Nagoya Mathematical Journal (2026), **261**:e17 1–32
doi:10.1017/nmj.2026.10101

NMJ

ARTICLE

Perfectoid towers generated from prisms

Ryo Ishizuka

Institute of Science Tokyo, Japan

Email: ishizuka.r.ac@m.titech.ac.jp

Received: 2 July 2025; **Revised:** 26 October 2025; **Accepted:** 26 October 2025

2020 Mathematics Subject Classification: *Primary* – 13B02, 14G45

Keywords: perfectoid rings; prisms; perfectoid towers; Frobenius lifts; δ-rings

Abstract

We present a unified construction of perfectoid towers from specific prisms which covers all the previous constructions of (p-torsion-free) perfectoid towers. By virtue of the construction, perfectoid towers can be systematically constructed for a large class of rings with Frobenius lift. Especially, any Frobenius lifting of a reduced $\mathbb{F}_p$-algebra has a perfectoid tower.

1. Introduction

Let p be a prime number. After André [1] and Bhatt [3], the theory of perfectoid rings has been applied to various fields of commutative algebra, especially in the study of mixed characteristic. Recently, Ishiro–Nakazato–Shimomoto [14] developed the notion of *perfectoid towers* (Definition 4.3), which is a "tower-theoretic" generalization of perfectoid rings and gives an axiomatic Noetherian approximation of perfectoid rings. The existence of perfectoid towers over a given (Noetherian local) ring is an extremely non-trivial problem and it is not known for general rings. Bhatt–Scholze [7] also generalized perfectoid rings by the notion of *prisms* (Definition 2.2), which is a "deperfection" of perfectoid rings and a building block of the theory of prismatic cohomology.

In this article, we connect those two objects by constructing perfectoid towers from specific prisms. All the previous constructions of (p-torsion-free) perfectoid towers can be unified by our construction. Those examples are in Section 6. Our main theorem is the following.

Theorem 1.1 (Special case of Theorem 4.5). *Let $(A, (d))$ be an (orientable) prism such that p, d is a regular sequence on A and A/pA is p-root closed*[1] *in $A/pA[1/d]$. Then the tower of rings*

$$R_0 \coloneqq A/dA \xrightarrow{\varphi} A/\varphi(d)A \xrightarrow{\varphi} A/\varphi^2(d)A \xrightarrow{\varphi} \dots \xrightarrow{\varphi} A/\varphi^i(d)A \xrightarrow{\varphi} \dots$$

induced from the Frobenius lift $\varphi\colon A \to A$ becomes a perfectoid tower (Definition 4.3) with injective transition maps. The p-completed colimit of the tower is isomorphic to the p-adic completion

[1] A ring A/pA is *p-root closed in $A/pA[1/d]$* if $x \in A/pA[1/d]$ satisfies $x^{p^n} \in A/pA$ for some $n \geq 1$, then $x \in A/pA$ holds.

© The Author(s), 2026. Published by Cambridge University Press on behalf of Foundation Nagoya Mathematical Journal. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

$(A_{\mathrm{perf}}/dA_{\mathrm{perf}})^{\wedge_p}$, *where* $A_{\mathrm{perf}} \coloneqq \operatorname{colim}_\varphi A$. *Furthermore, the tilt (Definition 4.4) of the perfectoid tower is isomorphic to*

$$A/pA \xrightarrow{F} A/pA \xrightarrow{F} A/pA \xrightarrow{F} \dots,$$

where $F\colon A/pA \to A/pA$ *is the Frobenius map.*

This shows that, once we have such a prism, we can construct a perfectoid tower systematically and its tilt can be obtained immediately. As a corollary, if a δ-ring R satisfies some (weaker) conditions, then R has a perfectoid tower.

Theorem 1.2 (Special case of Corollary 5.5). *Let R be a p-torsion-free p-adically complete δ-ring such that R/pR is reduced. Fix compatible sequences $\{p^{1/p^i}\}_{i\geq 0}$ and $\{\zeta_{p^i}\}_{i\geq 0}$ of p-power roots of p and unity in $\overline{\mathbb{Q}}$. Then the towers of rings*

$$R \hookrightarrow R^{1/p} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p}] \hookrightarrow \cdots \hookrightarrow R^{1/p^i} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^i}] \hookrightarrow \cdots, \text{ and}$$
$$R[\zeta_p] \hookrightarrow R^{1/p}[\zeta_{p^2}] \hookrightarrow \cdots \hookrightarrow R^{1/p^i}[\zeta_{p^{i+1}}] \hookrightarrow \cdots$$

are perfectoid towers arising from $(R,(p))$ and $(R[\zeta_p],(p))$, respectively, where $R^{1/p^i} \coloneqq \operatorname{colim}\{R \xrightarrow{\varphi} R \xrightarrow{\varphi} \dots \xrightarrow{\varphi} R\}$ is the colimit consisting of $i+1$ terms.

The p-completed colimits of the towers are isomorphic to $(R_{\mathrm{perf}} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^\infty}])^{\wedge_p}$ and $(R_{\mathrm{perf}} \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_{p^\infty}])^{\wedge_p}$, respectively, where $R_{\mathrm{perf}} \coloneqq \operatorname{colim}_\varphi R$. Moreover, their tilts are both isomorphic to the tower

$$R/pR[|T|] \xrightarrow{F} R/pR[|T|] \xrightarrow{F} R/pR[|T|] \xrightarrow{F} \dots,$$

where $R/pR[|T|]$ is the formal power series ring over R/pR with a variable T.

To accomplish this theorem, we prove that the tensor product of a perfectoid tower $(\{A/\varphi^i(I)A\},\{\varphi\})$ generated from a prism (A,I) and a tower of rings $(\{R^{1/p^i}\},\{\varphi_R\})$ generated from a δ-ring R is again a perfectoid tower (Theorem 5.4). The degree of the generic extension of those transition maps is also determined in Section 5.3 in some cases.

This theorem says that (the p-adic completion of) any Frobenius lifting of a reduced Noetherian ring of characteristic p has a perfectoid tower (Remark 5.6). Typical algebraic examples are the completion of any Stanley–Reisner ring $\mathbb{Z}_p[|\underline{T}|]/I_\Delta$ over $\mathbb{Z}_p$, that is, quotients of square-free monomial ideals (Example 6.12) and the completion of any affine semigroup ring $\mathbb{Z}_p[|H|]$ over $\mathbb{Z}_p$ (Proposition 6.5). Moreover, geometrically, we can take a ring of sections $R(\mathcal{X},\mathcal{L})$ of a canonical lift $\mathcal{A}$ of ordinary Abelian variety A and some ample line bundle $\mathcal{L}$ on $\mathcal{A}$.

In addition to these examples, we use a method of δ-stabilization of ideals (Definition 6.13) and give a sufficient condition that the quotient of a formal power series ring by a δ-stable ideal has a perfectoid tower in Proposition 6.9. This condition is in a form that can be determined by hand or computer algebra systems.

All previous examples of Noetherian perfectoid towers in [14] arise from regular local rings or local log-regular rings, which are Cohen–Macaulay and normal domains. However, based on our results, the following examples of perfectoid towers are given.

Example 1.3. We can get perfectoid towers arising from a Noetherian ring which is:

- a ramified complete intersection but not an integral domain (Example 6.11),
- a non-Cohen–Macaulay non-normal complete local domain (Example 6.7),
- a non-regular complete intersection domain but not normal (Example 6.8),
- a reduced non-Cohen–Macaulay and non-integral domain (Example 6.12),
- an unramified complete intersection domain but not log-regular with $p=2$ (Example 6.21 following Corollary 6.20), or
- a non-Cohen–Macaulay normal domain (Example 6.4).

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

The first one is done by our first main theorem (Theorem 1.1). The next three examples are combinatorial examples, such as affine semigroup rings and Stanley–Reisner rings. The fifth one is given by a (computer) calculation of δ-stabilization of ideals. The last example is given by a ring of sections $R(\mathcal{X}, \mathcal{L})$ of geometric objects as explained above.

Very recently, Ishiro–Shimomoto [15] study a relation between perfectoid towers and lim Cohen–Macaulay sequences introduced by Bhatt–Hochster–Ma [4] and give another way to construct perfectoid towers from certain δ-rings. Our work is independent of theirs, but both works show that perfectoid towers can be constructed from various rings with, at least, Frobenius lift.

2. Prisms

In this section, we recall the notion of prisms and fix some terminology of towers of (δ-)rings and prisms.

Definition 2.1 [7, Definition 2.1]. Let A be a ring. A *δ-structure on A* is a map of sets $\delta\colon A \to A$ such that $\delta(0) = \delta(1) = 0$ and

$$\delta(a+b) = \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p}; \delta(ab) = a^p\delta(b) + b^p\delta(a) + p\delta(a)\delta(b),$$

for all $a, b \in A$. A *δ-ring* is a pair (A, δ) of a ring A and a δ-structure on A. We often omit the δ-structure δ and simply say that A is a δ-ring. An element $d \in A$ is called a *distinguished element* if $\delta(d)$ is invertible in A.

On a δ-ring A, a map of sets $\varphi\colon A \to A$ is defined as

$$\varphi(a) \coloneqq a^p + p\delta(a)$$

for all $a \in A$. By the definition of δ, the map φ gives a ring endomorphism and we call it the *Frobenius lift* on the δ-ring A. This induces the Frobenius map on A/pA.

We often use the symbol $\varphi_*(-)$ and $F_*(-)$ as the restriction of scalars along φ and F, respectively.

Definition 2.2 [7]. A *preprism* is a pair (A, I), where A is a δ-ring and I is an invertible ideal of A. A preprism (A, I) is a *prism* (resp. *Zariskian prism*)[2] if the following holds:

1. A is derived (p, I)-complete (resp. (p, I)-Zariskian).
2. $p \in I + \varphi(I)A$.

A (pre)prism (A, I) is called:

1. *perfect* if A is a perfect δ-ring, that is, φ is an isomorphism,
2. *bounded* if A/I has bounded p^∞-torsion,
3. *orientable* if I is a principal ideal of A,
4. *crystalline* if $I = (p)$, or
5. *transversal* if (A, I) is orientable and p, d is a regular sequence on A for some orientation d of I (or satisfies some equivalent conditions such as [16, Lemma 2.9]). The transversal property was originally introduced in [2].

We will use the following simple example of Zariskian prisms in one of our main theorem (Corollary 5.5).

Example 2.3.

1. Set a δ-structure on $\mathbb{Z}_{(p)}[T]$ by $\delta(T) = 0$. Then the pair $((1+(T))^{-1}\mathbb{Z}_{(p)}[T], (p-T))$ is an orientable bounded Zariskian prism.

[2]The terminology *Zariskian prism* is non-standard and temporary, but we need to emphasize to only assume the Zariskian property instead of (derived) completeness.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

2. The q-crystalline prism $(\mathbb{Z}_p[|q-1|], ([p]_q))$ [7, Example 1.3(4)] is an orientable bounded prism. Here, $\mathbb{Z}_p[|q-1|]$ is the $(p, q-1)$-adic completion of the polynomial ring $\mathbb{Z}[q]$ with the δ-structure $\delta(q) = 0$ and $[p]_q \in \mathbb{Z}[q]$ is the q-analog of p, that is, $[p]_q \coloneqq (q^p - 1)/(q-1) = 1 + q + \cdots + q^{p-1}$.
3. Similarly, the pair $((1 + (q-1))^{-1}\mathbb{Z}_{(p)}[q], ([p]_q))$ is an orientable bounded Zariskian prism.

By easy observation, p, $p - T$ (resp. p, $[p]_q$) is a regular sequence in $\mathbb{Z}_{(p)}[T]$ (resp. $\mathbb{Z}_{(p)}[q]$) and $\mathbb{F}_p[T]$ (resp. $\mathbb{F}_p[q]$) is p-root closed in $\mathbb{F}_p[T][1/T]$ (resp. $\mathbb{F}_p[q][1/[p]_q]$ because of $[p]_q \equiv (q-1)^{p-1}$ modulo p).

Next, we fix some terminology of towers and those morphisms.

Definition 2.4.

1. A *tower of rings* is a sequence of rings $A_0 \xrightarrow{\iota_0} A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} \dots$, where these ι_i are maps of rings. We often denote $(\{A_i\}_{i\geq 0}, \{\iota_i\}_{\geq 0})$ as $(\{A_i\}, \{\iota_i\})$ (more simply, $(\{A_i\})$).
2. A *tower of δ-rings* is a tower of rings $(\{A_i\}_{i\geq 0}, \{\iota_i\}_{\geq 0})$, where each A_i is a δ-ring and ι_i is a map of δ-rings.
3. A *tower of preprisms* is a pair $(\{A_i\}_{i\geq 0}, \{\iota_i\}_{\geq 0}, I)$, where $(\{A_i\}_{i\geq 0}, \{\iota_i\}_{\geq 0})$ is a tower of δ-rings and (A_0, I) is a preprism. We often denote $(\{A_i\}_{i\geq 0}, \{\iota_i\}_{\geq 0}, (A_0, I))$ as $(\{A_i\}, I)$.
4. A *tower of prisms* is a tower of preprisms $(\{A_i\}, I)$ such that each A_i is a derived (p, I)-complete δ-A-algebra and the preprism (A_0, I) is a prism.

Remark 2.5. For any tower of (pre)prisms $(\{A_i\}, I)$, the base change $I_{A_i} \coloneqq I \otimes^L_{A_0} A_i \to A_i$ gives an *animated* (pre)prism $(I_i \to A_i)$ over the discrete prism (A_0, I) by [5, Corollary 2.10]. So the tower of (pre)prisms gives a tower consisting of *animated* (pre)prisms over (A_0, I) whose underlying δ-rings are discrete.

Note that even if A_i are all discrete δ-rings, each $(I_{A_i} \to A_i)$ is only an animated (pre)prism (see [5, Remark 2.8]). In the transversal case, this becomes an honest (pre)prism by Proposition 3.2. The image of $I_{A_i} \to A_i$ is an ideal IA_i of A_i generated by the image of $I \subseteq A_0$ in A_i.

Definition 2.6. Let $(\{A_i\})$ and $(\{A'_i\})$ be towers of rings.

1. A sequence of maps $f = (\{f_i\}_{i\geq 0})\colon (\{A_i\}) \to (\{A'_i\})$ is a *map of towers of rings* if $f_i\colon A_i \to A'_i$ is a map of rings and compatible with ι_i and ι'_i for each $i \geq 0$.
2. If $(\{A_i\})$ and $(\{A_i\})$ are towers of δ-rings, a map of towers of rings $f\colon (\{A_i\}) \to (\{A'_i\})$ is a *map of towers of* δ-rings if each f_i is a map of δ-rings.
3. A *map of towers of (pre)prisms* $f\colon (\{A_i\}, I) \to (\{A'_i\}, I')$ is a map of towers of δ-rings $f\colon (\{A_i\}) \to (\{A'_i\})$ such that $f_0\colon A_0 \to A'_0$ induces a map of (pre)prisms $(A_0, I) \to (A'_0, I')$.

3. Construction of towers

In this section, we construct towers of rings from a given preprism (Construction 3.3) and the Frobenius projection of the tower (Construction 3.11).

3.1. Construction of towers from prisms

In positive characteristic, the Frobenius map F on a reduced ring R induces a tower of rings $R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \dots$, which is called a *perfect tower*, and this is a perfectoid tower arising from (R, pR) [14, Definition 3.2 and Example 3.23(3)]. Similarly, we can construct a "perfect tower" from a given (pre)prism, which is the main subject in this section.

Construction 3.1. Let (A, I) be a preprism. We denote $\varphi\colon A \to A$ a Frobenius lift as usual. We have a tower of δ-rings

$$A_0 \xrightarrow{\varphi} A_1 \xrightarrow{\varphi} A_2 \xrightarrow{\varphi} \dots \xrightarrow{\varphi} A_i \xrightarrow{\varphi} \dots,$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

where A_i is the same as A and φ is the Frobenius lift on A. The map of rings $\varphi\colon A_i \to A_{i+1}$ induces the following maps of rings:

$$\overline{\varphi}_I^{(i)} \coloneqq \varphi/\varphi^i(I)\colon A_i/\varphi^i(I)A_i \to A_{i+1}/\varphi^{i+1}(I)A_{i+1},$$
$$\overline{\varphi}_{(p,I)}^{(i)} \coloneqq \varphi/(p, I^{[p^i]})\colon A_i/(p, I^{[p^i]})A_i \to A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1},$$

where $\varphi^i(I)A_i$ is the ideal of A_i generated by the image $\varphi^i(I) \subseteq A_i$ of $I \subseteq A_i$. Note that $\overline{\varphi}_{(p,I)}^{(i)}$ is the pth power map $A_i/(p, I^{[p^i]})A_i \xrightarrow{a\mapsto a^p} A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$. Those maps give three towers of rings:

$$(\{A_i\}, \{\varphi\}) = A_0 \xrightarrow{\varphi} A_1 \xrightarrow{\varphi} A_2 \xrightarrow{\varphi} \dots,$$
$$(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\}) = A_0/\varphi(I)A_0 \xrightarrow{\overline{\varphi}_I^{(0)}} A_1/\varphi(I)A_1 \xrightarrow{\overline{\varphi}_I^{(1)}} A_2/\varphi^2(I)A_2 \xrightarrow{\overline{\varphi}_I^{(2)}} \dots,$$
$$(\{A_i/(p, I^{[p^i]})\}, \{\overline{\varphi}_{(p,I)}^{(i)}\}) = A_0/(p, I)A_0 \xrightarrow{\overline{\varphi}_{(p,I)}^{(0)}} A_1/(p, \varphi(I))A_1 \xrightarrow{\overline{\varphi}_{(p,I)}^{(1)}} A_2/(p, \varphi^2(I))A_2 \xrightarrow{\overline{\varphi}_{(p,I)}^{(2)}} \dots.$$

The first tower becomes a tower of (pre)prism $(\{A_i\}, \{\varphi\}, I)$.

As mentioned in Remark 2.5, in the transversal case, the tower of prisms becomes a tower consisting of discrete prisms.

Proposition 3.2. *If a prism (A, I) is transversal in the sense of Definition 2.2, then the animated prism $(I \otimes^L_{A,\varphi^i} A_i \to A_i) = (I \xrightarrow{\varphi^i} A_i)$ is a (discrete) orientable transversal prism $(A_i, \varphi^i(I)A_i)$.*

Proof. Fix an orientation d of I. To prove the animated prism $(I \xrightarrow{\varphi^i} A_i)$ is a discrete prism, it suffices to show that $\varphi^i(d)$ is a non-zero-divisor in $A_i = A$ by [5, Lemma 2.13], which follows from the transversal property of (A, I). The transversal property of $(A_i, \varphi^i(I)A_i)$ also follows since $p, \varphi^i(d)$ becomes a regular sequence in A for each $i \geq 0$. □

Next, we show another representation of towers $(\{A_i\})$, $(\{A_i/\varphi^i(I)A_i\})$, and $(\{A_i/(p, I^{[p^i]})A_i\})$. In concrete examples as in Section 6, this representation is useful to understand the structure of the tower.

Construction 3.3. Let (A, I) be a preprism. For each $i \geq 0$, by [7, Remark 2.7] (or [5, Proposition A.20(1)]), we can define a δ-A-algebra A^{1/p^i} as the (finite) colimit in the category of δ-rings

$$A^{1/p^i} \coloneqq \operatorname{colim}\{A \xrightarrow{\varphi} A \xrightarrow{\varphi} \dots \xrightarrow{\varphi} A\},$$

which is the colimit consisting of $(i+1)$-copies of A with the Frobenius lift φ. The underlying ring of A^{1/p^i} is the colimit of A, that is, $A^{1/p^i} \cong \operatorname{colim}\{A \xrightarrow{\varphi} A \xrightarrow{\varphi} \dots \xrightarrow{\varphi} A\}$ as rings. In particular, if A is derived (p, I)-complete, A^{1/p^i} is a derived (p, I)-complete δ-ring by Lemma 3.4 below.

We denote the canonical map $A \to A^{1/p^i}$ of δ-rings from the jth term of the colimit as c^i_j for each $0 \leq j \leq i$, namely, we have a map of δ-rings

$$c^i_j\colon A \to A^{1/p^i}.$$

In the following, A^{1/p^i} is a δ-A-algebra via $c^i_0\colon A \to A^{1/p^i}$. Maps $c^{i+1}_0, \dots, c^{i+1}_i$ of δ-rings uniquely induce a map of δ-A-algebras

$$t_i\colon A^{1/p^i} \to A^{1/p^{i+1}}.$$

Set the ideal

$$I_i \coloneqq c^i_0(I)A^{1/p^i} \subseteq A^{1/p^i},$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

then t_i induces a map of δ-A-algebras

$$t_{i,I} \coloneqq t_i/I \colon A^{1/p^i}/I_i \to A^{1/p^{i+1}}/I_{i+1},$$
$$t_{i,(p,I)} \coloneqq t_i/(p,I) \colon A^{1/p^i}/(p,I_i) \to A^{1/p^{i+1}}/(p,I_{i+1}).$$

Consequently, we have a tower of preprisms $(\{A^{1/p^i}\},\{t_i\},I)$ and two towers of rings $(\{A^{1/p^i}/I_i\}, \{t_{i,I}\})$ and $(\{A^{1/p^i}/(p,I_i)\},\{t_{i,(p,I)}\})$. If (A,I) is a prism, $(\{A^{1/p^i}\},\{t_i\},I)$ is a tower of prisms.

By using commutativity of filtered colimits and tensor products, we have the following isomorphisms of A-algebras:

$$\begin{aligned} A^{1/p^i}/I_i &\cong \operatorname{colim}\{A/I \xrightarrow{\overline{\varphi}_I^{(0)}} A/\varphi(I)A \xrightarrow{\overline{\varphi}_I^{(1)}} \dots \xrightarrow{\overline{\varphi}_I^{(i-1)}} A/\varphi^i(I)A\}, \\ A^{1/p^i}/(p,I_i) &\cong \operatorname{colim}\{A/(p,I)A \xrightarrow{\overline{\varphi}_{(p,I)}^{(0)}} A/(p,\varphi(I))A \xrightarrow{\overline{\varphi}_{(p,I)}^{(1)}} \dots \xrightarrow{\overline{\varphi}_{(p,I)}^{(i-1)}} A/(p,\varphi^i(I))A\} \\ &\cong (A/pA)^{1/p^i}/I(A/pA)^{1/p^i}. \end{aligned}$$

As in $c_j^i \colon A \to A^{1/p^i}$, there exist the canonical maps of rings

$$c_{j,I}^i \colon A/\varphi^j(I)A \to A^{1/p^i}/I_i,$$
$$c_{j,(p,I)}^i \colon A/(p,I^{[p^j]})A \to A^{1/p^i}/(p,I_i)$$

for each $i \geq 0$ and $0 \leq j \leq i$.

Lemma 3.4. *Let (A,I) be a preprism. Then the maps c_i^i, $c_{i,I}^i$, and $c_{i,(p,I)}^i$*

$$c_i^i \colon A \xrightarrow{\cong} A^{1/p^i},$$
$$c_{i,I}^i \colon A/\varphi^i(I)A \xrightarrow{\cong} A^{1/p^i}/I_i,$$
$$c_{i,(p,I)}^i \colon A/(p,I^{[p^i]})A \xrightarrow{\cong} A^{1/p^i}/(p,I_i)$$

are isomorphisms of A-algebras, where the A-algebra structure on the left-hand sides is induced by the Frobenius lift φ. In particular, if A is derived (p,I)-complete, A^{1/p^i} (resp. $A^{1/p^i}/I_i$) is also derived (p,I)-complete (resp. derived p-complete).

Proof. Those isomorphisms follow from the definition of colimits. Since A (resp. $A_i/\varphi^i(I)A_i$) is derived $\varphi^i(p,I)$-complete (resp. derived p-complete), the completeness also follows. □

Lemma 3.5. *Let (A,I) be a preprism. We can get isomorphisms of towers of rings between Constructions 3.1 and 3.3 as follows:*

$$\{c_i^i\} \colon (\{A_i\},\{\varphi\}) \xrightarrow{\cong} (\{A^{1/p^i}\},\{t_i\})$$
$$\{c_{i,I}^i\} \colon (\{A_i/\varphi^i(I)A_i\},\{\overline{\varphi}_I^{(i)}\}) \xrightarrow{\cong} (\{A^{1/p^i}/I_i\},\{t_{i,I}\})$$
$$\{c_{i,(p,I)}^i\} \colon (\{A_i/(p,I^{[p^i]})A_i\},\{\overline{\varphi}_{(p,I)}^{(i)}\}) \xrightarrow{\cong} (\{A^{1/p^i}/(p,I_i)\},\{t_{i,(p,I)}\}).$$

In particular, the first isomorphism $\{c_i^i\}$ is an isomorphism of towers of preprisms between $(\{A_i\},\{\varphi\},I)$ and $(\{A^{1/p^i}\},\{t_i\},I)$.

Proof. The first isomorphism is because the maps c_i^i are compatible with t_i by those constructions. The second and third isomorphisms follow from the same argument. □

By those isomorphisms, we have the following equivalences of injectivity of t_i, $t_{i,I}$, and $t_{i,(p,I)}$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Corollary 3.6. *Let (A, I) be a preprism and fix $i \geq 0$.*

1. *The map of δ-rings $t_i \colon A^{1/p^i} \to A^{1/p^{i+1}}$ is injective if and only if the Frobenius lift $\varphi \colon A \to A$ is injective. In this case, A is p-torsion free but the converse is not true in general (see [7, Lemma 2.28]).*
2. *The map of rings $\overline{\varphi}_I^{(i)} \colon A/\varphi^i(I)A \to A/\varphi^{i+1}(I)A$ is injective if and only if the map $t_{i,I} \colon A^{1/p^i}/I_i \to A^{1/p^{i+1}}/I_{i+1}$ is injective.*
3. *The pth power map $\overline{\varphi}_{(p,I)}^{(i)} \colon A/(p, I^{[p^i]})A \to A/(p, I^{[p^{i+1}]})A$ is injective if and only if the map of rings $t_{i,(p,I)} \colon A^{1/p^i}/(p, I_i) \to A^{1/p^{i+1}}/(p, I_{i+1})$ is injective.*

The injectivity of φ on A holds under some assumptions as follows.

Lemma 3.7. *Let A be a p-adically separated δ-ring. If A is p-torsion-free and A/pA is reduced, then the Frobenius lift $\varphi \colon A \to A$ is injective.*

Proof. If $\varphi(x) = 0$ for $x \in A$, then $\overline{x}^p = 0$ in the reduced ring A/pA and thus there exists $x_1 \in A$ such that $x = px_1$. Since A is p-torsion-free, the equation $0 = \varphi(x) = p\varphi(x_1)$ implies $\varphi(x_1) = 0$ in A. Repeating this argument, x is contained in $\cap_{n\geq 0} p^n A = 0$. This shows the injectivity of φ. □

The injectivity of $\overline{\varphi}_I^{(i)} \colon A/\varphi^i(I)A \to A/\varphi^{i+1}(I)A$ in Corollary 3.6 (2) follows under assumptions that are also assumed in our main theorem (Theorem 1.1).

Lemma 3.8. *Let $(A, (d))$ be an orientable preprism such that p, d is a regular sequence on A and A/pA is p-root closed in $A/pA[1/d]$. Fix $i \geq 0$. If $A/\varphi^{i+1}(I)A$ is p-adically separated,[3] then the map of rings $\overline{\varphi}_I^{(i)} \colon A/\varphi^i(I)A \to A/\varphi^{i+1}(I)A$ is injective.*

Proof. Since p, d is a regular sequence on A, so is $p, \varphi^{i+1}(d)$ and especially $A/\varphi^{i+1}(d)A$ is p-torsion-free. Take an element $a \in A$ such that $\varphi(a) \in \varphi^{i+1}(d)A$. Taking modulo p, this implies that $\overline{a}^p = \overline{d}^{p^{i+1}} A/pA$. The p-root closed assumption says that $\overline{a} \in \overline{d}^{p^i} A/pA$ and there exist elements $a_1, b_1 \in A$ such that $a = pa_1 + \varphi^i(d)b_1 \in (p, \varphi^i(d))A$. Applying $\varphi(-)$ to this equation, we have $\varphi^{i+1}(d)A \ni \varphi(a) = p\varphi(a_1) + \varphi^{i+1}(d)\varphi(b_1)$ and thus $\varphi(a_1) \in \varphi^{i+1}(d)A$, since $A/\varphi^{i+1}(d)A$ is p-torsion-free. The same argument shows that there exist $a_2, b_2 \in A$ such that $a_1 = pa_2 + \varphi^i(d)b_2 \in (p, \varphi^i(d))A$ and thus $a \in (p^2, \varphi^i(d))A$. Repeating this process, we have $a \in (p^j, \varphi^i(d))A$ for all $j \geq 0$ and thus $\overline{a} \in \bigcap_{j\geq 0} p^j A/\varphi^i(d)A = 0$ by the p-adic separatedness of $A/\varphi^i(d)A$. This shows the injectivity. □

3.2. Construction of Frobenius projections

We next construct the "Frobenius projection" which plays a crucial role in the theory of perfectoid towers. The existence of the Frobenius projection of a given tower is a key property in the theory of perfectoid towers (or p-purely inseparable tower as below). The following observation and lemma (Lemma 3.10) show that the Frobenius projection of the tower $(\{A/\varphi^i(I)A\})$ generated from a prism (A, I) is quite easily understood.

Definition 3.9. Let (A, I) be a preprism. Set the canonical surjection

$$\pi_i \colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \twoheadrightarrow A_i/(p, I^{[p^i]})A_i, \tag{3.1}$$

induced from the identity map $A_{i+1} \xrightarrow{\mathrm{id}} A_i$ of $A = A_i = A_{i+1}$. We call the map π_i the *ith Frobenius projection* of the tower of prisms $(\{A_i\}, I)$ (or of the tower of rings $(\{A_i/\varphi^i(I)A_i\})$, see Lemma 3.14).

[3] If A is derived p-complete, which is satisfied when A is a prism, then $A/\varphi^i(I)A$ is p-adically separated for any $i \geq 0$. This is a consequence that $A/\varphi^i(I)A$ is derived p-complete and p-torsion-free.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

The following obvious lemma is a key observation in the theory of perfectoid towers.

Lemma 3.10. *The Frobenius map F on $A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ factors through π_i as follows:*

$$\begin{array}{ccc} A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} & \xrightarrow{\quad F \quad} & A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}. \\ & {\scriptstyle \pi_i} \searrow \quad \nearrow {\scriptstyle \overline{\varphi}^{(i)}_{(p,I)}} & \\ & A_i/(p, I^{[p^i]})A_i. & \end{array}$$

Note that the map $\overline{\varphi}^{(i)}_{(p,I)} \colon A_i/(p, I^{[p^i]})A_i \to A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is the pth power map defined in Construction 3.1.

The Frobenius projections of the tower of prisms $(\{A^{1/p^i}\})$ are also constructed as follows (Construction 3.11) and we record the compatibility under the isomorphisms $\{c^i_i\}$ as in Lemma 3.5 (Observation 3.12). However, this is a little bit complicated and thus the reader may skip to Definition 3.13 and Lemma 3.14.

Construction 3.11. Let (A, I) be a preprism. Note that any ring A_i is A itself. Considering the following commutative diagram of rings:

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{\varphi} & A_1 & \xrightarrow{\varphi} & \cdots & \xrightarrow{\varphi} & A_i & \xrightarrow{\varphi} & A_{i+1} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & & & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle \mathrm{id}_{A_{i+1}}} & \\ A_0 & \xrightarrow{\varphi} & A_1 & \xrightarrow{\varphi} & \cdots & \xrightarrow{\varphi} & A_i & & \\ \downarrow{\scriptstyle \mathrm{id}_{A_0}} & & \downarrow{\scriptstyle \mathrm{id}_{A_1}} & & & & \downarrow{\scriptstyle \mathrm{id}_{A_i}} & & \\ A_0 & \xrightarrow{\varphi} & A_1 & \xrightarrow{\varphi} & \cdots & \xrightarrow{\varphi} & A_i & \xrightarrow{\varphi} & A_{i+1}. \end{array} \tag{3.2}$$

Taking the colimits for the horizontal directions and taking care of those A-algebra structures, the Frobenius lift φ on the δ-ring $A^{1/p^{i+1}}$ factors through t_i in the category of A-algebras as follows:

$$\begin{array}{ccc} A^{1/p^{i+1}} & \xrightarrow{\quad \varphi \quad} & A^{1/p^{i+1}} \\ & {\scriptstyle \varphi_i} \searrow^{\cong} \quad \nearrow {\scriptstyle t_i} & \\ & A^{1/p^i} & \end{array}$$

We call the isomorphism φ_i of A-algebras the *ith Frobenius lift projection* of the tower of prisms $(\{A^{1/p^i}\}, I)$.

Taking the quotient in the category of rings (not of A-algebras), the Frobenius lift φ of $A^{1/p^{i+1}}$ induces the following commutative diagrams of rings:

$$\begin{array}{ccccc} & & F & & \\ & \overset{\frown}{\quad} & & \overset{\frown}{\quad} & \\ A^{1/p^{i+1}}/(p, I_{i+1}) & \xrightarrow{\quad \varphi/(p,I) \quad} & A^{1/p^{i+1}}/(p, I^{[p]}_{i+1}) & \twoheadrightarrow & A^{1/p^{i+1}}/(p, I_{i+1}) \\ {\scriptstyle \varphi_i/(p,I)} \searrow^{\cong} & & \nearrow {\scriptstyle t_i/(p,I^{[p]})} & & \nearrow {\scriptstyle t_{i,(p,I)}} \\ & A^{1/p^i}/(p, I_i^{[p]}) & \twoheadrightarrow & A^{1/p^i}/(p, I_i). & \end{array}$$

Here, by the diagram (3.2), note that the left lower isomorphism $\varphi_i/(p, I)$ is deduced from $I_i = c^0_i(I)A^{1/p^i}$ and $\varphi_i(I_{i+1})A^{1/p^i} = \varphi_i(c^0_{i+1}(I)A^{1/p^{i+1}})A^{1/p^i} = c^0_i(\varphi(I))A^{1/p^i} = \varphi(I_i)A^{1/p^i}$ in A^{1/p^i} by

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A \\ {\scriptstyle c_0^{i+1}}\downarrow & & \downarrow{\scriptstyle c_0^i} \\ A^{1/p^{i+1}} & \xrightarrow[\cong]{\varphi_i} & A^{1/p^i}. \end{array}$$

The lower surjective map

$$\varphi_{i,(p,I)} \colon A^{1/p^{i+1}}/(p, I_{i+1}) \xrightarrow{\varphi_i/(p,I)} A^{1/p^i}/(p, I_i^{[p]}) \twoheadrightarrow A^{1/p^i}/(p, I_i) \tag{3.3}$$

becomes a map of A-algebras, and we call it the *ith Frobenius projection* of the tower of prisms $(\{A^{1/p^i}\}, I)$ (or of the tower of rings $(\{A^{1/p^i}/I_i\}, \{t_{i,(p,I)}\})$, see Lemma 3.14).

Observation 3.12. Let (A, I) be a preprism. Through the isomorphism of towers of δ-rings $\{c_i^i\}$ (Lemma 3.5), we have a commutative diagram in the category of δ-rings

$$\begin{array}{ccc} A^{1/p^{i+1}} & \xrightarrow{\varphi_i} & A^{1/p^i} \\ {\scriptstyle c_{i+1}^{i+1}}\uparrow\cong & & \cong\uparrow{\scriptstyle c_i^i} \\ A_{i+1} & \xrightarrow{\mathrm{id}_A} & A_i. \end{array}$$

In particular, we can regard the Frobenius lift projection $\varphi_i \colon A^{1/p^{i+1}} \to A^{1/p^i}$ as the identity map $\mathrm{id}_A \colon A_{i+1} \to A_i$. Furthermore, the isomorphism of towers of rings $\{c_{i,(p,I)}^i\}$ gives a commutative diagram of rings

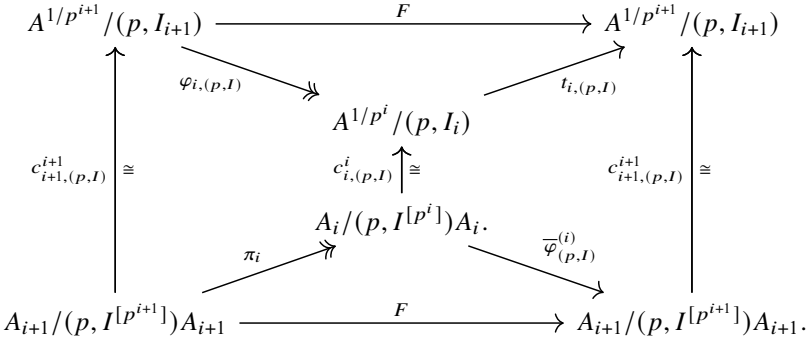
$$\begin{array}{ccccc} & & \overset{\varphi_{i,(p,I)}}{\frown} & & \\ A^{1/p^{i+1}}/(p, I_{i+1}) & \xrightarrow[\cong]{\varphi_i/(p,I)} & A^{1/p^i}/(p, I_i^{[p]}) & \twoheadrightarrow & A^{1/p^i}/(p, I_i) \\ {\scriptstyle c_{i+1,(p,I)}^{i+1}}\uparrow\cong & & \uparrow\cong & & {\scriptstyle c_{i+1,(p,I)}^{i+1}}\uparrow\cong \\ A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} & \xrightarrow[\cong]{\mathrm{id}} & A_i/(p, I^{[p^{i+1}]})A_i & \twoheadrightarrow & A_i/(p, I^{[p^i]})A_i, \end{array}$$

where the middle vertical isomorphism is induced from $c_i^i \colon A_i \to A^{1/p^i}$. In particular, we can regard the ith Frobenius projection $\varphi_{i,(p,I)} \colon A^{1/p^{i+1}}/(p, I_{i+1}) \twoheadrightarrow A^{1/p^i}/(p, I_i)$ as the natural surjection of rings

$$\pi_i \colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} = A/(p, I^{[p^{i+1}]})A \twoheadrightarrow A/(p, I^{[p^i]})A = A_i/(p, I^{[p^i]})A_i$$

defined in Definition 3.9. The Frobenius maps F on $A^{1/p^{i+1}}/(p, I_{i+1})$ and $A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ factor through $\varphi_{i,(p,I)}$ and π_i as follows:

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press



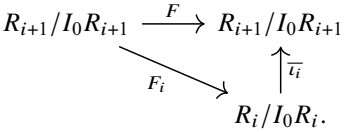
(3.4)

At the end of this section, we give the definition of p-purely inseparable towers and show that the tower of rings generated from prisms is a p-purely inseparable tower in many cases.

Definition 3.13 [14, Definition 3.4]. Let R be a ring and let I_0 be an ideal of R. A tower $(\{R_i\}, \{\iota_i\})$ of rings is called a *p-purely inseparable tower arising from* (R, I_0) if the following conditions hold:

(a) $R_0 \cong R$ and $p \in I_0$.
(b) The induced map $\overline{\iota_i}\colon R_i/I_0R_i \to R_{i+1}/I_0R_{i+1}$ from ι_i is injective for all $i \geq 0$.
(c) The image of the Frobenius map $F\colon R_{i+1}/I_0R_{i+1} \to R_{i+1}/I_0R_{i+1}$ is contained in the image of $\overline{\iota_i}$ for all $i \geq 0$.

Under these assumptions, the Frobenius map $F\colon R_{i+1}/I_0R_{i+1} \to R_{i+1}/I_0R_{i+1}$ factors through $\overline{\iota_i}$ as follows:



The map $F_i\colon R_{i+1}/I_0R_{i+1} \to R_i/I_0R_i$ is called the *ith Frobenius projection* of the tower $(\{R_i\}, \{\iota_i\})$.

Lemma 3.14. *Let (A, I) be a preprism. Assume that the pth power map $\overline{\varphi}^{(i)}_{(p,I)}\colon A_i/(p, I^{[p^i]})A_i \to A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is injective for all $i \geq 0$.*[4] *Then the tower of rings $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}^{(i)}_I\})$ is a p-purely inseparable tower arising from $(A/I, (p))$ and its Frobenius projection is nothing but the canonical surjection $\pi_i\colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \twoheadrightarrow A_i/(p, I^{[p^i]})A_i$ in Definition 3.9.*

In particular, the tower of rings $\{A^{1/p^i}/I_i, \{t_{i,I}\}\}$ is also a p-purely inseparable tower arising from $(A/I, (p))$ and its Frobenius projection is the Frobenius projection $\varphi_{i,(p,I)}\colon A^{1/p^{i+1}}/(p, I_{i+1}) \to A^{1/p^i}/(p, I_i)$ constructed in Construction 3.11.

Proof. This is clear by the construction of the Frobenius projection in Definition 3.9 (and Construction 3.11). □

4. Construction of perfectoid towers from prisms

This section is devoted to our first main result. We show that the tower of rings $(\{A_i/\varphi^i(I)A_i\}) \cong (\{A^{1/p^i}/IA^{1/p^i}\})$ generated from prism (A, I) becomes a perfectoid tower under mild assumptions and prove its properties (Theorem 4.5). To do this, we need some lemmas.

[4] The necessity of this injectivity is a little bit subtle (see Remark 4.6).

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Lemma 4.1. *Let (A, I) be an orientable preprism and fix an orientation d of I. Set elements*

$$f_0 := \overline{\varphi(d)}^I := \varphi(d) + I \in A/I,$$
$$f_1 := \overline{d}^{\varphi(I)} := d + \varphi(I)A \in A/\varphi(I)A.$$

If (A, I) is a Zariskian prism (Definition 2.2), then the following hold:

1. *The ideal $(f_0) \subseteq A/I$ generated by f_0 in A/I is the same as the ideal $(p) \subseteq A/I$.*
2. *There exists a unit element $u \in (A/\varphi(I)A)^\times$ such that $f_1^p = u \cdot \overline{\varphi}_I^{(0)}(f_0) \in A/\varphi(I)A$. Here, $\overline{\varphi}_I^{(0)}(f_0) = \varphi(f_0) + \varphi(I)A$ in $A/\varphi(I)A$ by the definition (Construction 3.1).*

Proof. Since A is (p, I)-Zariskian and p belongs to $(d, \varphi(d))A$, d is a distinguished element of A by [7, Lemma 2.25].

(1): Passing the equation $\varphi(d) = d^p + p\delta(d)$ to the quotient A/I, we have $f_0 = \overline{\varphi(d)}^I = \overline{p\delta(d)}^I$ in A/I. Since d is a distinguished element, we are done.

(2): We consider the following equations:

$$f_1^p = \overline{d^p}^{\varphi(I)} = \overline{\varphi(d) - p\delta(d)}^{\varphi(I)} = p \cdot \overline{-\delta(d)}^{\varphi(I)} \in A/\varphi(I)A$$
$$\overline{\varphi}_I^{(0)}(f_0) = \overline{\varphi}_I^{(0)}(\overline{\varphi(d)}^I) = \overline{\varphi}_I^{(0)}(\overline{p\delta(d)}^I) = p \cdot \overline{\varphi(\delta(d))}^{\varphi(I)} \in A/\varphi(I)A.$$

Since $\delta(d)$ is invertible in A, we can take $u := \overline{-\delta(d)\varphi(\delta(d))^{-1}}^{\varphi(I)} \in (A/\varphi(I)A)^\times$ and we are done. □

Lemma 4.2. *Let (A, I) be a preprism. For the Frobenius projection $\pi_i \colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \to A_i/(p, I^{[p^i]})A_i$ defined in (3.9), we have the following:*

1. *The kernel of π_i is $\ker(\pi_i) = I^{[p^i]}A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$. Via the isomorphism c_{i+1}^{i+1} in Lemma 3.4, we also have $\ker(\varphi_{i,(p,I)}) = c_{1,(p,I)}^{i+1}(I)A^{1/p^{i+1}}/(p, I)A^{1/p^{i+1}}$.*
2. *The induced isomorphism $(A^{1/p^{i+1}}/(p, I)A^{1/p^{i+1}})/\ker(\varphi_{i,(p,I)}) \cong A^{1/p^i}/(p, I)A^{1/p^i}$ from $\varphi_{i,(p,I)}$ is equivalent to the identity map $A_{i+1}/(p, I^{[p^{i+1}]}, I^{[p^i]})A_{i+1} \xrightarrow{\mathrm{id}} A_i/(p, I^{[p^i]})A_i$ via the isomorphism $\{c_i^i\}$ in Lemma 3.5.*
3. *If (A, I) is orientable, then the kernel $\ker(\pi_i) \subseteq A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is generated by $\overline{f_1}^{p^i} \in A_{i+1}/(p, I^{[p^{i+1}]})$, the image of $f_1 \in A_1/\varphi(I)A_1$ via the composition $A_1/\varphi(I)A_1 \twoheadrightarrow A_1/(p, I^{[p]})A_1 \xrightarrow{\overline{\varphi}_{(p,I)}^{(i)} \circ \cdots \circ \overline{\varphi}_{(p,I)}^{(1)}} A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$.*
4. *If (A, I) is orientable and the pth power map $\overline{\varphi}_{(p,I)}^{(i)} \colon A_i/(p, I^{[p^i]})A_i \to A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is injective, then the kernel of the Frobenius map F on $A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is generated by $\overline{f_1}^{p^i}$ as in (3).*

Proof. (1): The kernel of $\pi_i \colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \twoheadrightarrow A_i/(p, I^{[p^i]})A_i$ is nothing but the ideal generated by $I^{[p^i]}$ in $A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$. We have the commutative diagram

$$\begin{array}{ccccc}
A^{1/p}/(p, I)A^{1/p} & \xrightarrow{t_{i,(p,I)} \circ \cdots \circ t_{1,(p,I)}} & A^{1/p^{i+1}}/(p, I)A^{1/p^{i+1}} & \xrightarrow{\varphi_{i,(p,I)}} & A^{1/p^i}/(p.I)A^{1/p^i} \\
{\scriptstyle c_{1,(p,I)}^1}\uparrow\cong & \nearrow_{c_{1,(p,I)}^{i+1}} & \cong\uparrow{\scriptstyle c_{i+1,(p,I)}^{i+1}} & & \cong\uparrow{\scriptstyle c_{i,(p,I)}^{i}} \\
A_1/(p, I^{[p]})A_1 & \xrightarrow[\overline{\varphi}_{(p,I)}^{(i)} \circ \cdots \circ \overline{\varphi}_{(p,I)}^{(1)}]{} & A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} & \xrightarrow{\pi_i} & A_i/(p, I^{[p^i]})A_i.
\end{array} \tag{4.1}$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

The image of $IA_1/(p, I^{[p]})A$ under the lower horizontal map generates $\ker(\pi_i) = I^{[p^i]}A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$. So the kernel of the Frobenius projection $\varphi_{i,(p,I)} : A^{1/p^{i+1}}/(p, I)A^{1/p^i} \twoheadrightarrow A^{1/p^i}/(p, I)A^{1/p^i}$ is

$$c^{i+1}_{i+1,(p,I)}(I^{[p^i]}A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}) = c^{i+1}_{1,(p,I)}(I)A^{1/p^{i+1}}/(p, I)A^{1/p^{i+1}}.$$

(2): This is clear by the commutative diagram (3.4).

(3): By (1) and our assumption, the kernel $\ker(\pi_i)$ is generated by $\overline{d}^{p^i} \in A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$, where d is an orientation of I.

(4): By Lemma 3.10, if $\overline{\varphi}^{(i)}_{(p,I)}$ is injective, the kernel of the Frobenius map F on $A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$ is the same as $\ker(\pi_i)$ and we are done by (3). □

Under these lemmas, we can show that the tower of rings generated from prisms becomes a perfectoid tower. Before that, we recall the definition of perfectoid towers and its related concepts.

Definition 4.3 (Perfectoid towers; [14, Definition 3.21]). Let R be a ring and let I_0 be an ideal of R. A tower $(\{R_i\}, \{\iota_i\})$ of rings is called a *perfectoid tower arising from* (R, I_0) if it is p-purely inseparable tower arising from (R, I_0) (Definition 3.13) and satisfies the following conditions:

(d) The ith Frobenius projection $F_i : R_{i+1}/I_0R_{i+1} \to R_i/I_0R_i$ is surjective for all $i \geq 0$.
(e) Each R_i is I_0-Zariskian.
(f) The ideal I_0 is a principal ideal in R and there exists a principal ideal I_1 of R_1 such that $I_1^{[p]} = I_0R_1$ and the kernel $\ker(F_i)$ of the ith Frobenius projection is generated by the image of I_1 via $R_1 \twoheadrightarrow R_1/I_0R_1 \to R_{i+1}/I_0R_{i+1}$ for all $i \geq 0$.
(g) Any I_0-power-torsion element of R_i is I_0-torsion, that is, $R_i[I_0^\infty] = R_i[I_0]$. Furthermore, there exists a bijective map $F_{i,\mathrm{tor}} : R_{i+1}[I_0^\infty] \to R_i[I_0^\infty]$ of sets such that the following diagram commutes:

$$\begin{array}{ccc} R_{i+1}[I_0^\infty] & \longrightarrow & R_{i+1}/I_0R_{i+1} \\ {\scriptstyle F_{i,\mathrm{tor}}}\downarrow & & \downarrow{\scriptstyle F_i} \\ R_i[I_0^\infty] & \longrightarrow & R_i/I_0R_i. \end{array} \tag{4.2}$$

Such a principal ideal I_1 is uniquely determined and is called the *first perfectoid pillar* of this perfectoid tower
[14, Definition 3.25]. Furthermore, we can take a sequence of principal ideal $\{I_i \subseteq R_i\}_{i\geq 2}$ which satisfies $F_i(I_{i+1} \cdot R_{i+1}/I_0R_{i+1}) = I_i \cdot R_i/I_0R_i$ for each $i \geq 0$. Such a sequence of principal ideals $\{I_i\}_{i\geq 2}$ is uniquely determined and I_i is called the *ith perfectoid pillar* of this perfectoid tower [14, Definition 3.27].

Definition 4.4 (Tilts of perfectoid towers; [14, Definition 3.34]). Let $(\{R_i\}, \{\iota_i\})$ be a perfectoid tower arising from (R, I_0). The *ith small tilt* $(R_i)^{s.\flat}_{I_0}$ (or simply $R_i^{s.\flat}$) of $(\{R_i\}, \{\iota_i\})$ associated with (R, I_0) is the inverse limit

$$R_i^{s.\flat} := \lim_{k\geq 0}\{\cdots \xrightarrow{F_{i+k+1}} R_{i+k+1}/I_0R_{i+k+1} \xrightarrow{F_{i+k}} R_{i+k}/I_0R_{i+k} \xrightarrow{F_{i+k-1}} \cdots \xrightarrow{F_i} R_i/I_0R_i\}$$

for each $i \geq 0$. The transition map $\iota_i^{s.\flat} : R_i^{s.\flat} \to R_{i+1}^{s.\flat}$ is the inverse limit of the maps $\overline{\iota_{i+k}} : R_{i+k}/I_0R_{i+k} \to R_{i+k+1}/I_0R_{i+k+1}$ for $k \geq 0$. The *tilt* of the perfectoid tower $(\{R_i\}, \{\iota_i\})$ is the tower $(\{R_i^{s.\flat}\}, \{\iota_i^{s.\flat}\})$. The *ith small tilt* $I_i^{s.\flat}$ of the perfectoid pillar I_i is the kernel

$$I_i^{s.\flat} := \ker(R_i^{s.\flat} \xrightarrow{\Phi_0^{(i)}} R_i/I_0R_i \twoheadrightarrow R_i/I_iR_i)$$

for each $i \geq 0$, where $\Phi_0^{(i)} : R_i^{s.\flat} \to R_i/I_0R_i$ is the first projection. By [14, Proposition 3.41], the tilt $(\{R_i^{s.\flat}\}, \{\iota_i^{s.\flat}\})$ becomes a perfect(oid) tower arising from $(R_0^{s.\flat}, (I_0^{s.\flat}))$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Our goal is to construct a perfectoid tower from a class of prisms as follows.

Theorem 4.5. *Let (A, I) be an orientable Zariskian prism with an orientation $d \in I$. Assume that p, d is a regular sequence on A,[5] and A/pA is p-root closed in $A/pA[1/d]$. Then the following assertions hold:*

(1) $(\{R_i\}, \{\iota_i\}) := (\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\}) \cong (\{A^{1/p^i}/IA^{1/p^i}\}, \{t_{i,I}\})$ *is a perfectoid tower arising from $(A/I, (p))$ whose terms $A_i/\varphi^i(I)A_i$ are p-torsion-free. If A is derived p-complete or is Noetherian, then each transition map $\overline{\varphi}_I^{(i)}$ and $t_{i,I}$ is injective.*

(2) *Its tilt $(\{(A_i/\varphi^i(I)A_i)^{s.\flat}\}, \{(\overline{\varphi}_I^{(i)})^{s.\flat}\})$ is isomorphic to the perfect tower $(\{(A/pA)^{\wedge_d}\}, \{F\})$, where $(-)^{\wedge_d}$ is the d-adic completion.*

(3) *The p-adic completion $\widehat{R_\infty}$ of the colimit $R_\infty := \operatorname{colim}_i R_i$ is isomorphic to the quotient of the perfection of the prism (A, I), that is, isomorphic to the perfectoid ring $A_\infty/IA_\infty \cong (A_{\mathrm{perf}}/IA_{\mathrm{perf}})^{\wedge_p}$, where A_∞ is the (p, I)-adic completion of the colimit $A_{\mathrm{perf}} := \operatorname{colim}_\varphi A$.*

(4) *The first perfectoid pillar I_1 of the tower is $f_1A_1/\varphi(I)A_1$, where $f_1 = \overline{d}^{\varphi(I)} \in A_1/\varphi(I)A_1$ as in Lemma 4.1.*

(5) *The ith perfectoid pillar I_i of the tower is $f_iA_i/\varphi^i(I)A_i$, where $f_i := \overline{d}^{\varphi^i(I)} = d + \varphi^i(I)A_i \in A_i/\varphi^i(I)A_i$ for each $i \geq 2$.*

(6) *The small tilt $I_i^{s.\flat} \subseteq (A_i/\varphi^i(I)A_i)^{s.\flat}$ of I_i is isomorphic to $(d) \subseteq A/pA$ for each $i \geq 0$.*

Note that these statements also hold for any crystalline Zariskian prism $(A, (p))$.

Proof. In (1), if A is derived p-complete or Noetherian, then $A/\varphi^i(I)A$ is p-adically separated for any $i \geq 0$ under our assumption and then the injectivity of $\overline{\varphi}_I^{(i)}$ follows from Lemma 3.8. We check the axiom from (a) to (g) of p-purely inseparable towers (Definition 3.13) and perfectoid towers (Definition 4.3).

(a): This is clear.

(b): We must show that the pth power map $\overline{\varphi}_{(p,I)}^{(i)}$

$$A_i/(p, I^{[p^i]})A_i \cong (A/pA)/d^{p^i}(A/pA) \xrightarrow{a \mapsto a^p} (A/pA)/d^{p^{i+1}}(A/pA) \cong A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1}$$

is injective for all $i \geq 0$. This condition is equivalent to the condition that A/pA is p-root closed in $A/pA[1/d]$ which we assume now. Here, we use the assumption that d is a regular element of A/pA because we need the injection $A/pA \hookrightarrow A/pA[1/d]$.

(c): This is already proved in Lemma 3.14.

(d): The surjectivity of the Frobenius projection $\pi_i \colon A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \to A_i/(p, I^{[p^i]})A_i$ is clear.

(e): This follows from the $(p, I^{[p^i]})$-Zariskian property of A_i.

(f): The principality of (p) in A/I is clear. By Lemma 4.1 (1), the principal ideal (p) is the same as the principal ideal (f_0) in A/I. We set $I_1 := f_1A/\varphi(I)A \subseteq A_1/\varphi(I)A_1$, where $f_1 = \overline{d}^{\varphi(I)} \in A_1/\varphi(I)A_1$ is defined in Lemma 4.1. By Lemma 4.1 (2), we have $I_1^p = pA_1/\varphi(I)A_1$. By Lemma 4.2 (3), the kernel $\ker(\pi_i)$ of the Frobenius projection π_i is generated by the image of I_1 via $A_1/\varphi(I)A_1 \twoheadrightarrow A_1/(p, I^{[p]})A_1 \xrightarrow{\overline{\varphi}_{(p,I)}^{(i)} \circ \cdots \circ \overline{\varphi}_{(p,I)}^{(1)}} A_i/(p, I^{[p^i]})A_i$.

(g): Since p, d is regular on A, so is $p, \varphi^i(d)$ for each $i \geq 0$. This shows that p is a non-zero-divisor of $A_i/\varphi^i(d)A_i$ by a simple calculation. Thus, $(A_i/\varphi^i(d)A_i)[p^\infty]$ is zero and the condition (g) is clear. This shows (1).

[5] Because of the lack of the derived (p, I)-completeness of A, we do not know whether the regular sequence p, d on A is permutable or not (see the proof of [16, Lemma 2.9]). However, the two conditions in the statement do not depend on the choice of the orientation d.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

We compute the tilt of the perfectoid tower $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\})$ arising from $(A/I, (p))$. By Lemma 3.14, the ith small tilt is

$$\begin{aligned}(A_i/\varphi^i(I)A_i)^{s.\flat} &\cong \lim\{\cdots \xrightarrow{\pi_{i+2}} A_{i+2}/(p, I^{[p^{i+2}]})A_{i+2} \xrightarrow{\pi_{i+1}} A_{i+1}/(p, I^{[p^{i+1}]})A_{i+1} \xrightarrow{\pi_i} A_i/(p, I^{[p^i]})A_i\} \\ &\cong (A/pA)^{\wedge_d},\end{aligned}$$

where the symbol $(-)^{\wedge_d}$ is the d-adic completion. The transition map $(t_{i,I})^{s.\flat}\colon (A/pA)^{\wedge_d} \to (A/pA)^{\wedge_d}$ is induced from the inverse limit of pth power maps $\overline{\varphi}^{(i)}_{(p,I)}$. Then the transition map $(t_{i,I})^{s.\flat}$ is nothing but the Frobenius map F on $(A/pA)^{\wedge_d}$ and we show (2).

The colimit $R_\infty \coloneqq \operatorname{colim}_i R_i$ is isomorphic to $\operatorname{colim}_\varphi(A_i/\varphi^i(I)A_i) \cong A_{\mathrm{perf}}/IA_{\mathrm{perf}}$, where $A_{\mathrm{perf}} \coloneqq \operatorname{colim}_\varphi A$ as in [7, Lemma 3.9]. The desired consequence (3) is obtained because $(A_{\mathrm{perf}}/IA_{\mathrm{perf}})^{\wedge_p}$ and A_∞/IA_∞ are isomorphic.

By the above proof of (1), the first perfectoid pillar I_1 is $f_1A_1/\varphi(I)A_1$, where $f_1 = \overline{d}^{\varphi(I)} \in A_1/\varphi(I)A_1$ as in Lemma 4.1. The ith perfectoid pillar $I_i \subset A_i/\varphi^i(I)A_i$ is $f_iA_i/\varphi^i(I)A_i$, where $f_i = \overline{d}^{\varphi^i(I)} = d + \varphi^i(I)A_i \in A_i/\varphi^i(I)A_i$ for each $i \geq 2$ by the definition of the Frobenius projection π_i. This shows (4) and (5).

The small tilt $I_i^{s.\flat} \subseteq (A_i/\varphi^i(I)A_i)^{s.\flat}$ of the ith perfectoid pillar I_i is the kernel of the horizontal map of

$$\begin{array}{ccccc}(A_i/\varphi^i(I)A_i)^{s.\flat} & \xrightarrow{\Phi_0^{(i)}} & A_i/(p, I^{[p^i]})A_i & \longrightarrow & A_i/(I_i, I^{[p^i]})A_i \\ \cong\downarrow & & \| & & \| \\ (A/pA)^{\wedge_d} & \longrightarrow & A/(p, d^{[p^i]})A & \longrightarrow & A/(p,d)A,\end{array}$$

where the left two horizontal maps are the first projection and the right two horizontal maps are the canonical surjection. This shows (6): $I_i^{s.\flat} \cong (d) \subseteq A/pA$ for any $i \geq 0$. □

Remark 4.6. In the definition of p-purely inseparable towers (and of course perfectoid towers), the injectivity axiom (b) in Definition 3.13 is used to ensure that the tower has the Frobenius projection. However, in our case, the tower of rings $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\})$ has the Frobenius projection π_i without the assumption of the injectivity of the pth power map (Lemma 3.10). Furthermore, all the axioms of perfectoid towers without the axiom (b) are satisfied by the tower $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\})$ for *any* orientable Zariskian prism (A, I) with an orientation $d \in I$ such that p, d is a regular sequence on A. By the proof of Theorem 4.5, the tower $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\})$ is a perfectoid tower if and only if A/pA is p-root closed in $A/pA[1/d]$. Although the axiom (b) is necessary to give a good theory of perfectoid towers, if the existence of Frobenius projection is merely necessary, our construction covers it.

Based on the above theorem, we come up with the following question which is not clear yet.

Question 1. For a given perfectoid tower $(\{R_i\}, \{t_i\})$, can we construct a prism (A, I) such that the perfectoid tower $(\{A_i/\varphi^i(I)A_i\}, \{\overline{\varphi}_I^{(i)}\})$ is isomorphic to $(\{R_i\}, \{t_i\})$? More optimistically, can we construct a one-to-one correspondence between the set of (p-torsion-free) perfectoid towers and the set of (specific) prisms by using the above construction?

5. Perfectoid towers from δ-rings

In this section, for any δ-ring R, we make two perfectoid towers $\{R_i\}_{i\geq 0}$ satisfying $R_0 \cong R$ or $R_0 \cong R[\zeta_p]$, where ζ_p is a pth root of unity. The one is obtained by adjoining p-power roots of p to R and the other is obtained by adjoining p-power roots of unity. Recall that $R^{1/p^i} \coloneqq \operatorname{colim}\{R \xrightarrow{\varphi_R} R \xrightarrow{\varphi_R} \cdots \xrightarrow{\varphi_R} R\}$ is the colimit consisting of $(i+1)$-terms, where $\varphi_R\colon R \to R$ is the Frobenius lift of R.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

First, we prove a more general result (Theorem 5.4) which says that the base change of the tower $R \to R^{1/p} \to R^{1/p^2} \to \cdots$ along a perfectoid tower induced from a prism is a perfectoid tower.

5.1. Base change by perfectoid towers

Notation 5.1. In this section, we follow the notation below:

- Let V be an absolute unramified discrete valuation δ-ring of mixed characteristic $(0, p)$.
- Let (A, I) be an orientable bounded Zariskian prism with an orientation $d \in I$.
- Let R be a p-torsion-free p-Zariskian δ-ring.
- Write the Frobenius lift on A by φ and the Frobenius lift on R by φ_R.
- Assume that both A and R are p-torsion-free δ-V-algebra.

In this case, we can identify $R \otimes_V I$ and $I(R \otimes_V A)$ as ideals of $R \otimes_V A$ and the tensor product $R \otimes_V A$ is flat over R and A. In particular, it is p-torsion-free. The tensor product $R \otimes_V A$ and its (p, I)-Zariskization[6] $(1 + (p, I))^{-1}(R \otimes_V A)$ inherit the unique δ-structure compatible with the maps from R and A. Here, we implicitly use $\varphi((p, I)) = (p, d^p) \subseteq (p, I)$ in $R \otimes_V A$ and [7, Lemma 2.15]. We take the flat δ-(R, A)-algebra

$$P_{A,V}(R) \coloneqq (1 + (p, I))^{-1}(R \otimes_V A). \tag{5.1}$$

The Frobenius lift on $P_{A,V}(R)$ is denoted by φ_P, which is defined as the localization of $\varphi_R \otimes_{\varphi_V} \varphi$ on $R \otimes_V A$.

Lemma 5.2. *Keep the notation as in Notation 5.1. Then the pair $(P_{A,V}(R), IP_{A,V}(R))$ is an orientable bounded Zariskian prism. Moreover, if $R \otimes_V (A/\varphi^i(I)A)$ is p-Zariskian for any $i \geq 0$, there exist natural isomorphisms*

$$P_{A,V}(R)/\varphi^i(I)P_{A,V}(R) \cong R \otimes_V (A/\varphi^i(I)A)$$
$$P_{A,V}(R)^{1/p^i}/IP_{A,V}(R)^{1/p^i} \cong R^{1/p^i} \otimes_{V^{1/p^i}} (A^{1/p^i}/IA^{1/p^i})$$

for each $i \geq 0$ which are compatible with φ_P and $\varphi_R \otimes_{\varphi_V} \overline{\varphi}_I^{(i)}$. In particular, if $A/\varphi^i(I)A$ is integral over V, this isomorphism holds.

Proof. We already know that $P_{A,V}(R)$ is a δ-ring and $p \in IP_{A,V}(R) + \varphi(I)P_{A,V}(R)$. Since $P_{A,V}(R)$ is flat over A, the ideal $IP_{A,V}(R)$ is generated by a non-zero-divisor $1 \otimes d$ in $P_{A,V}(R)$. The p^∞-torsion part of $P_{A,V}(R)$ is the base change of the p^∞-torsion part of A because of the flatness of R over V. The construction of $P_{A,V}(R)$ ensures that it is (p, I)-Zariskian and therefore the pair $(P_{A,V}(R), IP_{A,V}(R))$ is an orientable bounded Zariskian prism.

If $R \otimes_V (A/\varphi^i(I)A)$ is p-Zariskian, then the isomorphism $P_{A,V}(R)/\varphi^i(I)P_{A,V}(R) \cong (1+I)^{-1}(R \otimes_V (A/\varphi^i(I)A))$ holds. It suffices to show that the p-Zariskian ring $R \otimes_V (A/\varphi^i(I)A)$ is I-Zariskian but this follows from the equation $(p, \varphi^i(I)) = (p, d^{p^i})$ on A. □

Our second main result is Theorem 5.4 below. This says that the base change of a perfectoid tower $(\{A_i/\varphi^i(I)A_i\})$ along a tower $\{R^{1/p^i}\}$ of a δ-ring R is again a perfectoid tower. To check the p-root closed property, we need the following lemma.

Lemma 5.3. *Let A be a ring and let $t \in A$ be a non-zero-divisor. Take a multiplicative closed subset $S \subseteq A$ consisting of some non-zero-divisors. If A is p-root closed in $A[1/t]$, then $S^{-1}A$ is p-root closed in $S^{-1}A[1/t]$.*

[6] More precisely, this is the localization of $R \otimes_V A$ by $1 + J$, where $J \subseteq R \otimes_V A$ is the ideal of $R \otimes_V A$ generated by p and I. This localization does not necessarily coincide with the localization of $R \otimes_V A$ by the image of $1 + (p, I)A$ in A through $A \to R \otimes_V A$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Proof. By our assumption, we have injections $A \hookrightarrow A[1/t] \hookrightarrow S^{-1}A[1/t]$. Let $a/st^n \in S^{-1}A[1/t]$ be an element satisfying that $(a/st^n)^p = a'/s' \in S^{-1}A$. Then we have $(as'/t^n)^p = s^p a'(s')^{p-1} \in A$ and thus $as'/t^n \in A$ by the p-root closedness of A in $A[1/t]$. This shows that $a/st^n = 1/ss' \cdot as'/t^n \in S^{-1}A$ and the lemma is proved. □

Theorem 5.4. *Keep the notation as in Notation 5.1 and assume that R/pR is reduced. If further $(A,(d))$ satisfies the assumptions of Theorem 4.5,[7] then so does the orientable Zariskian prism $(P_{A,V}(R), IP_{A,V}(R))$ in Lemma 5.2. In particular, the following assertions hold:*

1. *The Zariskian prism $(P_{A,V}(R), IP_{A,V}(R))$ gives a p-torsion-free perfectoid tower $(\{(1+(p))^{-1}(R\otimes_V (A/\varphi^i(I)A))\}, \{\varphi_R\otimes_{\varphi_V}\overline{\varphi}_I^{(i)}\}) \cong (\{(1+(p))^{-1}(R^{1/p^i}\otimes_{V^{1/p^i}}(A^{1/p^i}/IA^{1/p^i}))\}, \{\varphi_R\otimes_{\varphi_V} t_{i,I}\})$ arising from $((1+(p))^{-1}(R\otimes_V (A/I)),(p))$.*
2. *Its tilt is isomorphic to the perfect tower $(\{(R/pR\otimes_{V/pV} A/pA)^{\wedge_d}\},\{F\})$.*
3. *The p-completed colimit of the perfectoid tower is isomorphic to the p-completed tensor product $R_{\mathrm{perf}}\widehat{\otimes}_V A_{\mathrm{perf}}/IA_{\mathrm{perf}}$.*
4. *The ith perfectoid pillar of the tower is generated by $1\otimes f_i \in R\otimes_V A/\varphi^i(I)A$, where $f_i = \overline{d}^{\varphi^i(I)} = d+\varphi^i(I)A \in A/\varphi^i(I)A$ as in Theorem 4.5 for $i\geq 1$.*
5. *The ith small tilt of the ith perfectoid pillar is isomorphic to $(1\otimes d)\subseteq (R/pR\otimes_{V/pV} A/pA)^{\wedge_d}$ for $i\geq 0$.*

Moreover, if $R\otimes_V (A/\varphi^i(I)A)$ is p-Zariskian for all $i\geq 0$, then we have a perfectoid tower $(\{R\otimes_V (A/\varphi^i(I)A)\}, \{\varphi_R\otimes\overline{\varphi}_I^{(i)}\}) \cong (\{R^{1/p^i}\otimes_V (A^{1/p^i}/IA^{1/p^i})\}, \{\varphi_R\otimes t_{i,I}\})$ arising from $(R\otimes_V A/I,(p))$.

Proof. By Theorem 4.5 and Lemma 5.2, it is enough to show that p,d is a regular sequence on $P_{A,V}(R)$ and $P_{A,V}(R)/(p) \cong (1+I)^{-1}(R/pR\otimes_{V/pV} A/pA)$ is p-root closed in $P_{A,V}(R)/(p)[1/d]$. The first assertion follows from the regularity of p,d on A and the flatness of $R\otimes_V A$ over A. In the second assertion, Lemma 5.3 implies that it is enough to show that $R/pR\otimes_{V/pV} A/pA$ is p-root closed in $(R/pR\otimes_{V/pV} A/pA)[1/d]$. As in the proof of Theorem 4.5, this is equivalent to the injectivity of the pth power map $(R/pR\otimes_{V/pV} A/pA)/(d) \xrightarrow{x\mapsto x^p} (R/pR\otimes_{V/pV} A/pA)/(d^p)$. This map can be factorized as the composition

$$R/pR\otimes_{V/pV} A/(p,d)A \xrightarrow{\mathrm{id}\otimes\overline{\varphi}^{(0)}_{(p,I)}} R/pR\otimes_{V/pV} A/(p,d^p)A \xrightarrow{\mathrm{Frob}\otimes\mathrm{id}} R/pR\otimes_{V/pV} A/(p,d^p)A.$$

The former map is injective by the p-root closedness of A/pA in $A/pA[1/d]$ and the flatness of R over V. Since V is an absolute unramified discrete valuation ring, V/pV is a field and thus the latter map is injective by the reducedness of R/pR.

By Lemma 5.2, we have a natural isomorphism $(1+(p,I))^{-1}(R\otimes_V (A/\varphi^i(I)A)) \cong R\otimes_V (A/\varphi^i(I)A)$. So we can deduce the last assertion from the first assertion. □

5.2. Adjoining p-power roots of p and unity

Based on Theorem 5.4, we can construct perfectoid towers by adjoining p-power roots of p and unity to δ-rings.

Corollary 5.5. *Let R be a p-torsion-free p-Zariskian δ-ring such that R/pR is reduced. Fix compatible sequences $\{p^{1/p^i}\}_{i\geq 0}$ and $\{\zeta_{p^i}\}_{i\geq 0}$ of p-power roots of p and unity in $\mathbb{Q}$. Then we have the following p-torsion-free perfectoid towers:*

$$R \to R^{1/p}\otimes_{\mathbb{Z}}\mathbb{Z}[p^{1/p}] \to \cdots \to R^{1/p^i}\otimes_{\mathbb{Z}}\mathbb{Z}[p^{1/p^i}] \to \cdots \tag{5.2}$$

$$R\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_p] \to R^{1/p}\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_p^{1/p}] \to \cdots \to R^{1/p^i}\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_p^{1/p^i}] \to \cdots \tag{5.3}$$

[7] Namely, p,d is a regular sequence on A and A/pA is p-root closed in $A/pA[1/d]$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

arising from $(R,(p))$ *and* $(R\otimes_{\mathbb{Z}}\mathbb{Z}[\zeta_p],(p))$*, respectively. If* R *is* p*-adically separated, their transition maps are injective. Their* p*-completed colimits are isomorphic to* $(R_{\mathrm{perf}}\otimes_{\mathbb{Z}}\mathbb{Z}[p^{1/p^\infty}])^{\wedge_p}$ *and* $(R_{\mathrm{perf}}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\zeta_{p^\infty}])^{\wedge_p}$*, respectively. Their tilts are isomorphic to*

$$R/pR[|T|]\xrightarrow{F}R/pR[|T|]\xrightarrow{F}R/pR[|T|]\xrightarrow{F}\dots,$$

where F *is the Frobenius map on the formal power series ring* $R/pR[|T|]$*.*

If further R *is* p*-adically separated,* R *and* R/pR *are integral domains and the Frobenius lift* φ_R *is finite, then the perfectoid towers (5.2) and (5.3) are isomorphic to the towers of subrings*

$$R\hookrightarrow R^{1/p}[p^{1/p}]\hookrightarrow R^{1/p^2}[p^{1/p^2}]\hookrightarrow\cdots\hookrightarrow R^{1/p^i}[p^{1/p^i}]\hookrightarrow\cdots$$
$$R\hookrightarrow R^{1/p}[\zeta_p]\hookrightarrow R^{1/p^2}[\zeta_{p^2}]\hookrightarrow\cdots\hookrightarrow R^{1/p^i}[\zeta_{p^i}]\hookrightarrow\cdots$$

in a fixed absolute integral closure R^+ *of* R*. Here, we take an embedding of a finite extension* $R^{1/p^i}:=\mathrm{colim}\{R\xrightarrow{\varphi}R\xrightarrow{\varphi}\dots\xrightarrow{\varphi}R\}$ *of* R *into* R^+ *for each* $i\geq 0$*.*

Proof. The orientable bounded Zariskian prisms $((1+(T))^{-1}\mathbb{Z}_{(p)}[T],(p-T))$ and $((1+(q-1))^{-1}\mathbb{Z}_{(p)}[q],([p]_q))$ in Example 2.3 satisfy the assumption of Theorem 4.5. Applying Theorem 5.4 for $V:=\mathbb{Z}_{(p)}$, the isomorphisms $\mathbb{Z}[T]/(p-T^{p^i})\cong\mathbb{Z}[p^{1/p^i}]$ and $\mathbb{Z}[q]/([p]_q)\cong\mathbb{Z}[\zeta_p]$ yield the perfectoid towers and their tilts are isomorphic to the tower $(\{R/pR[|T|]\},\{F\})$. If R is p-adically separated, Lemma 3.7 implies that the Frobenius lift φ_R is injective. Since $\mathbb{Z}\hookrightarrow\mathbb{Z}[p^{1/p^i}]$ and $\mathbb{Z}\hookrightarrow\mathbb{Z}[\zeta_{p^i}]$ are flat, the transition maps in the towers are injective.

We next prove the last assertion. The isomorphism $R^{1/p^i}/pR^{1/p^i}\cong(R/pR)^{1/p^i}$ holds for each $i\geq 0$ and thus p is a prime element of R^{1/p^i}. Since φ_R is finite injective, the canonical map $R^{1/p^i}\hookrightarrow R^{1/p^{i+1}}$ is also finite injective. This gives a sequence of finite extensions of integral domains

$$R\hookrightarrow R^{1/p}\hookrightarrow R^{1/p^2}\hookrightarrow\cdots\hookrightarrow R^{1/p^i}\hookrightarrow\cdots.$$

This sequence is contained in a fixed absolute integral closure R^+ of R. By (5.2) and (5.3), we need to show that the canonical maps $R^{1/p^i}[T]/(p-T^{p^i})\cong R^{1/p^i}\otimes_{\mathbb{Z}}\mathbb{Z}[p^{1/p^i}]\twoheadrightarrow R^{1/p^i}[p^{1/p^i}]$ and $R^{1/p^i}[q^{1/p^i}]/([p]_q)\cong R^{1/p^i}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\zeta_{p^i}]\twoheadrightarrow R^{1/p^i}[\zeta_{p^i}]$ are isomorphisms. It is enough to show that $T^{1/p^i}-p$ and $[p]_q$ are irreducible polynomials in $R^{1/p^i}[T]$ and $R^{1/p^i}[q^{1/p^i}]$, respectively. This follows from the fact that p is a prime element of R^{1/p^i} and the Eisenstein criterion for $T^{1/p^i}-p$ and the variable transformation $((q+1)^p-1)/q$ of $[p]_q$. □

Remark 5.6. In the Noetherian case, such a δ-ring R in Corollary 5.5 relates the Frobenius liftable singularities: Let A be a reduced Noetherian local ring over a perfect field k of characteristic p. If A is Frobenius liftable, namely, there exists a flat $W(k)$-algebra R with a Frobenius lift φ_R such that $R/pR\cong A$, then there exists a perfectoid tower arising from $(\widehat{R},(p))$ and its tilt is the perfect tower arising from $(A[|T|],(p))$, where $\widehat{R}$ is the p-adic completion of R. In fact, $\widehat{R}$ is a p-torsion-free p-adically complete Noetherian local δ-ring such that $R/pR\cong A$ is reduced (here, we use [22, Tag 0G5H]) and we can apply Corollary 5.5. We use this observation in Proposition 6.9.

5.3. Generic ranks of transition maps of perfectoid towers.

If we know the generic rank of transition maps of a perfectoid tower is p-power, then some étale cohomology of mixed characteristic can be captured by the one of positive characteristic (see [14, Proposition 4.7]). In general, generic ranks are not easily computable, but we give some sufficient conditions in an algebraic situation and a geometric situation. One of the most simple case is the following.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Lemma 5.7. *Let (A, I) be an orientable Zariskian prism with an orientation $d \in I$. If φ on A is finite free of degree* $\deg \varphi$ *and $A/\varphi^i(I)A$ is an integral domain for each $i \geq 0$, then* $\deg \varphi$ *is the degree of the generic extension of the transition map $\overline{\varphi}_I^{(i)} : A/\varphi^i(I)A \hookrightarrow A/\varphi^{i+1}(I)A$ for each $i \geq 0$.*

In the case of δ-rings, there is a similar result.

Lemma 5.8. *Let R be a p-Zariskian p-adically separated δ-ring such that R and R/pR are integral domains and the Frobenius lift φ_R is finite. Let* $\deg \varphi_R$ *be the degree of the generic extension of the Frobenius lift φ_R on R. Then the degree of the generic extension of $R^{1/p^i}[p^{1/p^i}] \hookrightarrow R^{1/p^{i+1}}[p^{1/p^{i+1}}]$ is $p \cdot \deg \varphi_R$ for any $i \geq 0$.*

Proof. Set $K := \mathrm{Frac}(R)$. First, we can show that K and $\mathbb{Q}[p^{1/p^i}]$ are linearly independent over $\mathbb{Q}$ in an algebraic closure $\overline{K}$ of K: If elements $x_0, \dots, x_i$ in K satisfy $x_0 + x_1 p^{1/p^i} + \cdots + x_{p^i-1} p^{(p^i-1)/p^i} = 0$ in $\overline{K}$, then we may assume that $x_0, \dots, x_i$ are in R and so $x_0 + x_1 T + \cdots + x_{p^i-1} T^{p^i-1} = 0$ in $R[T]$ since we have an isomorphism $R[T]/(p - T^{p^i}) \cong R[p^{1/p^i}]$ as in the proof of Corollary 5.5. Therefore, $x_0 = \cdots = x_{p^i-1} = 0$ in R and so in K. This shows the linear independence. Especially, this implies that $\mathrm{Frac}(R[p^{1/p^i}]) \cong K[p^{1/p^i}] \cong K \otimes_{\mathbb{Q}} \mathbb{Q}[p^{1/p^i}]$. Since we have an isomorphism $R^{1/p^i} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^i}] \cong R^{1/p^i}[p^{1/p^i}]$, it suffices to consider the generic rank of the map $R \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^i}] \to (\varphi_* R) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^{i+1}}]$. Since the degree $[K(\varphi_* R) : K(R)]$ is $\deg \varphi_R$, we have

$$\mathrm{Frac}((\varphi_* R) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^{i+1}}]) \cong K(\varphi_* R) \otimes_{\mathbb{Q}} \mathbb{Q}[p^{1/p^{i+1}}] \cong K^{\oplus \deg \varphi_R} \otimes_{\mathbb{Q}} \mathbb{Q}[p^{1/p^{i+1}}]$$

and this is a $(p \cdot \deg \varphi_R)$-dimensional $(K \otimes_{\mathbb{Q}} \mathbb{Q}[p^{1/p^i}])$-vector space. □

We give a sufficient condition of p-power generic rank of φ in both algebraic and geometric situations.

Proposition 5.9. *Let R be a δ-ring such that R and R/pR are integral domains and the Frobenius lift φ_R on R is finite. If R is Noetherian and normal, then the generic rank of φ_R is p-power.*

Proof. By Lemma 3.7, the Frobenius lift φ_R is finite injective. Since $\mathfrak{p} := pR$ is a prime ideal of R and R is Noetherian normal, taking the localization at p, we have a finite injective map $\varphi_{\mathfrak{p}} : R_{\mathfrak{p}} \hookrightarrow R_{\mathfrak{p}}$ between discrete valuation rings. In particular, this map $\varphi_{\mathfrak{p}}$ is a finite flat endomorphism of a discrete valuation ring. Therefore, $\varphi_{\mathfrak{p},*} R_{\mathfrak{p}}$ is a finite free $R_{\mathfrak{p}}$-module. Thus, the generic rank of $\varphi_{\mathfrak{p}}$ is the same as the generic rank of the Frobenius map on $K(R/pR) = R_{\mathfrak{p}}/pR_{\mathfrak{p}}$ which is p-power. Since the generic rank of φ_R is the same as that of $\varphi_{\mathfrak{p}}$, the generic rank of φ_R is p-power. □

Proposition 5.10. *Let $R = \oplus_{i \geq 0} R_i$ be a p-torsion-free graded $W(k)$-algebra with an endomorphism φ on R which induces the Frobenius map on R/pR and $\varphi \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ sends $(R/p^n R)_i$ to $(R/p^n R)_{ip}$ for any $n \geq 0$ and $i \geq 0$. Assume that $\mathcal{X} := \mathrm{Proj}(R)$ is a smooth projective $W(k)$-scheme. Then the generic rank of the Frobenius lift φ on R is $[K(X)^{1/p} : K(X)]$, where $X := \mathcal{X}_{p=0}$ and in particular p-power.*

Proof. Recall that the Frobenius lift φ_R induces an endomorphism of schemes $\widetilde{F}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ and this is compatible with the Frobenius lift on $\mathrm{Spec}(W(k))$: φ_R induces a morphism of schemes $U \to \mathcal{X} = \mathrm{Proj}(R)$ from an open subscheme $U := \{\mathfrak{p} \in \mathcal{X} \mid \varphi_R^{-1}(\mathfrak{p}) \not\supseteq R_+\}$ in $\mathcal{X}$ [9, (13.2.4)]. Since U contains the special fiber $X := \mathcal{X} \times_{W(k)} k$, U should be the whole $\mathcal{X}$ and thus $\widetilde{F}_{\mathcal{X}}$ is well-defined.

Taking the special fiber, the Frobenius map on R/pR induces the Frobenius morphism $F_X : X \to X$ of a smooth k-scheme X (through the Frobenius map on k) and in particular this is a finite morphism. This says that the restriction of $\widetilde{F}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ to X has zero-dimensional fibers and this property extends to $\mathcal{X}$. Consequently, $\widetilde{F}_{\mathcal{X}}$ is a finite morphism [22, Tag 02OG] and thus $\widetilde{F}_{\mathcal{X}}$ is flat by miracle flatness [22, Tag 00R4]. In particular, $\widetilde{F}_{\mathcal{X}}$ is finite locally free by [9, Proposition 12.19]. By [9, Proposition 13.37(2)], we have $R_f \cong R_{(f)}[T, T^{-1}]$ for any homogeneous element $f \in R_1$. Since $\widetilde{F}_{\mathcal{X}}$ induces a finite free map $R_{(f)} \to R_{(\varphi(f))}$ between regular rings for some $f \in R_1$, the generic rank $\deg \varphi$ is the same as the rank of the finite free map $R_f \to R_{\varphi(f)}$ induced from φ. This is the same as the rank of the

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Frobenius map $(R/pR)_f \xrightarrow{F} (R/pR)_f$ since $D_+(f) \cap \mathcal{X}_{p=0} \neq \emptyset$. Therefore, the generic degree $\deg \varphi$ is the p-power $[K(X)^{1/p} : K(X)]$. □

6. Examples

We present some examples of perfectoid towers generated from prisms. While the known examples do not arise from mild singularities, such as (log-)regularity, the first term of the following examples is not necessarily log-regular. Before giving new examples, we reconstruct the previous examples.

6.1. *Previous examples of perfectoid towers*

These examples were calculated separately in [14], but now can be treated in a unified and simpler fashion using our first main theorem (Theorem 4.5). The first two examples are generalized in Proposition 6.9 later. Note that the first two examples are only examples of Noetherian perfectoid towers previously known in [14] and these are perfectoid towers arising from Cohen–Macaulay normal domains.

Example 6.1 (Regular rings; [14, Example 3.62(1)]). Any complete regular local ring R of dimension d with residue field k of characteristic $p > 0$ can be represented as A/I for some complete regular local prism (A, I) (see [16, Corollary 3.8]). In this case, $A \cong C(k)[|T_1, \dots, T_d|]$ and $I = (p - f)$, where $C := C(k)$ is the Cohen ring of k equipped with a δ-ring structure and $f \in (T_1, \dots, T_n)$ (see [16, Lemmas 2.6 and 5.1]). We denote the colimit $C^{1/p^i} := \operatorname{colim}\{C \xrightarrow{\varphi} C \xrightarrow{\varphi} \dots \xrightarrow{\varphi} C\}$ consisting of $i + 1$ terms. It gives extensions of integral domains $C \hookrightarrow C^{1/p} \hookrightarrow C^{1/p^2} \hookrightarrow$. Applying Theorem 4.5, we obtain a perfectoid tower

$$R \cong C[|T_1, \dots, T_d|]/(p - f) \xrightarrow{t_{1,I}} \dots \xrightarrow{t_{i,I}} C^{1/p^i}[|T_1^{1/p^i}, \dots, T_d^{1/p^i}|]/(p - f) \xrightarrow{t_{i+1,I}} \dots,$$

whose transition maps are injective and its tilt is isomorphic to the perfect tower

$$k[T_1, \dots, T_d] \hookrightarrow k^{1/p}[|T_1^{1/p}, \dots, T_d^{1/p}|] \hookrightarrow \cdots \hookrightarrow k^{1/p^i}[|T_1^{1/p^i}, \dots, T_d^{1/p^i}|] \hookrightarrow \cdots.$$

If $f = T_d$, the quotient $R \cong A/I$ is $C[|T_1, \dots, T_{d-1}|]$ and the above perfectoid tower is the same as the perfectoid tower generated from the δ-ring $C[|T_1, \dots, T_{d-1}|]$ (Corollary 5.5).

Example 6.2 (Local log-regular rings; [14, Section 3.6]). More generally, our construction covers the case of complete local log-regular rings: let C be the Cohen ring of a field k of positive characteristic p and let Q be a fine sharp saturated monoid. Fix a δ-ring structure of C as above. Then the complete Noetherian local domain $C[|Q|]$ has a δ-structure given by the Frobenius lift $e^q \mapsto (e^q)^p$ and $(C[|Q|], (p - f))$ becomes an orientable prism for any $f \in C[|Q|]$ which has no non-zero constant term (or $f = 0$) as above. By Kato's structure theorem, any complete local log-regular ring (R, Q, α) of residue characteristic p can be represented as $C[|Q|]/(p - f)$ for some $f \in C[|Q|]$ which has no non-zero constant term (see, e.g., [14, Theorem 2.22]).

Since $C[|Q|]/(p-f)$ is a complete local log-regular ring, $(C[|Q|], (p-f))$ is transversal and $k[|Q|]$ is p-root closed in $k[|Q|][1/\overline{f}]$. Then by Theorem 4.5, the tower

$$C[|Q|]/(p - f) \hookrightarrow C^{1/p}[|Q^{(1)}|]/(p - f) \hookrightarrow \cdots \hookrightarrow C^{1/p^i}[|Q^{(i)}|]/(p - f) \hookrightarrow \cdots$$

is a perfectoid tower arising from $(C[|Q|]/(p-f), (p))$ and those transition maps are injective. Its tilt is

$$k[|Q|] \hookrightarrow k[|Q^{(1)}|] \hookrightarrow \cdots,$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

since the Frobenius map on $k[|Q|]$ is compatible with the inclusion $k[|Q|] \hookrightarrow k[|Q^{(1)}|]$. The resulting perfectoid tower and its tilt are the same as those in [14, Proposition 3.58, Lemma 3.59, Theorem 3.61, and Example 3.62].

The above examples appear in commutative ring theory and the following examples are related to arithmetic geometry, in particular, prismatic theory.

Example 6.3 (Perfectoid rings; [7, Theorem 3.10]). Let R be a p-torsion-free perfectoid ring. Then there exists a unique transversal perfect prism $(A, (\xi))$ such that $R \cong A/(\xi)$. The assumption of Theorem 4.5 is satisfied because of the p-torsion-free property of A and perfectness of A/pA. Since the canonical map $c_0^i \colon A \to A^{1/p^i}$ is an isomorphism, we have $A^{1/p^i}/IA^{1/p^i} \cong A/I$. By Theorem 4.5, the following tower is a perfectoid tower:

$$R \cong A/I \xrightarrow{\mathrm{id}} A/I \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} A/I \xrightarrow{\mathrm{id}} \dots,$$

and its tilt is

$$R^\flat \cong A/pA \xrightarrow{F} A/pA \xrightarrow{F} \dots \xrightarrow{F} A/pA \xrightarrow{F} \dots.$$

This is the case of the perfectoid tower from (p-torsion-free) perfectoid rings [14, Example 3.53].

6.2. Examples from geometric Frobenius lifts

The next example is generated from a more geometric method, namely, the Frobenius lift on an Abelian variety. That tower is an example that the generic rank of those transition maps is p-power. More general theories of Frobenius lifts on smooth projective varieties and its relation to perfectoid towers are developed in [15], [17]. This is one of the examples of perfectoid towers arising from non-Cohen–Macaulay normal domains, which does not appear in [14].

Example 6.4. Let A be an ordinary Abelian variety over a perfect field k of characteristic p and L be an ample line bundle on A. Then we can take the canonical lift $\mathcal{A}$ and an ample line bundle $\mathcal{L}$ on $\mathcal{A}$ such that the ring of sections $R(\mathcal{A}, \mathcal{L}) := \oplus_{m \geq 0} H^0(\mathcal{A}, \mathcal{L}^{\otimes m})$ is a normal domain but not Cohen–Macaulay (as in [18, Lemma 4.11] and [6, Example 7.7]). Then this ring $R(\mathcal{A}, \mathcal{L})$ satisfies the condition of Corollary 5.5 and Proposition 5.10. So we have a perfectoid tower

$$R(\mathcal{A}, \mathcal{L}) \hookrightarrow R(\mathcal{A}, \mathcal{L})^{1/p}[p^{1/p}] \hookrightarrow \dots \hookrightarrow R(\mathcal{A}, \mathcal{L})^{1/p^i}[p^{1/p^i}] \hookrightarrow \dots$$

arising from $(R(\mathcal{A}, \mathcal{L}), (p))$ and the generic rank of transition maps are $p[K(A)^{1/p} : K(A)]$ which is p-power. Its tilt is

$$R(A, L)[|T|] \hookrightarrow R(A, L)^{1/p}[|T^{1/p}|] \hookrightarrow \dots \hookrightarrow R(A, L)^{1/p^i}[|T^{1/p^i}|] \hookrightarrow \dots.$$

In particular, we have an inequality

$$|H^i(\mathrm{Spec}(R(\mathcal{A}, \mathcal{L}))_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(\mathrm{Spec}(R(A, L)[|T|])_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|$$

by [14, Proposition 4.7].

6.3. Examples from affine semigroups

The next example is generated from an affine semigroup ring following Corollary 5.5. This is based on examples of local log-regular rings in Example 6.2. This construction gives a lot of examples of perfectoid towers arising from integral domains and therefore this is one way to construct perfectoid towers easily (but this is closely related to local log-regular rings).

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Proposition 6.5 (e.g., [20, Definition 7.1]). *Let $\mathbf{a}_1, \dots, \mathbf{a}_r$ be a set of elements of $\mathbb{Z}^n_{\geq 0}$ for some $n > 0$. Let H be a submonoid of $\mathbb{Z}^n_{\geq 0}$ generated by $\mathbf{a}_1, \dots, \mathbf{a}_r$. Then the affine semigroup ring $\mathbb{Z}_p[H]$ is a $\mathbb{Z}_p$-subalgebra of a polynomial ring $\mathbb{Z}_p[t_1, \dots, t_n]$ which is generated by $\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_r}$ as a $\mathbb{Z}_p$-algebra. This is a p-torsion-free finitely generated $\mathbb{Z}_p$-algebra and the formula $\mathbf{t}^h \mapsto \mathbf{t}^{ph}$ extends to a Frobenius-lift of the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[H]$. Then the $(p, \mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_r})$-adic completion[8] $\mathbb{Z}_p[|H|]$ of $\mathbb{Z}_p[H]$ satisfies the assumption of Corollary 5.5. So we have a perfectoid tower*

$$\mathbb{Z}_p[|H|] \hookrightarrow \mathbb{Z}_p[p^{1/p}][|H^{1/p}|] \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_p[p^{1/p^i}][|H^{1/p^i}|] \hookrightarrow \cdots$$

arising from $(\mathbb{Z}_p[|H|], (p))$, where H^{1/p^i} is the submonoid of $(1/p^i \cdot \mathbb{Z}_{\geq 0})^n$ generated by $1/p^i \cdot \mathbf{a}_1, \dots, 1/p^i \cdot \mathbf{a}_r$. Its tilt is

$$\mathbb{F}_p[|H|][|T|] \hookrightarrow \mathbb{F}_p[|H^{1/p}|][|T|] \hookrightarrow \cdots \hookrightarrow \mathbb{F}_p[|H^{1/p^i}|][|T|] \hookrightarrow \cdots,$$

where T is a new variable.

To analyze some ring properties of affine semigroup rings over $\mathbb{Z}_p[H]$ not only over a field, we record the following lemma.

Lemma 6.6. *Let H be a submonoid of $\mathbb{Z}^n_{\geq 0}$ generated by $\mathbf{a}_1, \dots, \mathbf{a}_r$ for some $n > 0$. Assume that H is simplicial, namely, the convex rational polyhedral cone spanned by $\mathbf{a}_1, \dots, \mathbf{a}_r$ in $\mathbb{Q}^n$ is spanned by r-elements in H, where r is* $\operatorname{rank}_{\mathbb{Z}} H\mathbb{Z}^n$.

1. *If $\mathbb{F}_\ell[H]$ is Cohen–Macaulay (resp. Gorenstein) for a prime ℓ, then $\mathbb{Z}_p[H]$ and $\mathbb{Z}_p[|H|]$ are Cohen–Macaulay (resp. Gorenstein) for any prime p.*
2. *If $\mathbb{F}_\ell[H]$ is normal for a prime ℓ, then $\mathbb{Z}_p[H]$ and $\mathbb{Z}_p[|H|]$ are normal and Cohen–Macaulay for any prime p. Moreover, $\mathbb{Z}_p[|H|]$ with $H \hookrightarrow \mathbb{Z}_p[|H|]$ is a local log-regular ring.*

In particular, such properties can be tested in the case of $\ell = 2$ only.

Proof. (1): Since Cohen–Macaulayness and Gorensteinness are stable under the quotient by a non-zero-divisor and the completion, it is enough to check such properties for the affine semigroup ring $\mathbb{F}_p[H]$ over $\mathbb{F}_p$. By [10, Theorems (1) and (2)], such properties only depend on the simplicial monoid H and this proves (1).

(2): By [20, Proposition 7.25], the normality of an affine semigroup ring $k[H]$ over a field k depends only on the monoid H. So the assumption implies that $\mathbb{Q}_p[H]$ is normal (in other words, the monoid H is normal). We know that the polynomial ring $\mathbb{Z}_p[\underline{t}]$ is integrally closed in $\mathbb{Q}_p[\underline{t}]$ and the equation $\mathbb{Z}_p[H] = \mathbb{Z}_p[\underline{t}] \cap \mathbb{Q}_p[H]$ holds. The normality of $\mathbb{Q}_p[H]$ leads to the normality of $\mathbb{Z}_p[H]$. Moreover, in this case, the normality of $\mathbb{Q}_p[H]$ is stable under the completion due to [23, Théorème 2]. So a similar argument shows that $\mathbb{Z}_p[|H|]$ is normal.

Using [12, Theorem 1] (see also [20, Corollary 13.43]), the normality of H implies that $\mathbb{Z}_p[H]$ and $\mathbb{Z}_p[|H|]$ are Cohen–Macaulay. The log-regularity follows from the definition of local log-regular rings: Normal affine semigroup H is fine, sharp, and saturated. □

By using this Lemma 6.6 and a computer algebra system such as [11], we can give the following examples: The first one is non-Cohen–Macaulay and the second one is Cohen–Macaulay but not normal.

Example 6.7. Take an affine semigroup ring $\mathbb{Z}_p[s, st, st^3, st^4]$ whose Frobenius lift is induced from $s \mapsto s^p, t \mapsto t^p$. This is a non-Cohen–Macaulay integral domain of dimension 3 for any prime p and so is the completion $\mathbb{Z}_p[|s, st, st^3, st^4|]$ with respect to the prime ideal (s, st, st^3, st^4). By Proposition 6.5,

[8] See [20, Lemma 8.15] for an explicit representation of $\mathbb{Z}_p[|H|]$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

we have a perfectoid tower

$$\mathbb{Z}_p[|s,st,st^3,st^4|] \hookrightarrow \mathbb{Z}_p[p^{1/p}][|s^{1/p},s^{1/p}t^{1/p},s^{1/p}t^{3/p},s^{1/p}t^{4/p}|] \hookrightarrow \\ \cdots \hookrightarrow \mathbb{Z}_p[p^{1/p^i}][|s^{1/p^i},s^{1/p^i}t^{1/p^i},s^{1/p^i}t^{3/p^i},s^{1/p^i}t^{4/p^i}|] \hookrightarrow \cdots$$

arising from $(\mathbb{Z}_p[|s,st,st^3,st^4|],(p))$ whose first term $\mathbb{Z}_p[|s,st,st^3,st^4|]$ is a non-Cohen–Macaulay and non-normal complete local domain of dimension 3.

Example 6.8. Take an affine semigroup ring $\mathbb{Z}_p[s^2,s^3]$ whose Frobenius lift is induced from $s \mapsto s^p$. This is a Cohen–Macaulay non-normal domain of dimension 2 for any prime p and so is the completion $\mathbb{Z}_p[|s^2,s^3|]$ with respect to the prime ideal (s^2,s^3). By Proposition 6.5, we have a perfectoid tower

$$\mathbb{Z}_p[|s^2,s^3|] \hookrightarrow \mathbb{Z}_p[p^{1/p}][|s^{2/p},s^{3/p}|] \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_p[p^{1/p^i}][|s^{2/p^i},s^{3/p^i}|] \hookrightarrow \cdots$$

arising from $(\mathbb{Z}_p[|s^2,s^3|],(p))$ whose first term $\mathbb{Z}_p[|s^2,s^3|]$ is a non-normal local complete intersection domain of dimension 2.

6.4. Examples from δ-stable ideals

Next, we give examples of Noetherian perfectoid towers arising from δ-rings by using δ-stable ideals.

First one is a tower of rings generated from a δ-stable ideal which generalizes examples arising from regular and log-regular rings (Examples 6.1 and 6.2). Especially, the completion of any Stanley–Reisner ring has a perfectoid tower. This is based on an example (Example 6.12) taught to the author by Shinnosuke Ishiro. One advantage of this construction is that the conditions are easy to check by hand (or using computer algebra systems) and therefore this gives a practical way to construct (a little bit complicated) perfectoid towers.

Proposition 6.9 (δ-stable ideals). *Let k be a perfect field of characteristic p and set $W \coloneqq W(k)$[9] with the unique Frobenius lift. Let $W[\underline{T}] \coloneqq W[T_1,\dots,T_n]$ be a polynomial ring over W and let J be an ideal of $W[\underline{T}]$ which is contained in $(p,T_1,\dots,T_n)$. Assume that there exists a δ-structure on $W[\underline{T}]$ such that compatible with the Frobenius lift on W and $\delta(J) \subseteq J$.*[10] *Then $W[\underline{T}]/J$ itself, its T-adic completion $W[|\underline{T}|]/J$, and its localization $W[\underline{T}]_{(p,\underline{T})}/J$ with respect to the maximal ideal $(p,\underline{T}) \subseteq W[\underline{T}]$ have unique δ-$W[\underline{T}]$-algebra structures [7, Lemmas 2.9 and 2.15]. Take a distinguished element d in the δ-ring $W[\underline{T}]$ and fix a generator $J=(f_1,\dots,f_r)$. Then we have the following:*

1. *If p,d is a regular sequence on $W[\underline{T}]/J$ and the pth power map $W[\underline{T}]/(J,p,d) \xrightarrow{a\mapsto a^p} W[\underline{T}]/(J,p,d^p)$ is injective, then there exists a perfectoid tower arising from $(W[|\underline{T}|]/(J,d),(p))$ with injective transition maps and its tilt is isomorphic to the perfect tower arising from $(k[|\underline{T}|]/J,(p))$.*
2. *If $W[\underline{T}]/J$ is p-torsion-free and $W[\underline{T}]/(p,J)$ is reduced, then there exists a perfectoid tower arising from $(W[|\underline{T}|]/J,(p))$ (resp. $(W[\underline{T}]_{(p,\underline{T})}/J,(p))$) with injective transition maps and their tilts are isomorphic to the perfect tower arising from $(k[|\underline{T},T'|]/J,(p))$, where T' is a new variable. If moreover $W[|\underline{T}|]/J$ (resp. $W[\underline{T}]_{(p,\underline{T})}/J$) is a normal domain and $W[|\underline{T}|]/(p,J)$ (resp. $W[\underline{T}]_{(\underline{T})}/(p,J)$) is an integral domain, then the generic extension of those transition maps has p-power degree.*

Proof. In (1), we have a bounded prism $(W[|\underline{T}|]/J,(d))$. Taking the $(\underline{T})$-adic completion, p,d is also a regular sequence on $W[|\underline{T}|]/J$. To apply Theorem 4.5, we need to show that $W[|\underline{T}|]/(p,J)$ is p-root closed in $W[|\underline{T}|]/(p,J)[1/f]$. As in the proof of Theorem 4.5, this

[9] Even if $W=\mathbb{Z}$, the conclusions (1) and (2) also hold for the $(p,\underline{T})$-adic completion $\mathbb{Z}_p[|\underline{T}|]/J$.

[10] This δ-structure is not necessarily defined as $\delta(T_i)=0$. If $W[\underline{T}]/J$ is p-torsion-free, the existence of such a δ-structure on $W[\underline{T}]$ is equivalent to the existence of a Frobenius lift on $W[\underline{T}]/J$. So this relates the Frobenius liftability of a singularity in positive characteristic as mentioned in Remark 5.6.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

is equivalent to showing that the pth power map $W[|\underline{T}|]/(J,p,d) \xrightarrow{a\mapsto a^p} W[|\underline{T}|]/(J,p,d^p)$ is injective. This follows from the assumption and the faithfully flat property of the $(\underline{T})$-adic completion.

Next, we show (2). As above, $W[|\underline{T}|]/J$ (resp. $W[\underline{T}]_{(p,\underline{T})}/J$) is p-torsion-free by taking the $(\underline{T})$-adic completion (resp. (p)-Zariskization). By using analytically unramified property of finitely generated k-algebras, $W[|\underline{T}|]/(p,J)$ is reduced (see, e.g., [21, Theorem 4.6.3, Proposition 9.1.3, and Theorem 9.2.2]). Also, the p-Zariskization $(1+(p))^{-1}W[\underline{T}]/(p,J) \cong W[\underline{T}]/(p,J)$ is reduced. By Corollary 5.5, we have a perfectoid tower arising from $(W[|\underline{T}|]/J,(p))$ (resp. $(W[\underline{T}]_{(p,\underline{T})}/J,(p))$) with injective transition maps. If we assume that $W[|\underline{T}|]/J$ (resp. $W[\underline{T}]_{(p,\underline{T})}/J$) is a normal domain and $W[|\underline{T}|]/(p,J)$ (resp. $W[\underline{T}]_{(p,\underline{T})}/(p,J)$) is an integral domain, then the generic extension of those transition maps has p-power degree. □

Corollary 6.10. *Keep the notation of Proposition 6.9 and set $f_k^{1/p^i} \in W[T^{1/p^i}]$ be the corresponding polynomial of f_k along the isomorphism $W[T] \xrightarrow{\cong} W[T^{1/p^i}]$ which sends T_j to T_j^{1/p^i} and $a \in W$ to $\varphi_W(a) \in W$ by the Frobenius lift φ_W on W. If $\delta(T_i)=0$, then we also have the following:*

(1') *The distinguished element d is written by $p-f$ for some $f \in (T_1,\dots,T_n)$. If p,f is a regular sequence on $W[\underline{T}]/J$ and the pth power map $W[\underline{T}]/(J,p,f) \xrightarrow{a\mapsto a^p} W[\underline{T}]/(J,p,f^p)$ is injective, then the tower*

$$W[|\underline{T}|]/(f_1,\dots,f_r,p-f) \hookrightarrow \cdots \hookrightarrow W[|\underline{T}^{1/p^i}|]/(f_1^{1/p^i},\dots,f_r^{1/p^i},p-f) \hookrightarrow \tag{6.1}$$

is a perfectoid tower arising from $(W[|\underline{T}|]/(J,p-f),(p))$ and its tilt is isomorphic to the perfect tower

$$k[|\underline{T}|]/(f_1,\dots,f_r) \hookrightarrow \cdots \hookrightarrow k[|\underline{T}^{1/p^i}|]/(f_1^{1/p^i},\dots,f_r^{1/p^i}) \hookrightarrow \cdots \tag{6.2}$$

arising from $(k[|\underline{T}|]/J,(p))$.

(2') *If $W[\underline{T}]/J$ is p-torsion-free and $W[\underline{T}]/(p,J)$ is reduced, then the towers*

$$W[|\underline{T}|]/(f_1,\dots,f_r) \hookrightarrow \cdots \hookrightarrow W[|\underline{T}^{1/p^i}|][p^{1/p^i}]/(f_1^{1/p^i},\dots,f_r^{1/p^i}) \hookrightarrow \cdots \text{ and} \tag{6.3}$$

$$W[\underline{T}]_{(p,\underline{T})}/(f_1,\dots,f_r) \hookrightarrow \cdots \hookrightarrow W[\underline{T}^{1/p^i}]_{(p,\underline{T}^{1/p^i})}[p^{1/p^i}]/(f_1^{1/p^i},\dots,f_r^{1/p^i}) \hookrightarrow \cdots \tag{6.4}$$

are perfectoid towers arising from $(W[|\underline{T}|]/J,(p))$ and $W[\underline{T}]_{(p,\underline{T})}/(J,(p))$, respectively, and their tilts are isomorphic to the perfect tower

$$k[|\underline{T},T'|]/(f_1,\dots,f_r) \hookrightarrow \cdots \hookrightarrow k[|\underline{T}^{1/p^i},T'^{1/p^i}|]/(f_1^{1/p^i},\dots,f_r^{1/p^i}) \hookrightarrow \cdots \tag{6.5}$$

arising from $(k[|\underline{T},T'|]/J,(p))$, where T' is a new variable.

Proof. In (1'), any distinguished element of a complete Noetherian local δ-ring $W[|\underline{T}|]/J$ with $\delta(T_i)=0$ can be written as $p-f$ for some $f \in (T_1,\dots,T_n)$ up to unit by the proof of [16, Lemma 5.1]. Other assertions follow from (1) and (2) in Proposition 6.9. □

One typical example of δ-stable ideals is the ideal generated by a square-free monomial. First, we give an example based on (6.1).

Example 6.11. Set a δ-structure on $\mathbb{Z}_p[|X,Y,Z,W|]$ by $\delta(X)=\delta(Y)=\delta(Z)=\delta(W)=0$. Take a δ-stable ideal (XY) in $\mathbb{Z}_p[|X,Y,Z,W|]$ and a distinguished element $p-ZW$ as above (Corollary 6.10 (a')). Then p,ZW is a regular sequence on $\mathbb{Z}_p[X,Y,Z,W]/(XY)$ and the pth power map $\mathbb{F}_p[X,Y,Z,W]/(XY,ZW) \xrightarrow{a\mapsto a^p} \mathbb{F}_p[X,Y,Z,W]/(XY,Z^pW^p)$ is injective. Therefore,

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Corollary 6.10 (a') tells us that the tower

$$\mathbb{Z}_p[|X,Y,Z,W|]/(XY,p-ZW) \hookrightarrow \mathbb{Z}_p[|X^{1/p},Y^{1/p},Z^{1/p},W^{1/p}|]/(X^{1/p}Y^{1/p},p-ZW) \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_p[|X^{1/p^i},Y^{1/p^i},Z^{1/p^i},W^{1/p^i}|]/(X^{1/p^i}Y^{1/p^i},p-ZW) \hookrightarrow \cdots$$

is a perfectoid tower arising from $(\mathbb{Z}_p[|X,Y,Z,W|]/(XY,p-ZW),(p))$. The first term is a ramified complete intersection but not an integral domain.

Secondly, we give an example based on (6.3).

Example 6.12 (Square-free monomial case). Set a δ-structure on $\mathbb{Z}_p[|X,Y,Z|]$ by $\delta(X)=\delta(Y)=\delta(Z)=0$ and take a δ-stable ideal (XY,YZ) in $\mathbb{Z}_p[|X,Y,Z|]$. Then we can show that the tower

$$\mathbb{Z}_p[|X,Y,Z|]/(XY,YZ) \hookrightarrow \mathbb{Z}_p[|p^{1/p},X^{1/p},Y^{1/p},Z^{1/p}|]/(X^{1/p}Y^{1/p},Y^{1/p}Z^{1/p}) \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_p[|p^{1/p^i},X^{1/p^i},Y^{1/p^i},Z^{1/p^i}|]/(X^{1/p^i}Y^{1/p^i},Y^{1/p^i}Z^{1/p^i}) \hookrightarrow \cdots$$

is a perfectoid tower arising from $(\mathbb{Z}_p[|X,Y,Z|]/(XY,YZ),(p))$ by Corollary 6.10 (b'). This example gives a perfectoid tower whose first term $\mathbb{Z}_p[|X,Y,Z|]/(XY,YZ)$ is not Cohen–Macaulay. The same argument works for any quotient of $\mathbb{Z}_p[|X_1,\dots,X_n|]$ by square-free monomial ideals, that is, the $(\underline{T})$-completion of any Stanley–Reisner ring $\mathbb{Z}_p[\underline{T}]/I_\Delta$ over $\mathbb{Z}_p$ for any simplicial complex Δ has a perfectoid tower arising from $(\mathbb{Z}_p[|\underline{T}|]/I_\Delta,(p))$ (for the notion of Stanley–Reisner rings and simplicial complexes, see, e.g., [8]).

6.5. Examples from δ-stabilization of ideals

Any ideal J can be extended to the δ-stabilization J_δ of J, the universal δ-stable ideal containing J, as in [7, Example 2.10]. At least $p=2$ or $p=3$, we give more complicated examples (Corollary 6.20) than above by using the δ-stabilization of ideals. For convenience, we define the notion of δ-height of an ideal of δ-rings.

Definition 6.13. Let R be a δ-ring and let J be an ideal of R. The *δ-height* of J is defined as

$$\mathrm{ht}^\delta(J) := \inf\{n\geq 0 \mid \delta^{n+1}(J)\subseteq (\cup_{0\leq i\leq n}\delta^i(J))R\},$$

where $(\cup_{0\leq i\leq n}\delta^i(J))R$ is the ideal of R generated by $\cup_{0\leq i\leq n}\delta^i(J)$. We assume that the value is ∞ if the sequence of ideals $\{(\cup_{0\leq i\leq n}\delta^i(J))R\}_{n\geq 0}$ does not stabilize although if R is Noetherian, this value is finite. If it is finite, then the δ-stabilization I_δ is the same as the ideal generated by $\cup_{0\leq i\leq \mathrm{ht}^\delta(J)}\delta^i(J)$. We denote by $\mathrm{ht}^\delta(f)$ for an element f of R the δ-height of the ideal (f).

In the principal ideal case, there is an upper bound of the δ-height of an element.

Lemma 6.14. *Let R be a δ-ring and let f be an element of R. Take a ϕ-monomial decomposition $f=\sum_{i=1}^m k_iM_i$ of f with $k_i\in\mathbb{Z}$ and ϕ-monomials $M_i\in R$ in the sense of [19, Definition 3.12], namely, $\varphi(M_i)=M_i^p$.*[11] *Then we have $\mathrm{ht}^\delta(f)\leq \mathrm{ht}^\delta(k_1t_1+\cdots+k_mt_m)$, where $t_i\in\mathbb{Z}[t_1,\dots,t_m]$ is the variable of the δ-ring $\mathbb{Z}[t_1,\dots,t_m]$ with $\delta(t_i)=0$.*

Proof. Set $\overline{f} := k_1t_1+\cdots+k_mt_m$ in $\mathbb{Z}[t_1,\dots,t_m]$. Since $\delta(M_i)=0$ holds for any ϕ-monomials M_i, we can take a map of δ-rings $\mathbb{Z}[t_1,\dots,t_m]\to R$ sending t_i to M_i. If $\delta^{n+1}(\overline{f})$ belongs to the ideal $(\overline{f},\delta(\overline{f}),\dots,\delta^n(\overline{f}))\mathbb{Z}[t_1,\dots,t_m]$, then we have $\delta^{n+1}(f)\in(f,\delta(f),\dots,\delta^n(f))R$ and thus $\mathrm{ht}^\delta(f)\leq n$. □

[11] For example, any element f of a polynomial ring $\mathbb{Z}[X_1,\dots,X_n]$ has a ϕ-monomial decomposition $f=\sum_{i=1}^m k_iM_i$ with ϕ-monomials M_i because any polynomial over $\mathbb{Z}$ can be written as a sum of monomials with coefficients 1 or -1.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

By using a computer algebra system such as Macaulay2 [11], we can compute the δ-stabilization of a given ideal in a polynomial ring and check the sufficient conditions in Corollary 6.10. Based on observations of computer calculations, we give a general construction of perfectoid towers from δ-stabilization in Corollary 6.20 when $p = 2, 3$. Hereafter, we will prepare its proof.

Lemma 6.15. *Let m be an integer greater than or equal to 3. Set a δ-structure on $\mathbb{Z}_p[|X_1, \dots, X_m|]$ by $\delta(X_i) = 0$. Take $f := X_1^{n_1} + \cdots + X_m^{n_m}$ for $n_i \geq 1$.*

1. *In $\mathbb{Z}_p[X_1, \dots, X_m]/(f)$, the element $\delta(f)$ is the same as a non-zero polynomial*

$$\beta^{(m)} = \beta^{(m)}(X_2^{n_2}, \dots, X_m^{n_m}) = \beta_p^{(m)} X_2^{n_2 p} + \beta_{p-1}^{(m)} X_2^{n_2(p-1)} + \beta_{p-2}^{(m)} X_2^{n_2(p-2)} + \cdots + \beta_0^{(m)} \tag{6.6}$$

   for some polynomials $\beta_i^{(m)} = \beta_i^{(m)}(X_2^{n_2}, \dots, X_m^{n_m})$ in $\mathbb{Z}_p[X_3, \dots, X_m]$.
2. *Explicitly, $\beta_p^{(m)} = (1 + (-1)^p)/p$ and $\beta_{p-1}^{(m)} = (-1)^p(X_3^{n_3} + \cdots + X_m^{n_m})$ hold. The last term $\beta_0^{(m)} = \beta_0^{(m)}(X_2^{n_2}, \dots, X_m^{n_m})$ is the same as $\beta^{(m-1)}(X_3^{n_3}, \dots, X_m^{n_m})$ for any $m \geq 4$.*
3. *Let Λ be $\mathbb{Z}_p$ or $\mathbb{F}_p$ and let $g = g_1 f + g_2 \beta^{(m)}$ be a polynomial in $\Lambda[X_1, \dots, X_m]$ with $g_1, g_2 \in \Lambda[X_1, \dots, X_m]$. If the degree of X_1 in g is strictly less than n_1, then g_1 is contained in the ideal $(\beta^{(m)})$ in $\Lambda[X_1, \dots, X_m]$.*

Proof. Under modulo $f = X_1^{n_1} + \cdots + X_m^{n_m}$, we have

$$\begin{aligned}\delta(f) &\equiv -\sum_{\substack{0 \leq e_1, \dots, e_m \leq p-1 \\ e_1 + \cdots + e_m = p}} \frac{(p-1)!}{e_1! \dots e_m!} (-1)^{e_1} (X_2^{n_2} + \cdots + X_m^{n_m})^{e_1} X_2^{n_2 e_2} \dots X_m^{n_m e_m} \\ &= -\sum_{\substack{0 \leq f_2, \dots, f_m, e_2, \dots, e_m \leq p-1 \\ f_2 + \cdots + f_m + e_2 \cdots + e_m = p \\ f_2 + \cdots + f_m \leq p-1}} \frac{(p-1)!}{f_2! \dots f_m! e_2! \dots e_m!} (-1)^{f_2 + \cdots + f_m} X_2^{n_2(e_2+f_2)} \dots X_m^{n_m(e_m+f_m)}, \end{aligned} \tag{6.7}$$

which is an element of $\mathbb{Z}[X_2, \dots, X_m]$. We will compute the coefficient of $X_2^{pn_2}$ and $X_2^{n_2(p-1)}$ in the summation (6.7).

In the case of $e_2 + f_2 = p$, the limitation of the sum in (6.7) implies that $e_3 = \cdots = e_m = 0$, $f_3 = \cdots = f_m = 0$, and $e_2 > 0$. So the coefficient of $X_2^{n_2 p}$ is

$$\begin{aligned}\sum_{e_2=1}^{p-1} \frac{(p-1)!}{(p-e_2)! e_2!} (-1)^{p-e_2} &= \frac{1}{p} \left(\sum_{e_2=0}^{p} \frac{p!}{(p-e_2)! e_2!} (-1)^{p-e_2} - \left((-1)^{p-0} + (-1)^{p-p} \right) \right) \\ &= -\frac{(-1)^p + 1}{p} \end{aligned}$$

and this is -1 if $p = 2$ and 0 if $p > 2$.

Next one is $e_2 + f_2 = p - 1$ and we can assume that $e_i + f_i = 1$ for some $i > 2$. Then $e_j = f_j = 0$ for $j \neq 2, i$ and there are two cases: $e_i = 0$ and $f_i = 1$, or $e_i = 1$ and $f_i = 0$. In the former case, the coefficient of $X_2^{n_2(p-1)} X_i^{n_i}$ is

$$\begin{aligned}\sum_{e_2=1}^{p-1} \frac{(p-1)!}{(p-1-e_2)! 1! e_2!} (-1)^{(p-1-e_2)+1} &= -\left(\sum_{e_2=0}^{p-1} \frac{(p-1)!}{(p-1-e_2)! e_2!} (-1)^{p-1-e_2} - (-1)^{p-1} \right) \\ &= (-1)^{p-1}. \end{aligned}$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

In the latter case, the coefficient of $X_2^{n_2(p-1)}X_i^{n_i}$ is

$$\sum_{e_2=0}^{p-1}\frac{(p-1)!}{(p-1-e_2)!e_2!1!}(-1)^{p-1-e_2}=0.$$

Therefore, the terms in (6.7) of degree $n_2(p-1)$ in X_2 can be summed up as $(-1)^pX_2^{n_2(p-1)}(X_3^{n_3}+\cdots+X_m^{n_m})$. Consequently, the summation (6.7) can be written as

$$\beta^{(m)}:=\frac{1+(-1)^p}{p}X_2^{n_2p}+(-1)^pX_2^{n_2(p-1)}(X_3^{n_3}+\cdots+X_m^{n_m})+\beta_{p-2}^{(m)}X_2^{n_2(p-2)}+\cdots+\beta_0^{(m)} \tag{6.8}$$

for some polynomials $\beta_i^{(m)}$ in $X_3,\dots,X_m$ and set $\beta_p^{(m)}:=(1+(-1)^p)/p$ and $\beta_{p-1}^{(m)}:=(-1)^p(X_3^{n_3}+\cdots+X_m^{n_m})$. In particular, $\beta_{p-1}^{(m)}$ is a non-zero element and so is $\beta^{(m)}$. The last term $\beta_0^{(m)}$ is the case that $e_2+f_2=0$ in (6.7) and this is the same as $\beta^{(m-1)}(X_3^{n_3},\dots,X_m^{n_m})$ for any $m\geq 4$. This shows the assertions (1) and (2).

We show the next assertion (3). Let $g=g_1f+g_2\beta^{(m)}$ be a polynomial in $\Lambda[X_1,\dots,X_m]$ with $g_1,g_2\in\Lambda[X_1,\dots,X_m]$ and the degree of X_1 in g is strictly less than n_1. Write the polynomials $g_1=a_NX_1^N+a_{N-1}X_1^{N-1}+\cdots+a_1X_1+a_0$ and $g_2=b_MX_1^M+b_{M-1}X_1^{M-1}+\cdots+b_1X_1+b_0$ such that a_i and b_j are in $\Lambda[X_2,\dots,X_m]$ and a_N and b_M are non-zero. Since g has the degree of X_1 strictly less than n_1, we can deduce that $N+n_1=M$. For simplicity, set $\alpha:=X_2^{n_2}+\cdots+X_m^{n_m}$ and $\beta:=\beta^{(m)}$. So we have $g=g_1(X_1^{n_1}+\alpha)+g_2\beta$, namely,

$$\begin{aligned}g=&(a_NX_1^{N+n_1}+\cdots+a_{N-n_1}X_1^{N-n_1+n_1}+\cdots+a_0X_1^{n_1})+(\alpha a_NX_1^N+\cdots+\alpha a_0)+\\&+(\beta b_MX_1^M+\cdots+\beta b_NX_1^N+\cdots+\beta b_0).\end{aligned} \tag{6.9}$$

It suffices to show that a_i are divisible by β in $\Lambda[X_2,\dots,X_m]$ for all $0\leq i\leq N$. If $n_1>N$, the above equation shows

$$a_iX_1^{i+n_1}+\beta b_{i+n_1}X_1^{i+n_1}=0 \tag{6.10}$$

for all $0\leq i\leq N$ since the degree of X_1 in g is strictly less than n_1. This implies that a_i is divisible by β for all $0\leq i\leq N$.

If $n_1\leq N$, the same vanishing (6.10) holds for any $N-n_1+1\leq i\leq N$ and thus a_i is divisible by β for all $N-n_1+1\leq i\leq N$. The equation (6.9) gives

$$a_{i-n_1}X_1^i+\alpha a_iX_1^i+\beta b_iX_1^i=0 \tag{6.11}$$

for any $n_1\leq i\leq N$ because of $n_1\leq N$. Whether $N-n_1+1$ is smaller than or greater than n_1, setting a_i as 0 for $i<0$, we have the same equation (6.11) for any $N-n_1+1\leq i\leq N$. Then the divisibility of a_i shows that a_{i-n_1} is divisible by β for any $N-n_1+1\leq i\leq N$, that is, a_i is divisible by β for any $N-2n_1+1\leq i\leq N-n_1$. Using (6.11) again, a_{i-n_1} is divisible by β for any $N-2n_1+1\leq i\leq N-n_1$, that is, a_i is divisible by β for any $N-3n_1+1\leq i\leq N-2n_1$. Repeating this argument, a_i is divisible by β for all $0\leq i\leq N$. Consequently, in both cases, g_1 is contained in (β) in $\Lambda[X_1,\dots,X_m]$. □

Example 6.16. In Lemma 6.18 later, we use $\beta^{(p+1)}=\beta^{(p+1)}(X_2^{n_2},\dots,X_{p+1}^{n_{p+1}})$ for prime p. We give explicit representations of $\beta^{(p+1)}$ for $p=2$ and $p=3$:

$$\begin{aligned}\beta^{(2+1)}&=X_2^{2n_2}+X_2^{n_2}X_3^{n_3}+X_3^{2n_3},\\\beta^{(3+1)}&=(X_2^{n_2}+X_3^{n_3})(X_3^{n_3}+X_4^{n_4})(X_4^{n_4}+X_2^{n_2}).\end{aligned}$$

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

This calculation follows from $\beta^{(p+1)}(t_2,\dots,t_{p+1})$ and substitution $t_i = X_i^{n_i}$ for $i = 2,\dots,p+1$ as in Lemma 6.14.

Corollary 6.17. *Keep the notation in Lemma 6.15.*

1. *The sequence* $p, f, \delta(f)$ *is a regular sequence on* $\mathbb{Z}_p[|X_1,\dots,X_m|]$ *and especially* $\mathrm{ht}^{\delta}(f) \geq 1$.
2. *The initial ideal of* $(f,\delta(f))$ *in* $\Lambda[X_1,\dots,X_m]$ *with respect to the lexicographic order* $X_1 > X_2 > \cdots > X_m$ *is* $(X_1^{n_1}, X_2^{2n_2})$ *if* $p = 2$ *and* $(X_1^{n_1}, X_2^{n_2(p-1)}X_3)$ *if* $p > 2$.

Proof. For the first statement, it is enough to show that f is a non-zero-divisor on $\mathbb{F}_p[X_1,\dots,X_m]/(\beta^{(m)})$ since $\delta(f)$ and $\beta^{(m)}$ are equivalent under modulo f by Lemma 6.15(1). If a polynomial g_1 in $\mathbb{F}_p[X_1,\dots,X_m]$ satisfies $g_1 f \in (\beta^{(m)})$, then g_1 is contained in $(\beta^{(m)})$ by using Lemma 6.15(3).

For the initial ideal, take $g \in (f,\delta(f)) = (f,\beta^{(m)})$. Dividing by f, we may assume that the degree of X_1 in g is strictly less than n_1. By Lemma 6.15(3) again, g is contained in $(\beta^{(m)})$ in $\Lambda[X_1,\dots,X_m]$ and thus its initial term is generated by the initial term of $\beta^{(m)}$, which is $X_2^{2n_2}$ if $p = 2$ and $X_2^{n_2(p-1)}X_3$ if $p > 2$. □

Lemma 6.18. *Set a* δ*-structure on* $\mathbb{Z}_p[|X_1,\dots,X_{p+1}|]$ *by* $\delta(X_i) = 0$. *Take* $f := X_1^{n_1} + \cdots + X_{p+1}^{n_{p+1}}$ *and* $\beta := \beta^{(p+1)}(X_2^{n_2},\dots,X_{p+1}^{n_{p+1}})$ *in* $\mathbb{Z}_p[X_1,\dots,X_{p+1}]$ *for* $n_i \geq 1$. *If* n_1 *and* n_2 *are prime to* p *and* $\mathbb{F}_p[X_2,\dots,X_{p+1}]/(\beta)$ *is reduced, then so is* $\mathbb{F}_p[|X_1,\dots,X_{p+1}|]/(f,\delta(f))$.

Proof. Write $\alpha := X_2^{n_2} + \cdots + X_m^{n_m}$ and $\beta := \beta^{(p+1)}(X_2^{n_2},\dots,X_{p+1}^{n_{p+1}})$. We first show that $\mathbb{F}_p[X_1,\dots,X_{p+1}]/(f,\delta(f))$ is reduced. It is enough to show that $(X_1^{n_1}+\alpha,\beta) \cap \mathbb{F}_p[X_1^p,\dots,X_{p+1}^p]$ is contained in $((X_1^{n_1}+\alpha)^p,\beta^p)$ in $\mathbb{F}_p[X_1,\dots,X_{p+1}]$. Take $g \in (X_1^{n_1}+\alpha,\beta) \cap \mathbb{F}_p[X_1^p,\dots,X_{p+1}^p]$. Dividing by $(X_1^{n_1}+\alpha)^p$ in $\mathbb{F}_p[X_1^p,\dots,X_{p+1}^p]$, we may assume that the degree of X_1 in g is strictly less than pn_1.

It is enough to show that g is contained in (β^p) in $\mathbb{F}_p[X_1,\dots,X_{p+1}]$. Since g belongs to $(X_1^{n_1}+\alpha,\beta)$, dividing g by $X_1^{n_1}+\alpha$ and β and Corollary 6.17(2) yield a representation $g = g_1(X_1^{n_1}+\alpha) + g_2\beta$, where $g_1 = a_N X_1^N + a_{N-1}X_1^{N-1} + \cdots + a_1X_1 + a_0$ and $g_2 = b_M X_1^M + b_{M-1}X_1^{M-1} + \cdots + b_1X_1 + b_0$ are polynomials such that a_i and b_j are in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$, a_N and b_M are non-zero, and $N < n_1(p-1)$ and $M < n_1$ hold.

It is enough to show that $a_i \in (\beta^p)$ and $b_i \in (\beta^{p-1})$. Considering the same equation (6.9) and the condition $g \in \mathbb{F}_p[X_1^p,\dots,X_{p+1}^p]$, the initial degree $N+n_1$ is in $p\mathbb{Z}$ and then N is prime to p since n_1 is so. Similarly, because of $M < n_1$, the term $a_k X_1^{k+n_1}$ is zero if $k+n_1$ is prime to p and is contained in $\mathbb{F}_p[X_1^p,\dots,X_{p+1}^p]$ if $k+n_1$ is divisible by p for any $n_1 \leq k+n_1 \leq N+n_1$. Namely, we have

$$a_k = \begin{cases} 0 & \text{if } k \not\equiv -n_1 \pmod p, \\ \in \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p] & \text{if } k \equiv -n_1 \pmod p \end{cases}$$

for any $0 \leq k \leq N$. If $N > M$, the term $\alpha a_N X_1^N$ is the only term whose degree of X_1 is N which is prime to p and it should be zero, but this contradicts $\alpha a_N \neq 0$. So we have $n_1 > M \geq N$ and the same argument for b_M shows that the only cases are $M = N$ or $M \in p\mathbb{Z}$. In both case, similarly as above, considering $\alpha a_k X_1^k + \beta b_k X_1^k$ and $\beta b_k X_1^k$, we have

$$\alpha a_k + \beta b_k = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod p \text{ and } 0 \leq k \leq N, \\ \in \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p] & \text{if } k \equiv 0 \pmod p \text{ and } 0 \leq k \leq N \end{cases}$$

$$\beta b_k = \begin{cases} 0 & \text{if } k \not\equiv 0 \pmod p \text{ and } N < k \leq M, \\ \in \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p] & \text{if } k \equiv 0 \pmod p \text{ and } N < k \leq M. \end{cases}$$

Clearly, if $N < k \leq M$, then we have $\beta b_k \in \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p]$. If $k \not\equiv -n_1 \bmod p$ and $0 \leq k \leq N$, then $\beta b_k \in \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p]$ also holds.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

Our assumption says that $\mathbb{F}_p[X_2,\dots,X_{p+1}]/(\beta)$ is reduced and this implies the containment $(\beta)\cap \mathbb{F}_p[X_2^p,\dots,X_{p+1}^p]\subseteq(\beta^p)$. Therefore, any k such that $k\not\equiv -n_1 \bmod p$ and $0\le k\le M$ satisfies $a_k=0\in(\beta^p)$ and $b_k\in(\beta^{p-1})$. If $k\equiv -n_1 \bmod p$, then a_k belongs to $\mathbb{F}_p[X_2^p,\dots,X_{p+1}^p]$ and $\alpha a_k+\beta b_k=0$. Using Claim 6.19 below, we can deduce that $a\in(\beta)\cap\mathbb{F}_p[X_2^p,\dots,X_{p+1}^p]$ and thus $a\in(\beta^p)$. This also implies that $b_k\in(\beta^{p-1})$. Consequently, g is contained in (β^p) in $\mathbb{F}_p[X_1,\dots,X_{p+1}]$ and this shows the reduced property of $\mathbb{F}_p[X_1,\dots,X_{p+1}]/(f)_\delta$ and this implies the reduced property of $\mathbb{F}_p[|X_1,\dots,X_{p+1}|]/(f)_\delta$ as in the proof of Proposition 6.9 (2). □

Claim 6.19. Set $\alpha=X_2^{n_2}+\cdots+X_{p+1}^{n_{p+1}}$ and $\beta=\beta^{(p+1)}(X_2^{n_2},\dots,X_{p+1}^{n_{p+1}})$ in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$ such that n_2 is prime to p. If a and b in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$ satisfy $\alpha a+\beta b=0$, then a is divisible by β and b is divisible by α in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$.

Proof.
Since n_2 is prime to p, the polynomial $\alpha=X_2^{n_2}+X_3^{n_3}+\cdots+X_{p+1}^{n_{p+1}}$ in $\mathbb{F}_p(X_3,\dots,X_{p+1})[X_2]$ is separable. Each root $r_1,\dots,r_{n_2}$ of this polynomial satisfies $r_i^{n_2}+X_3^{n_3}+\cdots+X_{p+1}^{n_{p+1}}=0$ and then $\beta|_{X_2=r_i}\cdot b|_{X_2=r_i}=0$ in $\mathbb{F}_p[X_3,\dots,X_{p+1}]$. By the construction of β in Lemma 6.15(1), the polynomial $\beta|_{X_2=r_i}$ is

$$
\begin{aligned}
&-\sum_{\substack{0\le e_1,\dots,e_{p+1}\le p-1\\ e_1+\cdots+e_{p+1}=p}}\frac{(p-1)!}{e_1!\dots e_{p+1}!}(-1)^{e_1}(r_i^{n_2}+X_3^{n_3}\cdots+X_{p+1}^{n_{p+1}})^{e_1}r_i^{n_2e_2}X_3^{n_3e_3}\dots X_{p+1}^{n_{p+1}e_{p+1}}\\
&=-\sum_{\substack{0\le e_2,\dots,e_{p+1}\le p-1\\ e_2+\cdots+e_{p+1}=p}}\frac{(p-1)!}{e_2!\dots e_{p+1}!}(-1)^{e_2}(X_3^{n_3}\cdots+X_{p+1}^{n_{p+1}})^{e_2}X_3^{n_3e_3}\dots X_{p+1}^{n_{p+1}e_{p+1}},
\end{aligned}
$$

and this is the same as the non-zero element $\beta^{(p)}(X_3^{n_3},\dots,X_{p+1}^{n_{p+1}})$ by the definition of $\beta^{(p)}$ in Lemma 6.15. This implies that $b|_{X_2=r_i}=0$ for any i and thus b is divisible by $\alpha=\prod_{i=1}^{n_2}(X_2-r_i)$ in $\overline{\mathbb{F}_p(X_3,\dots,X_{p+1})}[X_2]$, where $\overline{\mathbb{F}_p(X_3,\dots,X_{p+1})}$ is the algebraic closure of $\mathbb{F}_p(X_3,\dots,X_{p+1})$. Take an element $b'=b'_MX_2^M+\cdots+b'_0$ in $\overline{\mathbb{F}_p(X_3,\dots,X_{p+1})}[X_2]$ such that $b=\alpha b'=(X_2^{n_2}+a')b'$ for $a'\in\mathbb{F}_2[X_3,\dots,X_{p+1}]$. Since b is in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$, comparing the coefficients shows that each b'_i belongs to $\mathbb{F}_p[X_3,\dots,X_{p+1}]$ and then b is divisible by α in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$. This also implies that a is divisible by β in $\mathbb{F}_p[X_2,\dots,X_{p+1}]$. ■□

We give an example of non-monomial δ-stable ideals. This tower arises from a complete intersection but not a log-regular ring unlike Example 6.2.

Corollary 6.20. *Let p be 2 or 3 and let f be an element $X_1^{n_1}+X_2^{n_2}+X_3^{n_3}$ in $\mathbb{Z}_2[X_1,X_2,X_3]$ or $X_1^{n_1}+X_2^{n_2}+X_3^{n_3}+X_4^{n_4}$ in $\mathbb{Z}_3[X_1,X_2,X_3,X_4]$ for $n_i\ge 1$, respectively. Then we have the following:*

1. *The δ-height $\mathrm{ht}^\delta(f)$ is 1 and the sequence $p,f,\delta(f)$ is a regular sequence on $\mathbb{Z}_p[|X_1,\dots,X_{p+1}|]$.*
2. *The mod p reduction $\mathbb{F}_p[|X_1,\dots,X_{p+1}|]/(f)_\delta$ is reduced if and only if at least two of $(n_1,\dots,n_{p+1})$ are prime to p.*
3. *If at least two of (n_1,n_2,n_3) are odd and $X_2^{2n_2}+X_2^{n_2}X_3^{n_3}+X_3^{2n_3}$ is an irreducible polynomial[12] in $\mathbb{Z}_2[X_2,X_3]$ (resp. in $\mathbb{F}_2[X_2,X_3]$), then the quotient $\mathbb{Z}_2[X_1,X_2,X_3]/(f)_\delta$ (resp. $\mathbb{F}_2[X_1,X_2,X_3]/(f)_\delta$) is an integral domain.*

In particular, when at least two of $(n_1,\dots,n_{p+1})$ are prime to p, we have a perfectoid tower arising from $(\mathbb{Z}_p[|X_1,\dots,X_{p+1}|]/(f)_\delta,(p))$ as (6.3) and the first term $\mathbb{Z}_p[|X_1,\dots,X_{p+1}|]/(f)_\delta$ is a p-torsion-free complete intersection of dimension p by (1). If $p=2$, it is not isomorphic to any local log-regular ring.

Proof. The last claim follows from (1), (2), and Lemma 6.22 below since $\mathbb{Z}_2[|X_1,X_2,X_3|]/(f)_\delta$ is a two-dimensional non-regular unramified Gorenstein complete local ring and its mod 2-reduction is

[12] This condition does not hold in general, for example, $n_1=n_2=n_3=5$.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

reduced if at least two of n_1, n_2, or n_3 are odd. Note that the non-regularity follows from the following argument: Take the maximal ideal $\mathfrak{m} := (2, X_1, X_2, X_3)$ of $\mathbb{Z}_2[|X_1, X_2, X_3|]$. If any n_i is strictly greater than 1, then $(f)_\delta$ is contained in $\mathfrak{m}^2$ and thus the embedding dimension of $\mathbb{Z}_2[|X_1, X_2, X_3|]/(f)_\delta$ is 4. If some n_i is equal to 1, for example, $n_1 = 1$, then $\mathbb{Z}_2[|X_1, X_2, X_3|]/(f)_\delta$ is isomorphic to $\mathbb{Z}_2[|X_2, X_3|]/(X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3})$ but still the embedding dimension is 3 since the defining ideal is contained in $(2, X_2, X_3)^2$. Therefore, in any case, $\mathbb{Z}_2[|X_1, X_2, X_3|]/(f)_\delta$ is not regular.

(1): Combining Corollary 6.17 and Lemma 6.14, it is enough to show that $\delta^2(t_1 + t_2 + t_3)$ (resp. $\delta^2(t_1 + t_2 + t_3 + t_4)$) is contained in the ideal $(t_1 + t_2 + t_3, \delta(t_1 + t_2 + t_3))$ (resp. $(t_1 + t_2 + t_3 + t_4, \delta(t_1 + t_2 + t_3 + t_4))$). This follows from a direct computation based on Example 6.16. In particular, $(f)_\delta = (f, \beta^{(p+1)}(X_2^{n_2}, \dots, X_{p+1}^{n_{p+1}}))$ by Lemma 6.15(1).

(2): Let us prove the only if part. We prove the contrapositive, namely, we assume that at most one of $(n_1, \dots, n_{p+1})$ is prime to p and show that $\mathbb{F}_p[|X_1, \dots, X_{p+1}|]/(f)_\delta$ is not reduced. Since the ideal $(f)_\delta = (f, \beta^{(p+1)}(X_2^{n_2}, \dots, X_{p+1}^{n_{p+1}}))$ is independent on the order of $X_1, \dots, X_{p+1}$, we can change the indices so that n_2, n_3 (resp. n_2, n_3, and n_4) are divisible by p.

First, assume that $p = 2$ and n_2 and n_3 are even. Set $n_2 = 2n_2'$ and $n_3 = 2n_3'$ for some positive integer n_2', n_3'. Then we have

$$(X_2^{n_2} + X_2^{n_2'}X_3^{n_3'} + X_3^{n_3})^2 = X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3} = 0$$

in $\mathbb{F}_p[|X_1, X_2, X_3|]/(f)_\delta$. By Corollary 6.17(2), this element $X_2^{n_2} + X_2^{n_2'}X_3^{n_3'} + X_3^{n_3}$ is non-zero in $\mathbb{F}_p[|X_1, X_2, X_3|]/(f)_\delta$ because of the initial degree $n_2 < 2n_2$. Therefore, $\mathbb{F}_p[|X_1, X_2, X_3|]/(f)_\delta$ is not reduced.

Next we prove when $p = 3$. Assume that n_2, n_3, and n_4 are divisible by 3. Set $n_2 = 3n_2'$, $n_3 = 3n_3'$, and $n_4 = 3n_4'$ for some positive integer n_2', n_3', and n_4'. Based on Example 6.16, we have

$$((X_2^{n_2'} + X_3^{n_3'})(X_2^{n_2'} + X_4^{n_4'})(X_4^{n_4'} + X_3^{n_3'}))^3 = 0$$

in $\mathbb{F}_3[|X_1, X_2, X_3, X_4|]/(f)_\delta$. This element $(X_2^{n_2'} + X_3^{n_3'})(X_2^{n_2'} + X_4^{n_4'})(X_4^{n_4'} + X_3^{n_3'})$ is non-zero because of the initial degree $2n_2'$ of X_2 is strictly less than the degree $2n_2$ of X_2 in the initial ideal in Corollary 6.17 and thus $\mathbb{F}_3[|X_1, X_2, X_3, X_4|]/(f)_\delta$ is not reduced.

Next, we prove if part, namely, we assume that at least two of $(n_1, \dots, n_{p+1})$ are prime to p and show that $\mathbb{F}_p[|X_1, \dots, X_{p+1}|]/(f)_\delta$ is reduced. Since $(f)_\delta$ does not change by changing the indices of $X_1, \dots, X_{p+1}$, we may assume that n_1, n_2 are prime to 2 (resp. 3). By (1) together with Lemma 6.18 and Example 6.16, it is enough to show that

$$\mathbb{F}_2[X_2, X_3]/(X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3}) \quad \text{and}$$
$$\mathbb{F}_3[X_2, X_3, X_4]/((X_2^{n_2} + X_3^{n_3})(X_2^{n_3} + X_4^{n_4})(X_4^{n_4} + X_3^{n_3}))$$

are reduced. This follows if we show $(\beta) \cap \mathbb{F}_p[X_2^p, \dots, X_{p+1}^p] \subseteq (\beta^p)$ in $\mathbb{F}_p[X_2, \dots, X_{p+1}]$.

In case of $p = 2$, we already assume that n_1 and n_2 are odd. Take $c = \beta b \in (\beta) \cap \mathbb{F}_2[X_2^2, X_3^2]$. Dividing b by $\beta = X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3}$ in $\mathbb{F}_2[X_3][X_2]$, we can write $b = \beta b' + r$ for some b' and r in $\mathbb{F}_2[X_2, X_3]$, where the degree of X_2 in r is strictly less than $2n_2$. Suppose that X_2 has an odd degree in b' and let K be the largest odd degree of X_2 in b' with a non-zero coefficient. Then the leading term of $c = \beta^2 b' + \beta r$ whose degree of X_2 is $K + 4n_2$ is non-zero because of the leading term of $\beta^2 b'$. This contradicts the assumption that c belongs to $\mathbb{F}_2[X_2^2, X_3^2]$. So $c - \beta^2 b' = \beta r \in (\beta) \cap \mathbb{F}_2[X_2^2, X_3]$ holds. Therefore, replacing b by r, we have $c = \beta b \in (\beta) \cap \mathbb{F}_2[X_2^2, X_3]$ and the degree of X_2 in b is strictly less than $2n_2$. It is enough to show that $b = 0$. Assume the converse, namely, we write $b = b_M X_2^M + \cdots + b_1 X_2 + b_0$, where $b_i \in \mathbb{F}_2[X_3]$, $b_M \neq 0$, and $M < 2n_2$. By Example 6.16, we have

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

$\beta = \beta^{(2+1)} = X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3}$ and thus

$$c = \beta b = (b_M X_2^{M+2n_2} + \cdots + b_0 X_2^{2n_2}) + (b_M X_2^{M+n_2} X_3^{n_3} + \cdots + b_{n_2} X_2^{2n_2} X_3^{n_3} + \cdots + b_0 X_2^{n_2} X_3^{n_3}) + (b_M X_2^{M} X_3^{2n_3} + \cdots + b_{n_2} X_2^{n_2} X_3^{2n_3} + \cdots + b_0 X_3^{2n_3}).$$

Since this belongs to $\mathbb{F}_2[X_2^2, X_3]$, the leading degree $M + 2n_2$ is even because of $b_M \neq 0$ and thus M is even. If $M < n_2$, then $b_M X_2^{M+n_2} X_3^{n_3}$ is the only non-zero term whose degree of X_2 is $M + n_2$. This number $M+n_2$ becomes odd and thus b_M should vanish but this contradicts the assumption. If $M \geq n_2$, then

$$b_k X_2^{k+2n_2} + b_{k+n_2} X_2^{k+2n_2} X_3^{n_3} = 0 \tag{6.12}$$

holds for any odd number k in $0 \leq k \leq M - n_2$ since we assume $M < 2n_2$. Because of $b_k X_2^k X_3^{2n_3} = 0$ for any odd number k in $0 \leq k \leq n_2 - 1$, the equality (6.12) implies $b_{k+n_2} = 0$ for any odd number k in $0 \leq k \leq \min\{M - n_2, n_2 - 1\}$, that is, $b_l = 0$ for any even number l in $n_2 \leq l \leq \min\{M, 2n_2 - 1\}$. Since M is even and strictly less than $2n_2$, this vanishing implies $b_M = 0$ but this contradicts $b_M \neq 0$.

For the case of $p = 3$, we already assume that n_1 and n_2 are prime to 3. Take $c = \beta b \in (\beta) \cap \mathbb{F}_3[X_2^3, X_3^3, X_4^3]$. Since $X_2^{n_2} + X_3^{n_3}$ and $X_2^{n_2} + X_4^{n_4}$ are separable in $\mathbb{F}_3(X_3, X_4)[X_2]$, so is their product β. Taking the different roots $s_1, \ldots, s_{2n_2}$ of β in $\overline{\mathbb{F}_3(X_3, X_4)}$, we have $c|_{X_2 = s_i} = 0$. Since c belongs to $\mathbb{F}_3[X_2^3, X_3^3, X_4^3]$ and s_i are different from each other, c is divisible by $(X_2 - s_i)^3$ and thus c is divisible by β^3 in $\overline{\mathbb{F}_3(X_3, X_4)}[X_2]$. Therefore, since c and β are both in $\mathbb{F}_3[X_3, X_4][X_2]$, as in the proof of Claim 6.19, comparing the coefficients shows that c is divisible by β^3 in $\mathbb{F}_3[X_2, X_3, X_4]$.

(3): Set Λ to be $\mathbb{Z}_2$ or $\mathbb{F}_2$ and $f_2 \coloneqq X_2^{2n_2} + X_2^{n_2}X_3^{n_3} + X_3^{2n_3}$. Note that $(f)_\delta = (f, f_2)$ in $\Lambda[X_1, X_2, X_3]$ by (1). If n_2 and n_3 are odd numbers, then $X_2^{n_2} + X_3^{n_3}$ has no factors with multiplicity strictly greater than 1 in $\Lambda[X_2, X_3]$ (this follows from, e.g., taking partial derivation) and Eisenstein criterion implies that $X_1^{n_1} + X_2^{n_2} + X_3^{n_3}$ is irreducible in $\Lambda[X_1, X_2, X_3]$. Other cases also follow from the same argument and so f is a monic irreducible polynomial in $\Lambda[X_1, X_2, X_3]$. Then we can take a finite free extension $\Lambda[X_2, X_3] \hookrightarrow \Lambda[X_1, X_2, X_3]/(f)$ and its base change

$$R \coloneqq \Lambda[X_2, X_3]/(f_2) \hookrightarrow \Lambda[X_1, X_2, X_3]/(f, f_2) =: S$$

is also a finite free extension from an integral domain R since f_2 is irreducible in $\Lambda[X_2, X_3]$ by our assumption. Especially, S is R-torsion-free and induces an injection $S \hookrightarrow (R \setminus \{0\})^{-1}S$. Take any non-zero element s of S and we can find a monic polynomial $s^m + r_1 s^{m-1} + \cdots + r_m = 0$ with $r_i \in R$. Then $s(s^{m-1} + r_1 s^{m-2} + \cdots + r_{m-1}) = -r_m$ in S holds and thus s is a unit in $(R \setminus \{0\})^{-1}S$. This implies that $(R \setminus \{0\})^{-1}S$ is a field and thus the subring $S = \Lambda[X_1, X_2, X_3]/(f, f_2)$ is also an integral domain. □

Example 6.21. Set a δ-structure on $\mathbb{Z}_2[X, Y, Z]$ by $\delta(X) = \delta(Y) = \delta(Z) = 0$. Assume $p = 2$ and take $f \coloneqq X^3 + Y^4 + Z^5$ in $\mathbb{Z}_2[X, Y, Z]$. Then Lemma 6.18 tells us that

$$(f)_\delta = (X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6)$$

holds in $\mathbb{Z}_2[X, Y, Z]$. By Corollary 6.20, the (X, Y, Z)-adic completion (resp. localization at (p, X, Y, Z)) of the tower

$$\mathbb{Z}_2[X, Y, Z]/(X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6) \hookrightarrow \cdots \hookrightarrow \mathbb{Z}_2[2^{1/2^i}][X^{1/2^i}, Y^{1/2^i}, Z^{1/2^i}]/(X^{3/2^i} + Y^{4/2^i} + Z^{5/2^i}, Y^{8/2^i} + X^{3/2^i}Y^{4/2^i} + X^{6/2^i}) \hookrightarrow \cdots$$

is a perfectoid tower arising from $(\mathbb{Z}_2[|X, Y, Z|]/(X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6), (2))$ (resp. $(\mathbb{Z}_2[X, Y, Z]_{(2,X,Y,Z)}/(X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6), (2))$). The first term of the tower is a complete intersection (resp. complete intersection domain) but they are not log-regular rings.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

We provide a sufficient condition for the non-log-regularity of the first term of the tower in Corollary 6.20. This proof is based on a private communication with Shinnosuke Ishiro.

Lemma 6.22. *Let R be a complete Noetherian local ring of dimension 2 with residue field k of mixed characteristic $(0, p)$. Assume that R is unramified Gorenstein but not regular and R/pR is reduced. Then R is not isomorphic to any local log-regular ring.*

Proof. If R is a local log-regular ring, there exists a log structure $Q \xrightarrow{\alpha} R$ from a fine, sharp, and saturated monoid Q such that R/I_α is regular and $2 = \dim(R) = \dim(R/I_\alpha) + \dim(Q)$ (see [14, Definition 2.19]). Since R is not regular, I_α is non-zero and thus $\dim(Q) > 0$. By the structure theorem of local log-regular ring [14, Theorem 2.22], $R \cong \mathbb{Z}_p[|Q \oplus \mathbb{N}^r|]/(p - f)$ holds, where $r := \dim(R/I_\alpha)$ and f has no non-zero constant term. If $\dim(Q) = 1$, then Q should be equal to $\mathbb{N}$. So $R \cong \mathbb{Z}_p[|\mathbb{N}^2|]/(\theta)$ is regular but this contradicts the non-regularity of R. So $\dim(Q)$ should be 2. Since R is Gorenstein, [13, Remark 2.2 and Corollary 4.11] shows that R is isomorphic to $\mathbb{Z}_p[|s^{n+1}, st, t^{n+1}|]/(p - f) \cong \mathbb{Z}_p[|x, y, z|]/(xz - y^{n+1}, p - g)$ for some $g \in (x, y, z)$ and $n \geq 2$. Since R is unramified, g is x or z and thus R is isomorphic to $\mathbb{Z}_p[|s^{n+1}, st, t^{n+1}|]/(p - s^{n+1})$. Taking modulo p, this should be a non-reduced ring $\mathbb{F}_p[|s^{n+1}, st, t^{n+1}|]/(s^{n+1})$ but this contradicts the reduced property of R/pR. □

Acknowledgements. The author would like to thank Shinnosuke Ishiro, Kei Nakazato, and Kazuma Shimomoto for their valuable discussions about perfectoid towers, Shou Yoshikawa for his pointing out the usage of ϕ-monomials, and Sora Miyashita for the many things he taught me about affine semigroup rings. Anonymous referees gave helpful comments and suggestions, for example, adding more examples, examining the generic degrees of the transition maps, and pointing out some mistakes.

Funding statement. This work was supported by JSPS KAKENHI Grant Number 24KJ1085.

References

[1] Y. André, *La conjecture du facteur direct*, Publ. Math. l'IHÉS 127 (2018), no. 1, 71–93.
[2] J. Anschütz and A.-C. Le Bras, *The p-completed cyclotomic trace in degree2*, Ann. K-Theory 5 (2020), no. 3, 539–580.
[3] B. Bhatt, *On the direct summand conjecture and its derived variant*, Invent. Math. 212 (2018), no. 2, 297–317.
[4] B. Bhatt, M. Hochster and L. Ma, *Lim Cohen-Macaulay sequences of modules*, preprint, arXiv:2410.18372, 2024.
[5] B. Bhatt and J. Lurie, *The prismatization of p-adic formal schemes*, preprint, arXiv:2201.06124, 2022.
[6] B. Bhatt, L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron and J. Witaszek, *Perfectoid pure singularities*, preprint, arXiv:2409.17965, 2024.
[7] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. Math. 196 (2022), no. 3, 1135–1275.
[8] C. A. Francisco, J. Mermin and J. Schweig, "A survey of Stanley–Reisner theory" in S.M. Cooper, S. Sather-Wagstaff (Eds.), *Connections Between Algebra, Combinatorics, and Geometry*, Springer, New York, NY, 2014, pp. 209–234.
[9] U. Görtz and T. Wedhorn, *Algebraic geometry I: Schemes: With examples and exercises*, 2nd ed., Springer Studium Mathematik—Master, 2, Wiesbaden: Springer Spektrum, 2020.
[10] S. Goto, N. Suzuki and K. Watanabe, *On affine semigroup rings*, Jpn. J. Math. New Ser. 2 (1976), 1–12.
[11] D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry. http://www2.macaulay2.com
[12] M. Hochster, *Rings of invariants of Tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. Math. 96 (1972), no. 2, 318–337.
[13] S. Ishiro, *Local log-regular rings vs. toric rings*, Commun. Algebra, 53 (2025), no. 4, 1640–1655.
[14] S. Ishiro, K. Nakazato and K. Shimomoto, *Perfectoid towers and their tilts: With an application to the Étale cohomology groups of local log-regular rings*, Algebra Number Theory 19 (2025), no. 12, 2307–2358.
[15] S. Ishiro and K. Shimomoto, *δ -rings, perfectoid towers, and Lim Cohen-Macaulay sequences*, preprint, arXiv:2509.06527, 2025.
[16] R. Ishizuka and K. Nakazato, *Prismatic Kunz's theorem*, J. Algebra 693 (2026), 732–769.
[17] R. Ishizuka and K. Shimomoto, *Quasi-canonical liftings of smooth projective varieties and Frobenius lifts*, preprint, arXiv:2506.01345, 2025.
[18] T. Kawakami and T. Takamatsu, *On Frobenius liftability of surface singularities*, preprint, arXiv:2402.08152, 2024.
[19] T. Kawakami, T. Takamatsu and S. Yoshikawa, *Fedder type criteria for quasi-F-splitting I*, preprint, arXiv:2204.10076, 2022.

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press

[20] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, 227, New York: Springer-Verlag, 2005.
[21] I. Swanson and C. Huneke, *Integral closure of ideals, rings, and modules*, Cambridge: Cambridge University Press, 2006.
[22] The Stacks Project Authors, *Stacks project*. https://stacks.math.columbia.edu, 2018.
[23] O. Zariski, *Sur la normalité analytique des variétés normales*, Ann. Inst. Fourier 2 (1950), 161–164.

Cite this article: R. Ishizuka, *Perfectoid towers generated from prisms*. Nagoya Mathematical Journal (2026);261, 1–32. https://doi.org/10.1017/nmj.2026.10101

https://doi.org/10.1017/nmj.2026.10101 Published online by Cambridge University Press