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Codimension n flips

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Abstract. The generic unfolding of codimension n flips (one eigenvalue -1 and the others with norm different from 1) embedded in a Morse–Smale diffeomorphism is analyzed. Local and global bifurcations are described.

1. *Introduction*

One of the principal objectives in the theory of dynamical systems (diffeomorphisms, endomorphisms, vector fields) is to give a dynamical classification of all systems. Two dynamical systems belong to the same class if they have the same dynamical behaviour. A dynamical system is called stable if it belongs to the interior of its class. Morse–Smale systems are the simplest among the stable ones [**PSm**] and they play an important role in the development of modern dynamics. Generically, one can bifurcate from a Morse–Smale diffeomorphism, through an n-parameter family, either losing hyperbolicity of one of the periodic orbits or creating non-transversal intersections between stable and unstable manifolds of saddles. Bifurcations of one-parameter families starting at Morse–Smale are investigated in [**NPT, NP**].

In this paper we investigate the unfolding of a diffeomorphism F_0 on a compact manifold M, which is Morse–Smale, except for one periodic orbit being a *codimension n flip* (one eigenvalue equal to -1 and the others off the unit circle). This diffeomorphism is considered as the starting diffeomorphism in a generic n-parameter C^∞ family of C^∞ diffeomorphisms F_μ, $\mu \in U$ where U is an open neighborhood of $(0, 0)$ in $\mathbb{R}^n$. The set of diffeomorphisms of this type constitute a codimension n subset of $\mathrm{Dif}(M)$ inside the boundary of the Morse–Smale ones. They appear in a natural way when one considers n-parameter families of diffeomorphisms starting at a Morse–Smale and crossing the boundary. This approach is a natural continuation of the study of the codimension one flip starting at Morse–Smale done in [**NPT**].

The notions of stability and weak (mild) stability for families are the standard ones (see [**NPT**] or §2). We recall the notions of cycle and criticality. A *cycle* is a sequence $p_0, \ldots, p_k$ of periodic orbits with $p_i \neq p_j$ for $0 \leq i < j < k$, $p_0 = p_k$, $p_{i+1} \subset \overline{W^u(p_i)}$ for $0 \leq i < k$. Let p be the codimension n flip orbit. Suppose F_0 restricted to the

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central manifold has a sink at p. Then there exists a unique F_0-invariant differentiable foliation $\mathcal{F}^{ss}$ of $W^s(p, F_0)$ containing $W^{ss}(p, F_0)$ as a leaf. We say that p is s-critical if there exists a hyperbolic periodic orbit q such that $W^u(q, F_0)$ intersects some leaf of $\mathcal{F}^{ss}$ non-transversally. Analogously, if F_0 restricted to the central manifold has a source at p, we define *u-criticality*. We say that p is *critical* if it is either s- or u-critical. We say that p is *non-critical* if it is not critical. The following theorem gives conditions for global stability.

THEOREM A.

(a) *F is stable if $n \leq 3$ and p is non-critical.*

(b) *F is weakly stable (and is not stable) if $1 < n \leq 3$ and p is critical.*

(c) *F is weakly stable (and is not locally stable) if $n > 3$.*

(d) ***[NPT]** F is stable if $n = 1$.*

Let V be a one-dimensional C^k manifold, $k > 1$, and let f be a C^k diffeomorphism defined in V. We say that a vector field X is *adapted to* f if $f = X_1$ (the time 1 map of X). In cases of stability, with the exception of case (d), the conjugacy between F_μ and a nearby family G_μ must preserve the unique C^1 vector field adapted to F_0 restricted to the central manifold. This rigidity property is analogous to the codimension one saddle-node case [**NPT**].

This work points in the direction of trying to get a global picture of all the codimension two bifurcations for diffeomorphisms. In this area other research has been developed, such as the deep local analysis of the codimension two Hopf bifurcation done in [**C**], the semi-local stability analysis of a saddle-node with quadratic tangency done in [**BP**], the global analysis of codimension two reflection done in [**G1**] and the analysis of the codimension n saddle-node done in [**G2**].

Definitions used throughout this paper are given in §2. In §3 we describe the local unfolding of the codimension n flip and we analyze its local stability. In §4 we describe the global unfolding and study its global stability.

These results were announced in [**G3**].

2. *Definitions*

Throughout this paper, M = compact C^∞ manifold. $\mathrm{Dif}(M) = \{C^\infty$ diffeomorphisms of M with the C^∞ topology$\}$.

Let $F_0 \in \mathrm{Dif}(M)$ and p be a fixed point. Suppose $DF_0(p)$ has all eigenvalues with norm different from 1 except one. We say that p is a *codimension n flip* if this eigenvalue is -1 and there exists a F_0-invariant curve $\mathcal{C}$ through p such that $F_0^2/\mathcal{C}$ has contact of order $2n$ but not of order $(2n+1)$ with the identity at p. For periodic orbits there are analogous definitions.

U = open neighborhood of $0 \in \mathbb{R}^n$ (parameter space). $\mathcal{F}_U(M) = \{C^\infty$ maps $F : U \times M \mapsto U \times M$ such that $F(\mu, p) = (\mu, F_\mu(p))$ and $F_\mu \in \mathrm{Dif}(M)\}$. We endow $\mathcal{F}_U(M)$ with the C^∞ weak topology (compact-open topology). Elements of $\mathcal{F}_U(M)$ are denominated n-parameter families of diffeomorphisms and are denoted by $\{F_\mu\}$ or F.

$\mathcal{A} = \{F \in \mathcal{F}_U(M)$ such that F_0 has finite hyperbolic limit set except for one periodic orbit p being a codimension n flip. Transversality between stable, unstable manifolds

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of the periodic orbits and strong stable, strong unstable manifolds of p and generic unfolding (see §3) of the non-hyperbolic orbit are required}.

Let F and G be elements of $\mathcal{F}_U(M)$. They are *topologically conjugate* (or, simply, *conjugate*) if there are open neighborhoods U_F and U_G of 0, contained in U, and a homeomorphism $H : U_F \times M \mapsto U_G \times M$ such that:

(a) $H(\mu, p) = (\phi(\mu), H_\mu(p))$, where $\phi : U_F \mapsto U_G$ and $H_\mu : M \mapsto M$ are homeomorphisms;

(b) $H \circ F = G \circ H$.

We say that F and G are *weakly conjugate* if in the previous definition H_μ does not depend continuously on μ. We say that F at (μ_0, p) is *locally conjugate* to G at $(\tilde{\mu}_0, \tilde{p})$ if there are neighborhoods, V of (μ_0, p) and $\tilde{V}$ of $(\tilde{\mu}_0, \tilde{p})$ contained in $U \times M$, and a conjugacy $H : V \mapsto \tilde{V}$ between F/V and $G/\tilde{V}$. Analogously we define *local weak conjugacies*.

Conjugacy, weak conjugacy, local conjugacy and local weak conjugacy are equivalence relations. F is *stable* if it belongs to the interior of its equivalence class by conjugacy. Similarly, we define *weak stability*. F is *locally stable* at (μ_0, p) if for any neighborhood V of (μ_0, p), in $U \times M$ there exists a neighborhood W of F in $\mathcal{F}_U(M)$ such that for every G in W there exist neighborhoods of (μ_0, p), $V_F \subset V$, $V_G \subset V$, and a homeomorphism $H : V_F \mapsto V_G$ of the type $H(\mu, q) = (\phi(\mu), H_\mu(q))$, such that $H \circ (F/V_F) = (G/V_G) \circ H$. If H_μ does not depend continuously on μ we say that F is *locally weakly stable*.

$o(\|(x, y)\|^k)$ is a term that if divided by $\|(x, y)\|^k$ remains bounded above when $(x, y) \to (0, 0)$. Analogous definitions hold for $o(\|\mu\|^k)$.

$\mathrm{Id}_{\mathbb{R}^n}$ stands for the identity in $\mathbb{R}^n$.

3. *Local unfolding*

3.1. *Introduction.* In the next two sections we shall consider germs of n-parameter families of diffeomorphisms of $\mathbb{R}$ (with coordinate x), $\{F_\mu\}$ such that F_0 has a codimension n flip at 0. These germs are generated when one restricts families of $\mathcal{A}$ to the central manifold, as in [**NPT**].

In the next section the codimension two flip is analyzed. The stability results carry over to the codimension three case. They constitute part of the codimension n case investigation done in the subsequent section.

3.2. *The codimension two flip.* Let $U \subset \mathbb{R}^2$ be an open neighborhood of 0. Here we consider $\{F_\mu\}$ with $n = 2$ and $\mu \in U$. For any $k < \infty$ we may assume that F is C^k [**S**]. We assume that 0 is a source of F_0. The other case occurs for F_0^{-1} and is completely analogous. After a conjugacy by a local change of coordinates, F_μ takes the form

$$F_\mu(x) = -x + a_1(\mu)x + a_2(\mu)x^3 + x^5 + o_\mu(\|x\|^7),$$

where $\{a_i : U \to \mathbb{R},\ a_i(0) = 0\}$ are differentiable functions. We say that $F = \{F_\mu\}$ is a *generic unfolding* of the codimension two flip if $\mathrm{DET}(\partial a_i(0)/\partial \mu_j) \neq 0$. Assuming this condition, after a conjugacy by a local change of coordinates F_μ^2 takes the form

$$F_\mu^2(x) = x + \mu_1 x + \mu_2 x^3 - x^5 + o_\mu(\|x\|^7). \tag{1}$$

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The local dynamics of F_μ is determined by the set Ω_μ of periodic orbits of F_μ. This set is constituted by a unique fixed point, the origin, and a set of periodic orbits with period two. Ω_μ coincides with the zeros of the function $G(\mu, x) = F_\mu^2(x) - x$. Let

$$P(\mu, x) = \mu_1 x + \mu_2 x^3 - x^5.$$

Near the origin the set $G(\mu, x) = 0$ consists of the plane $x = 0$ and a surface Σ having the same shape as $P(\mu, x)/x = 0$. Let $\pi_\mu : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(\mu, x) \to \mu$. The set $\Lambda_G = \{\mu$ such that G_μ has zeros with multiplicity greater than one$\}$ is equal to the projection by π_μ of the non-regular points of Σ (critical points of $\pi_\mu : \Sigma \to \mathbb{R}^2$).

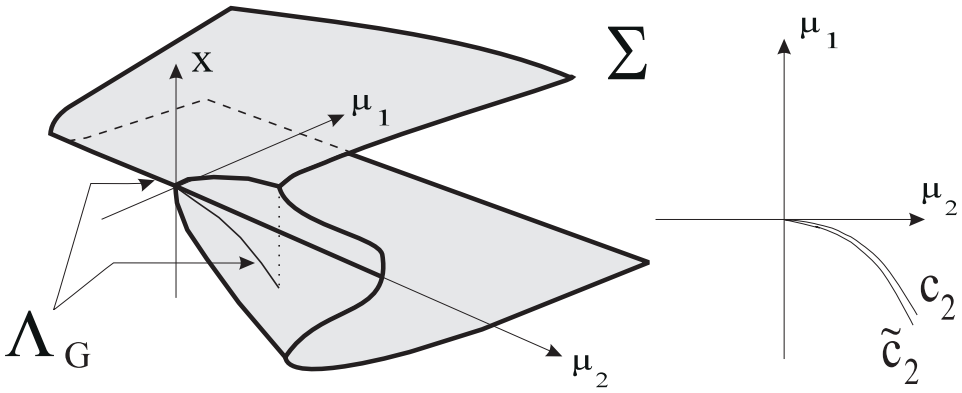


FIGURE 1.

PROPOSITION 3.1. *Λ_G and $\mathbb{R}^2 - \Lambda_G$ constitute (locally) a Whitney stratification of the parameter space. There exists a C^1 local diffeomorphism φ mapping this stratification onto the corresponding one for P.*

Proof. Using the implicit function theorem and direct calculations one can prove that Λ_G is the union of

$$\begin{cases} c_1(\mu_2) = (0, \mu_2), & \mu_2 \neq 0 \\ c_2(\mu_2) = \left(-\dfrac{\mu_2^2}{4} + o(|\mu_2|^3), \mu_2\right), & \mu_2 > 0 \\ 0. \end{cases}$$

Similarly, Λ_P is the union of

$$\begin{cases} \tilde{c}_1(\mu_2) = (0, \mu_2), & \mu_2 \neq 0 \\ \tilde{c}_2(\mu_2) = \left(-\dfrac{\mu_2^2}{4}, \mu_2\right), & \mu_2 > 0 \\ 0. \end{cases}$$

Let $\xi : \mathbb{R} \to [0, 1]$ be a C^∞ function such that $\xi/\mathbb{R} - (-1, 1) \equiv 0$ and $\xi/(-1/2, 1/2) \equiv 1$. Let $U \subset \mathbb{R}^2$ be a small neighborhood of 0 and $X : U \to \mathbb{R}^2$ be defined by

$$\begin{aligned} X(\mu_1, \mu_2) &= 0, \quad \text{if } \mu_2 \leq 0 \\ X(\mu_1, \mu_2) &= \xi\left(\frac{\mu_1 + \mu_2^2/4}{\mu_2^2/8}\right) \cdot (\tilde{c}_2(\mu_2) - c_2(\mu_2)), \quad \text{if } \mu_2 > 0. \end{aligned}$$

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We define $\varphi = X_1$ (X_1 stands for the time 1 map of the vector field X). It can be checked that φ satisfies the proposition. □

When we consider F_0 a diffeomorphism of the real line that has a codimension two flip at 0, the origin is either a sink or a source of F_0. We shall always assume it to be a sink. The other case is completely analogous. Let $U \subset \mathbb{R}^2$ be a small open set containing the origin.

THEOREM 3.2. *Let $\{F_\mu\}$ and $\{\tilde{F}_\mu\}$, $\mu \in U$, be two germs such that F_0 and $\tilde{F}_0$ have a codimension two flip at the origin. Suppose also that both germs unfold generically the codimension two flip. Then they are locally conjugate at $\mu = 0$ and $x = 0$.*

COROLLARY 3.3. *Let $\{F_\mu\} \in \mathcal{F}_U(M)$ be a germ such that $0 \in U$ and F_0 has a codimension two flip at $p \in M$. Suppose $\{F_\mu\}$ unfolds generically p. Then $\{F_\mu\}$ is locally stable at $(0, p)$.*

Proof of Theorem 3.2. Assume the flip is a fixed point. The case of periodic orbit with period k is analogous if we consider $\{F_\mu^k\}$. After a conjugacy by a local change of coordinates, F_μ^2 and $\tilde{F}_\mu^2$ can be written in the form (1). By Proposition 3.1 there exists a germ of a C^1 diffeomorphism $\xi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $D\xi(0) = \mathrm{Id}_{\mathbb{R}^2}$ and ξ maps the stratification of F onto the stratification of $\tilde{F}$ in the parameter space. Moreover, ξ preserves the second coordinate.

We may suppose $U \subset \mathbb{R}^2$ is a small open disk with center at 0. Let $I = [-\varepsilon, \varepsilon] \subset \mathbb{R}$ be a small interval. Let $W \subset U \times I$ and $\tilde{W} \subset U \times I$ be defined by

$$W : \begin{cases} x \le \varepsilon \\ F_\mu^2(x) - x \le 0 \\ x \ge -c\mu_1 \\ x \ge 0 \end{cases} \qquad \tilde{W} : \begin{cases} x \le \varepsilon \\ \tilde{F}_\mu^2(x) - x \le 0 \\ x \ge -c\mu_1 \\ x \ge 0. \end{cases}$$

In W and in $\tilde{W}$ we consider families X_μ and $\tilde{X}_\mu$ of C^1 vector fields adapted to F_μ^2 and $\tilde{F}_\mu^2$, respectively. They are C^{k-1} outside the periodic orbits of F_μ, $\tilde{F}_\mu$, respectively. Moreover, they depend continuously on μ in the C^1 topology. The existence of vector fields with these properties can be proved as in [**Y**].

Let $c_2 : \{\mu_2 \ge 0\} \to U$ and $\tilde{c}_2 : \{\mu_2 \ge 0\} \to U$ be the saddle-node curves of F, $\tilde{F}$, respectively. Define the surfaces $S \subset U \times I$ and $\tilde{S} \subset U \times I$ by

$$S : \begin{cases} \mu = c_2(\mu_2), & \text{if } \mu_2 > 0 \\ \mu_1 = 0, & \text{if } \mu_2 \le 0 \end{cases} \qquad \tilde{S} : \begin{cases} \mu = \tilde{c}_2(\mu_2), & \text{if } \mu_2 > 0 \\ \mu_1 = 0, & \text{if } \mu_2 \le 0. \end{cases}$$

For $i = 0, 2$ let $W_i \subset W - S$ be the region where F_μ^2 has i periodic orbits. Then $W - S = W_0 \cup W_2$. Analogously $\tilde{W} - \tilde{S} = \tilde{W}_0 \cup \tilde{W}_2$. The vector fields X_μ and $\tilde{X}_\mu$ are unique in the closure of W_2 and $\tilde{W}_2$, respectively [**Ko, Sz**].

Suppose there exists a local conjugacy $H : U \times I \to U \times I$ between F and $\tilde{F}$ such that $H(\mu, x) = (\phi(\mu), h_\mu(x))$. As conjugacies preserve dynamics, the parameter curve c_2, associated to the existence of saddle-nodes, must be mapped by ϕ onto the curve $\tilde{c}_2$. Varying parameter μ_1 from any point $\mu \in c_2$ a codimension one saddle-node bifurcation occurs and as in [**NPT**] h_μ is a C^k map and $(h_\mu)_* X_\mu = \tilde{X}_{\phi(\mu)}$, outside the

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saddle-node. The continuity of h_μ, X_μ and $\tilde{X}_\mu$ with respect to μ implies that h_0 is C^k and $(h_0)_* X_0 = \tilde{X}_0$, outside the origin. Hence this rigidity is a necessary condition and shall be considered in the construction of the conjugacy.

We proceed to construct H. We begin constructing $\phi : U \to U$. Let $\pi_1 : U \to \mathbb{R}$ be the canonical projection on the first coordinate. Let $c_2^-(\mu_2) = c_2(\mu_2) - (\mu_2^2/40, 0)$ and $c_2^{--}(\mu_2) = c_2(\mu_2) - (\mu_2^2/20, 0)$. Define a homeomorphism $\sigma : U \to U$ such that

$$\sigma(\mu) = \mu \quad \text{if } \mu_2 \le 0 \text{ or } \mu_1 \ge \pi_1 \circ \tilde{c}_2(\mu_2) \text{ or } \mu_1 \le \pi_1 \circ \xi \circ c_2^{--}(\mu_2).$$

If $\mu = (\mu_1, \mu_2)$ satisfies $(\mu_2 > 0)$ and $(\pi_1 \circ c_2^-(\mu_2) < \mu_1 < \pi_1 \circ c_2(\mu_2))$, we consider two cases.

(a) $\exists n_\mu \in \mathbb{Z}^+$ (positive integers) such that $F_\mu^{2n_\mu}(\varepsilon) = -c\mu_1$. Let $\tilde{\mu} = (\tilde{\mu}_1, \mu_2)$ be defined by $\tilde{F}_{\tilde{\mu}}^{2n_\mu}(\varepsilon) = -c\tilde{\mu}_1$. The uniqueness of this definition comes from (1). Define σ at $\xi(\mu)$ by $\sigma(\xi(\mu)) = (\tilde{\mu}_1, \mu_2)$.

(b) μ not in case (a). Consider $n_\mu = \max\{n \in \mathbb{Z}^+ \mid F_\mu^{2n}(\varepsilon) > -c\mu_1\}$. Let $\bar{\mu} = (\bar{\mu}_1, \mu_2)$ be defined by the equality $F_{\bar{\mu}}^{2(n_\mu+1)}(\varepsilon) = -c\bar{\mu}_1$. Let $R_\mu = (F_\mu^{2n_\mu}(\varepsilon) + c\mu_1)/(F_{\bar{\mu}}^{2n_\mu}(\varepsilon) + c\bar{\mu}_1)$. From (1), $\bar{\mu}_1 > \mu_1$ and for a fixed $\mu_2 > 0$ the function $\mu_1 \to R_{(\mu_1,\mu_2)}$ is defined at a countable collection of open intervals, is strictly increasing in each interval and the image of each interval is the open interval $(0, 1)$. Let $\hat{\mu}_1 = \pi_1 \circ c_2(\mu_2)$. There exists a unique $0 < t_\mu < 1$ such that

$$\frac{(X_{c_2(\mu_2)})_{-t_\mu}(-c\hat{\mu}_1) + c\hat{\mu}_1}{F_{c_2(\mu_2)}^{-2}(-c\hat{\mu}_1) + c\hat{\mu}_1} = R_\mu.$$

Let $\hat{\hat{\mu}}_1 = \pi_1 \circ \tilde{c}_2(\mu_2)$. Define

$$\tilde{R}_\mu = \frac{\left(\tilde{X}_{\tilde{c}_2(\mu_2)}\right)_{-t_\mu}\left(-c\hat{\hat{\mu}}_1\right) + c\hat{\hat{\mu}}_1}{\tilde{F}_{\tilde{c}_2(\mu_2)}^{-2}\left(-c\hat{\hat{\mu}}_1\right) + c\hat{\hat{\mu}}_1}.$$

For a fixed μ_2 define $\bar{\bar{\mu}} = (\bar{\bar{\mu}}_1, \mu_2)$ and $\bar{\bar{\bar{\mu}}} = (\bar{\bar{\bar{\mu}}}_1, \mu_2)$ such that $\tilde{F}_{\bar{\bar{\mu}}}^{2n_\mu+1}(\varepsilon) = -c\bar{\bar{\mu}}_1$ and $\tilde{F}_{\bar{\bar{\bar{\mu}}}}^{2n_\mu}(\varepsilon) = -c\bar{\bar{\bar{\mu}}}_1$. Let $\tilde{\mu} = (\tilde{\mu}_1, \mu_2)$ be the unique μ satisfying

$$\bar{\bar{\bar{\mu}}}_1 < \tilde{\mu}_1 < \bar{\bar{\mu}}_1 \quad \text{and} \quad \frac{\tilde{F}_{\tilde{\mu}}^{2n_\mu}(\varepsilon) + c\tilde{\mu}_1}{\tilde{F}_{\bar{\bar{\mu}}}^{2n_\mu}(\varepsilon) + c\bar{\bar{\mu}}_1} = \tilde{R}_\mu.$$

Define σ at $\xi(\mu)$ by $\sigma(\xi(\mu)) = \tilde{\mu}$. For a fixed $\mu_2 > 0$ let $\sigma_{\mu_2} = \sigma \mid \{(\mu_1, \mu_2), \mu_1 \in \mathbb{R}\}$. Then σ_{μ_2} is already defined as a homeomorphism of the line minus an interval $[a, b]$. One can check that $\sigma_{\mu_2}(a) < \sigma_{\mu_2}(b)$ (if not take c_2^- closer to c_2). We extend each σ_{μ_2} to a homeomorphism of the line varying continuously with the parameter μ_2. The construction of σ is finished. Define $\phi := \sigma \circ \xi$.

Consider Λ_F as in Proposition 3.1. $U - \Lambda_F = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where $\mathcal{R}_i = \{\mu \in U$ such that F_μ has i hyperbolic periodic orbits$\}$. Analogously, $\xi(U) - \Lambda_{\tilde{F}} = \tilde{\mathcal{R}}_1 \cup \tilde{\mathcal{R}}_2 \cup \tilde{\mathcal{R}}_3$ and $\xi(\mathcal{R}_i) = \tilde{\mathcal{R}}_i$. Let $\varphi : \overline{\mathcal{R}_3 \cup \mathcal{R}_2} \to \mathbb{R}$ be a C^0 map satisfying:

(a) Let $c_2 \subset U$ be the saddle-node curve. For $\mu \in c_2$ let s_μ be the positive point of the saddle-node orbit of F_μ. Define $\varphi(\mu) := s_\mu$. Analogously, define $\tilde{s}_\mu$, $\mu \in \phi(c_2)$.

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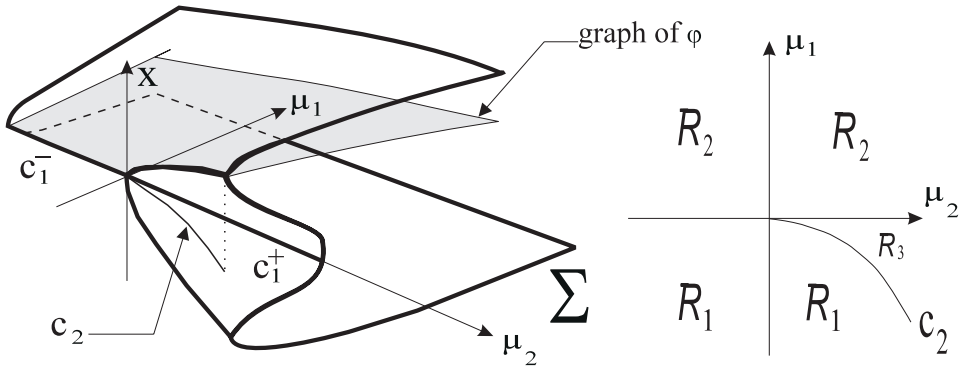


FIGURE 2.

(b) Let $c_1^- = \{\mu \in U \mid \mu_1 = 0 \wedge \mu_2 \leq 0\}$. Define $\varphi \mid c_1^- = 0$.

(c) For $\mu \in (\overline{\mathcal{R}_2 \cup \mathcal{R}_3} - (c_2 \cup c_1^-))$, let q_μ be the positive point of the attractive periodic orbit of F_μ with period two. For $\mu \in \mathcal{R}_3$ let r_μ be the positive point of the repulsive periodic orbit of F_μ with period two. Analogously, we define $\tilde{r}_\mu$ and $\tilde{q}_\mu$, considering $\tilde{F}_\mu$. If $\mu \in \mathcal{R}_3$ we require $r_\mu < \varphi(\mu) < q_\mu$. If $\mu \in \overline{\mathcal{R}_2} - c_1^-$ we require $0 < \varphi(\mu) < q_\mu$.

The construction of a φ satisfying these conditions is straightforward and is left to the reader. Analogously, one can construct $\tilde{\varphi} : \overline{\tilde{\mathcal{R}}_3 \cup \tilde{\mathcal{R}}_2} \to \mathbb{R}$ associated to $\tilde{F}$, changing F to $\tilde{F}$ and $\mathcal{R}_i$ to $\tilde{\mathcal{R}}_i$ in conditions (a)–(c).

For $\mu \in \mathcal{R}_3 \cup c_2$ define $p_\mu^1 := (-c\mu_1, \mu_2)$. For $\mu \in \overline{\mathcal{R}_2 \cup \mathcal{R}_3}$ define $p_\mu^2 := \varphi(\mu)$. For $\mu \in \overline{\tilde{\mathcal{R}}_2 \cup \tilde{\mathcal{R}}_3}$ define $\tilde{p}_\mu^2 := \tilde{\varphi}(\mu)$. Let X_μ^t be the time t map of X. Let $c_1^+ = \{\mu \in U \mid \mu_1 = 0 \wedge \mu_2 > 0\}$. We shall define h_μ for $x \geq 0$. We divide into five cases:

(a) $\mu \in \mathcal{R}_1$:
$$h_\mu(\varepsilon) = \varepsilon, \quad h_\mu(X_\mu^t(\varepsilon)) = \tilde{X}_{\phi(\mu)}^t(\varepsilon), \quad h_\mu(0) = 0$$

(b) $\mu \in c_2$:
$$h_\mu(\varepsilon) = \varepsilon, \quad h_\mu(X_\mu^t(\varepsilon)) = \tilde{X}_{\phi(\mu)}^t(\varepsilon), \quad h_\mu(p_\mu^1) = p_{\phi(\mu)}^1,$$
$$h_\mu(X_\mu^t(p_\mu^1)) = \tilde{X}_{\phi(\mu)}^t(p_{\phi(\mu)}^1), \quad h_\mu(s_\mu) = \tilde{s}_\mu, \quad h_\mu(0) = 0$$

(c) $\mu \in c_1^-$:
$$h_\mu(\varepsilon) = \varepsilon, \quad h_\mu(X_0^t(\varepsilon)) = \tilde{X}_0^t(\varepsilon), \quad h_\mu(0) = 0$$

(d) $\mu \in \mathcal{R}_3$:
$$h_\mu(\varepsilon) = \varepsilon, \quad h_\mu(X_\mu^t(\varepsilon)) = \tilde{X}_{\phi(\mu)}^t(\varepsilon), \quad h_\mu(p_\mu^2) = \tilde{p}_{\phi(\mu)}^2,$$
$$h_\mu(F_\mu^2(p_\mu^2)) = \tilde{F}_{\phi(\mu)}^2(\tilde{p}_{\phi(\mu)}^2),$$
$h_\mu \mid [p_\mu^2, F_\mu^2(p_\mu^2)]$ is a linear map onto $[\tilde{p}_{\phi(\mu)}^2, \tilde{F}_{\phi(\mu)}^2(\tilde{p}_{\phi(\mu)}^2)]$,
extend it by the conjugacy equation to the interval (r_μ, q_μ),
$$h_\mu(q_\mu) = \tilde{q}_\mu, \quad h_\mu(r_\mu) = \tilde{r}_\mu, \quad h_\mu(0) = 0,$$
$$h_\mu(p_\mu^1) = p_{\phi(\mu)}^1, \quad h_\mu(X_\mu^t(p_\mu^1)) = \tilde{X}_{\phi(\mu)}^t(p_{\phi(\mu)}^1)$$

(e) $\mu \in c_1^+ \cup \mathcal{R}_2$:
$$h_\mu(\varepsilon) = \varepsilon, \quad h_\mu(X_\mu^t(\varepsilon)) = \tilde{X}_{\phi(\mu)}^t(\varepsilon), \quad h_\mu(p_\mu^2) = \tilde{p}_{\phi(\mu)}^2,$$
$h_\mu \mid [p_\mu^2, F_\mu^2(p_\mu^2)]$ is a linear map onto $[\tilde{p}_{\phi(\mu)}^2, \tilde{F}_{\phi(\mu)}^2(\tilde{p}_{\phi(\mu)}^2)]$,
extend it by the conjugacy equation to the interval $(0, q_\mu)$,
$$h_\mu(q_\mu) = \tilde{q}_\mu, \quad h_\mu(0) = 0.$$

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For all $\mu \in U$ we extend the definition of h_μ to $[\varepsilon, -\varepsilon]$ using the conjugacy equation. Define $H(\mu, x) := (\phi(\mu), h_\mu(x))$. The careful construction of ϕ implies the continuity of H at the saddle-node curve. □

Proof of Corollary 3.3. Assume, without loss of generality, that p is a fixed point. By the implicit function theorem, there exist neighborhoods V_1 of F_0 in $\mathrm{Dif}(M)$ and V_2 of p in M such that the set $\Upsilon = \{g \in V_1 \mid g \text{ has a codimension two flip in } V_2\}$ is a codimension two submanifold of $\mathrm{Dif}(M)$. The generic unfolding of the codimension two flip is equivalent to the map $\Lambda_F : U \to \mathrm{Dif}(M)$, $\Lambda_F(\mu) = F_\mu$ being transversal to Υ. Take $\{\tilde{F}_\mu\}$ near $\{F_\mu\}$. From openness of transversality and continuity of $F \to \Lambda_F$, there exist μ_0 near 0 and $\tilde{p} \in V_2$ such that $\tilde{F}_{\mu_0}$ unfolds generically the codimension two flip $\tilde{p}$. Let W^c and $\tilde{W}^c$ be the central manifolds of F and $\tilde{F}$ at $(0, p)$, $(\mu_0, \tilde{p})$ respectively. From the facts above and Theorem 3.2 it follows that there exists a neighborhood $0 \in U_2 \subset U$ such that $\{F_\mu\}_{\mu \in U_2}$ and $\{\tilde{F}_\mu\}_{\mu \in \mu_0 + U_2}$ restricted to its central manifolds are locally conjugate at $\mu = 0$ and p. From [**PT**] this conjugacy can be extended to a local conjugacy at a neighborhood of $(0, p) \in U \times M$. □

3.3. *The codimension n flip.* Let $U \ni 0$ be an open set of $\mathbb{R}^n$ and $F \in \mathcal{F}_U(M)$ such that F_0 has a codimension n flip at p. After restriction of this family to the central manifold at $(0, p)$ we obtain a C^k n-parameter family of diffeomorphisms of the real line. The $(2n+1)$-jet of F_0^2 at p takes the form $x + cx^{2n+1}$ for some constant $c \neq 0$. We suppose $c < 0$. The other case is analogous. F_μ^2, after conjugacy by a local change of coordinates, takes the form

$$F_\mu^2(x) = x + a_1(\mu)x + a_2(\mu)x^3 + \cdots + a_n(\mu)x^{2n-1} - x^{2n+1} + o_\mu(\|x\|^{2n+2}), \quad (2)$$

where $a_i(0) = 0$. We say that $F = \{F_\mu\}$ is a *generic unfolding* of the codimension n flip p if $\mathrm{DET}(\partial a_i(0)/\partial \mu_j) \neq 0$.

Let V_1 be a neighborhood of F_0 in $\mathrm{Dif}(M)$ and V_2 a neighborhood of p in M. Let $\phi_F : U \to V_1$ be the C^∞ map $\phi_F(\mu) = F_\mu$.

PROPOSITION 3.4. *There exists neighborhoods V_1 of f and V_2 of p in M such that the set of maps $f \in V_1$ that have a codimension n flip in V_2 is a codimension n submanifold Γ_n of V_1.*

PROPOSITION 3.5. *The condition of ϕ_F being transversal to Γ_n is equivalent to F being a generic unfolding of the codimension n flip.*

These two propositions follow from (2), the implicit function theorem and transversality theory.

COROLLARY 3.6. *Let $F \in \mathcal{F}_U(M)$ be such that F_0 has a codimension n flip at p. Suppose F is a generic unfolding of p. Given neighborhoods V_1 of F_0 in $\mathrm{Dif}(M)$, V_2 of p in M, V_3 of 0 in U, there exists a neighborhood V_4 of F in $\mathcal{F}_U(M)$ such that for every $G \in V_4$ there exists a unique $\bar{\mu} \in V_3$ and a unique $\bar{p} \in V_2$ satisfying $G_{\bar{\mu}} \in V_1$, $\bar{p}$ is a codimension n flip and G is a generic unfolding of $\bar{p}$.*

Proof. Openness of transversality and continuity of the map $F \to \phi_F$. □

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Consider F as in Corollary 3.6.

THEOREM 3.7. *F is locally weakly stable at $(0, p)$.*

This is one of the main results of this section. In order to prove it we pursue the following steps. First we turn the periodic orbits of F_μ symmetric and construct displacement functions, adapting the techniques developed in [**T**]. Then we characterize the displacement functions of F and a nearby G as universal deformations of isomorphic germs. We apply the equivalence of universal deformations from singularities of differentiable functions theory to obtain the equivalence of the displacement functions. Then we prove local weak stability of F restricted to the central manifold. This implies local weak stability of F by [**PT**]. These steps will be divided into several lemmas.

After applying a convenient conjugacy to F (restricted to the central manifold) we may suppose F_μ has the local C^k form

$$F_\mu(x) = -x + b_1(\mu)x + b_2(\mu)x^3 + \cdots + b_n(\mu)x^{2n-1} + b_{n+1}(\mu)x^{2n+1} + o_\mu(\|x\|^{2n+2}), \quad (3)$$

where $b_i(0) = 0$, if $i < n+1$, $b_{n+1}(0) = \pm\frac{1}{4}$ and $\mathrm{DET}(\partial b_i(0)/\partial \mu_j) \neq 0$.

We consider the case $b_{n+1}(0) = \frac{1}{4}$. The minus case is analogous.

LEMMA 3.8. ([**T**]) *There exists $h : U \times \mathbb{R} \to U \times \mathbb{R}$ a C^k local diffeomorphism at $(0, 0)$ such that $h(\mu, x) = (\mu, h_\mu(x))$, $h_\mu(0) = 0$ and $h_\mu \circ F_\mu \circ h_\mu^{-1}$ has symmetric local periodic orbits ($F_\mu(p) = -p$, for p a local periodic orbit).*

Proof. Consider $h_\mu(x) = (x - F_\mu(x))/2$. □

Let us fix some basic notation from singularities of differentiable functions theory. One can consult [**M**] for further information. Define the following:

E_n^k: the space of germs of C^k real functions defined in an open neighborhood of $0 \in \mathbb{R}^n$.
$M_k^i \subset E_n^k$: the ideal of C^k germs that are $(i-1)$ flat at 0.

Consider F_μ with symmetric local periodic orbits, i.e. after applying Lemma 3.8.

LEMMA 3.9. *F_μ has the local form*

$$F_\mu(x) = -x + c_1(\mu)x + \cdots + c_n(\mu)x^{2n-1} + c_{n+1}(\mu)x^{2n+1} + o_\mu(\|x\|^{2n+2}),$$

where $c_i(\mu) = b_i(\mu) + h_i(\mu)$, $h_i(\mu) \in M_{k-2n}^2$ if $i \leq n$ and $h_{n+1}(\mu) \in M_{k-2n}^1$. Moreover, $c_1(\mu), \ldots, c_n(\mu)$ generate M_{k-2n}^1 and $\mathrm{DET}(\partial c_i(0)/\partial \mu_j) \neq 0$.

Proof. The local form follows from direct calculations. The condition $\mathrm{DET}(\partial b_i(0)/\partial \mu_j) \neq 0$ implies $b_1(\mu), \ldots, b_n(\mu)$ generate M_{k-2n}^1. This and the fact that $h_i(\mu) \in M_{k-2n}^2$ give the generation of M_{k-2n}^1 by $c_1(\mu), \ldots, c_n(\mu)$. Consequently, the determinant must be different from zero. □

Let U be an open neighborhood of $0 \in \mathbb{R}^n$. Let $F : U \times \mathbb{R} \to \mathbb{R}$ be a C^k family of local diffeomorphisms F_μ at $0 \in \mathbb{R}$ ($F(\mu, x) = F_\mu(x)$ and $F_\mu(0) = 0$). We assume each F_μ has symmetric local periodic orbits. We say a C^k function $D : U \times \mathbb{R} \to \mathbb{R}$ is a *displacement function* for F if locally:

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(a) $D(\mu, x) = -D(\mu, -x)$ for (μ, x) in a neighborhood of $(0, 0)$.
(b) $D(\mu, 0) = 0$ for every $\mu \in U$.
(c) $D(\mu, x) = 0$ if and only if x is a periodic orbit of F_μ.
(d) If ($x > 0$ and $D(\mu, x) > 0$) or ($x < 0$ and $D(\mu, x) < 0$), then the ω-limit, in absolute value, is greater than x and the α-limit, in absolute value, is smaller than x. If ($x > 0$ and $D(\mu, x) < 0$) or ($x < 0$ and $D(\mu, x) > 0$) we change greater for smaller.
(e) If x_0 is a periodic orbit of F_μ the order of the first non-null derivative of $D_\mu(x)$ at x_0 is the multiplicity of x_0 as a zero of $F_\mu^m(x) - x$, where m is the period of x_0. Moreover, the signal of this derivative must coincide with the signal of the first non-null derivative of $F_\mu^m(x) - x$ at x_0.

For any $F : U \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions of the previous definition, we define $D(\mu, x) = F_\mu^2(x) - F_\mu^2(-x) - 2x$. It is easy to check that D is a displacement function for F.

Consider F as in Lemma 3.9 and its corresponding displacement function D.

LEMMA 3.10. *D has the local form*

$$D(\mu, x) = d_1(\mu)x + d_2(\mu)x^3 + \cdots + d_{n+1}(\mu)x^{2n+1} + o_\mu(\|x\|^{2n+3}),$$

where $d_i(\mu) = -4b_i(\mu) + \tilde{h}_i(\mu)$, $\tilde{h}_i(\mu) \in M^2_{k-2n}$ if $i \le n$ and $\tilde{h}_{n+1}(\mu) \in M^1_{k-2n-2}$.

Proof. Direct calculations. □

Thus $D(\mu, x) = xE(\mu, x)$ where:
(a) $E(\mu, x) = d_1(\mu) + d_2(\mu)x^2 + \cdots + d_{n+1}(\mu)x^{2n} + o_\mu(\|x\|^{2n+2})$;
(b) $E(\mu, x) = E(\mu, -x)$;
(c) $E(\mu, x)$ is C^{k-1}.

$[x]$ denotes the greatest integer smaller than or equal to x.

LEMMA 3.11. *There exists a C^l local diffeomorphism $\mathcal{G}$ at $(0, 0)$, $l = [k/2] - 1$, such that $E(\mu, x) = \mathcal{G}(\mu, x^2)$.*

Proof. This is a direct application of the C^k division theorem [**L**]. □

The local form of E leads to the local form of $\mathcal{G}$:

$$\mathcal{G}(\mu, x) = d_1(\mu) + d_2(\mu)x + \cdots + d_{n+1}(\mu)x^n + o_\mu(\|x\|^{n+1}),$$

where $d_i(0) = 0$, $i \le n$, $d_{n+1}(0) = -1$ and

$$\mathrm{DET}\left(\frac{\partial d_i}{\partial \mu_j}(0)\right) \neq 0, \quad 1 \le i \le n, 1 \le j \le n.$$

Let $f(x) = \mathcal{G}(0, x)$. There exists a C^{l-n} local diffeomorphism $\Lambda : \mathbb{R} \to \mathbb{R}$, $\Lambda(0) = 0$, such that $f \circ \Lambda(x) = -x^n$ (see [**M**]). Consider the C^{l-n} map $H : U \times \mathbb{R} \to \mathbb{R}$ defined by $H(\mu, x) = \mathcal{G}(\mu, \Lambda(x))$. Direct calculations lead to the local form

$$H(\mu, x) = e_1(\mu) + e_2(\mu)x + \cdots + e_{n+1}(\mu)x^n + o_\mu(\|x\|^{n+1}), \tag{4}$$

where $e_i(0) = 0$, $i \le n$, $e_{n+1}(0) = -1$ and $o_0(\|x\|^{n+1}) = 0, \forall x$.

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LEMMA 3.12.

$$\mathrm{DET}\left(\frac{\partial e_i}{\partial \mu_j}(0)\right) \neq 0, \quad 1 \leq i \leq n, 1 \leq j \leq n.$$

Proof. Direct calculations. □

Let $J(f)$ be the Jacobian ideal of f. Let $f \in E_n^k$ and let $\mathrm{cod}(f)$ be the codimension of $J(f)$ in E_n^{k-1}. In the subsequent steps we recall some fundamental concepts of singularities of differentiable functions theory. One can consult [**M**] and [**V**] as reference. We shall use the concept of equivalence to mean right equivalence.

Let $f \in E_i^k$. An *n-deformation of* f is a germ $F \in E_{n+i}^k$ such that $F(0, x) = f(x)$. Let F and G be two deformations of f. We say F and G are *s-equivalent* if there exist a C^s local diffeomorphism $\Phi : (\mathbb{R}^n \times \mathbb{R}^i, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^i, (0,0))$ such that $\Phi(\mu, x) = (\phi(\mu), \Phi_\mu(x))$, where each $\Phi_\mu : \mathbb{R}^i \to \mathbb{R}^i$ is a local diffeomorphism, $\Phi_0 = \mathrm{Id}_{\mathbb{R}^i}$ and $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a local diffeomorphism and $F = G \circ \Phi$. Let F be an n-deformation of f. We say F is a C^s *universal deformation* if any other deformation G of f is s-isomorphic to the pull-back of F by a C^s map from the parameter space of G to $\mathbb{R}^n$. Let

$$s_k = \min\left\{k - \mathrm{cod}(f) - 3, k - 4, \left[\frac{k-3}{\mathrm{cod}(f)+2}\right] - 1\right\}.$$

We call

$$\left\{\dot{F}_i = \frac{\partial F}{\partial \mu_i}\bigg|_{\mu=0}\right\}$$

the *initial velocities of* F. We state a particular version of a fundamental theorem of singularity theory for C^k unfoldings.

PROPOSITION 3.13. *Let* $f \in E_1^k$ *be such that* $\mathrm{cod}(f) \leq (k-7)/2$. *A* C^k *n-deformation* F *of* f *is a* C^{s_k} *universal deformation if after projection in* $E_1^{k-1}/J(f)$ *the initial velocities of* F *generate* $E_1^{k-1}/J(f)$.

We will need the following well-known result about equivalence of universal deformations.

PROPOSITION 3.14. *Let* $f \in E_i^k$ *be a germ of finite codimension. Two* C^s *universal n-deformations of* f *are s-equivalent.*

Consider H given by (4).

LEMMA 3.15. H *is a* $C^{s_{l-n}}$ *universal n-deformation of* $-x^n$.

Proof. After projection in $E_1^{l-n-1}/J(-x^n)$ the initial velocities $\dot{H}_j$ are mapped into

$$\dot{h}_j(x) = \frac{\partial e_1}{\partial \mu_j}(0) + \frac{\partial e_2}{\partial \mu_j}(0)x + \cdots + \frac{\partial e_{n-1}}{\partial \mu_j}(0)x^{n-2}.$$

By Lemma 3.12 the $\dot{h}_j(x)$ generate $E_1^{l-n-1}/J(-x^n)$. We apply Proposition 3.13 to finish the proof. □

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Let W be the canonical n-deformation of $-x^n$,

$$W(\mu, x) = \mu_1 + \mu_2 x + \cdots + \mu_n x^{n-1} - x^n.$$

By Proposition 3.13 W is a $C^{s_{l-n}}$ universal n-deformation of $-x^n$. By Proposition 3.14 W and H are s_{l-n}-equivalent. Let Φ be the $C^{s_{l-n}}$ local diffeomorphism such that $W \circ \Phi = H$, $\Phi(\mu, x) = (\phi(\mu), \Phi_\mu(x))$, $\Phi_0 = \mathrm{Id}_{\mathbb{R}}$, $\phi(0) = 0$. We modify Φ_μ to map zero to zero. Let $\tilde{\Phi}_\mu(x) = \Phi_\mu(x) - \Phi_\mu(0)$, $\tilde{W}(\mu, x) = W(\mu, x + \Phi_{\phi^{-1}(\mu)}(0))$, $\tilde{\Phi}(\mu, x) = (\phi(\mu), \tilde{\Phi}_\mu(x))$. Then $\tilde{W} \circ \tilde{\Phi} = W \circ \Phi$. Take $f(\mu) = \Phi_{\phi^{-1}(\mu)}(0)$.

$$\tilde{W}(\mu, x) = a_1(\mu) + a_2(\mu)x + \cdots + a_n(\mu)x^{n-1} - x^n,$$

where $a_i(\mu) = \mu_i + b_i(\mu)$, $b_i(\mu) \in M^2_{s_{l-n}-n}$ if $i \leq n-1$ and $a_n(\mu) = \mu_n - nf(\mu)$.

Let $\lambda(\mu) = \tilde{\Phi}'_\mu(0)$. Consider the equation $\tilde{W} \circ \tilde{\Phi} = H$. Taking i-order partial derivatives with respect to x at $(\mu, 0)$ for $i = 0, 1, 2, \ldots, n-1$, we obtain

$$\begin{pmatrix} e_1(\mu) \\ \vdots \\ e_n(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda(\mu) & 0 & \ldots & 0 \\ 0 & g_{32}(\mu) & \lambda^2(\mu) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n2}(\mu) & g_{n3}(\mu) & \ldots & \lambda^{n-1}(\mu) \end{pmatrix} \cdot \begin{pmatrix} a_1(\phi(\mu)) \\ \vdots \\ a_n(\phi(\mu)) \end{pmatrix}.$$

For the above equation we use the simple notation that $e(\mu) = A(\mu) \cdot a(\phi(\mu))$. Then $a(\mu) = A(\phi^{-1}(\mu))^{-1} \cdot e(\phi^{-1}(\mu))$. So

$$\frac{\partial a}{\partial \mu_j}(0) = A^{-1}(0) \cdot \frac{\partial e(\phi^{-1}(\mu))}{\partial \mu_j}(0).$$

Lemma 3.12 and $D\phi^{-1}(0)$ being an isomorphism yield $\mathrm{DET}(\partial a_i(0)/\partial \mu_j) \neq 0$.

We consider $\Lambda : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $\Lambda(\mu) = (a_1(\mu), \ldots, a_n(\mu))$. It follows that Λ is a $C^{s_{l-n}-n}$ local diffeomorphism. $\tilde{W}(\mu, x) = W(\Lambda(\mu), x)$. We define $\hat{\Phi}(\mu, x) = (\Lambda \circ \phi(\mu), \tilde{\Phi}_\mu(x))$. Then $W \circ \hat{\Phi} = \tilde{W} \circ \tilde{\Phi} = W \circ \Phi = H$. Therefore we have proved the following.

LEMMA 3.16. *Let H be a C^l n-deformation of $-x^n$ such that its Taylor polynomial of order n satisfies the condition of Lemma 3.12. Then H and the canonical n-deformation W are $C^{s_{l-n}-n}$-equivalent. The equivalence $\Phi(\mu, x) = (\phi(\mu), \Phi_\mu(x))$ can be chosen such that $\Phi_\mu(0) = 0$ for all μ.*

We point out that l must be chosen sufficiently big so that $s_{l-n} - n$ is positive.

Proof of Theorem 3.7. We take F and G as in Corollary 3.6. We take central manifolds for F at $(0, p)$ and for G at $(\bar{\mu}, \bar{p})$. After restriction to these central manifolds we may suppose F and G take the form (4). By Lemma 3.8 we can consider that F_μ and G_μ have symmetric local periodic orbits. We repeat the construction after Lemma 3.9 to obtain displacement functions $D^F : U \times \mathbb{R} \to \mathbb{R}$ and $D^G : U \times \mathbb{R} \to \mathbb{R}$ for F and G. By Lemma 3.10 $D^F(\mu, x) = xE^F(\mu, x)$ and $D^G(\mu, x) = xE^G(\mu, x)$. By Lemma 3.11 $E^F(\mu, x) = \mathcal{G}^F(\mu, x^2)$ and $E^G(\mu, x) = \mathcal{G}^G(\mu, x^2)$. By the construction after Lemma 3.11, Lemma 3.12 and Lemma 3.15 there exist local diffeomorphisms

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$\Lambda^F : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $\Lambda^G : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $H^F(\mu, x) = \mathcal{G}^F(\mu, \Lambda^F(x))$ and $H^G(\mu, x) = \mathcal{G}^G(\mu, \Lambda^G(x))$ are universal n-deformations of $-x^n$. By Lemma 3.16 there exist Φ^F, equivalence between H^F and W, the canonical n-deformation and Φ^G, equivalence between H^G and W. So $(\Phi^F)^{-1} \circ \Phi^G$ is an equivalence between H^F and H^G. We define $\tilde{\Lambda}^F : U \times \mathbb{R} \to U \times \mathbb{R}$ and $\tilde{\Lambda}^G : U \times \mathbb{R} \to U \times \mathbb{R}$ by $\tilde{\Lambda}^F(\mu, x) = (\mu, \Lambda^F(x))$ and $\tilde{\Lambda}^G(\mu, x) = (\mu, \Lambda^G(x))$. Let $\xi = \tilde{\Lambda}^F \circ (\Phi^F)^{-1} \circ \Phi^G \circ (\tilde{\Lambda}^G)^{-1}$. Then ξ is an equivalence between $\mathcal{G}^F$ and $\mathcal{G}^G$. We remark that $\xi(\mu, x) = (\phi(\mu), \xi_\mu(x))$, where $\xi_\mu(0) = 0$ for all $\mu \in U$. Moreover, $\xi_\mu : \mathbb{R} \to \mathbb{R}$ preserves orientation. If $x \geq 0$ then $E^G(\mu, x) = \mathcal{G}^G(\mu, x^2) = \mathcal{G}^F(\phi(\mu), \xi_\mu(x^2)) = E^F(\phi(\mu), \sqrt{\xi_\mu(x^2)})$. Analogously, if $x < 0$ then $E^G(\mu, x) = E^F(\phi(\mu), -\sqrt{\xi_\mu(x^2)})$. We define $\Phi : U \times \mathbb{R} \to U \times \mathbb{R}$ by

$$\Phi(\mu, x) = \begin{cases} (\phi(\mu), \sqrt{\xi_\mu(x^2)}), & \text{for } x \geq 0 \\ (\phi(\mu), -\sqrt{\xi_\mu(x^2)}), & \text{for } x < 0 \end{cases} \qquad \Phi(\mu, x) = (\phi(\mu), \Phi_\mu(x)).$$

Φ is a $C^{s_{l-n}-n-1}$ local diffeomorphism, where $l = [k/2] - 1$ and k is the order of the differentiability of F and G restricted to their central manifolds. $E^F \circ \Phi = E^G$. It is easy to check that for all $\mu \in U$, Φ_μ preserves orientation, $\Phi_\mu(-x) = -\Phi_\mu(x)$ and Φ_μ map the zeroes of D^G_μ into the zeroes of D^F_μ preserving their multiplicity and their topological character (attractor, repellor, saddle-node). Moreover, in the case of a saddle-node orbit Φ_μ preserves the sides of attraction and repulsion.

We proceed to construct a C^0 conjugacy $H : U \times \mathbb{R} \to U \times \mathbb{R}$, $H(\mu, x) = (\phi(\mu), H_\mu(x))$ between F and G restricted to their central manifolds. For each $\mu \in U$ we define H_μ equal to Φ_μ at the local periodic orbits of G_μ. Let $0 = p^1_\mu < p^2_\mu < \cdots < p^{k_\mu}_\mu$, $1 \leq k_\mu \leq n + 1$, be the k_μ non-negative periodic points of G_μ. Let $\tilde{p}^i_\mu = H_\mu(p^i_\mu)$, for $1 \leq i \leq k_\mu$. Let $q^i_\mu = (p^i_\mu + p^{i+1}_\mu)/2$, for $1 \leq i < k_\mu$. Let $\tilde{q}^i_\mu = (\tilde{p}^i_\mu + \tilde{p}^{i+1}_\mu)/2$. We take a small $\varepsilon > 0$ such that $\varepsilon > \max\{p^{k_\mu}_\mu, \tilde{p}^{k_\mu}_\mu\}$. We define $q^{k_\mu}_\mu = \tilde{q}^{k_\mu}_\mu = \varepsilon$. We define $H_\mu(q^i_\mu) = \tilde{q}^i_\mu$ and $H_\mu(G^2_\mu(q^i_\mu)) = F^2_\mu(\tilde{q}^i_\mu)$, for $1 \leq i \leq k_\mu$. Let J^i_μ be the closed interval whose extremities are q^i_μ and $G^2_\mu(q^i_\mu)$. Let $\tilde{J}^i_\mu$ be the closed interval whose extremities are $\tilde{q}^i_\mu$ and $F^2_\mu(\tilde{q}^i_\mu)$. Let I^i_μ be the open interval (p^i_μ, p^{i+1}_μ). Let $\tilde{I}^i_\mu = (\tilde{p}^i_\mu, \tilde{p}^{i+1}_\mu)$. We extend H_μ to a C^k diffeomorphism preserving orientation $H_\mu : J^i_\mu \to \tilde{J}^i_\mu$. Then we use the conjugacy equation $H_\mu = (F^2_\mu)^{-m} \circ H_\mu \circ (G^2_\mu)^m$, $m \in \mathbb{Z}$, to extend it to a C^k diffeomorphism $H_\mu : I^i_\mu \to \tilde{I}^i_\mu$. Details can be seen in [**G4**]. So $H_\mu : [0, \varepsilon] \to [0, \varepsilon]$ is well defined. We extend it to $H_\mu : [G_\mu(\varepsilon), \varepsilon] \to [F_\mu(\varepsilon), \varepsilon]$ using the conjugacy equation $H_\mu \circ G_\mu = F_\mu \circ H_\mu$. We remark that each H_μ is a continuous map. H_μ varies continuously with μ, for $\mu \in U - \{\mu \mid G_\mu$ has an odd codimension saddle-node$\}$. By [**PT**] this conjugacy can be extended to a neighborhood of $(\bar{\mu}, \bar{p})$ in $U \times M$. □

The next result gives information about the local stability of F.

THEOREM 3.17. *Let $F \in \mathcal{F}_U(M)$ be such that F_0 has a codimension n flip at p. Suppose F is a generic unfolding of p. Then F is locally stable at $(0, p)$ if and only if $n \leq 3$.*

Before proving Theorem 3.17 we will establish a necessary condition for the existence of continuous conjugacies. Let F be as in Theorem 3.17. Consider F restricted to the (local) central manifold $W^c(p)$ at $(0, p)$. F is C^k for big k. Let

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$W^c_\mu(p) = W^c(p) \cap \{\mu \times M\}$. Using the same notation as before, suppose $p^i_{\mu_0}$ is a codimension one saddle-node of $F^2_{\mu_0}$. Let $p^{i-1}_{\mu_0} < p^i_{\mu_0} < p^{i+1}_{\mu_0}$ be consecutive fixed points of F^2_μ (if one of these points does not exist we change it for $\pm\varepsilon$). Let $H : U \times \mathbb{R} \to U \times \mathbb{R}$, $H(\mu, x) = (\phi(\mu), h_\mu(x))$ be a continuous local conjugacy between F and a nearby $\tilde{F}$. There exists a unique C^1 vector field X_{μ_0} adapted to $F^2_{\mu_0} \mid (p^{i-1}_{\mu_0}, p^{i+1}_{\mu_0})$. It is C^{k-1} off the fixed point $p^i_{\mu_0}$ [**Ko, Sz**]. Analogously, there exists a unique C^1 vector field $\tilde{X}_{\mu_0}$ adapted to $\tilde{F}^2_{\phi(\mu_0)} \mid (h_{\mu_0}(p^{i-1}_{\mu_0}), h_{\mu_0}(p^{i+1}_{\mu_0}))$ that is C^{k-1} off $h_{\mu_0}(p^i_{\mu_0})$.

LEMMA 3.18. *$h_{\mu_0} \mid ((p^{i-1}_{\mu_0}, p^{i+1}_{\mu_0}) - \{p^i_{\mu_0}\})$ is a C^{k-1} diffeomorphism and $(h_{\mu_0})_* X_{\mu_0} = \tilde{X}_{\mu_0}$.*

Proof. Let $\delta > 0$ be a small number and let $c : [-\delta, \delta] \to U$ be a C^∞ curve such that $c(0) = \mu_0$ and $f_t = F_{c(t)}$ satisfies $\partial f^2_t(p^i_{\mu_0})/\partial t \neq 0$. The existence of this curve is a consequence of the fact that F^2_μ is a versal unfolding of $F^2_{\mu_0}$ if we consider $F^2_{\mu_0}$ as a germ at the saddle-node. Without loss of generality, we assume that for $t > 0$ the saddle-node disappears and for $t < 0$ it breaks into two hyperbolic fixed points for f^2_t. From the existence of the conjugacy H, the same is true for $\tilde{f}^2_t$ if we define $\tilde{f}_t = \tilde{F}_{\phi(c(t))}$. Let $J \subset I = F^2_{\mu_0} \mid (p^{i-1}_{\mu_0}, p^{i+1}_{\mu_0})$ be an interval a little smaller than I containing $p^i_{\mu_0}$ in its interior such that for $t > 0$, $f^2_t \mid J$ does not contain any fixed point. By [**Y**] there exists, for each $t \geq 0$, C^1 vector fields X_t and $\tilde{X}_t$ adapted to $f^2_t \mid J$ and to $\tilde{f}^2_t \mid h_{\mu_0}(J)$ varying continuously with t. The result follows using the same type of argument as in Theorem 3.2 of [**NPT**]. □

PROPOSITION 3.19. *Consider $n > 1$. Let $F : U \times \mathbb{R} \to \mathbb{R}$ and $\tilde{F} : U \times \mathbb{R} \to \mathbb{R}$ be two C^k generic unfoldings of the codimension n flip 0 (at $\mu = 0$). Let X and $\tilde{X}$ be, respectively, the adapted vector fields to F^2_0 and to $\tilde{F}^2_0$. Let $H(\mu, x) = (\phi(\mu), h_\mu(x))$ be a continuous local conjugacy between F and $\tilde{F}$. Then $h_0 \mid \mathbb{R} - \{0\}$ is C^{k-1} and $(h_0)_* X = \tilde{X}$.*

Proof. By Lemma 3.15 and Proposition 3.14 there exist a codimension one submanifold $\Gamma \subset U$ defined by $\Gamma = \{\mu \in U \mid F^2_\mu$ has as fixed points a codimension one saddle-node p_μ and n hyperbolic fixed points, p_μ is the biggest fixed point$\}$. The same definition holds for $\tilde{\Gamma} \subset U$ and $\tilde{F}$. It is clear that $\phi(\Gamma) = \tilde{\Gamma}$. These submanifolds accumulate at $0 \in U$. For $\mu \in \Gamma$ let $I_\mu = [p_\mu, \varepsilon]$ ($(-\varepsilon, \varepsilon)$ contains all the fixed points of F^2_μ). Let X_μ be the adapted vector field to $F^2_\mu \mid I_\mu$. Let $\tilde{X}_\mu$ be the adapted vector field to $f^2_{\phi(\mu)} \mid h_\mu(I_\mu)$. By Lemma 3.18 $(h_\mu)_* X_\mu = \tilde{X}_\mu$. Moreover, $\lim_{\mu \to 0} p_\mu = 0$, $\lim_{\mu \to 0} X_\mu = X \mid [0, \varepsilon]$ and $\lim_{\mu \to 0} \tilde{X}_\mu = \tilde{X} \mid [0, h_0(\varepsilon)]$. Thus $(h_0)_* X \mid [0, \varepsilon] = \tilde{X} \mid [0, h_0(\varepsilon)]$. Analogously, we prove $(h_0)_* X \mid [-\varepsilon, 0] = \tilde{X} \mid [h_0(-\varepsilon), 0]$. □

Proof of Theorem 3.17. Suppose $n \leq 3$. If $n = 1$ it is done in [**NPT**]. For $n = 2$ it is Corollary 3.3 of §3.2. For $n = 3$ we proceed as in the proof of Theorem 3.7 and we modify the conjugacy H_μ for μ in a neighborhood of the surface $\{\mu \mid F_\mu$ has a codimension one saddle-node$\}$. For μ in this surface we define H_μ preserving the adapted vector field. For μ near this surface we define H_μ in order to have continuity of H with respect to μ. We follow the same strategy and the same techniques used in

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the proof of Theorem 3.2 of §3.2. The construction, although more elaborate, does not involve any new ideas.

Suppose $n > 3$. Let $\Lambda = \{\mu \mid F_\mu$ has two consecutive codimension one saddle-nodes$\}$. Suppose $H(\mu, x) = (\phi(\mu), h_\mu(x))$ is a continuous local conjugacy between F and a nearby $\tilde{F}$. Suppose $\mu \in \Lambda$. Let X_μ and Y_μ be the adapted vector fields to F_μ^2 restricted to intervals that are maximal containing each saddle-node and no more fixed points. Analogously, consider $\tilde{X}_\mu$ and $\tilde{Y}_\mu$ the corresponding vector fields adapted to $\tilde{F}_\mu^2$ restricted to the image of these intervals by h_μ. By Lemma 3.18, $(h_\mu)_* X_\mu = \tilde{X}_\mu$ and $(h_\mu)_* Y_\mu = \tilde{Y}_\mu$. Both vector fields are defined in the open interval whose extremities are the two saddle-nodes. In general, they do not coincide [**Ko, Ma**] (the modulus of stability is infinite [**R**]). So there are two incompatible conditions that h_μ should satisfy. Therefore H cannot be continuous with respect to μ. □

4. *Global unfolding*

The main objective of this section is to prove Theorem A of §1. First we shall analyze the existence of a strong stable foliation at the center stable manifold of the codimension n flip. If $n \geq 2$, a conjugacy must preserve this foliation. We shall construct a compatible system of (strong) unstable foliations for the periodic orbits of F_μ, $\mu \in U$. This system will be used in the construction of conjugacies.

Let $F \in \mathcal{A}$. Let p be the codimension n flip of F. Let $W^{cs}(p)$ be the center stable manifold of F at p. We set $W^{cs}_\mu(p) = W^{cs}(p) \cap \{\mu\} \times M$. The manifold $W^{cs}(p)$ is C^{k+1} for k large. We assume, for all this section, that F_0 restricted to $W^{cs}_0(p)$ has a sink at p. The other case, p a source for $F_0 \mid W^{cs}_0(p)$, is completely analogous. For each $\mu \in U$, $W^{cs}_\mu(p)$ is the union of stable manifolds of (local) periodic orbits. Therefore $W^{cs}_\mu(p)$ is unique. In $W^{cs}(p)$ there exists a continuous F-invariant foliation with C^1 leaves, the tangent planes to the leaves depend continuously on the base point at $W^{cs}(p)$, each leaf intersects the central manifold $W^c(p)$ transversally at one point, the leaf that intersects p is the strong stable manifold of F_0 at p, $W^{ss}(p)$. This foliation is called the strong stable foliation and we denote it by $\mathcal{F}^{ss}(p)$. In general, $\mathcal{F}^{ss}(p)$ is not unique. In the case treated here $\mathcal{F}^{ss}(p)$ is unique and is C^k. We set $\mathcal{F}^{ss}_\mu(p) = \mathcal{F}^{ss}(p) \cap \{\mu\} \times M$. Suppose $\tilde{F}$ is a family near to F in $\mathcal{F}_U(M)$. Let $n \geq 2$. Let $\tilde{p}$ be the codimension n flip for $\tilde{F}$. Let $\mathcal{F}^{ss}(\tilde{p})$ be the corresponding strong stable foliation.

PROPOSITION 4.1. *Let $H : U \times M \to U \times M$, $H(\mu, x) = (\phi(\mu), h_\mu(x))$ be a conjugacy between F and $\tilde{F}$. Then h_0 maps $\mathcal{F}^{ss}_0(p)$ onto $\mathcal{F}^{ss}_{\phi(0)}(\tilde{p})$.*

Proof. Let $\Lambda_F = \{\mu \in U \mid F_\mu$ has only a codimension one saddle-node and a hyperbolic fixed point as periodic orbits unfolding from the codimension n flip$\}$. Λ_F is a codimension one submanifold of U and is part of a Whitney stratification of U consisting of $\{\mu \mid F_\mu$ has local periodic orbits with multiplicity greater than one$\}$. Moreover, $0 \in \overline{\Lambda_F}$. The case $n = 2$ is treated in §3.2. For $\tilde{F}$ there exists a corresponding $\Lambda_{\tilde{F}} \subset U$. As H is a conjugacy, $\phi(\Lambda_F) = \Lambda_{\tilde{F}}$. For $\mu \in \Lambda_F$, h_μ preserves the strong stable foliation. This is a direct consequence from Theorem 3.2 of [**NPT**]. As Λ_F accumulates at 0, by continuity of $\mathcal{F}^{ss}(p)$ and h_μ with respect to μ, we get that h_0 must map $\mathcal{F}^{ss}_0(p)$ onto $\mathcal{F}^{ss}_{\phi(0)}(\tilde{p})$. □

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Let $p_{(1,\mu)}, p_{(2,\mu)}, \ldots, p_{(j-1,\mu)}, \Lambda_\mu, p_{(j+1,\mu)}, \ldots, p_{(l,\mu)}$ be the periodic orbits of F_μ. Here Λ_μ denotes the set of local periodic orbits that unfold from the codimension n flip. $\Lambda_\mu = \{p_{((j,1),\mu)}, \ldots, p_{((j,n_\mu),\mu)}\}$. We set $W^s(\Lambda_\mu) = \bigcup_{1\le i\le n_\mu} W^s(p_{(j,i)})$ and $W^u(\Lambda_\mu) = \bigcup_{1\le i\le n_\mu} W^u(p_{(j,i)})$. We set the partial order $p_{i,\mu} \le p_{k,\mu}$ if $W^u(p_{i,\mu}) \cap W^s(p_{k,\mu}) \neq \emptyset$. For Λ_μ we consider $W^s(\Lambda_\mu)$ and $W^u(\Lambda_\mu)$. For $\mu = 0$ the set Λ_μ reduces to the codimension n flip $p = p_{j,0}$. In this case transversality of the stable and unstable manifolds and the finite nature of the limit set imply the non-existence of cycles (with respect to the partial order) for F_0 [**Sm**]. So we can take a total order of the periodic orbits with respect to this relation: $p_{1,0} < p_{2,0} < \cdots < p_{l,0}$. By [**Sh**] we can consider $\mathcal{M} = \{\emptyset = M_0 \subset M_1 \subset \cdots \subset M_l\}$ a filtration for F_0 such that $K_i(F_0) = \bigcap_{m\in\mathbb{Z}} F_0^m(M_i - M_{i-1}) = p_{i,0}$. We denote the non-wandering set of F_μ by $\Omega(F_\mu)$. Let V_i, $1 \le i \le l$, be small neighborhoods of $p_{i,0}$ in M. If we take U sufficiently small then $\Omega(F_\mu) \cap (M_i - M_{i-1}) \subset K_i(F_\mu) = \bigcap_{m\in\mathbb{Z}} F_\mu^m(M_i - M_{i-1}) \subset V_i$. The last subset relation yields $K_i(F_\mu) = p_{i,\mu}$ if $i \neq j$ and $K_j(F_\mu) = \Lambda_\mu$. As $\mathcal{M}$ is also a filtration for F_μ, there is not any cycle with periodic orbits outside Λ_μ. The following result excludes the possibility of a cycle involving periodic orbits of Λ_μ.

PROPOSITION 4.2. *Let $F \in \mathcal{A}$. If U is sufficiently small, then:*

(a) $\Omega(F_\mu) = \{$*periodic orbits of* $F_\mu\}$*;*

(b) F_μ *has no cycles;*

(c) *it is possible to take a total order of the periodic orbits of F_μ compatible with the order of the corresponding periodic orbits of F_0, if we consider Λ_μ as one periodic element.*

In order to prove Proposition 4.2, we need to analyze $W^u(\Lambda_\mu)$. Let $W^{cu}(p)$ and $W^c(p) = W^{cu}(p) \cap W^{cs}(p)$ be, respectively, the center unstable and the center manifolds of F at p. These are C^{k+1} manifolds for big k. These manifolds are not unique but $\bigcup_{\mu\in U} W^u(\Lambda_\mu) \subset W^{cu}(p)$ is unique. We set $W^{cu}_\mu(p) = W^{cu}(p) \cap \{\mu \times M\}$ and $W^c_\mu(p) = W^c(p) \cap \{\mu \times M\}$. In $W^c_\mu(p)$ we can consider the periodic orbits ordered by the length between q and $F_\mu(q)$. Let $\{a_\mu, b_\mu = F_\mu(a_\mu)\}$ be the maximal periodic orbit with respect to this order. There exists a C^k, F-invariant foliation $\mathcal{F}^{uu}(p)$ in $W^{cu}(p)$. It is called an unstable foliation. The leaf at $(0, p)$ is $W^u(p)$. For each μ the leaves that intersect $W^{cu}_\mu(p)$ belong to $W^{cu}_\mu(p)$. The restriction of the foliation $\mathcal{F}^{uu}(p)$ to $\bigcup_{\mu\in U} W^u(\Lambda_\mu)$ is unique. Each leaf instersects transversally $W^c(p)$ at one point. Let f_{a_μ} and f_{b_μ} be the leaves that contain a_μ and b_μ respectively. These leaves constitute the boundary of $W^u(\Lambda_\mu)$.

PROPOSITION 4.3. *Let V be a small open ball with center at $(0, p)$ in $U \times M$. Then*

$$\lim_{\mu\to 0} f_{a_\mu} \cap V = \lim_{\mu\to 0} f_{b_\mu} \cap V = W^u(p) \cap V$$

in the C^k topology.

Proof. It follows directly from the expression of F^2_μ restricted to $W^c(p)$ (see expression (2) in §3.3) and from the fact that $\mathcal{F}^{uu}(p)$ is C^k. □

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Remark. Let $N > 0$ and $W^u_N(\Lambda_\mu) = \bigcup_{1\le j\le N} F^j_\mu(W^u(\Lambda_\mu \cap V))$. Given $\varepsilon > 0$, we can take U sufficiently small in order that if $\mu \in U$ then $W^u_N(\Lambda_\mu)$ will be contained in the neighborhood of radius ε (points that are ε-close) of $\bigcup_{1\le j\le N} W^u(p)\cap V$. This is a direct consequence of Proposition 4.3.

Proof of Proposition 4.2. We have already proved item (a). Item (c) follows from item (b). Let $\mathcal{M}$ be a filtration for F_μ as we considered just before Proposition 4.2. From the construction of this filtration (see [**Sh**]) it is clear that $W^u(p)\cap M_j$ is compact. It follows from Proposition 4.3 and the subsequent remark that $W^u(\Lambda_\mu) \cap M_j$ is compact and stays close to $W^u(p) \cap M_j$. $W^u(p) \cap M_j$ does not intersect $W^{cs}_0(p) - p$. Therefore $W^u(\Lambda_\mu) \cap M_j$ does not intersect $W^{cs}_\mu(p) - p$. This fact and the order that the filtration imposes on $W^u(p_{i,\mu})$ imply item (b). □

According to Proposition 4.2 we can consider $p_{1,\mu} < p_{2,\mu} < \cdots < p_{j-1,\mu} < \Lambda_\mu < p_{j+1,\mu} < \cdots < p_{l,\mu}$ totally ordered. Let V_i, $i \neq j$, be neighborhoods of $p_{i,0}$. If we take U sufficiently small then $p_{i,\mu} \in V_i$ and we consider local unstable foliations (see [**NPT**]). Since these foliations are F_μ-invariant, we consider iterates F^m_μ, $m \in \mathbb{N}$, of the leaves. These iterates induce an unstable foliation, denoted by $\mathcal{F}^u(p_i)$, in $\{\mu\} \times \bigcup_{m\in\mathbb{N}} F^m_\mu(V_i)$ for $\mu \in U$. Alternatively, we denote $\mathcal{F}^{uu}(p)$ by $\mathcal{F}^u(p_j)$.

We recall the following definition

Definition 4.1. The system of foliations $\mathcal{F}^u(p_1), \mathcal{F}^u(p_2), \ldots, \mathcal{F}^u(p_{l,\mu})$ is compatible when:

(a) if a leaf F of $\mathcal{F}^u(p_i)$ intersects a leaf S of $\mathcal{F}^u(p_k)$, $i \le k$, then $F \supseteq S$;

(b) for all $i \le k$ the restriction of $\mathcal{F}^u(p_k)$ to each leaf of $\mathcal{F}^u(p_i)$ is a foliation of class C^k, $k \ge 2$.

PROPOSITION 4.4. *Let $F \in \mathcal{A}$. There exists a compatible system of unstable foliations for F_μ, $\mu \in U$.*

Proof. We follow the same ideas developed in the proof of Proposition 3.2 of [**NPT**]. The extra element here is the codimension n flip p. Let us construct a fundamental domain A_0 for $W^{cs}_0(p)$. From expression (3) in §3.3 we can express F at $W^{cs}_0(p)$ in a local C^{2n+2} system of coordinates as

$$F_0(x, y) = (-x + \tfrac{1}{4}x^{2n+1} + o(\|xy\|, \|x\|^{2n+2}), By + o(\|xy\|, \|y\|^2)),$$

where B is an $s \times s$ matrix, $\|B\| < 1$ and s is the number of eigenvalues of $DF_0(p)$ with negative real part. Here $y = 0$ represents the one-dimensional center manifold and $x = 0$ the s-dimensional strong stable manifold.

Consider $a \in W^c_0(p)$, $a < p$ near to p. Let L_a be the leaf of $\mathcal{F}^{ss}(p)$ through a. Take $b \in W^c_0(p)$, $F_0^{-1}(a) > b > F_0(a)$. Let L_b be the leaf of $\mathcal{F}^{ss}(p)$ through b. Consider two s-dimensional disks $D_a \in L_a$ and $D_b \in L_b$ with centers, respectively, in a and in b and radii equal to $|a - p|$. Let C be the closed set formed by D_a, D_b and an s-dimensional cylinder in $W^{cs}_0(p)$ bounded by the border of D_a union the border of D_b and generated by one-dimensional curves transversal to $\mathcal{F}^{ss}(p)$. These curves are taken almost parallel to $W^c_0(p)$. It is easy to check from the preceding expression that $F_0(C) \cap C = \emptyset$. For

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$\mu \in U$, $C_\mu = \{\mu\} \times C$ also crosses the strong stable manifolds of elements of Λ_μ transversally in $W^{cs}_\mu(p)$ and $F_\mu(C_\mu) \cap C_\mu = \emptyset$. We let A_μ be the closed region bounded by C_μ and $F_\mu(C_\mu)$. We take $A = \bigcup_{\mu \in U} A_\mu$. It is diffeomorphic to $U \times A_0$. With this fundamental domain for $W^{cs}(p)$ we can construct compatible unstable foliations exactly in the same way as was done in Proposition 3.2 of [**NPT**]. □

We call the codimension n flip p, *s-critical* or, simply, *critical* if there exists a hyperbolic periodic orbit q such that $W^u(q, F_0)$ intersects some leaf of $\mathcal{F}^{ss}_0(p)$ non-transversally. We say that p is *non-critical* if it is not critical.

In the construction of conjugacies we will need a one-dimensional F-invariant foliation, defined in $W^{cs}(p)$, transversal to $\mathcal{F}^{ss}(p)$. We denote it as the central foliation.

PROPOSITION 4.5. *If F_0 has a non-critical codimension n flip then there exists a central foliation $\mathcal{F}^c(p)$ defined in a neighborhood of p in $W^{cs}(p)$ with the following properties:*

(a) *leaves of $\mathcal{F}^c(p)$ are contained in $\{\mu\} \times M$ for some $\mu \in U$;*
(b) *$\mathcal{F}^c(p)$ is F_μ-invariant;*
(c) *$\mathcal{F}^c(p)$ is compatible with $\mathcal{F}^u(p_i)$, $i < j$, in the sense of Definition 4.1;*
(d) *leaves of $\mathcal{F}^c(p)$ are of class C^k and vary continuously in the C^k topology;*
(e) *$\mathcal{F}^c(p)$ is transversal to $\mathcal{F}^{ss}(p)$;*
(f) *The union of leaves of $\mathcal{F}^{uu}(p)$ that intersect a leaf of $\mathcal{F}^c(p)$ constitutes a C^k submanifold.*

Remark. The k here is the same k used in the C^{k+1} differentiability of $W^{cs}(p)$. We take $k \geq 1$.

Proof. Since $p_1 < p_2 < \cdots < p_{j-1}$ are all hyperbolic, in the construction of a compatible system of unstable foliations the leaves of $\mathcal{F}^u(p_i)$, $1 \leq i < j$, can be taken to be C^{k+1}. Let A be a fundamental domain for $W^{cs}(p)$ as in Proposition 4.4. For each $\mu \in U$ consider the leaf of $\mathcal{F}^{ss}(p)$ through the fixed point correspondent to the codimension n flip p. Let L be the $(s+1)$-dimensional manifold formed by these leaves. L splits $A = A_1 \cup A_2$. Let $B = A_1 \cup F(A_2)$. We will define $\mathcal{F}^c(p)$ in B as the set of integral curves of a vector field X. The vector field X satisfies the following properties. X is tangent to the leaves of $\mathcal{F}^u(p_i)$ and it is C^k along these leaves for $i < j$ such that $W^u(p_i) \cap W^{cs}(p) \neq \emptyset$. The vector field X is transversal to $\mathcal{F}^{ss}(p)$, it is of class C^0 and dF-invariant. To define X we proceed by induction taking $i = j-1, j-2, \ldots, 1$. If $W^u(p_{j-1})$ intersects $W^{cs}(p)$ we consider a C^k vector field X tangent to $W^u(p_{j-1})$ in a neighborhood of $W^u(p_{j-1}) \cap \partial_{\text{ext}} A \cap B$ in $W^u(p_{j-1})$. In a neighborhood of $W^u(p_{j-1}) \cap F(\partial_{\text{int}} A) \cap B$ in $W^u(p_{j-1})$ we consider $dF^2(X)$. If $W^u(p_{j-1}) \cap \partial_{\text{int}} A \cap L \neq \emptyset$, in a neighborhood of this intersection in $W^u(p_{j-1})$ we consider $-dF(X)$. It is easy to extend this vector field X to a neighborhood of $W^u(p_{j-1}) \cap B$ in $W^u(p_{j-1})$. The leaves of $\mathcal{F}^u(p_{j-1})$ are C^{k+1} imbeddings of a disk in $W^u(p_{j-1})$, continuously parametrized by their intersection with $W^s(p_{j-1})$. So we can project X into the leaves of $\mathcal{F}^u(p_{j-1})$ near $\partial_{\text{ext}} A \cap B$. We consider its image by $-dF$ near $\partial_{\text{int}} A \cap L$ and its image by dF^2. We extend it across B as above. We can take X transversal to $\mathcal{F}^{ss}(p)$ as p is non-critical. By induction we suppose X is already defined along the leaves of $\mathcal{F}^u(p_{i+1}), \ldots, \mathcal{F}^u(p_{j-1})$ near B. If $W^u(p_i) \cap B = \emptyset$ we proceed to the next folitation $\mathcal{F}^u(p_{i-1})$. If $W^u(p_i) \cap B \neq \emptyset$ but $W^u(p_i) \cap W^s(p_k) = \emptyset$ for $i < k < j$, then $W^u(p_i) \cap B$ is compact and disjoint from

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$W^u(p_k) \cap B$. So we can proceed as before. If $W^u(p_i) \cap B \neq \emptyset$ and $W^u(p_i) \cap W^s(p_{k_0}) \neq \emptyset$ for some $i < k_0 < j$, then as $\mathcal{F}^u(p_i)$ is compatible with $\mathcal{F}^u(p_k)$ for $i < k < j$ the vector field X is already defined in a neighborhood V of $\bigcup_{i<k<j} W^u(p_k) \cap B$ and is tangent to the leaves of $\mathcal{F}^u(p_i)$ in V. But $W^u(p_i) \cap (B - V)$ is compact. So we can proceed as before. Thus we can construct X with the above properties in a neighborhood of B in $W^{cs}(p)$. We take the integral curves of X and their positive iterates by F. By λ-lemma [**P**] the leaves that do not intersect L extend to center manifolds. Part (f) follows since each leaf of $\mathcal{F}^c(p)$ is C^k. We remark that the construction of X is continuous on μ and yields part (d). We can consider $\mathcal{F}^u(p_i)$ of class C^k outside $\bigcup_{i|p_i \text{ is not a source}} W^u(p_i)$. The vector field X will be C^k in this set.

PROPOSITION 4.6. *Let $F \in \mathcal{A}$. For every $\mu \in U$ and $W^s(p_{i,\mu})$ there is no non-transversal intersection between $W^s(p_{i,\mu})$ and $\mathcal{F}^{uu}(p)$ restricted to $W^u(\Lambda_\mu)$.*

Proof. For $\mu = 0$, $W^u(\Lambda_0)$ reduces to $W^u(p)$ and the result comes from transversality conditions that are part of the definition of $\mathcal{A}$. For $\mu \neq 0$ continuity of $W^s(p_{i,\mu})$ and $\mathcal{F}^{uu}(p)$ in the C^k topology, with respect to μ, Proposition 4.3 and the subsequent remark imply the result. □

Let $\mathcal{V} = \{\mu \in U \mid F_\mu$ has as periodic orbits near the codimension n flip p just a fixed point and a codimension one saddle-node periodic orbit$\}$. $\mathcal{V}$ is a connected codimension one submanifold.

PROPOSITION 4.7. *Let $H : U \times M \to U \times M$, $H(\mu, x) = (\phi(\mu), h_\mu(x))$ be a conjugacy between F and $\tilde{F}$. For all $\mu \in \mathcal{V}$, h_μ maps $\mathcal{F}^{uu}(p) \cap W^u(\Lambda_\mu)$ onto $\mathcal{F}^{uu}(p) \cap W^u(\tilde{\Lambda}_{\phi(\mu)})$.*

Proof. Let $q_{1,\mu}$ and $q_{2,\mu}$ be, respectively, the fixed point and the periodic saddle-node (with period two). Then $\mathcal{F}^{uu}(p) \cap W^u(\Lambda_\mu) = W^u(q_{1,\mu}) \cup (\mathcal{F}^{uu}(p) \cap W^u(q_{2,\mu}))$. It is easy to see that $W^u(q_{1,\mu})$ must be mapped onto $W^u(\tilde{q}_{1,\phi(\mu)})$, where $\tilde{q}_{1,\phi(\mu)}$ is the correspondent fixed point for $\tilde{F}_{\phi(\mu)}$. Let $\tilde{q}_{2,\phi(\mu)}$ be the correspondent saddle-node periodic orbit for $\tilde{F}_{\phi(\mu)}$. As in Lemma 3.18 of §3.3, we take a C^∞ curve $c(t)$ in U such that $c(0) = \mu$ and $f_t = F_{c(t)}$ unfolds generically the codimension one saddle-node. Then, as in [**NPT**], $\mathcal{F}^{uu}(p)$ restricted to $W^u(q_{2,\mu})$ must be mapped onto the correspondent $\mathcal{F}^{uu}(h_0(p))$ restricted to $W^u(\tilde{q}_{2,\phi(\mu)})$. □

Proposition 4.7 indicates that in the construction of conjugacies between F and a nearby $\tilde{F}$ the strong unstable foliation restricted to $W^u(\Lambda_\mu)$ should be preserved. Proposition 4.6 denotes the absence of u-criticality in $W^u(\Lambda_\mu)$.

PROPOSITION 4.8. *Let $F \in \mathcal{A}$. If $n \geq 2$ and p is critical then F is not stable.*

Proof. We follow the same ideas contained in the proof of Proposition 2.4, chapter III, of [**NPT**]. We consider a point q of quasi-transversal intersection between $W^u(p_i)$ and $\mathcal{F}_0^{ss}(p)$. We take a curve γ in $W^{cs}(p)$ such that γ contains q and is transversal to $\mathcal{F}^{ss}(p)$. Perturbing, if necessary, we may assume that all the eigenvalues of $dF_0(p_i)$ and of $d\tilde{F}_0(\tilde{p}_i)$ are distinct. Let α be the weakest contracting eigenvalue of $dF_0(p_i)$. Let $y \in \gamma$ be near q. Let A be a fundamental domain of $W^s(p_i)$ and V a small

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neighborhood of A. Then it is possible to choose γ so that $\rho(y,q) \sim |\alpha|^{n(y)}$. Here ρ denotes the distance with respect to some Riemannian metric and $\sim$ denotes equality up to a positive multiplicative factor, uniformly bounded and bounded away from zero. $n(y)$ is the least positive integer n such that $F_0^{-n}(y) \in V$. We take $\tilde{F}$ a perturbation of F such that $|\tilde{\alpha}| < |\alpha|$. Then $\rho(h_0(y), h_0(q)) < K|\tilde{\alpha}|^{n(y)}$ for some fixed $K > 0$. Thus $\lim_{y \to x} \rho(h_0(y), h_0(q))/\rho(y,q) = 0$. Since we can project these distances along $\mathcal{F}^{ss}(p)$ and $\mathcal{F}^{ss}(\tilde{p})$ onto the respective central manifolds, this gives a contradiction with Lemma 3.18 of §3.3. □

Proof of Theorem A. Consider a nearby family $\tilde{F} \in \mathcal{F}_U(M)$. By Corollary 3.6 of §3.3 there exist μ_0 near 0 and $\tilde{p}$ near p such that $\tilde{F}_{\mu_0}$ has a codimension n flip at $\tilde{p}$ and unfolds it generically. We proceed to construct a conjugacy $H : U \times M \to U \times M$, $H(\mu, x) = (\phi(\mu), h_\mu(x))$ between F and $\tilde{F}$. In cases (b) and (c) of Theorem A we define $\phi : (U, 0) \to (U, \mu_0)$ as done in the proof of Theorem 3.7 of §3.3. In case (a) we define ϕ as done in Theorem 3.17 of §3.3. By Proposition 4.2 we can consider $p_{1,\mu} < p_{2,\mu} < \cdots < p_{j-1,\mu} < \Lambda_\mu < p_{j+1,\mu} < \cdots < p_{l,\mu}$, the periodic orbits of F_μ, totally ordered. Here $\Lambda_\mu = \{p_{(j,1),\mu}, \ldots, p_{(j,n_\mu),\mu}\}$ is the set of local periodic orbits unfolding from p. We remark that $n_\mu \geq 1$. If $n_\mu = 1$ we denote Λ_μ as $p_{j,\mu}$. By Proposition 4.4 we can take a compatible system of (strong) unstable foliations $\mathcal{F}^u(p_1), \mathcal{F}^u(p_2), \ldots, \mathcal{F}^u(p_l)$. For $\tilde{F}_{\phi(\mu)}$ we can consider its corresponding periodic orbits $\tilde{p}_{i,\phi(\mu)}$ and $\tilde{\Lambda}_{\phi(\mu)}$ such that the correspondence $p_{i,\mu} \to \tilde{p}_{i,\phi(\mu)}$ is continuous and order preserving. This correspondence comes from the fact that for small μ the filtration for F_0 is also a filtration for $\tilde{F}_{\phi(\mu)}$ and $K_i(\tilde{F}_{\phi(\mu)}) = \bigcap_{m \in \mathbb{Z}} \tilde{F}^m_{\phi(\mu)}(M_i - M_{i-1}) = \tilde{p}_{i,\phi(\mu)}$ if $i \neq j$ and $K_j(\tilde{F}_{\phi(\mu)}) = \tilde{\Lambda}_{\phi(\mu)}$. For $\tilde{F}_{\phi(\mu)}$ we consider a corresponding (and nearby) system of (strong) unstable foliations $\mathcal{F}^u(\tilde{p}_1), \mathcal{F}^u(\tilde{p}_2), \ldots, \mathcal{F}^u(\tilde{p}_l)$.

We define $h_\mu(p_{i,\mu}) = \tilde{p}_{i,\phi(\mu)}$, for $i \neq j$. We define $h_\mu(\Lambda_\mu) = \tilde{\Lambda}_{\phi(\mu)}$, preserving the order of the real line. The global conjugacy H will be constructed, by induction, on the stable manifolds of the periodic orbits and on the center stable manifold of F at the codimension n flip. We consider fundamental domains $A_{i,\mu}$ for $W^s(p_{i,\mu})$ if $i \neq j$. We take $A_{j,\mu}$ as the fundamental domain for $W^{cs}(\Lambda_\mu)$ as in Proposition 4.4. Define $A_i = \bigcup_{\mu \in U} A_{i,\mu}$. The conjugacy H will have the following properties:

(a) $H(\{\mu\} \times M) = \{\phi(\mu)\} \times M$, for $\mu \in U$;

(b) leaves of $\mathcal{F}^u(p_i)$ are sent into leaves of $\mathcal{F}^u(\tilde{p}_i)$;

(c) if $W^u(p_i) \cap W^s(p_k) \neq \emptyset$, then for each leaf J of $\mathcal{F}^u(p_i)$, the restriction of H to $J \cap W^s(p_k)$ is differentiable and in a neighborhood of $J \cap A_k$ in $W^s(p_k)$ ($W^{cs}(p_k)$ if $k = j$), H is C^1 near the inclusion map.

H maps the sources of F_μ onto the sources of $\tilde{F}_\mu$. We set $p_i = p_{i,0}$ and $\tilde{p}_i = \tilde{p}_{i,0}$. Suppose H has already been defined on the stable (or center stable) manifolds of F at $p_1, p_2, \ldots, p_m$. As each leaf of $\mathcal{F}^u(p_i)$ intersects $W^s(p_i)$ in one point, H defines a map from the space of leaves of $\mathcal{F}^u(p_i)$ onto the space of leaves of $\mathcal{F}^u(\tilde{p}_i)$ for $1 \leq i \leq m$. We consider $W^s(p_{m+1})$ and $W^s(\tilde{p}_{m+1})$. We will define H from a neighborhood of A_{m+1} onto a neighborhood of $\tilde{A}_{m+1}$. There are five cases.

(1) $m + 1 < j$. In this case we proceed as in [**NPT**].

(2) $m+1 = j$, $n \leq 3$, and p_j is not s-critical. In this case it is possible to construct H at $W^{cs}(p_j)$ as the 'product' of two partial conjugacies, H^c defined at the center manifold

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of p_j and H^s defined at a disk D_μ contained in a leaf of $\mathcal{F}^{ss}(p_j)$. This approach can be seen in the proof of Theorem 4.3 of [**NPT**]. It can be adapted to this case. The differences are as follows. (i) If $n = 2$ the conjugacy H^c is the one constructed in the proof of Theorem 3.2 of §3.2. (ii) If $n = 3$ the conjugacy H^c is constructed as indicated in Theorem 3.17 of §3.3. (iii) The disk D_μ has its 'center' at a point $q_\mu \in W^c_\mu(p_j)$ that is smaller (by the order of the real line) than any local fixed point of F^2_μ. (iv) We work with F^2_μ to define h_μ at the left half of $W^{cs}(p_j)$ (the middle is determined by the leaf of $\mathcal{F}^{ss}(p_j)$ passing through the fixed point for F_μ). Then we extend h_μ to the other half using the conjugacy equation.

(3) $m + 1 = j$, $n \leq 3$, and p_j is s-critical. Let q_μ, r_μ be the smallest, respectively the biggest, local fixed point of F^2_μ restricted to $W^c_\mu(p_j)$ considering the order of the real line. In particular, $q_0 = r_0 = p_j$. Analogously, we consider $\tilde{q}_\mu$ and $\tilde{r}_\mu$ for $\tilde{F}_\mu$. Take C_μ and D_μ leaves of $\mathcal{F}^{ss}(p_j)$ such that $q_\mu \in C_\mu$ and $r_\mu \in D_\mu$. In a similar way consider $\tilde{C}_\mu$ and $\tilde{D}_\mu$. Let A_μ be a fundamental domain for $F^2_\mu \mid W^{cs}_\mu(p_j)$, as in the proof of Proposition 4.4. Let $\mathcal{R}_\mu \in W^{cs}_\mu(p_j)$ be the region delimited by $\partial_{\text{ext}} A_\mu$, C_μ and D_μ. Since $W^u(p_i)$ intersects $W^{ss}(p)$ transversally, for $i < j$, we can construct, as in Proposition 4.5, a central foliation $\mathcal{F}^c_\mu(p_j)$ in $\mathcal{R}_\mu$. This foliation is invariant by F_μ, transversal to $\mathcal{F}^{ss}(p_j)$, compatible with $\mathcal{F}^u(p_i)$ for $i < j$. It has C^k leaves (varying continuously with respect to the C^k topology). It is a classical C^k foliation outside $\bigcup_{p_i \text{ is not a source}} W^u(p_i)$. Analogously, we consider a central foliation $\mathcal{F}^c_\mu(\tilde{p}_j)$ in $\tilde{\mathcal{R}}_\mu$ that is invariant by $\tilde{F}_{\phi(\mu)}$. We define H in $\mathcal{R} = \bigcup_{\mu \in U} \mathcal{R}_\mu$ as the product of two partial conjugacies, H^c and H^s.

The first one is defined from $W^c(p_j)$ to $W^c(\tilde{p}_j)$, as constructed in Theorem 3.7 of §3.3. In fact we consider just $H^c_\mu : [q_\mu, r_\mu] \to [\tilde{q}_{\phi(\mu)}, \tilde{r}_{\phi(\mu)}]$.

The second conjugacy is defined at a fundamental domain for $F^2_\mu \mid C_\mu$. The leaf C_μ splits A_μ into two regions. Let $\Omega_\mu \subset A_\mu$ be the closure of the region that does not contain $\mathcal{R}_\mu$. We set $C^*_\mu = C_\mu \cap A_\mu$. In a neighborhood V_μ of $\partial_{\text{ext}} C^*_\mu$ in C_μ we consider a map "F_μ" defined by: if $x \in V_\mu$ then $x \in J$ where J is a leaf of $\mathcal{F}^c_\mu(p_j)$. We set "F_μ"$(x) = F_\mu(J) \cap C_\mu$. Analogously, we define "$\tilde{F}_\mu$" at a neighborhood $\tilde{V}_\mu$ of $\partial_{\text{ext}} \tilde{C}^*_\mu$ in $\tilde{C}_\mu$. In V_μ we define H^s inductively, mapping leaves of $\mathcal{F}^u(p_i)$, $i < j$, into correspondent leaves of $\tilde{\mathcal{F}}^u(\tilde{p}_i)$ intersected with $\tilde{V}_{\phi(\mu)}$. For each leaf this map is C^k near the inclusion. At "F_μ"$(V_\mu) \cup F^2_\mu(V_\mu)$ we extend H^s by the equations H^s"F_μ" $=$ "$\tilde{F}_{\phi(\mu)}$"H^s and $H^s F^2_\mu = F^2_{\phi(\mu)} H^s$, respectively. Using the isotopy extension theorem we extend the definition of H^s to the intersection of leaves of $\mathcal{F}^u(p_i)$ with C^*_μ. As $C^*_\mu \subset \bigcup_{i<j} W_\mu(p_i)$ the map H^s is already defined at a neighborhood of C^*_μ.

We set $\partial_{\text{ext}}\Omega_\mu = \partial\Omega \cap \partial_{\text{ext}} A_\mu$ and $\partial_{\text{int}}\Omega_\mu = \partial\Omega \cap \partial_{\text{int}} A_\mu$. Then $\partial\Omega_\mu = \partial_{\text{ext}}\Omega_\mu \cup \partial_{\text{int}}\Omega_\mu \cup C^*_\mu$. We define H at a neighborhood of $\partial_{\text{ext}}\Omega_\mu$ inductively at leaves of $\mathcal{F}^u(p_i)$, $i < j$. Near $\partial_{\text{int}}\Omega_\mu$, H is defined by $H F^2_\mu = \tilde{F}^2_{\phi(\mu)} H$. This definition is extended to the intersection of leaves of $\mathcal{F}^u(p_i)$ with Ω_μ using the isotopy extension theorem. Details of this type of construction can be seen in [**NPT**]. This extension is done compatible with the definition of H^s already done at C^*_μ. The construction of H at the closure of Ω_μ is finished.

For each μ we define H at $\mathcal{R}_\mu$ as the product of H^c and H^s (using the transversal

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foliations $\mathcal{F}^c_\mu(p_j)$ and $\mathcal{F}^{ss}(p_j)$). We extend the definition of H to all $W^{cs}_\mu(p_j)$ by $HF_\mu = \tilde{F}_{\phi(\mu)}H$.

(4) $m+1=j$, $n>3$. We proceed as in (3).

(5) $m+1>j$. Let $W^u_\mu = \bigcup_{p_{(j,i),\mu}\in\Lambda_\mu} W^u(p_{(j,i),\mu})$. We can view W^u_μ as the unstable manifold of Λ_μ. We now construct the conjugacy H on the fundamental domain $A_\mu = A_{m+1,\mu}$ of $W^s(p_{m+1,\mu})$. We proceed inductively (in descending order on i) constructing H on $W^u(p_{i,\mu}) \cap W^s(p_{m+1,\mu})$ and on $\mathcal{F}^u(p_{i,\mu}) \cap W^s(p_{m+1,\mu})$ for $i < m+1$. If $i \neq j$ this construction is equal to the one described in Theorem 4.3 of [**NPT**]. If $i=j$ there are two cases to consider. If $W^u(p_j) \cap (\bigcup_{j<l<m+1} W^s(p_l)) = \emptyset$ then by Proposition 4.3 each leaf of $\mathcal{F}^u(p_j) \cap W^u_\mu$ intersects A_μ in a compact set disjoint from neighborhoods of $W^u(p_l) \cap A_\mu$ for $j<l<m+1$. By Proposition 4.3 these intersections are C^k near to $W^u(p_j) \cap A_0$. We notice that H has been constructed on $W^{cs}(p_j)$ and $\mathcal{F}^u(p_j)$ is parametrized by the intersection of its leaves with $W^{cs}(p_j)$. Therefore H defines a map from the space of leaves of $\mathcal{F}^u(p_j)$ onto the space of leaves of $\mathcal{F}^u(\tilde{p}_j)$. Let S_μ be the leaf of $\mathcal{F}^u(p_j)$ passing through the unique fixed point of F_μ that unfolds from p_j. Analogously we define $\tilde{S}_\mu$ for $\tilde{F}_{\phi(\mu)}$. Let $(\phi(\mu), \tilde{S}_\mu) = H(\mu, S_\mu)$. We construct H at $S_\mu \cap A_\mu$ as a diffeomorphism near the identity of a neighborhood of $S_\mu \cap A_\mu$ onto a neighborhood of $\tilde{S}_\mu \cap \tilde{A}_\mu$. This construction maps leaves of $\mathcal{F}^u(p_j)$ onto corresponding leaves of $\mathcal{F}^u(\tilde{p}_j)$ and preserves the conjugacy at the border of A_μ. It is done using an extension of the isotopy extension theorem [**Pa, NPT**]. Consider $W^u(p_j) \cap (\bigcup_{j<l<m+1} W^s(p_l)) \neq \emptyset$. Since $\mathcal{F}^u(p_j)$ and $\mathcal{F}^u(p_l)$, $j<l<m+1$ are compatible, H is already defined on the leaves of $\mathcal{F}^u(p_j)$ in a neighborhood V of $A_\mu \cap (\bigcup_{j<l<m+1} W^u(p_{l,\mu}))$ in A_μ. But $Y = S_\mu \cap (A_\mu - V)$ is compact and, by Proposition 4.3, $W^u_\mu \cap (A_\mu - V)$ is also compact. Using the same method used in the previous case we can extend H to Y and to the leaves of $\mathcal{F}^u(p_j)$ in a neighborhood of Y in A_μ. This construction is continuous on μ.

In any case, we extend the construction of H on A_{m+1} to $W^s(p_{m+1})$ (or $W^{cs}(p_{m+1})$) using the conjugacy equation.

To prove the continuity of H one follows the same arguments as in Theorem 4.3 of [**NPT**]. The principal addition here is that the continuity on μ, implicit in item (a) of the theorem, is a direct consequence of the careful construction of H^c at the central manifold of p_j.

The statement that F is not stable if p is critical is given by Proposition 4.8. The statement that F is not locally stable if $n>3$ is part of Theorem 3.17 of §3.3. □

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