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Brill-Noether generality of binary curves

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Abstract. We show that the space $G^r_{\underline{d}}(X)$ of linear series of certain multi-degree $\underline{d} = (d_1, d_2)$ (including the balanced ones) and rank r on a general genus-g binary curve X has dimension $\rho_{g,r,d} = g - (r+1)(g-d+r)$ if nonempty, where $d = d_1 + d_2$. This generalizes Caporaso's result from the case $r \leq 2$ to arbitrary rank, and shows that the space of Osserman-limit linear series on a general binary curve has the expected dimension, which was known for $r \leq 2$. In addition, we show that the space $G^r_{\underline{d}}(X)$ is still of expected dimension after imposing certain ramification conditions with respect to a sequence of increasing effective divisors supported on two general points $P_i \in Z_i$, where $i = 1, 2$ and Z_1, Z_2 are the two components of X. Our result also has potential application to the lifting problem of divisors on graphs to divisors on algebraic curves.

1 Introduction

Let k be an algebraically closed field. For a curve C over k, a $\mathfrak{g}^r_d$ on C denotes a linear series on C of degree d and rank r. Brill–Noether theory states that if C is general, then the space $G^r_d(C)$ of $\mathfrak{g}^r_d$s on C has dimension

$$\rho_{g,r,d} = g - (r+1)(g-d+r).$$

This provides a bridge between the realms of abstract curves and curves in projective spaces; for a general curve, the theory tells us whether it is equipped with a nondegenerate map of a certain degree to a certain projective space. The result was first proved in [GH80] by degenerating to an irreducible curve C' with g nodes (the normalization of which is a rational curve), as in the left part of Figure 1, and showing that $G^r_d(C')$ has dimension at most $\rho_{g,r,d}$. After that, it has inspired a great deal of additional research, including re-proofs and new techniques such as limit linear series and tropical linear series; see [EH86, CDPR12, Oss19, AB15].

While the space $G^r_d(C)$ has been well studied for smooth curves (*cf.* [ACGH13, Ch.V]), little is known for reducible ones. In this paper, we consider a *binary curve* X of genus g, which, as in the right part of Figure 1, is obtained by glueing two copies Z_1, Z_2 of $\mathbb{P}^1_k$ along $g+1$ pairs of different points (Q^1_j, Q^2_j), where $Q^i_j \in Z_i$ for $0 \leq j \leq g$ and $1 \leq i \leq 2$. For a multi-degree $\underline{d} = (d_1, d_2)$ on (Z_1, Z_2) such that $d_1 + d_2 = d$, denote by $\mathfrak{g}^r_{\underline{d}}$ a linear series on X whose underlying line bundle has multi-degree $\underline{d}$. It was proved in [Cap10] that the space $G^r_{\underline{d}}(X)$ of $\mathfrak{g}^r_{\underline{d}}$s on a general binary curve X has dimension at most $\rho_{g,r,d}$ if $\underline{d}$ is balanced, namely $|d_1 - d_2| \leq g+1$, and $r \leq 2$. We extend the result of

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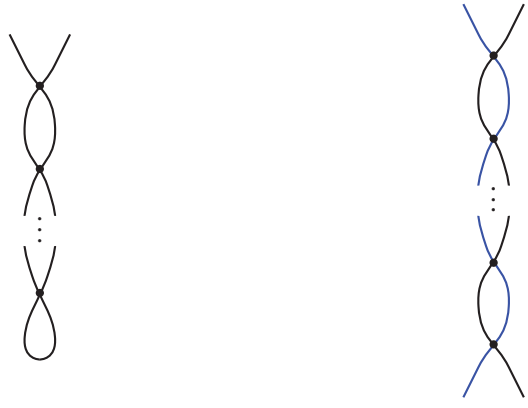


Figure 1: A genus g irreducible curve (resp. binary curve) with g (resp. $g+1$) nodes.

Caporaso to arbitrary rank r, and show that $G^r_{\underline{d}}(X)$ still has the expected dimension after imposing certain ramification conditions.

Theorem 1.1 *Let X be a genus-g binary curve obtained by glueing two copies Z_1, Z_2 of $\mathbb{P}^1_k$, and fix r and $\underline{d}$ as above. Let $P_i \in Z_i$ be two general points. Suppose $d_i \geq -1$ for $1 \leq i \leq 2$ and X is general. Then for a sequence $D_\bullet$ of increasing effective divisors supported on $\{P_1, P_2\}$ and an admissible sequence $a_\bullet$ along $D_\bullet$ satisfying certain conditions, the space of $\mathfrak{g}^r_{\underline{d}}$s with multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ is either empty or of expected dimension. In particular, $G^r_{\underline{d}}(X)$ is either empty or of dimension $\rho_{g,r,d}$.*

See Theorem 3.4 for details; see also Section 2.2 for the notions of admissible sequences and the corresponding ramification conditions. Our proof of Theorem 1.1 benefits from the idea of [GH80]. In their proof, the normalization $\widetilde{C}'$ of C' is realized as a rational normal curve in $\mathbb{P}^d_k$. Let $(N^1_j, N^2_j)_{1\leq j\leq g}$ be the preimage in $\widetilde{C}'$ of each node of C', as in Figure 2. The space $G^r_d(C')$ is identified with an open subset of the space of $(d-r-1)$ planes of $\mathbb{P}^d_k$ that pass though the g chords $\overline{N^1_j N^2_j}$ of $\widetilde{C}'$, where $\overline{N^1_j N^2_j}$ meets $\widetilde{C}'$ at N^1_j and N^2_j. Then it is enough to show that the intersection

$$\sigma(\overline{N^1_1 N^2_1}) \cap \cdots \cap \sigma(\overline{N^1_g N^2_g})$$

is dimensionally transverse. Here, $\sigma(\overline{N^1_j N^2_j}) \subset Gr(d-r, d+1)$ is the Schubert cycle of $(d-r-1)$-planes in $\mathbb{P}^d_k$ meeting $\overline{N^1_j N^2_j}$. They proceeded by induction, at each step degenerating N^1_m and N^2_m to N^1_1 and getting a union of Schubert varieties S_m of the flag

$$N^1_1 \subset \overline{2N^1_1} \subset \cdots \subset \overline{(d+1)N^1_1} = \mathbb{P}^d_k,$$

where $\overline{jN^1_1}$ denotes the j-th osculating plane of $\widetilde{C}'$ at N^1_1. Then the problem is reduced to the properness of the intersection $S_m \cap \sigma(\overline{N^1_{m+1} N^2_{m+1}}) \cap \cdots \cap \sigma(\overline{N^1_g N^2_g})$.

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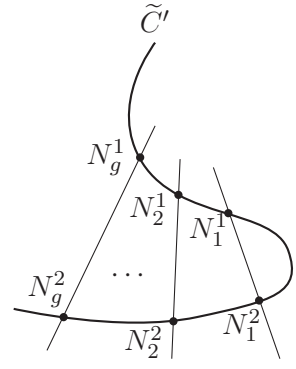


Figure 2:

In principle, the idea of degenerating the chords of the normalization of C' should also work in our context. However, we inevitably came across Schubert cycles with excess dimension during the degeneration; see Remark 3.6 for details. To resolve this issue, we adopt Esteves' compactification [Est01] of the Jacobian of X (see §2.1) and construct a compactification of $G^r_{\underline{d}}(X)$ accordingly. We observe that the Schubert condition of S_m induces a ramification condition at N_1^1 for linear series on $\widetilde{C}'$, and thus modify the proof of [GH80] to our case; see Section 3 for details.

As an application, our result shows that the space of Osserman limit linear series on a general binary curve has the expected dimension, which was known for $r \leq 2$ ([Oss16, Corollary 7.4]).

Corollary 1.2 *The space of Osserman-limit linear series on a general binary curve has the expected dimension. As a result, all limit linear series are smoothable to nearby smooth curves.*

Proof The first statement follows directly from Theorem 1.1 and [Oss16, Corollary 7.3], while the second one is a result of the main smoothing theorem of limit linear series carried out in [Oss19]. ∎

Moreover, we comment that the smoothing property we get for limit linear series also provides potential tools for lifting rank-r divisors on the graph G associated with a regular family $\mathcal{X}$ of curves over a discrete valuation ring to divisors on the generic fiber of $\mathcal{X}$ that admit the same rank. More precisely, this can be achieved by lifting the divisor on G to a limit linear series on the special fiber, which in our case is a general binary curve, and applying Corollary 1.2. A similar approach can be found in [He19, §5]; see also [BJ16, §10] for a survey on this problem, and [He18] for results of lifting divisors while preserving both the rank and ramifications. Note that the liftability is only completely solved when G is a chain of loops [CJP15].

Conventions. All curves we consider, unless otherwise specified, are assumed proper over an algebraically closed field k, reduced and connected, and at worst nodal.

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2 The Compactified Relative Jacobian and Moduli Scheme of Linear Series

In this section, we recall the notion of compactified relative Jacobian over a family of (possibly reducible) curves carried out in [Est01]. For each curve X in the family, the compactified Jacobian contains extra points corresponding to certain torsion free, rank one sheaves on X. Note that for a pair (I, V) consisting of a torsion free rank one sheaf I on X and a subspace $V \subset H^0(X, I)$, we can still talk about ramifications of V at nonsingular points of X. Then for the degeneration argument of the next section, we construct a family of (compactified) moduli spaces of linear series with imposed ramifications on binary curves as a closed subscheme of a relative Grassmannian over the compactified relative Jacobian. Note that this construction is well known; see, for example, [Oss06, §4] or [Oss, §2].

2.1 The Compactified Relative Jacobian

Let X be a curve and let I be a coherent sheaf on X. We say that I is *torsion free* if it has no embedded components and I has *rank one* if it has generic rank one at every irreducible component of X. We say that I is *simple* if $\mathrm{End}_X(I) = k$. According to [OS79, Proposition 10.1], each torsion free sheaf I of rank one can be written as the pushforward of a line bundle L_I on a partial normalization $n_I\colon X_I \to X$. A direct calculation shows that the natural map

$$n_{I*}\mathscr{E}\mathrm{nd}\,(L_I) \longrightarrow \mathscr{E}\mathrm{nd}\,(n_{I*}L_I)$$

is an isomorphism. Hence, a torsion free sheaf is simple if and only if the corresponding normalization X_I is connected.

Let I be a torsion free sheaf of rank one on X. If $Y \subset X$ is a subcurve, then we denote by I_Y the maximum torsion free quotient of $I|_Y$. Fix an integer d, and fix a line bundle $\mathscr{E}$ on X of degree $d - g + 1$. We say that $\mathscr{E}$ is a *polarization* on X. For every subcurve $Y \subset X$, let $e_Y := \deg_Y(\mathscr{E})$. We say that I is *stable* (resp. *semistable*) with respect to $\mathscr{E}$ if for every nonempty proper subcurve $Y \subsetneq X$, we have

$$\chi(I_Y) > e_Y \quad (\text{resp. } \chi(I_Y) \geq e_Y).$$

Let $P \in X$ be a non-singular point. We say that a semistable sheaf I on X is *P-quasistable* with respect to $\mathscr{E}$ if $\chi(I_Y) > e_Y$ for all proper subcurves Y of X containing P. We refer the reader to [Est01] for more general definitions of polarization and stability.

Example 2.1 Let X be a binary curve with components Z_1 and Z_2. Fix an integer y. Take $P \in Z_1$. Let $\mathscr{E} = E_{d,g,y}(X)$ be a line bundle on X with multi-degree $(\frac{d-g+y}{2}, \frac{d-g-y}{2} + 1)$ over (Z_1, Z_2) if $2|d - g - y$ and multi-degree $(\frac{d-g+y+1}{2}, \frac{d-g-y+1}{2})$ if $2 \nmid d - g - y$. Then a simple torsion free rank one sheaf I is P-quasistable if

$$\chi(I_{Z_1}) > \frac{d-g+y}{2} \quad \text{and} \quad \chi(I_{Z_2}) \geq \frac{d-g-y}{2} + 1$$

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when $2|d-g-y$ and

$$\chi(I_{Z_1}) > \frac{d-g+y+1}{2} \quad \text{and} \quad \chi(I_{Z_2}) \geq \frac{d-g-y+1}{2}$$

when $2 \nmid d-g-y$.

On the other hand, recall that in [Cap10], a multi-degree (d_1, d_2) on a genus-g binary curve is balanced if $|d_1 - d_2| \leq g+1$. We adapt this notion and say that (d_1, d_2) is g- *balanced* if $|d_1 - d_2| \leq g+1$. The advantage is that our adaption is independent of the underlying curve. If I is a line bundle on X with multi-degree (d_1, d_2) over (Z_1, Z_2) and I is P-quasistable with respect to $E_{d,g,0}(X)$, then (d_1, d_2) is g-balanced.

Let S be a locally Noetherian scheme over k. Let $\pi: \mathfrak{X} \to S$ be a projective, flat morphism whose fibers are curves of arithmetic genus g. Let $\mathcal{I}$ be an S-flat coherent sheaf on $\mathfrak{X}$. We say that $\mathcal{I}$ is *relatively torsion free* (resp. *rank one*, resp. *simple*) over S if $\mathcal{I}(s)$ is torsion free (resp. rank one, resp. simple) for every (closed) point $s \in S$. We say that $\mathcal{I}$ has *relative degree* d over S if $\mathcal{I}(s)$ has Euler characteristic $d-g+1$. Let $\mathcal{E}$ be a line bundle on $\mathfrak{X}$ of relative degree $d-g+1$ over S. We call such an $\mathcal{E}$ a *relative polarization* on $\mathfrak{X}$ over S. A relatively torsion free, rank one sheaf $\mathcal{I}$ on $\mathfrak{X}$ over S is *relatively stable* (resp. *relatively semistable*) with respect to $\mathcal{E}$ over S if $\mathcal{I}(s)$ is stable (resp. semistable) with respect to $\mathcal{E}(s)$ for all points $s \in S$. Let $\sigma: S \to \mathfrak{X}$ be a section of π through the smooth locus of $\mathfrak{X}$. A relatively torsion free, rank one sheaf $\mathcal{I}$ on $\mathfrak{X}$ over S is *relatively σ-quasistable* with respect to $\mathcal{E}$ over S if $\mathcal{I}(s)$ is $\sigma(s)$-quasistable with respect to $\mathcal{E}(s)$ for all points $s \in S$.

Let $\mathbf{J}^*$ denote the contravariant functor from the category of locally Noetherian S-schemes to sets, defined on an S-scheme T by

$$\mathbf{J}^*(T) = \{\text{relatively simple, torsion free, rank one sheaves on } \mathfrak{X} \times_S T \text{ over } T\}/\sim,$$

where " $\sim$ " is the following equivalence relation:

$$\mathcal{I}_1 \sim \mathcal{I}_2 \iff \text{There is an invertible sheaf } M \text{ on } T \text{ such that } \mathcal{I}_1 \cong \mathcal{I}_2 \otimes \pi^*(M).$$

Let $\mathbf{J}$ be the étale sheaf associated with $\mathbf{J}^*$. Then $\mathbf{J}$ is represented by an algebraic space J by [AK80]. Fix a degree d and a relative polarization $\mathcal{E}$ on $\mathfrak{X}$ of relative degree $d-g+1$. Fix a section σ of π. Let $J^\sigma_{\mathcal{E}}$ be the subspace of J parametrizing relatively simple torsion free rank one sheaves on $\mathfrak{X}$ over S that are relatively σ-quasistable with respect to $\mathcal{E}$. By [Est01, Proposition 27], $J^\sigma_{\mathcal{E}}$ is an open subspace of J.

Theorem 2.2 ([Est01, Theorem A (3)]) *$J^\sigma_{\mathcal{E}}$ is proper over S.*

We will use $J^\sigma_{\mathcal{E}}$ as the compactified relative Jacobian in the proof of our main theorem, and vary the polarization $\mathcal{E}$ for different situations.

2.2 Multi-vanishing Sequence and Ramification.

Let X be a curve of genus g. Fix integers $r, d > 0$ and non-singular points $P_1, \ldots, P_t$ of X. Let $D_\bullet = (D_0, D_1, \ldots)$ be an increasing sequence, possibly infinite, of effective divisors supported on $\{P_1, \ldots, P_t\}$.

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Definition 2.3 A *generalized linear series* of degree d and rank r on X, denoted by a "generalized $\mathfrak{g}^r_d$" on X, is a pair (I, V) consisting of a simple torsion free sheaf I of rank one on X such that $\chi(I) = d - g + 1$ and a subspace $V \subset H^0(X, I)$ of dimension $r + 1$.

We say that a generalized linear series (I, V) is *P-quasistable* with respect to a non-singular point $P \in X$ and a polarization $\mathscr{E}$ on X if I is so. Recall the notion of multi-vanishing sequence in [Oss19].

Definition 2.4 Let (I, V) be a generalized $\mathfrak{g}^r_d$ on X. Inserting $D_{-1} = 0$ to $D_\bullet$ if necessary, we can assume $D_0 = 0$. The *multi-vanishing order* of a section $s \in H^0(X, I)$ along $D_\bullet$ is the degree of D_l where l is the greatest number such that $s \in H^0(X, I(-D_l))$. The *multi-vanishing sequence* of (I, V) along $D_\bullet$ is the sequence

$$v_0 \le v_1 \le \cdots \le v_r,$$

where a value a appears in the sequence m times if for some j we have $\deg D_j = a$ and

$$\begin{cases} \dim V(-D_j) = m & \text{if } D_\bullet \text{ ends with } D_j, \\ \dim(V(-D_j)/V(-D_{j+1})) = m & \text{otherwise.} \end{cases}$$

Example 2.5 Let X be the projective line, $I = \mathcal{O}(4)$ and take a linear series V generated by x, x^3 and $x^4 - x^2$. Denote $P_1 = 0$, $P_2 = 1$, and let

$$D_\bullet = 0, P_1 + P_2, 2P_1 + P_2, 2P_1 + 2P_2, 3P_1 + 2P_2.$$

Then the multi-vanishing order of x (resp. x^3, resp. $x^4 - x^2$) is 0 (resp.0, resp. 3), and the multi-vanishing sequence of (I, V) is 0,2,3.

From now on, for simplicity, let X be a binary curve of genus g with components Z_1 and Z_2, and take $P_i \in Z_i$ for $1 \le i \le t = 2$. The multi-vanishing sequence of a generalized linear series along $D_\bullet$ satisfies the condition of being an admissible sequence, defined as follows.

Definition 2.6 A nondecreasing sequence $(a_j)_{0 \le j \le r}$ is called an *admissible sequence* along $D_\bullet$ if for each j, we have $a_j = \deg D_{l_j}$ for some l_j, and, whenever D_{l_j+1} is defined, the number of repetitions of a_j is at most $\deg(D_{l_j+1} - D_{l_j})$. For a multidegree $\underline{d} = (d_1, d_2)$ over (Z_1, Z_2), we say that an admissible sequence $a_\bullet$ along $D_\bullet$ is $\underline{d}$-*bounded* if

$$D_{l_r} \le (d_1 + 1)P_1 + (d_2 + 1)P_2,$$

and the number of repetitions of $\deg D_{l_r}$ in $a_\bullet$ is at most $d_1 + d_2 + 3 - \deg D_{l_r}$.

Remark 2.7 (i) Note that, for an admissible sequence $a_\bullet$ along $D_\bullet$, the generalized linear series (I, V) has multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ if and only if $\dim V(-D_{l_j}) \ge r + 1 - j$ for all $0 \le j \le r$, where l_j is the number such that $a_j = \deg D_{l_j}$, as in Definition 2.6. As a result, if $\widetilde{D}_\bullet$ is an extension of $D_\bullet$ and $a_\bullet$ is an admissible

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sequence along $\widetilde{D}_\bullet$, then (I, V) has multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ if and only if it has multi-vanishing sequence at least $a_\bullet$ along $\widetilde{D}_\bullet$.

(ii) Fix an admissible sequence $a_\bullet$ along $D_\bullet$. Extending $D_\bullet$ if necessary, we can assume that there exists a term in $D_\bullet$ that has degree strictly more than a_r. Set $a_{-1} = -1$. Let $D'_\bullet$ be the sequence of effective divisors obtained from $D_\bullet$ as follows: for all (l, j, c) such that

$$a_{j-1} < \deg D_l = a_j = a_{j+1} = \cdots = a_{j+c} < a_{j+c+1}$$

or

$$a_{j-1} < \deg D_l = a_j = a_{j+1} = \cdots = a_r \text{ and } j + c = r,$$

insert a sequence of divisors $D_{l,1}, \ldots, D_{l,c}$ to $D_\bullet$ such that

$$D_l < D_{l,1} < D_{l,2} < \cdots < D_{l,c} < D_{l+1} \text{ and } \deg D_{l,u} = \deg D_l + u \text{ for all } 1 \le u \le c.$$

Accordingly, for all (j, c) as above replacing a_{j+u} by $a_j + u$ for $1 \le u \le c$, we obtain an admissible sequence $a'_\bullet$ along $D'_\bullet$. Let $D''_\bullet$ be obtained from $D'_\bullet$ by removing divisors whose degree does not appear in $a'_\bullet$. Then (I, V) has multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ if and only if it has multi-vanishing sequence at least $a'_\bullet$ along $D'_\bullet$, which is equivalent to having multi-vanishing sequence at least $a'_\bullet$ along $D''_\bullet$. Note that $D''_\bullet = (D''_0, \ldots, D''_r)$ and $a'_j = \deg D''_j$ for $0 \le j \le r$. Moreover, if $a_\bullet$ is $\underline{d}$-bounded, we can also pick $D''_\bullet$ such that $D''_\bullet \le (d_1 + 1)P_1 + (d_2 + 1)P_2$.

2.3 A Family of Moduli Schemes of Generalized Linear Series with Ramifications

Consider the family $\pi\colon \mathfrak{X} = X \times \mathbb{A}^1_k \to \mathbb{A}^1_k$. Fix a polarization $\mathcal{E}$ of relative multi-degree (e_1, e_2) and a section σ of π through the smooth locus. It follows from [Est01, Theorem B] that $J^\sigma_{\mathfrak{X},\mathcal{E}} = J^\sigma_{\mathcal{E}}$ is a scheme. Let $\mathcal{L}^\sigma_{\mathfrak{X},\mathcal{E}}$ be the universal sheaf over $\mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}} = J^\sigma_{\mathfrak{X},\mathcal{E}} \times_{\mathbb{A}^1_k} \mathfrak{X}$. Let $\widetilde{\pi}\colon \mathfrak{X} \to X$ and $p_1\colon \mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}} \to J^\sigma_{\mathfrak{X},\mathcal{E}}$ and $p_2\colon \mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}} \to \mathfrak{X}$ be the natural maps as in diagram (2.1). For $1 \le i \le 2$, let D_i be an effective divisor on $Z_i \backslash (Z_1 \cap Z_2)$ of degree at least $g + 1 - e_i$, and let $\mathcal{D}'$ be the divisor $p_2^*(\widetilde{\pi}^*(D_1 + D_2))$ on $\mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}}$.

$$\begin{array}{ccccc} \mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}} & \xrightarrow{p_2} & \mathfrak{X} & \xrightarrow{\widetilde{\pi}} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} & & \\ J^\sigma_{\mathfrak{X},\mathcal{E}} & \longrightarrow & \mathbb{A}^1_k & & \end{array} \tag{2.1}$$

Note that the restriction of $\mathcal{L}^\sigma_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')$ at each fiber of $\mathfrak{J}^\sigma_{\mathfrak{X},\mathcal{E}}$ over $J^\sigma_{\mathfrak{X},\mathcal{E}}$ has a $(d - g + 1 + \deg(D_1 + D_2))$-dimensional space of global sections and trivial first cohomology group. Indeed, for each torsion free sheaf I in $J^\sigma_{\mathfrak{X},\mathcal{E}}$, we have

$$I \otimes \mathcal{O}_X(D_1 + D_2) = n_{I*}\big(L_I \otimes \mathcal{O}_{X_I}(D_1 + D_2)\big),$$

where n_I, X_I, and L_I are defined in Section 2.1. Since n_I is an affine morphism, by [Har77, Exercise 3.8.2], we have

$$h^i\big(I \otimes \mathcal{O}_X(D_1 + D_2)\big) = h^i\big(L_I \otimes \mathcal{O}_{X_I}(D_1 + D_2)\big) \text{ for } i = 0, 1.$$

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As $I_{Z_i} = L_I|_{Z_i}$ for each $1 \le i \le 2$, the line bundle $L_I \otimes \mathcal{O}_{X_I}(D_1 + D_2)$ has multi-degree at least (g, g) along (Z_1, Z_2) by semi-stability of I. Therefore, the conclusion follows from Riemann–Roch theorem.

By [FK94, §0.5], the pushforward $p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')$ is a vector bundle of rank $d - g + 1 + \deg(D_1 + D_2)$ on $J^{\sigma}_{\mathfrak{X},\mathcal{E}}$ whose fibers at a point are identified with the global sections of the restriction of $\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')$ to the fiber of $\mathfrak{J}^{\sigma}_{\mathfrak{X},\mathcal{E}}$ at that point.

Definition 2.8 The family $\overline{W}^r_d(\mathfrak{X}/\mathbb{A}^1_k)$ of Brill–Noether loci over $\mathbb{A}^1_k$ is the locus on $J^{\sigma}_{\mathfrak{X},\mathcal{E}}$ on which the map

$$p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}') \to p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')|_{\mathcal{D}'}$$

has kernel of dimension at least $r + 1$. Here, $\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')|_{\mathcal{D}'}$ is the restriction of $\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')$ on $\mathcal{D}'$.

In the definition above, $\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}$ is not a line bundle over $\mathfrak{J}^{\sigma}_{\mathfrak{X},\mathcal{E}}$, but we still have the exact sequence

$$0 \longrightarrow \mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}} \longrightarrow \mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}') \longrightarrow \mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')|_{\mathcal{D}'} \longrightarrow 0,$$

since $\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}$ is locally free at points where $\mathcal{O}_{\mathcal{D}'}$ is nonzero. Hence, the kernel of the map in Definition 2.8 at a point of $J^{\sigma}_{\mathfrak{X},\mathcal{E}}$ is identified with the space of global sections of the corresponding torsion free sheaf on X.

Definition 2.9 Let $p\colon Gr(r+1, p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')) \to J^{\sigma}_{\mathfrak{X},\mathcal{E}}$ be the relative Grassmannian over $J^{\sigma}_{\mathfrak{X},\mathcal{E}}$ and let $\mathscr{V} \to p^* p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')$ be the tautological subbundle. The family $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ of moduli spaces of generalized linear series on X over $\mathbb{A}^1_k$ is the locus on $Gr(r+1, p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}'))$ where the composed map

$$\mathscr{V} \longrightarrow p^* p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}') \longrightarrow p^* p_{1*}\mathcal{L}^{\sigma}_{\mathfrak{X},\mathcal{E}}(\mathcal{D}')|_{\mathcal{D}'}$$

is zero.

Let $\mathcal{D}$ be a divisor on $\mathfrak{X}$ supported on the smooth locus over $\mathbb{A}^1_k$. Suppose $\mathcal{O}_{\mathcal{D}} = \mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}(-\mathcal{D})$ is flat over $\mathbb{A}^1_k$. Then we can construct a family $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma, \mathcal{D}, b)$ of moduli space of generalized linear series on X over $\mathbb{A}^1_k$ with given ramification (corresponding to b) at $\mathcal{D}$ as the locus on $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ on which the induced map

$$\mathscr{V} \longrightarrow p^* p_{1*}(\mathcal{L}^{\sigma}_{X,\mathcal{E}}|_{p_2^*(\mathcal{D})})$$

has kernel of dimension at least b.

Definition 2.10 For $i = 1, 2$, let $\mathcal{P}_i \subset Z_i \times \mathbb{A}^1_k$ be two sections of π supported on the smooth locus of $\mathfrak{X}$. Let $\mathcal{D}_\bullet = a^1_\bullet \mathcal{P}_1 + a^2_\bullet \mathcal{P}_2$ be an increasing sequence of effective divisors and let $a_\bullet$ be an admissible sequence along $\mathcal{D}_\bullet$ (or along $(\mathcal{D}_\bullet)_z$ for each $z \in \mathbb{A}^1_k$). For each $0 \le j \le r$, take l_j such that $a_j = a^1_{l_j} + a^2_{l_j} = \deg \mathcal{D}_{l_j}$. The family $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma; \mathcal{D}_\bullet, a_\bullet)$ of moduli spaces over $\mathbb{A}^1_k$ of generalized linear series on

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X with ramification imposed by $(\mathcal{D}_\bullet, a_\bullet)$ is the intersection over all $0 \le j \le r$ of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma, \mathcal{D}_{l_j}, r+1-j)$ in $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$.

By [FK94, §0.5], our constructions above is compatible with base extension $T \to \mathbb{A}^1_k$. In particular, we have the following proposition.

Proposition 2.11 *The spaces $\overline{W}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$, $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$, $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma, \mathcal{D}, b)$, and $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma; \mathcal{D}_\bullet, a_\bullet)$ are all proper over $\mathbb{A}^1_k$. Moreover, we have the following:*

(i) *The fiber of $\overline{W}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ at $z \in \mathbb{A}^1_k$ parameterizes $\sigma(z)$-stable (with respect to $\mathcal{E}_z$) simple torsion free sheaves of rank one on X that have global sections of dimension at least $r+1$.*

(ii) *The fiber of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ at $z \in \mathbb{A}^1_k$ parameterizes $\sigma(z)$-quasistable generalized $\mathfrak{g}^r_d$s with respect to $\mathcal{E}_z$ on X.*

(iii) *The fiber of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma, \mathcal{D}, b)$ at $z \in \mathbb{A}^1_k$ parametrizes $\sigma(z)$-quasistable generalized $\mathfrak{g}^r_d$s (I, V) with respect to $\mathcal{E}_z$ on $\mathfrak{X}_z = X$ such that $\dim V(-\mathcal{D}_z) \ge b$.*

(iv) *The fiber of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma; \mathcal{D}_\bullet, a_\bullet)$ at $z \in \mathbb{A}^1_k$ parameterizes $\sigma(z)$-quasistable generalized $\mathfrak{g}^r_d$s with respect to $\mathcal{E}_z$ on $\mathfrak{X}_z = X$ whose multi-vanishing sequence along $(\mathcal{D}_\bullet)_z$ is at least $a_\bullet$.*

Since $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma; \mathcal{D}_\bullet, a_\bullet)$ is a Schubert subscheme (*cf.* [Oss, §C.2]) of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$, and $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ is a determinantal subscheme of $Gr(r+1, p_{1*}\mathcal{L}^\sigma_{\mathfrak{X},\mathcal{E}}(\mathcal{D}'))$, the fiber at each $z \in \mathbb{A}^1_k$ of $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma; \mathcal{D}_\bullet, a_\bullet)$ (resp. $\overline{G}^r_d(\mathfrak{X}/\mathbb{A}^1_k, \mathcal{E}, \sigma)$) is expected to have dimension

$$g - (r+1)(g-d+r) - \left(\sum_{j=0}^{r}(a_j - j) + \sum_{l\ge 0}\binom{r_l}{2}\right) \text{ (resp. } \rho_{g,r,d}).$$

Here, r_l is the number of repetitions of $\deg D_l$ in $a_\bullet$

2.4 A Decomposition of the Space of Generalized Linear Series

Now for a line bundle $\mathscr{E}$ on X and a non-singular point P of X, we denote by $\overline{W}^r_d(X, \mathscr{E}, P)$ (resp. $\overline{G}^r_d(X, \mathscr{E}, P)$) the space parameterizing P-quasistable simple torsion free, rank one sheaves (resp. generalized $\mathfrak{g}^r_d$s) on X. For a sequence $D_0, \dots, D_r$ of increasing effective divisors supported on $\{P_1, P_2\}$, where P_i is a non-singular point in Z_i, as in Section 2.2, denote by $\overline{G}^r_d(X, \mathscr{E}, P; D_\bullet, a_\bullet)$ the space parametrizing P-quasistable generalized $\mathfrak{g}^r_d$s with multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$.

On the other hand, fix a multi-degree $\underline{d} = (d_1, d_2)$ over (Z_1, Z_2). By a $\mathfrak{g}^r_{\underline{d}}$ on X, we mean a linear series (L, V) on X such that L has multi-degree $\underline{d}$ and V has dimension $r+1$. We denote by $W^r_{\underline{d}}(X)$ (resp. $G^r_{\underline{d}}(X)$, resp. $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$) the space parametrizing line bundles of multi-degree $\underline{d}$ on X whose space of global sections has dimension at least $r+1$ (resp. $\mathfrak{g}^r_{\underline{d}}$s on X, resp. $\mathfrak{g}^r_{\underline{d}}$s on X whose multi-vanishing sequence along $D_\bullet$ is at least $a_\bullet$).

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Moreover, note that the stability for a torsion free sheaf I of rank one on X with respect to $\mathscr{E}$ only depends on the multi-degree of L_I. Accordingly, we say that a multi-degree (d_1, d_2) is *stable* (resp. *semistable*, resp. *P-quasistable*) if $d_i + 1 > \deg \mathscr{E}|_{Z_i}$ for each i (resp. $d_i + 1 \geq \deg \mathscr{E}|_{Z_i}$ for each i, resp. $d_i + 1 > \deg \mathscr{E}|_{Z_i}$ if $P \in Z_i$ and $d_i + 1 \geq \deg \mathscr{E}|_{Z_i}$ otherwise) for convenience. Hence, I is P-quasistable with respect to $\mathscr{E}$ if and only if the multi-degree of L_I is P-quasistable with respect to $\mathscr{E}$. Note also that the global sections of I are identified with the global sections of L_I. As a result, we have the following proposition.

Proposition 2.12 *Let S be the set of singular points of X. For each subset $J \subsetneq S$, let X_J be the normalization of X at J. Then*

$$\overline{W}^r_d(X, \mathscr{E}, P) = \bigcup_{J \subsetneq S} \bigcup_{\underline{d}_J} W^r_{\underline{d}_J}(X_J),$$

$$\overline{G}^r_d(X, \mathscr{E}, P) = \bigcup_{J \subsetneq S} \bigcup_{\underline{d}_J} G^r_{\underline{d}_J}(X_J),$$

$$\overline{G}^r_d(X, \mathscr{E}, P; D_\bullet, a_\bullet) = \bigcup_{J \subsetneq S} \bigcup_{\underline{d}_J} G^r_{\underline{d}_J}(X_J; D_\bullet, a_\bullet).$$

Here, $\underline{d}_J$ runs over all multi-degrees (d^1_J, d^2_J) that are P-quasistable with respect to $\mathscr{E}$ such that $d^1_J + d^2_J = d - |J|$.

3 The Proof of the Main Theorem

In this section, we restate Theorem 1.1 and give a proof. Our proof proceeds by induction on genus. For the inductive step, we reduce the genus by imposing an extra ramification condition. For convenience, we specify the following notation.

Notation 3.1 *Let X be a binary curve of genus g with components Z_1 and Z_2. Let $D_\bullet$ (resp. $E_\bullet$) be an increasing sequence of effective divisors supported on $\{P_1, P_2\}$ (resp $\{P'_1, P'_2\}$), where P_i and P'_i are non-singular points of X in Z_i and $P_i \neq P'_i$ for $1 \leq i \leq 2$. Let $a_\bullet$ (resp. $a'_\bullet$) be an admissible sequence along $D_\bullet$ (resp. $E_\bullet$). Let $\underline{d}$ be a multi-degree over (Z_1, Z_2). Recall that $G^r_{\underline{d}}(X)$ (resp. $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$) denotes the space of $\mathfrak{g}^r_{\underline{d}}$s on X (resp. $\mathfrak{g}^r_{\underline{d}}$s on X with multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$). Let $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)^\circ$ be the subspace of $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ parametrizing pairs (L, V) such that $\dim h^0(X, L) = r + 1$. Let*

$$\partial G^r_{\underline{d}}(X; D_\bullet, a_\bullet) = G^r_{\underline{d}}(X; D_\bullet, a_\bullet) \backslash G^r_{\underline{d}}(X; D_\bullet, a_\bullet)^\circ.$$

Also, let $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)^{\circ\circ}$ be the subspace of $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)^\circ$ parametrizing $\mathfrak{g}^r_{\underline{d}}$s whose multi-vanishing sequence along $D_\bullet$ is exactly $a_\bullet$. Moreover, denote by

$$G^r_{\underline{d}}(X; D_\bullet, a_\bullet; D'_\bullet, a'_\bullet) = G^r_{\underline{d}}(X; D_\bullet, a_\bullet) \times_{G^r_{\underline{d}}(X)} G^r_{\underline{d}}(X; D'_\bullet, a'_\bullet)$$

the space of $\mathfrak{g}^r_d$s on X with multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ and $a'_\bullet$ along $D'_\bullet$.

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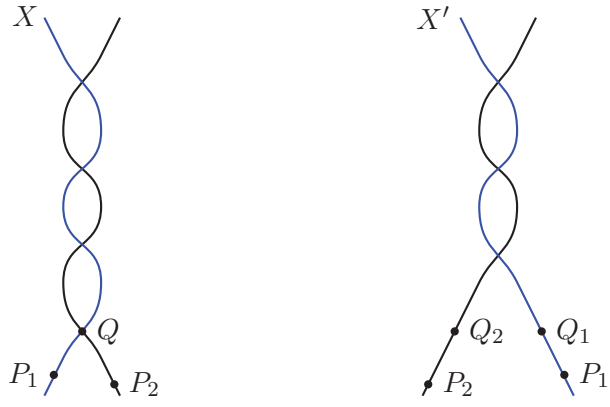


Figure 3:

Lemma 3.2 *Let (d_1, d_2) be a multi-degree over (Z_1, Z_2) such that either of the following is true:*

(i) *(d_1, d_2) is g-balanced as in Example 2.1;*
(ii) *$d_i \geq -1$ for each i;*
(iii) *$d_i \leq g + r$ for each i.*

Suppose $d < g + r$ and $G^r_{\underline{d}}(X)$ is nonempty. Then $r \leq d_i \leq d - r$ for each i.

Proof Take $(L, V) \in G^r_{\underline{d}}(X)$. If $d_1 \geq g$, then the restriction map

$$\phi\colon H^0(X, L) \longrightarrow H^0(Z_2, L|_{Z_2})$$

is surjective with kernel of dimension $d_1 - g$. As a result,

$$h^0(X, L) = d_1 - g + \max\{d_2 + 1, 0\} \leq r,$$

which is impossible. Hence, we conclude that $d_1 \leq g$, and similarly $d_2 \leq g$. It follows that ϕ is injective. Thus, $r + 1 \leq h^0(Z_2, L|_{Z_2}) = d_2 + 1$. As a result, $d_2 \geq r$ and $d_1 = d - d_2 \leq d - r$, and vice versa. ∎

Let X' be the normalization of X at a node Q, whose preimage in Z_i is Q_i, as in Figure 3. Let $E_0 = 0$ and $E_1 = Q_1 + Q_2$ be divisors on X'. Let $b_0 = 0$ and $b_j = 2$ for all $1 \leq j \leq r$. Denote $G^r_{\underline{d}}(X'; D_\bullet, a_\bullet; E_\bullet, b_\bullet)$ by G_{E_0}. By pulling back a $\mathfrak{g}^r_d$ on X to X', we obtain a natural map

$$\varphi\colon G^r_{\underline{d}}(X; D_\bullet, a_\bullet) \longrightarrow G_{E_0}.$$

Let G_{E_1} be the subspace of G_{E_0} consisting of $\mathfrak{g}^r_{\underline{d}}$s with base points Q_1 and Q_2. We have the following lemma.

Lemma 3.3 *The map φ above is injective over $G_{E_0} \backslash G_{E_1}$ and has fiber of dimension at most one over G_{E_1}.*

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Proof Let n_Q be the normalization map. We have a commutative diagram

$$\begin{array}{ccc} G^r_{\underline{d}}(X; D_\bullet, a_\bullet) & \xrightarrow{\varphi} & G_{E_0} \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \xrightarrow{n_Q^*} & \mathrm{Pic}(X'). \end{array}$$

Let (L', V') be a $\mathfrak{g}^r_d$ on X' and L be a line bundle on X such that $n_Q^*(L) = L'$. Since the induced map $H^0(X, L) \to H^0(X', L)$ is injective, there is at most one $(r+1)$-dimensional subspace V of $H^0(X, L)$ that maps to V'. Therefore, the fiber dimension of φ at G_{E_1} is at most one, since the fiber of n_Q^* has dimension one. If $(L', V') \notin G_{E_1}$, then we can take a section $s \in V'$ such that either $s(Q_1) \neq 0$ or $s(Q_2) \neq 0$. Then either there is no section of L that restricts to s or the glueing data of L at Q is determined by s. Hence, the fiber of φ at (L', V') consists of at most one point. ∎

We now restate and prove the main theorem.

Theorem 3.4 *Let X be a general genus g binary curve with components Z_1 and Z_2. Let $P_i \in Z_i$ be two general points. Let $\underline{d} = (d_1, d_2)$ be a multi-degree where $-1 \leq d_i$ for each i and $d_1 + d_2 = d$. Let $r \geq 0$ be an integer. Let $a_\bullet$ be an admissible sequence along an increasing sequence $D_\bullet$ of effective divisors supported on $\{P_1, P_2\}$. Suppose either (i) $d_i \leq g$ for each i or (ii) $a_\bullet$ is $\underline{d}$-bounded. Then the space $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ of linear series of multi-degree $\underline{d}$ and rank r that has multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$ is either empty or of dimension*

$$\rho = g - (r+1)(g - d + r) - \left(\sum_{j=0}^{r} (a_j - j) + \sum_{l \geq 0} \binom{r_l}{2} \right).$$

Here, r_l is the number of repetitions of $\deg D_l$ in $a_\bullet$. In particular, $G^r_{\underline{d}}(X)$ is either empty or of dimension $\rho_{g,r,d}$.

Proof According to Remark 2.7(ii), we can assume $D_\bullet = (D_0, \ldots, D_r)$, where $D_j = a^1_j P_1 + a^2_j P_2$, and $a_j = a^1_j + a^2_j = \deg D_j$ for each $0 \leq j \leq r$. Hence, $a^i_{j+1} \geq a^i_j \geq 0$ and $a_{j+1} > a_j$ for each i and j. Moreover, we have either (i) $d_i \leq g$ for each i or (ii) $a^i_r \leq d_i + 1$ for each i, and the expected dimension ρ becomes

$$\rho = g - (r+1)(g - d + r) - \sum_{j=0}^{r} (a_j - j).$$

Since $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ is a Schubert subscheme of $G^r_{\underline{d}}(X)$, which is a determinantal subscheme of the relative Grassmannian over $\mathrm{Pic}^{\underline{d}}(X)$, it has dimension at least ρ if nonempty. Hence, it remains to show that $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ has dimension at most ρ.

We prove this by induction on g. Since when $d_i \leq g$ for each i and $a^1_r > d_1 + 1$, the space $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ is empty, we can always assume $a^1_r \leq d_1 + 1$. Similarly assume $a^2_r \leq d_2 + 1$. For the $g = 0$ case, there is a unique line bundle $L_{\underline{d}}$ on X with multi-degree $\underline{d}$. Since $h^0(X, L_{\underline{d}}) = d + 1$, the space $G^r_{\underline{d}}(X)$ of linear series on X is identified with the

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Grassmannian $Gr(r+1, d+1)$, and $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ is a Schubert variety defined by the partial flag

$$H^0\big(X, L_{\underline{d}}(-D_r)\big) \subset H^0\big(X, L_{\underline{d}}(-D_{r-1})\big) \subset \cdots \subset H^0\big(X, L_{\underline{d}}(-D_0)\big).$$

Note that $H^0(X, L_{\underline{d}}(-D_j))$ has dimension exactly $d - a_j + 1$. Therefore, it is easy to verify that $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ has dimension as expected.

Suppose the conclusion is true for $g-1$. Let X', Q, Q_i, and G_{E_j} be as above. By Lemma 3.3, it is enough to show that G_{E_0} has dimension (at most) ρ, and G_{E_1} has dimension at most $\rho - 1$. If $a^i_r = d_i + 1$ for each i, then $G^r_{\underline{d}}(X'; D_\bullet, a_\bullet)$ is empty, so is G_{E_1}. If $a^1_r \le d_1$, then $G^r_{(d_1-1, d_2)}(X'; D_\bullet, a_\bullet)$ has dimension at most $\rho - 1$ by induction. Since it contains G_{E_1}, the dimension of G_{E_1} is at most $\rho - 1$, and similarly for the case $a^2_r \le d_2$. We next show that G_{E_0} has dimension at most ρ.

Consider the trivial family $\pi\colon \mathfrak{X}' = X' \times \mathbb{A}^1_k \to \mathbb{A}^1_k$. Denote by $\widetilde{\pi}\colon \mathfrak{X}' \to X'$ the other projection. Denote by $\widetilde{g} = g - 1$.

(a) Assume $d \le g + r - 2 = \widetilde{g} + r - 1$. By Lemma 3.2, we can assume $r \le d_i \le \widetilde{g} - 1$ for $1 \le i \le 2$. Take $\mathcal{E} = \widetilde{\pi}^*(\mathscr{E})$, where $\mathscr{E} = E_{d,\widetilde{g},0}(X')$ is defined in Example 2.1. Let $\sigma = P \times \mathbb{A}^1_k$ where $P \in Z_1$. It is easy to verify that G_{E_0} is contained in $\overline{G}^r_d(X', \mathscr{E}, P)$.

Consider the degeneration $Q_1 \to P_1$ as $z \to 0$. Let G be the limit of G_{E_0} in $\overline{G}^r_d(\mathfrak{X}'/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ at $z = 0$. Take $(I_0, V_0) \in G$. For each $0 \le l \le r$, we have

$$\dim V(-a^1_l P_1 - a^2_l P_2 - Q_1 - Q_2) \ge r - l \text{ for all } (I, V) \in G_{E_0}.$$

By Proposition 2.11(iii) (restrict to an open subvariety of $\mathbb{A}^1_k$ if necessary to make sure the limit $Q_1 \to P_1$ misses the singular points of each fiber of $\mathfrak{X}'$), it follows that

$$\dim V_0(-(a^1_l + 1)P_1 - a^2_l P_2 - Q_2) \ge r - l$$

for each l.

Let $m \in \{-1, 0, 1, \ldots, r\}$ be the unique number such that

$$\dim V_0\big(-(a^1_l + 1)P_1 - a^2_l P_2\big) \ge r - l + 1$$

for all $0 \le l \le m$ and

$$\dim V_0\big(-(a^1_l + 1)P_1 - a^2_l P_2\big) = r - l.$$

for $l = m + 1$ (if $m = r$, then this condition is waived).

If $m = r$, then

$$\dim V_0\big(-(a^1_l + 1)P_1 - a^2_l P_2\big) \ge r - l + 1$$

for all $0 \le l \le r$.

We next assume $m < r$. Then

$$\dim V_0(-a^1_{m+1} P_1 - a^2_{m+1} P_2) = r - m$$

and

$$\begin{aligned}\dim V_0(-(a^1_{m+1} + 1)P_1 - a^2_{m+1} P_2) &= r - m - 1 \\ &= \dim V_0(-(a^1_{m+1} + 1)P_1 - a^2_{m+1} P_2 - Q_2).\end{aligned}$$

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We thus have

$$V_0\big(-(a^1_{m+1}+1)P_1-a^2_{m+1}P_2\big)=V_0\big(-(a^1_{m+1}+1)P_1-a^2_{m+1}P_2-Q_2\big).$$

Namely, all sections in $V_0(-(a^1_{m+1}+1)P_1-a^2_{m+1}P_2)$ vanish at Q_2.

Let n be the smallest number such that $a^1_n \geq a^1_{m+1}+1$. Then for all $l \geq n$, we have

$$V_0(-a^1_lP_1-a^2_lP_2)\subset V_0\big(-(a^1_{m+1}+1)P_1-a^2_{m+1}P_2\big).$$

Hence, all sections of $V_0(-a^1_lP_1-a^2_lP_2)$ vanish at Q_2. As a result,

$$\dim V_0(-a^1_lP_1-a^2_lP_2-Q_2)=\dim V_0(-a^1_lP_1-a^2_lP_2)\geq r-l+1.$$

Let $\mathfrak{X}'_0$ be the fiber of $\mathfrak{X}'$ at $z=0$. It follows that G is contained in a union over all possible (m,n) of subspaces $G_{m,n}$ of $\overline{G}^r_d(\mathfrak{X}'_0,\mathscr{E},P)$ consisting of pairs (I_0,V_0) satisfying the flowing conditions:

$$\begin{cases}\dim V_0(-a^1_lP_1-a^2_lP_2-Q_2)\geq r-l+1, & l\geq n\\ \dim V_0(-(a^1_l+1)P_1-a^2_lP_2)\geq r-l+1, & l\leq m\\ \dim V_0(-a^1_lP_1-a^2_lP_2)\geq r-l+1, & 0\leq l\leq r\\ \dim V_0(-(a^1_l+1)P_1-a^2_lP_2-Q_2)\geq r-l, & 0\leq l\leq r.\end{cases}$$

Now let $Q_2\to P_2$; then the conditions above become (again, by Proposition 2.11(iii))

$$\begin{cases}\dim V_0(-a^1_lP_1-(a^2_l+1)P_2)\geq r-l+1, & l\geq n\\ \dim V_0(-(a^1_l+1)P_1-a^2_lP_2)\geq r-l+1, & l\leq m\\ \dim V_0(-a^1_lP_1-a^2_lP_2)\geq r-l+1, & 0\leq l\leq r\\ \dim V_0(-(a^1_l+1)P_1-(a^2_l+1)P_2)\geq r-l, & 0\leq l\leq r.\end{cases}$$

Let h be the smallest number such that $a^1_h=a^1_{m+1}$ and let $u\in\{h-1,h,\dots,n-1\}$ be the unique number such that

$$\dim V_0\big(-a^1_lP_1-(a^2_l+1)P_2\big)\geq r-l+1$$

for all $h\leq l\leq u$ and

$$\dim V_0\big(-a^1_lP_1-(a^2_l+1)P_2)\big)=r-l=\dim V_0\big(-(a^1_l+1)P_1-(a^2_l+1)P_2\big).$$

for $l=u+1$. Similarly as above, we have

$$V_0\big(-a^1_{u+1}P_1-(a^2_{u+1}+1)P_2)\big)=V_0\big(-(a^1_{u+1}+1)P_1-(a^2_{u+1}+1)P_2\big).$$

Now for all $u+1<l\leq n-1$, we have

$$V_0(-a^1_lP_1-a^2_lP_2)\subset V_0\big(-a^1_{u+1}P_1-(a^2_{u+1}+1)P_2\big);$$

thus,

$$V_0(-a^1_lP_1-a^2_lP_2)=V_0\big(-(a^1_l+1)P_1-a^2_lP_2\big).$$

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It follows that the limit of $G_{m,n}$ in $\overline{G}^r_d(\mathfrak{X}'/\mathbb{A}^1_k, \mathcal{E}, \sigma)$ as $Q_2 \to P_2$ is contained in the union over all possible (u, h) of $\overline{G}^r_d(X', \mathscr{E}, P; D'_\bullet, a'_\bullet)$, where

$$(a^1_j)' = \begin{cases} a^1_j + 1, & j < h \text{ or } u+1 < j < n, \\ a^1_j, & h \le j \le u+1 \text{ or } j \ge n, \end{cases}$$

and

$$(a^2_j)' = \begin{cases} a^2_j, & j < h \text{ or } u+1 \le j \le n-1, \\ a^2_j + 1, & h \le j \le u \text{ or } j \ge n, \end{cases}$$

and $D'_j = (a^1_j)'P_1 + (a^2_j)'P_2$ and $a'_j = \deg D'_j$ (it is possible that $D'_u = D'_{u+1}$, in this case we replace $(a^1_{u+1})'$ with $(a^1_{u+1})' + 1$) for $0 \le j \le r$. Note that we have $a'_j = a_j + 1$ unless $j = u+1$ and $D'_u \ne D'_{u+1}$, in which case $a'_j = a_j$.By Proposition 2.12, we have

$$\overline{G}^r_d(X', \mathscr{E}, P; D'_\bullet, a'_\bullet) = \bigcup_{J \subsetneq S} \bigcup_{\underline{d}_J} G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet),$$

where S is the set of nodes of X', and X'_J is the normalization of X' at J, and $\underline{d}_J$ runs over all P-quasistable multi-degrees (d^1_J, d^2_J) with respect to $\mathscr{E}$ such that $d_J := d^1_J + d^2_J = d - |J|$. Note that X'_J has genus $g_J = \widetilde{g} - |J|$. Straightforward calculation shows that (d^1_J, d^2_J) is g_J-balanced. Hence by Lemma 3.2, either $G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet)$ is empty or $r \le d^i_J \le g_J$ for $1 \le i \le 2$. By induction, $G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet)$ has dimension at most

$$\rho_J := g_J - (r+1)(g_J - d_J + r) - \sum_{j=0}^{r} (a'_j - j) \le \rho.$$

Hence, $G_{m,n}$ has dimension at most ρ, and so do G and G_{E_0}.

(b) Let $d = g + r - 1 = \widetilde{g} + r$. In this case, we cannot apply Lemma 3.2 for X' and $\widetilde{g}$ (or for any partial normalization X'_J of X' and g_J), as in part (a). Nevertheless, we can still apply this lemma for X and g and assume $r \le d_i \le \widetilde{g}$ for $1 \le i \le 2$. We can assume $a^i_r \le d_i$ for each i, since otherwise, $G^r_{\underline{d}}(X'; D_\bullet, a_\bullet)$ is empty. Without loss of generality, suppose $a^1_r \ge a^2_r$, and let $y = a^1_r - a^2_r \ge 0$. Take a line bundle $\mathscr{E}_y = E_{d,\widetilde{g},y}(X')$ on X' as in Example 2.1 and let $\mathcal{E}_y = \widetilde{\pi}^*(\mathscr{E}_y)$. As in part (a), let $\sigma = P \times \mathbb{A}^1_k$ where $P \in Z_1$. We first verify that $G^r_{\underline{d}}(X')$ is contained in $\overline{G}^r_d(X', \mathscr{E}_y, P)$. Indeed, we have

$$d_1 + 1 \ge a^1_r + 1 = \frac{1}{2}(a_r + y) + 1 \ge \frac{r+y}{2} + 1 > \frac{d - \widetilde{g} + y + 1}{2},$$
$$d_2 + 1 = d - d_1 + 1 \ge d - \widetilde{g} + 1 \ge \frac{d - \widetilde{g} - y}{2} + 1.$$

Hence, $G^r_{\underline{d}}(X') \subset \overline{G}^r_d(X', \mathscr{E}_y, P)$.

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Now let $Q_1 \to P_1$ and $Q_2 \to P_2$. By the argument in part (a), the limit of G_{E_0} is contained in a union of subspaces of $\overline{G}^r_d(X', \mathscr{E}_y, P)$ of the form

$$\overline{G}^r_d(X', \mathscr{E}_y, P; D'_\bullet, a'_\bullet) = \bigcup_{J \subsetneq S} \bigcup_{\underline{d}_J} G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet),$$

where $D'_\bullet$ and $a'_\bullet$ are as above and $\underline{d}_J$ runs over all P-quasistable multi-degrees with respect to $\mathscr{E}_y$. Note that

$$a^i_j \le (a^i_j)' \le a^i_j + 1 \text{ for each } i \text{ and } j.$$

It follows that $y - 1 \le (a^1_r)' - (a^2_r)' \le y + 1$.

For each $\underline{d}_J = (d^1_J, d^2_J)$, let $d_J = d^1_J + d^2_J$. If $(a^1_r)' > d^1_J + 1$, then

$$\begin{aligned} d^2_J = d_J - d^1_J &\le d_J - \frac{d - \widetilde{g} + y}{2} = d_J - (d - \widetilde{g}) + 1 + \frac{d - \widetilde{g} + y}{2} - (y+1) \\ &\le g_J + 1 + d^1_J - (y+1) < g_J + 1 + (a^1_r)' - (y+1) \le g_J + 1 + (a^2_r)'. \end{aligned}$$

Hence, there is no nonzero section s of a line bundle on X'_J with multi-degree $\underline{d}_J$ such that s has multi-vanishing order at least a'_r along $D'_\bullet$. Thus, $G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet)$ is empty. Similarly, if $(a^2_r)' > d^2_J + 1$, then

$$\begin{aligned} d^1_J = d_J - d^2_J &\le d_J - \frac{d - \widetilde{g} - y - 1}{2} < d_J - (d - \widetilde{g}) + 1 + \frac{d - \widetilde{g} - y - 1}{2} + 2 + (y-1) \\ &\le g_J + 1 + d^2_J + 2 + (y-1) \le g_J + 1 + (a^2_r)' + (y-1) \le g_J + 1 + (a^1_r)'. \end{aligned}$$

Hence, $G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet)$ is empty.

We now suppose $(a^i_r)' \le d^i_J + 1$ for each i, in other words $a'_\bullet$ is $\underline{d}_J$ bounded. By induction each $G^r_{\underline{d}_J}(X'_J; D'_\bullet, a'_\bullet)$ has dimension at most ρ_J, which is defined in part (a). Hence, the space $\overline{G}^r_d(X', \mathscr{E}_y, P; D'_\bullet, a'_\bullet)$ has dimension at most ρ, and so does G_{E_0}.

(c) Let $d = g + r$. Recall that we assumed $a^i_r \le d_i + 1$ for each i in the beginning of the proof. We extend $D_\bullet$ to an infinite sequence $\widehat{D}_\bullet$ such that $\deg \widehat{D}_j = j$ for all $j \ge 0$ and $\widehat{D}_{d+2} = (d_1 + 1)P_1 + (d_2 + 1)P_2$. Then

$$G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet) = G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$$

by Remark 2.7(i).

(c.1) We first show that $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^\circ$ has dimension at most ρ. If $a_r = r$, then $\rho = g$, and the conclusion is obvious. We can assume $a_r > r$. Let λ be the smallest number such that $a_\lambda > \lambda$. Consider an injective map

$$G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^{\circ\circ} \longrightarrow G^{r-\lambda}_{\underline{d}'}(X; D''_\bullet, a''_\bullet)^{\circ\circ},$$

where (L, V) is mapped to $(L(-D_\lambda), V(-D_\lambda))$. Here, $d'_i = d_i - a^i_\lambda$ and $a''_j = a_{j+\lambda} - a_\lambda$ and $D''_j = D_{j+\lambda} - D_\lambda$ for $1 \le i \le 2$ and $0 \le j \le r - \lambda$, and $\underline{d}' = (d'_1, d'_2)$. Note that $G^{r-\lambda}_{\underline{d}'}(X; D''_\bullet, a''_\bullet)$ is expected to have dimension ρ. Note also that $d' := d'_1 + d'_2 < g + r - \lambda$, and

$$d'_1 = d_1 - a^1_\lambda \le d_1 - \lambda - 1 + a^2_\lambda \le d_1 - \lambda + d_2 = g + r - \lambda,$$

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and, similarly, $d'_2 \le g + r - \lambda$. By Lemma 3.2, either $0 \le d'_i \le g$ for $1 \le i \le 2$ or $G^{r-\lambda}_{\underline{d}'}(X; D''_\bullet, a''_\bullet)$ is empty. Hence by part (a) and (b), $G^{r-\lambda}_{\underline{d}'}(X; D''_\bullet, a''_\bullet)^{\circ\circ}$ is either empty or of dimension ρ, and so is $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^{\circ\circ}$.

Now let $\widetilde{a}_\bullet \ge a_\bullet$ be another admissible sequence along $\widehat{D}_\bullet$. Note that $\widetilde{a}_j = \deg \widehat{D}_{\widetilde{a}_j}$ for all j. Let $\widehat{D}_{\widetilde{a}_r} = \widetilde{a}^1_r P_1 + \widetilde{a}^2_r P_2$. By construction of $\widehat{D}_\bullet$, if $\widetilde{a}^1_r > d_1 + 1$, then $\widetilde{a}^2_r \ge d_2 + 1$. Hence, $G^r_{\underline{d}}(X; \widehat{D}_\bullet, \widetilde{a}_\bullet)^{\circ\circ}$ is empty. We can assume $\widetilde{a}^i_r \le d_i + 1$ for each i. The same argument as above shows that $G^r_{\underline{d}}(X; \widehat{D}_\bullet, \widetilde{a}_\bullet)^{\circ\circ}$ is either empty or of expected dimension. Moreover, there are only finitely many $\widetilde{a}_\bullet \ge a_\bullet$ such that $G^r_{\underline{d}}(X; \widehat{D}_\bullet, \widetilde{a}_\bullet)^{\circ\circ}$ is nonempty. Now it follows from

$$G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^\circ = \bigcup_{\widetilde{a}_\bullet \ge a_\bullet} G^r_{\underline{d}}(X; \widehat{D}_\bullet, \widetilde{a}_\bullet)^{\circ\circ}$$

that $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^\circ$ is either empty or of expected dimension.

(c.2) We show that $\partial G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)$ has dimension less than ρ. There is a map

$$\partial G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet) \longrightarrow \bigcup_{\widetilde{r} \ge r+1} \bigcup_{\widehat{a}_\bullet} G^{\widetilde{r}}_{\underline{d}}(X; \widehat{D}_\bullet, \widehat{a}_\bullet)^{\circ\circ}$$

defined by $(L, V) \mapsto (L, \Gamma(X, L))$. Here, the second union is over all $\widehat{a}_\bullet$ such that there is a sequence $0 \le l_0 < l_1 < \cdots < l_r \le \widetilde{r}$ such that $\widehat{a}_{l_j} \ge a_j$ for all $0 \le j \le r$. Since $d < g + \widetilde{r}$ for all $\widetilde{r} \ge r + 1$, every $G_{\widetilde{r},\widehat{a}_\bullet} := G^{\widetilde{r}}_{\underline{d}}(X; \widehat{D}_\bullet, \widehat{a}_\bullet)^{\circ\circ}$ is either empty or has expected dimension by part (a) and part (b), and the union above is a finite union. On the other hand, consider a tuple $(\widetilde{r}, \widehat{a}_\bullet)$ and take $\{l_j\}_{0 \le j \le r}$ as above, which minimizes the sum $\sum_{0 \le j \le r} l_j$. Denote by $\lambda_j = \widehat{a}_{l_j}$ for $0 \le j \le r$. Then for all $(\widetilde{L}, \widetilde{V}) \in G_{\widetilde{r},\widehat{a}_\bullet}$, the preimage of $(\widetilde{L}, \widetilde{V})$ is a Schubert variety in the Grassmannian Gr $(r+1, \widetilde{V})$ consisting of spaces $V \subset \widetilde{V}$ such that

$$\dim(V \cap \widetilde{V}(-\widehat{D}_{\lambda_j})) \ge r + 1 - j \text{ for } 0 \le j \le r,$$

which has dimension

$$(\widetilde{r} - r)(r+1) - \sum_{j=0}^{r} (l_j - j).$$

It follows that the preimage of $G_{\widetilde{r},\widehat{a}_\bullet}$ has dimension

$$\begin{aligned}
&g - (\widetilde{r}+1)(g - d + \widetilde{r}) - \sum_{j=0}^{\widetilde{r}} (\widehat{a}_j - j) + (\widetilde{r} - r)(r+1) - \sum_{j=0}^{r} (l_j - j) \\
&\quad \le g - (\widetilde{r}+1)(g - d + \widetilde{r}) - \sum_{j=0}^{r} (\widehat{a}_{l_j} - l_j) + (\widetilde{r} - r)(r+1) - \sum_{j=0}^{r} (l_j - j) \\
&\quad = \rho - (\widetilde{r} - r)(g - d + \widetilde{r}) - \sum_{j=0}^{r} (\widehat{a}_{l_j} - a_j) \\
&\quad \le \rho - (\widetilde{r} - r)(g - d + \widetilde{r}) < \rho.
\end{aligned}$$

Hence, $\partial G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)$ has dimension less than ρ.It follows that $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)$ is either empty or of dimension ρ, and so is $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$.

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(d) Assume $d > g + r$. In this case, $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)^\circ$ and $G^{\widetilde{r}}_{\underline{d}}(X; \widehat{D}_\bullet, \widehat{a}_\bullet)^{\circ\circ}$ are empty for all $r < \widetilde{r} < d - g$ by Riemann–Roch theorem. Here, $\widehat{D}_\bullet$ and $\widehat{a}_\bullet$ are the same as in part (c.2). Then by the same argument in part (c.2) and the conclusion of part (c), $\partial G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)$ has dimension at most

$$\max_{\widetilde{r} \geq d-g} \{\rho - (\widetilde{r} - r)(g - d + \widetilde{r})\} = \rho.$$

Hence, $G^r_{\underline{d}}(X; \widehat{D}_\bullet, a_\bullet)$ is either empty or of dimension at most ρ, and so is $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$. ∎

The following example shows that our restrictions on the multi-degree and the ramification condition are necessary.

Example 3.5 Let X be an arbitrary binary curve.

(i) Suppose $d_1 \geq g$ and $d_2 \leq -2$, and $d \geq g + r - 1$. As discussed in Lemma 3.2, each line bundle of multi-degree $\underline{d} = (d_1, d_2)$ has $(d_1 - g)$-dimensional space of global sections. Hence,

$$\dim G^r_{\underline{d}}(X) = g + (r+1)(d_1 - g - r - 1) > \rho(g, r, d)$$

as $d_1 - 1 > d$.

(ii) Suppose $d_1 \geq g + 1$ and $d \geq g + r$. Let $D_j = a^1_j P_1 + a^2_j P_2$ and $a_j = \deg D_j$ for $0 \leq j \leq r$, where $a^1_j = 0$ for $0 \leq j \leq r$ and $a^2_j = j$ for $0 \leq j \leq r - 1$. Suppose $a^2_r > \rho_{g,r,d} + r$. Then $a_\bullet$ is not $\underline{d}$-bounded, and the expected dimension of $G^r_{\underline{d}}(X; D_\bullet, a_\bullet)$ is negative. However, each line bundle L of multi-degree $\underline{d}$ contains a non-zero global section s vanishing on Z_2. Hence, any $(r+1)$-dimensional subspace of $H^0(X, L)$ containing s, which exists by Riemann–Roch theorem, has multi-vanishing sequence at least $a_\bullet$ along $D_\bullet$.

We end this paper with explaining the obstruction of applying the idea in [GH80] directly to our case and the connection between this idea and our proof of Theorem 3.4.

Remark 3.6 Suppose X is obtained from glueing $Q^1_j \in Z_1$ to $Q^2_j \in Z_2$ for $0 \leq j \leq g$. Let $\mathbb{P}^{d_1}_k = \{(x_0, \ldots, x_{d_1}, 0, \ldots, 0)\}$ and $\mathbb{P}^{d_2}_k = \{(0, \ldots, 0, x_{d_1+1}, \ldots, x_{d+1})\}$ be two disjoint planes in $\mathbb{P}^{d+1}_k$. As pointed out by Eric Larson, after identifying Z_i with a rational normal curve in $\mathbb{P}^{d_i}_k$, as in the right part of Figure 4, the space $G^r_{\underline{d}}(X)$ corresponds to the space of $(d-r)$-planes passing though the $g+1$ lines $\overline{Q^1_j Q^2_j}$ modulo the k^* action

$$y \in k^* : (x_0, \ldots, x_{d+1}) \longmapsto (x_0 \cdot y, \ldots, x_{d_1} \cdot y, x_{d_1+1}, \ldots, x_{d+1}).$$

Here, $\overline{Q^1_j Q^2_j}$ meets Z_i at Q^i_j. Moreover, for a $(d-r)$-plane Λ that intersects $\overline{Q^1_j Q^2_j} \backslash \{Q^1_j, Q^2_j\}$ for each $0 \leq j \leq g$, if $\Lambda \cap \overline{Q^1_{j_0} Q^2_{j_0}}$ only contains one point, then this point determines the glueing data of a line bundle L of multi-degree $\underline{d}$ on X at the

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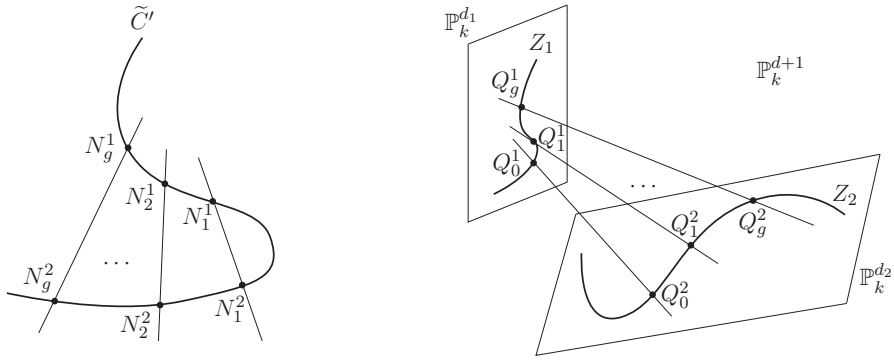


Figure 4:

corresponding node, and each hyperplane in $\mathbb{P}_k^{d+1}$ that contains Λ induces a global section of L up to scaling.

However, unlike in [GH80] the intersection $\Sigma = \sigma(\overline{Q_0^1 Q_0^2}) \cap \cdots \cap \sigma(\overline{Q_g^1 Q_g^2})$ in $Gr(d-r+1, d+2)$ is not necessarily dimensionally transverse, where $\sigma(\overline{Q_j^1 Q_j^2})$ is the space of $(d-r)$-planes in $\mathbb{P}_k^{d+1}$ that meets $\overline{Q_j^1 Q_j^2}$: if g is big enough, then the expected dimension of the intersection is negative, but it always contains the $(d-r)$-planes that contain either the given $\mathbb{P}_k^{d_1}$ or $\mathbb{P}_k^{d_2}$. Even if we ignore these "extra" $(d-r)$-planes, the degeneration process did not work well as in [GH80]. Using the same symbol as in the introduction (or see the left part of Figure 4), the main fact in [GH80] is that for general points N_{m+1}^i, if a $(d-r-1)$-plane Λ' in $\mathbb{P}_k^d$ satisfies $\dim(\Lambda' \cap \overline{aN_1^1}) \geq \lambda$ for some $a \leq d-1$ and $\Lambda' \cap \overline{N_{m+1}^1 N_{m+1}^2} \neq \emptyset$, then $\dim(\Lambda' \cap \overline{aN_1^1, N_{m+1}^1, N_{m+1}^2}) \geq \lambda + 1$. Here, $\overline{aN_1^1, N_{m+1}^1, N_{m+1}^2}$ denotes the linear subspace of $\mathbb{P}_k^d$ spanned by $\overline{aN_1^1}$ and N_{m+1}^i, and recall that $\overline{aN_1^1}$ is the a-th osculating plane of $\widetilde{C}'$ at N_1^1. In our case, after degenerating $Q_1^i, \ldots, Q_m^i$ to Q_0^i for $i = 1, 2$, from the first $m+1$ items of the intersection of Σ, we get Schubert varieties of flags of the form

$$\overline{b_0^1 Q_0^1, b_0^2 Q_0^2} \subsetneq \cdots \subsetneq \overline{b_{d+1}^1 Q_0^1, b_{d+1}^2 Q_0^2}.$$

Note that $b_\bullet^1$ and $b_\bullet^2$ are non-decreasing and non-negative integers bounded above by $d_1 + 1$ and $d_2 + 1$, respectively, and again, $\overline{b_j^1 Q_0^1, b_j^2 Q_0^2}$ denotes the space spanned by $\overline{b_j^1 Q_0^1}$ and $\overline{b_j^2 Q_0^2}$. It is not true in general that $\dim(\Lambda \cap \overline{b_l^1 Q_0^1, b_l^2 Q_0^2}) \geq \lambda$ for some $l \leq d$ and $\Lambda \cap \overline{Q_{m+1}^1 Q_{m+1}^2} \neq \emptyset$ implies

$$\dim(\Lambda \cap \overline{b_l^1 Q_0^1, b_l^2 Q_0^2, Q_{m+1}^1, Q_{m+1}^2}) \geq \lambda + 1, \tag{3.1}$$

due to the possible existence of planes Λ containing Q_{m+1}^i when $b_l^i = d_i + 1$ for either $i = 1$ or $i = 2$. Moreover, the locus of Λ not satisfying (i) may have dimension more than expected. For example, consider the Schubert cycle S consisting of $(d-r)$-planes that meet $\mathbb{P}_k^{d_1}$ in dimension at least x. The intersection of S with $\sigma(\overline{Q_{m+1}^1 Q_{m+1}^2})$ is

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expected to have codimension $\gamma = r + (x+1)(r+1+x-d_1)$ in $Gr(d-r+1, d+2)$. But the subset of S of planes containing Q^1_{m+1} has codimension $r+1+x(r+1+x-d_1)$, which is less than γ if $r+x > d_1$.

It turns out that the space Σ/k^*, as a potential compactification of $G^r_{\underline{d}}(X)$, is too big for the degeneration argument, and we need to rule out the "bad planes": $(d-r)$-planes in Σ that contain exactly one of $\{Q^1_j, Q^2_j\}$ for some j. Note that intuitively, "bad planes" correspond to torsion free sheaves on X that are not locally free at one node or more. We thus adopt Esteve's compactification of the Jacobian of X, which is obtained by adding certain controllable torsion free sheaves.

Now the Schubert conditions above for a $(d-r)$-plane $\Lambda \subset \mathbb{P}^{d+1}_k$ induce ramification conditions for the corresponding linear series. More precisely, let $\widetilde{X}$ be the normalization of X and (L, V) be a linear series on $\widetilde{X}$ induced by a Λ. Then $\dim(\Lambda \cap \overline{b^1_l Q^1_0, b^2_l Q^2_0}) \geq \lambda$ if and only if

$$\dim\left(V(-b^1_l Q^1_0 - b^2_l Q^2_0)\right) \geq r+1-(b^1_l + b^2_l - 1) + \lambda,$$

and $\Lambda \cap \overline{Q^1_{m+1} Q^2_{m+1}} \neq \emptyset$ if and only if $\dim V(-Q^1_{m+1} - Q^2_{m+1}) \geq r$. These two inequalities imply immediately that

$$\dim V(-b^1_l Q^1_0 - b^2_l Q^2_0 - Q^1_{m+1} - Q^2_{m+1}) \geq r - (b^1_l + b^2_l - 1) + \lambda,$$

which is equivalent to (i) if $b^i_l \leq d_i$ for each i. Therefore, we are able to modify the proof in [GH80] and apply it to our case by considering linear series with imposed ramification.

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