d$ parts of the polynomials do not have a common point of vanishing away from 0 over $\overline{\mathbb{Q}}$. For $n = 1$, this is always satisfied, as the polynomials are of the form $a_k x^k + \cdots + a_0$ and $a_k x^k$ vanishes only at 0. For $n > 1$, this is only sometimes satisfied; examples include the Veronese surface and $\Gamma = \{(x, y, x^2 + y^2, x^2 - y^2)\}$. On the other hand, it is never satisfied for hypersurfaces when $n > 1$: the zero locus of a single non-constant polynomial in $n$ variables over an algebraically closed field has dimension $n - 1$ and hence cannot be a point.

One could ask if the theorem could be generalized to the case where this condition is not satisfied. The key ingredient is the estimate $H(F(\mathbf{r})) \gg H(\mathbf{r})^d$, which we derive from Theorem 11. But, if $F$ does not extend to a morphism over $\overline{\mathbb{Q}}$, this theorem does not hold and explicit counterexamples can be constructed. It is even possible that the conclusion of Theorem 11 always has counterexamples when the map is not a morphism, though the author has not been able to prove or disprove this. When the conclusion of Theorem 11 fails we get additional rational points of low height on $\Gamma$, and the present argument fails. It seems unlikely that the main theorem of this paper still holds in this case.

*Acknowledgement.* I would like to thank my advisor, Simon Kristensen, for suggesting the problem and for productive discussions.

## *References*

1. V. Beresnevich, Rational points near manifolds and metric Diophantine approximation. *Ann. of Math.* (2) **175**(1) (2012), 187–235.
2. V. Beresnevich, D. Dickinson and S. Velani, Measure theoretic laws for lim sup sets. *Mem. Amer. Math. Soc.* **179**(846) (2006), 1–91.
3. V. Beresnevich and S. Velani, A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures. *Ann. of Math.* (2) **164**(3) (2006), 971–992.
4. V. I. Bernik and M. M. Dodson, *Metric Diophantine Approximation on Manifolds* (*Cambridge Tracts in Mathematics* **137**), Cambridge University Press (Cambridge, 1999).
5. N. Budarina, D. Dickinson and J. Levesley, Simultaneous Diophantine approximation on polynomial curves. *Mathematika* **56**(1) (2010), 77–85.
6. H. Dickinson and M. M. Dodson, Simultaneous Diophantine approximation on the circle and Hausdorff dimension. *Math. Proc. Cambridge Philos. Soc.* **130**(3) (2001), 515–522.

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press

**7.** K. Falconer, *Fractal Geometry* (*Mathematical Foundations and Applications*), 3rd edn., John Wiley (Chichester, 2014).
**8.** L. Fishman, D. Kleinbock, K. Merrill and D. Simmons, Intrinsic Diophantine approximation on quadric hypersurfaces. *Preprint*, 2015, arXiv:1405.7650.
**9.** L. Fishman, D. Kleinbock, K. Merrill and D. Simmons, Intrinsic Diophantine approximation on manifolds: general theory. *Preprint*, 2015, arXiv:1509.05439.
**10.** A. Ghosh, A. Gorodnik and A. Nevo, Metric Diophantine approximation on homogeneous varieties. *Compositio Math.* **150**(8) (2014), 1435–1456.
**11.** M. Hindry and J. H. Silverman, *Diophantine Geometry: An Introduction* (*Graduate Texts in Mathematics* **201**), Springer (New York, 2000).
**12.** D. Y. Kleinbock and G. A. Margulis, Flows on homogeneous spaces and Diophantine approximation on manifolds. *Ann. of Math.* (2) **148**(1) (1998), 339–360.
**13.** D. Kleinbock and K. Merrill, Rational approximation on spheres. *Israel J. Math.* **209** (2015), 293–322, doi:10.1007/s11856-015-1219-z.
**14.** J. Schleischitz, Rational approximation to surfaces defined by polynomials in one variable. *Preprint*, 2016, arXiv:1601.02810.

Morten Hein Tiljeset,
Department of Mathematical Sciences,
Faculty of Science,
University of Aarhus,
Ny Munkegade 118,
DK-8000 Aarhus C,
Denmark
E-mail: mortil@math.au.dk

https://doi.org/10.1112/S0025579316000243 Published online by Cambridge University Press