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ON PÓLYA'S 4D RANDOM WALK CONSTANT

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Abstract

The return probability in the simple symmetric random walk on the four-dimensional lattice $\mathbb{Z}^4$ is evaluated in closed form in terms of the generalised hypergeometric function.

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1. Introduction

A celebrated theorem of Pólya [13] states that the simple symmetric random walk on $\mathbb{Z}^d$ is recurrent in dimensions 1 and 2, but transient in dimension $d \geq 3$. A fundamental problem is to calculate the return probability, which we denote by $p(d)$, for $d \geq 3$ (see [4, Section 5.9] for an overview of some of the contributions to this problem). Exact formulae for $p(3)$, the return probability in dimension 3, were given in [2, 9, 14], and this line of research culminated in the simple closed-form formula

$$p(3) = 1 - 1/u(3) = 0.3405373295\ldots,$$

where

$$u(3) = \frac{\sqrt{6}}{32\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right) \tag{1.1}$$

$$= \frac{\sqrt{3}-1}{32\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2 \tag{1.2}$$

$$= 1.5163860591\ldots.$$

Equation (1.1) was derived in [6], but the value printed there included an erroneous additional factor of 384π, as pointed out in [16]. The simplified expression (1.2) was

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obtained in [7] through an application of a Gamma function identity given in [1] to (1.1). The numerical values presented above for $p(3)$ and $u(3)$ are given in [4, page 322], whilst numerical evaluations for $p(d)$ for $4 \leq d \leq 8$ are tabulated in [4, page 323].

More generally, in dimension $d \geq 3$, [5] derived a closed-form expression for the return probability $p(d)$ in terms of the Lauricella function of type C: for $d \geq 3$,

$$p(d) = 1 - 1/u(d),$$

where

$$u(d) = F_C^{(d)}\Big(1, \frac{1}{2}; 1, \ldots, 1; \frac{1}{d^2}, \ldots, \frac{1}{d^2}\Big), \tag{1.3}$$

and the Lauricella function F_C (see [3, 8]) is defined by

$$F_C^{(d)}(a, b; c_1, \ldots, c_d; x_1, \ldots, x_d) = \sum_{k_1 \geq 0} \cdots \sum_{k_d \geq 0} \frac{(a)_{k_1+\cdots+k_d}(b)_{k_1+\cdots+k_d}}{(c_1)_{k_1} \cdots (c_d)_{k_d}} \frac{x_1^{k_1} \cdots x_d^{k_d}}{k_1! \cdots k_d!},$$

with $(v)_0 = 1$ and $(v)_k = v(v+1)\cdots(v+k-1)$, $k \geq 1$, denoting the Pochhammer symbol.

Now that a closed-form formula for the return probability for general $d \geq 3$ has been achieved in terms of a known special function, a natural further question is whether one can obtain closed-form expressions in terms of a simpler special function in a dimension greater than 3. In this note, we address this problem by obtaining closed-form expressions for the return probability in dimension 4 in terms of the ${}_{q+1}F_q$ generalised hypergeometric function, defined via the power series

$${}_{q+1}F_q\Big(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix} \Big| x\Big) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{q+1})_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, \tag{1.4}$$

which is absolutely convergent for $|x| < 1$, and is also absolutely convergent for $|x| = 1$ if $\mathrm{Re}(\sum_{i=1}^{q} b_i - \sum_{j=1}^{q+1} a_j) > 0$ (see [12, Section 16.2]). Our formulae for the return probability offer a considerable simplification of the formula for $p(4)$ obtained from simply setting $d = 4$ in the general formula (1.3), since the generalised hypergeometric function is defined through a single infinite series, and is a more elementary and widely used special function than the Lauricella function of type C.

2. Results and proofs

The following theorem is the main result of this note.

THEOREM 2.1. *In dimension 4, the return probability is given by*

$$p(4) = 1 - 1/u(4) = 0.1932016732\ldots,$$

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where

$$u(4) = \frac{3\Gamma(\frac{1}{3})^9}{16\pi^6}\,{}_4F_3\left(\begin{matrix}\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2}\\ \frac{2}{3},\frac{5}{6},\frac{5}{6}\end{matrix}\,\middle|\,1\right) - \frac{64\sqrt{3}\,\pi^3}{3\Gamma(\frac{1}{3})^9}\,{}_4F_3\left(\begin{matrix}\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6}\\ \frac{7}{6},\frac{7}{6},\frac{4}{3}\end{matrix}\,\middle|\,1\right) \tag{2.1}$$

$$= \frac{\Gamma(\frac{1}{3})^6}{2^{5/3}\pi^4}\,{}_5F_4\left(\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{5}{6},\frac{5}{6},1\end{matrix}\,\middle|\,1\right) \tag{2.2}$$

$$= 1.2394671218\ldots.$$

PROOF. We shall prove the theorem by combining several integral identities from the existing literature. By [11, (2.11a)], the return probability is given by $p(4) = 1 - 1/u(4)$, where

$$u(4) = \frac{1}{4\pi^4}\int_{(-\pi,\pi)^4} \frac{dx_1\,dx_2\,dx_3\,dx_4}{4 - \sum_{k=1}^4 \cos x_k}. \tag{2.3}$$

From [15, (2.1) and (2.3)],

$$\frac{1}{4\pi^4}\int_{(-\pi,\pi)^4} \frac{dx_1\,dx_2\,dx_3\,dx_4}{4 - \sum_{k=1}^4 \cos x_k} = \frac{16}{\pi^2}\int_0^\infty tI_0^3(t)K_0(t)^3\,dt, \tag{2.4}$$

whilst we also have the integral identity

$$\int_0^\infty tI_0^3(t)K_0(t)^3\,dt = \frac{3}{\pi^2}\int_0^\infty tI_0(t)K_0(t)^5\,dt \tag{2.5}$$

(see [15, page 471]). Here, $I_0(\cdot)$ and $K_0(\cdot)$ are the zeroth-order modified Bessel functions of first and second kinds, respectively (see [12, Ch. 10]). Combining (2.3), (2.4) and (2.5) now gives

$$u(4) = \frac{48}{\pi^4}\int_0^\infty tI_0(t)K_0(t)^5\,dt. \tag{2.6}$$

Applying the formula of [15, Proposition 2.3] for the integral $\int_0^\infty tI_0(t)K_0(t)^5\,dt$ to (2.6) now yields (2.1).

We now derive (2.2). From [15, Propositions 2.3 and 2.4],

$$u(4) = \frac{48}{\pi^4}\int_0^\infty tI_0(t)K_0(t)^5\,dt = \frac{4\sqrt{3}}{9\pi}\int_0^1 \frac{1}{\sqrt{1-x}}\left[{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\,\middle|\,x\right)\right]^2 dx,$$

and evaluating the latter integral using the integral [15, (2.28) of Proposition 3.1] yields the expression (2.2). This completes the proof. □

REMARK 2.2. Equation (2.2) can alternatively be deduced from (2.1) via an application of [10, (2.3)]. The direct application of [10, (2.3)] yields the expression

$$u(4) = \frac{\Gamma(\frac{1}{3})^6}{2^{5/3}\pi^4}\,{}_7F_6\left(\begin{matrix}\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{6}\\ \frac{1}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},1,\frac{7}{6}\end{matrix}\,\middle|\,1\right),$$

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which reduces to the ${}_5F_4$ generalised hypergeometric function given in (2.2) due to the common parameters $1/6$ and $7/6$ (which is clear from the series representation (1.4) of the generalised hypergeometric function).

REMARK 2.3. In the proof of Theorem 2.1, we arrived at several integral representations of $u(4)$. However, in terms of numerical computation, the representation (2.2) of $u(4)$ in terms of a single ${}_5F_4$ function is particularly convenient since the generalised hypergeometric function is implemented in standard computational algebra packages.

It is natural to ask whether one can further simplify the expressions (2.1) and (2.2) for $u(4)$. The author did not find any such simplifications in the literature; however, this does not rule out the possibility of further simplifications. It is also natural to ask whether one can obtain a representation of $u(d)$ for $d \geq 5$ in terms of generalised hypergeometric functions or other special functions that are simpler than the Lauricella function of type C. The following integral representations were obtained by [11]: for $d \geq 3$,

$$\begin{aligned} u(d) &= \frac{d}{(2\pi)^d} \int_{(-\pi,\pi)^d} \left(d - \sum_{k=1}^{d} \cos x_k\right)^{-1} dx_1\, dx_2 \cdots dx_d \\ &= \int_0^\infty \left[I_0\left(\frac{x}{d}\right)\right]^d e^{-x}\, dx. \end{aligned}$$

However, the author was unable to find evaluations of these integrals in the literature for $d \geq 5$ (other than the known evaluation in terms of the Lauricella function of type C given by [5]).

We end with the following curiosity in which it is noted that in dimension 3, the return probability can also be expressed in terms of the generalised hypergeometric function with unit argument.

PROPOSITION 2.4. *In dimension 3,*

$$u(3) = \frac{3\sqrt{2}}{\pi}\, {}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{1}{2} \\ \frac{25}{24}\end{matrix} \,\middle|\, 1\right) \tag{2.7}$$

$$= \frac{1+\sqrt{3}}{2}\, {}_3F_2\left(\begin{matrix}\frac{1}{12}, \frac{1}{2}, \frac{11}{12} \\ 1, 1\end{matrix} \,\middle|\, 1\right). \tag{2.8}$$

PROOF. We first derive (2.7). By Gauss's multiplication formula [12, (5.5.6)],

$$\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{13}{24}\right)\Gamma\left(\frac{7}{8}\right) = \frac{2\pi}{3^{1/8}}\Gamma\left(\frac{5}{8}\right), \quad \Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{23}{24}\right) = \frac{2\pi}{3^{3/8}}\Gamma\left(\frac{7}{8}\right).$$

Applying these equations to (1.1) gives

$$u(3) = \frac{3\sqrt{2}}{\pi}\frac{\Gamma(\frac{11}{24})\Gamma(\frac{25}{24})}{\Gamma(\frac{13}{24})\Gamma(\frac{23}{24})} = \frac{\sqrt{2}}{8\pi}\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{13}{24})\Gamma(\frac{23}{24})}, \tag{2.9}$$

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where in the second step, we used the standard identity $\Gamma(x+1) = x\Gamma(x)$. Equation (2.7) now follows by applying the identity

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \tag{2.10} $$

(see [12, (15.4.20)]) to (2.9).

We now prove (2.8). By the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ (see [12, (5.5.3)]),

$$ \Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{11}{24}\Big) = \frac{\pi^2}{\sin(\frac{\pi}{24})\sin(\frac{11\pi}{24})\Gamma(\frac{13}{24})\Gamma(\frac{23}{24})} = \frac{8\pi^2}{(\sqrt{6}-\sqrt{2})\Gamma(\frac{13}{24})\Gamma(\frac{23}{24})}, \tag{2.11} $$

where we used $2\sin(\pi/24)\sin(11\pi/24) = \cos(5\pi/12) = (\sqrt{6}-\sqrt{2})/4$. Substituting (2.11) into (1.2) and simplifying,

$$ \begin{aligned} u(3) = \frac{1+\sqrt{3}}{2}\bigg[\frac{\sqrt{\pi}}{\Gamma(\frac{13}{24})\Gamma(\frac{23}{24})}\bigg]^2 &= \frac{1+\sqrt{3}}{2}\bigg[{}_2F_1\bigg(\begin{matrix}\frac{1}{24},\frac{11}{24}\\ 1\end{matrix}\,\bigg|\,1\bigg)\bigg]^2 \\ &= \frac{1+\sqrt{3}}{2}\,{}_3F_2\bigg(\begin{matrix}\frac{1}{12},\frac{1}{2},\frac{11}{12}\\ 1,1\end{matrix}\,\bigg|\,1\bigg), \end{aligned} $$

where we obtained the second equality using (2.10) and the last equality was obtained by an application of Clausen's formula (see [12, (16.12.2)]). This completes the proof. □

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