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A note on iterative arguments in a topos

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Intuitively, transfinite iteration is a repetitive process, which eventually reaches completion, but might need to progress through an infinite chain of steps before finally doing so. But whereas such a chain is always readily at hand in classical set theory in the form of ordinals, iterative arguments involving sets (i.e. objects) in a general topos have to depend on some intrinsic or naturally available inductive structure, say algebraic, which might not be associated with a (well-ordered) chain.

As an example of this – already in the realm of countability – we recall the definition of Kuratowski-finiteness, an intrinsic notion of finiteness in a topos (which, we recall, is not required to have a natural numbers object). Given a set A, the power set $P(A)$ is a semilattice with respect to finite union. A is said to be Kuratowski-finite if it is a member of the subsemilattice $K(A)$ of $P(A)$ generated by the singletons, in other words, if A can be constructed from its singleton subsets by iterating empty and binary union. It follows that $K(A)$ consists of all Kuratowski-finite subsets of A, and arguments involving these are naturally proved by semilattice induction.

We would like to present here three 'inductive principles', valid in any topos, which enabled us to carry chain-type inductive arguments through in particular situations, but which also seemed to be of independent interest. The three principles (Propositions 5, 7 and 10) are closely related (in fact the last subsumes the first two, although we shall not prove this), and are deduced from a simple technical result in the theory of sup-lattices (Lemma 3 below). Before doing so, we recall a few facts and definitions; a good reference for our purposes is [**4**], chapter 1.

Given a complete lattice M, a closure operator j_φ on M can always be specified in terms of a set $\varphi \subseteq M \times M$ of (closure-)conditions. The set of fixed points of j_φ is

$$M_\varphi = \{q \in M \mid \forall \langle c, p\rangle \in \varphi; p \leqslant q \Rightarrow c \leqslant q\};$$

and one says $q \in M_\varphi$ is closed under the conditions φ. The infimum of a set of φ-closed elements is again φ-closed; by definition j_φ assigns to $x \in M$ the last φ-closed element above x,

$$j_\varphi(x) = \bigwedge\{q \geqslant x \mid q \in M_\varphi\}.$$

Since j_φ is idempotent, the set M_φ is exactly the image of j_φ, and the co-restriction $q_\varphi : M \to M_\varphi$ of j_φ, which is left adjoint to the inclusion $M_\varphi \hookrightarrow M$, defines a quotient in the category of sup-lattices (complete lattices and supremum-preserving maps). More specifically, a sup-lattice map $f : M \to M'$ factors (necessarily uniquely) through $M \to M_\varphi$ if and only if f forces the conditions φ to become inclusions:

$$f(c) \leqslant f(p) \quad \text{for all } \langle c, p\rangle \in \varphi.$$

The least element $j_\varphi(\bot)$ of M_φ is often singled out for particular attention; following [1] we denote it by $I(\varphi)$.

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A basic example is provided by any monotone operation $\omega: M^n \to M$ on M: the conditions $\varphi_\omega \equiv \{\langle \omega\langle x, \ldots, x\rangle, x\rangle \mid x \in M\}$ give

$$M_\omega \equiv M_{\varphi_\omega} = \{q \in M \mid \omega\langle q, \ldots, q\rangle \leqslant q\}$$

and $j_\omega(x) \equiv j_{\varphi_\omega}(x)$ is the least fixed point above x of the operation $y \mapsto y \vee \omega\langle y, \ldots, y\rangle$; we call j_ω the closure operator induced by ω. In particular, if ω is a unary operator $M \to M$, $I(\varphi_\omega) = \mathrm{fix}(\omega)$, the (Tarski [5]) least fixed-point of ω.

Conditions φ on a power set $P(A)$ may be regarded as a collection of inductive rules. $I(\varphi)$ is then the subset of A inductively defined by these rules (with terminology as in [1]). Explicitly, one specifies for each $\langle C, P\rangle \in \varphi$ that

$$\mathrm{Rule}_{\langle C, P\rangle}: P \subseteq I(\varphi) \Rightarrow C \subseteq I(\varphi),$$

and states that members of $I(\varphi)$ are obtained by applying these rules only. If π is then any property of elements of A which is preserved under the rules (i.e. π holds for all members of C whenever it holds for all members of P), then the set $\{x \in A \mid \pi(x)\}$ is φ-closed and therefore contains $I(\varphi)$. One says that the property π is true for the members of $I(\varphi)$ by induction on the rules φ (cf. [1]).

Inductive arguments often exploit the presence of a degree of 'commutativity' between various (inductive) structures. Given two monotone operations $\sigma, \tau: M \to M$, σ is said to be distributive over τ if $\sigma \circ \tau \leqslant \tau \circ \sigma$. One checks that for two sets φ, φ' of conditions on M, distributivity of j_φ over $j_{\varphi'}$ amounts to the requirement that φ-closed elements of M remain φ-closed after application of $j_{\varphi'}$; equivalently, $j_{\varphi \cup \varphi'} = j_{\varphi'} \circ j_\varphi$.

Algebraic operations which commute, or are compatible in some other way, are a typical source of distributivity of closure operators. To make this statement precise, we recall that the category of sup-lattices has tensor products. That is, for each pair M, M' of complete lattices there is a universal sup-lattice bimap (function preserving suprema in each variable separately) $M \times M' \to M \otimes M'$ ($\langle x, x'\rangle \mapsto x \otimes x'$); any given bimap $M \times M' \to M''$ arises from a uniquely determined sup-lattice homomorphism $b: M \otimes M' \to M''$ as $\langle x, x'\rangle \mapsto b(x \otimes x')$. Tensoring with a fixed object M' defines an endofunctor on the category of sup-lattices. It is left adjoint to the endofunctor which assigns to M the (complete lattice of) supremum-preserving functions $M' \to M$. Using this adjunction, we conclude that the following result holds:

LEMMA 1. *For a given set φ of conditions on M, the bimap $b: M \otimes M' \to M''$ factors through $q_\varphi \otimes \mathrm{id}: M \otimes M' \to M_\varphi \otimes M'$ if and only if*

$$f(c \otimes x') \leqslant f(p \otimes x') \quad \textit{for all } \langle c, p\rangle \in \varphi,\ x' \in M'. \qquad \blacksquare$$

A power set $P(A)$ is freely generated as a sup-lattice by the singleton subsets of A. That is, the (covariant) power set functor (associating with $f: A \to A'$ the image mapping $\mathrm{Im}: P(A) \to P(A')$) is left adjoint to the function forgetting the sup-lattice structure. It transforms cartesian products of sets into tensor products of their power sets. In this way, any finitary operation $\alpha: A \times \ldots \times A \to A$ gives rise to a 'power operation' $P(\alpha): P(A) \otimes \ldots \otimes P(A) \to P(A)$; explicitly,

$$P(\alpha)(S_0 \otimes \ldots \otimes S_{n-1}) \mapsto \{\alpha\langle s_0, \ldots, s_{n-1}\rangle \mid s_0 \in S_0, \ldots, s_{n-1} \in S_{n-1}\}.$$

$J \subseteq A$ is fixed by the induced closure operator $j_{P(\alpha)}: P(A) \to P(A)$ precisely when J is closed under α in the standard sense of algebra.

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We say that a monotone operation $\omega : M^n \to M$ is compatible with a set of conditions φ if

$$\omega\langle x_0, \ldots, x_{k-1}, c, x_{k+1}, \ldots, x_{n-1}\rangle \leqslant j_\varphi(\omega\langle x_0, \ldots, x_{k-1}, p, x_{k+1}, \ldots, x_{n-1}\rangle)$$

for all $\langle c, p\rangle \in \varphi$, $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1} \in M$, $0 \leqslant k < n$. We also say that ω is compatible with a monotone operation $\sigma : M \to M$ whenever ω is compatible with φ_σ, i.e. if

$$\omega\langle x_0, \ldots, x_{k-1}\, \sigma(x), x_{k+1}, \ldots, x_{n-1}\rangle \leqslant j_\sigma(\omega\langle x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_{n-1}\rangle)$$

for all $x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_{n-1} \in M$, $0 \leqslant k < n$. We note that if ω preserves suprema in each variable separately, then it is enough to check the above relations for $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}$ in some subset which generates M under suprema, e.g. singleton subsets if M is a power set. We shall use the following consequence of Lemma 1.

LEMMA 2. *Suppose that $\omega : M^n \to M$ preserves suprema in each variable separately, say $\omega\langle x_0, \ldots, x_{n-1}\rangle = \tilde{\omega}\langle x_0 \otimes \ldots \otimes x_{n-1}\rangle$, where $\tilde{\omega} : M \otimes \ldots \otimes M \to M$. Then the following are equivalent:*

(i) *$\tilde{\omega}$ co-restricts to M_φ;*
(ii) *ω is compatible with j_φ;*
(iii) *ω is compatible with φ.*

Proof. (i) $\Rightarrow$ (ii) Let $q_\varphi \circ \tilde{\omega} = \tilde{\omega}_\varphi \circ q_\varphi \otimes \ldots \otimes q_\varphi$, where $\tilde{\omega}_\varphi : M_\varphi \otimes \ldots \otimes M_\varphi \to M_\varphi$. Then

$$\begin{aligned}
\omega\langle x_0, \ldots, j_\varphi(x), \ldots, x_{n-1}\rangle &\leqslant j_\varphi\, \omega\langle x_0, \ldots, j_\varphi(x), \ldots, x_{n-1}\rangle \\
&= \tilde{\omega}_\varphi(j_\varphi(x_0) \otimes \ldots \otimes j_\varphi^2(x) \otimes \ldots \otimes j_\varphi(x_{n-1})) \\
&= \tilde{\omega}_\varphi(j_\varphi(x_0) \otimes \ldots \otimes j_\varphi(x) \otimes \ldots \otimes j_\varphi(x_{n-1})) \\
&= j_\varphi\, \omega\langle x_0, \ldots, x, \ldots, x_{n-1}\rangle.
\end{aligned}$$

(ii) $\Rightarrow$ (iii) This implication is immediate from $\langle c, p\rangle \in \varphi \Rightarrow c \leqslant j_\varphi(p)$.
(iii) $\Rightarrow$ (i) This follows on repeatedly applying Lemma 1. ∎

The following fact links 'algebraic' commutativity with 'inductive' commutativity.

LEMMA 3. *Let M be a sup-lattice and $\omega : M \times \ldots \times \to M$ an operation preserving suprema in each variable separately. If ω is compatible with some set of conditions φ on M, then j_ω distributes over j_φ.*

Proof. If ω is compatible with φ, then for any j_ω-closed $q \in M$,

$$\begin{aligned}
\omega(j_\varphi(q) \otimes \ldots \otimes j_\varphi(q)) &\leqslant j_\varphi(\omega(q \otimes \ldots \otimes q)) && \text{by repeated application of Lemma 2(ii)} \\
&\leqslant j_\varphi(q) && \text{by the assumption on } q.
\end{aligned}$$

∎

COROLLARY 4. *Let τ, $\sigma : M \to M$ be increasing monotone operations on a complete lattice, where τ preserves suprema. If τ is distributive over σ, then j_τ is distributive over j_σ.* ∎

Our first induction principle says that Kuratowski-finite sets may in fact still be constructed from the empty set in the 'elementwise' fashion of iteratively adding a singleton at a time; formally, we have

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PROPOSITION 5. *The Kuratowski-finite subsets $K(A)$ of a set A are inductively defined by the rules*

(F0) $\varnothing \in K(A)$;

(F1) $F \in K(A) \Rightarrow F \cup \{x\} \in K(A)$, *for each* $x \in A$;

(F2) *all members of* $K(A)$ *are obtained by applying* (F0) *or* (F1).

This result has an intuitively straightforward proof when one has recourse to a free algebra with one unary operation on one generator (in other words an indexing set for natural numbers). Nevertheless, the freeness is not essential, and it is not difficult to see that Proposition 5 is an instance of the following, more general fact, for which Lemma 3 furnishes a 'natural numbers free' proof:

LEMMA 6. *In a commutative monoid* $\langle C, 0, + \rangle$, *the submonoid generated by* $S \subseteq C$ *is obtained by adding the identity* 0, *and closing under the operation* $x \mapsto s + x$ *for each* $s \in S$.

Proof. For $c \in C$, let t_c denote the operation $x \mapsto c + x$ of translation by c. By associativity and commutativity, $t_c(x) + y = t_c(x + y) = x + t_c(y)$ for any $c \in C$. These identities are inherited by the power operations, and imply that $P(+)$ is compatible with $P(t_c)$ for each $c \in C$. By Lemma 3 the closure of 0 under $x \mapsto s + x$, $s \in S$, is a submonoid of C: since it contains S and is clearly contained in any other submonoid containing C, the statement follows. ∎

Our second induction principle could be viewed as a weak variant of transfinite induction, which in [**6**] (though in a slightly disguised form) was found to be strong enough to carry a number of chain-type iterative arguments through. To formulate it, we consider a monotone operation $\rho : M \to M$ on a complete lattice, and inductively define a set R of approximations to fix (ρ) from below by the following rules:

(R0) $\bot \in R$;

(R1) $x \vee y \in R \Rightarrow x \vee \rho(x) \vee y \in R$, for all $x, y \in M$;

(R2) $D \subseteq R \Rightarrow \bigvee D \in R$, whenever D is directed;

(R3) all members of R are obtained using one of (R0), (R1) or (R2).

(We recall that a partially ordered set is said to be directed if it is inhabited, with an upper bound for each pair of its elements.) In other words, $R = I(\varphi)$, where φ has a pair $\langle \{x \vee \rho(x) \vee y \mid z = x \vee y\}, \{z\} \rangle$ for each $z \in M$ and $\langle \bigvee D, D \rangle$ for each directed $D \subseteq M$. It is clear that the property of being below fix (ρ) is preserved by these rules, and so by induction this is true for all members of R. Moreover, each rule attempts to create a larger, and therefore improved approximation to fix (ρ) from already existing ones: (R1) produces a set of 'successor'-approximations, while (R2) gives a 'limit'-approximation. We have

PROPOSITION 7. *For any monotone operator* ρ *on a sup-lattice* M, *the rules* (R0)–(R3) *are sufficient to produce the least fixed point of* ρ; *that is*, fix $(\rho) \in R$.

Proof. We shall be done if R is closed under binary suprema. For then R is directed and by (R2) contains its own supremum p; since $R \subseteq \downarrow(\text{fix}\,(\rho))$, $p \leqslant \text{fix}\,(\rho)$ while, since $\rho(p) \in R$, $\rho(p) \leqslant p$. But this means $p \in M_\rho$, i.e. $p \geqslant \text{fix}\,(\rho)$. To show that R is indeed closed under binary joins, we note that the power operation $P(\vee) : P(M) \otimes P(M) \to P(M)$ is compatible with (R1) and (R2): compatibility of $\vee$ with (R1) follows from

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the fact that if q is of the form $\rho(x) \vee y$ for some y, the same holds for $q \vee z$, any z; compatibility with (R2) is due to distributivity of binary joins over directed joins. By Lemma 3, $P(\vee)$ distributes over j_φ. But then, since the singleton $\{\bot\}$ is closed under binary joins, so is its closure under (R1) and (R2), which is R. ▮

We note that the proof does not work if rule (R1) is weakened to produce only a single 'successor'-approximation at each step:

(R1′) $\quad x \in R \Rightarrow x \vee \rho(x) \in R$, for all $x \in M$.

We therefore cannot claim that Proposition 7 represents a topos-theoretic version of transfinite induction.

We next strengthen Corollary 4 for increasing operations. But first we need another lemma. We say that operation $\omega : M^n \to M$ is increasing if $x_0 \vee \ldots \vee x_{n-1} \leqslant \omega\langle x_0, \ldots, x_{n-1}\rangle$ for all $x_1, \ldots, x_{n-1} \in M$. Then we have

LEMMA 8. *Let M be a sup-lattice and $\omega : M \times M \to M$ an increasing binary operation preserving directed suprema in each variable separately. Then for any increasing monotone operator $\sigma : M \to M$ with which ω is compatible, j_ω is distributive over j_σ.*

Proof. The proof is based on the same idea as that of Proposition 7. Consider the rules φ on $P(M)$, where $\langle \downarrow \sigma(x), x\rangle \in \varphi$ for each $x \in M$ and $\langle \{\bigvee D\}, D\rangle \in \varphi$ for each directed $D \subseteq M$. Then the power operation $P(\omega) : P(M) \otimes P(M) \to P(M)$ is compatible with φ (using monotonicity and distributivity over σ of ω for the first set of conditions, and the preservation of directed suprema by ω for the second) and so, by Lemma 3, $j_{P(\omega)}$ is distributive over j_φ. Given now $q \in M$ with q j_ω-closed, we are required to show that $j_\sigma(q)$ remains j_ω-closed. Since ω is increasing, $\{q\}$ is closed under the operation ω; by the distributivity just shown, the same is true for $j_\varphi(\{q\})$. But this means, again since ω is increasing, that $j_\varphi(\{q\})$ is in fact directed, and therefore, by being closed under directed suprema, contains its own supremum p. It follows that $\sigma(p) = p = \omega(p, p)$, which implies that p is j_ω-closed and that $j_\sigma(q) \leqslant p$. Since the down-set of $j_\sigma(q)$ is closed under the rules φ and contains q, it contains $j_\varphi(\{q\})$ and therefore also p. That is, $p \leqslant j_\sigma(q)$, which completes the proof. ▮

We note that Lemma 8 still makes sense, and indeed remains true, if we only require M to have suprema over directed subsets. We now have the following result which may be compared with Corollary 4.

COROLLARY 9. *Let τ, $\sigma : M \to M$ be increasing monotone operations on a complete lattice, where τ preserves directed suprema. If τ is distributive over σ, then j_τ is distributive over j_σ.*

Proof. Let $\omega : M \times M \to M$ be the increasing operation $\langle x, y\rangle \mapsto \tau(x \vee y)$. Then ω preserves directed suprema. Also, since $\tau(\sigma(x) \vee y) \leqslant \tau(\sigma(x \vee y)) \leqslant \sigma \circ \tau(x \vee y)$, for all $x, y \in M$, ω is compatible with σ. Therefore, by Lemma 8, $j_\tau = j_\omega$ is distributive over j_σ. ▮

Our third inductive principle was motivated by the type of transfinite argument employed in [3]. It says that the use of transfinite iteration as there (cf. [2] for another example) is in fact 'harmless' from a (higher order) constructive point of view:

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PROPOSITION 10. *Let A be a set, and φ a set of inductive rules on $P(A)$ with finitary premises, that is, for each $\langle C, P\rangle \in \varphi$, P is (Kuratowski-)finite. Suppose $\sigma : P(A) \to P(A)$ is a monotone operation such that $T \subseteq \sigma(T)$ for all $T \in P(A)$, and which has the property that $\sigma(J)$ remains closed under the rules φ whenever J is closed under φ. Then if J is φ-closed, so is the least fixed set of σ containing J; that is, the set S inductively defined by the rules*

(S0) $J \subseteq S$;
(S1) $T \subseteq S \Rightarrow \sigma(T) \subseteq S$, *for all* $T \subseteq A$;
(S2) *all elements of S are obtained using either* (S0) *or* (S1).

Proof. In view of Lemma 9, we only remark that for a set of rules φ which is finitary in the above sense, the closed sets are stable under directed unions, which means that j_φ preserves directed suprema. ∎

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