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An analytic counterexample to the KAM theorem

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Abstract. We study the non-existence of KAM tori for quasi-integrable, analytic Lagrangians. Let $\mathcal{L}\colon \mathbf{T}^m \times \mathbf{R}^m \to \mathbf{R}$, $\mathcal{L}(Q, \dot{Q}) = \frac{1}{2}|\dot{Q}|^2 + h(Q)$ and let $\bar{\omega} \in \mathbf{R}^m$ be a frequency exponentially close to resonances. We find h analytic of norm arbitrarily small such that $\mathcal{L}$ has no invariant torus of frequency $\bar{\omega}$ projecting diffeomorphically on $\mathbf{T}^m$.

0. *Introduction*

One of the problems in the theory of Hamiltonian systems is to understand which are the frequencies for which the KAM theorem holds. To be more precise, let us suppose we are given the Lagrangian

$$\mathcal{L}\colon \mathbf{T}^m \times \mathbf{R}^m \to \mathbf{R}, \quad \mathcal{L}(Q, \dot{Q}) = \tfrac{1}{2}|\dot{Q}|^2 + h(Q, \dot{Q}),$$

where $\mathbf{T}^m = \mathbf{R}^m/2\pi\mathbf{Z}^m$ is the m-dimensional torus. The KAM theorem asserts that, if h is sufficiently small in a certain norm (C^k or analytic) then there is a class of $\omega \in \mathbf{R}^m$ for which there is a smooth embedding $i\colon \mathbf{T}^m \to \mathbf{T}^m \times \mathbf{R}^m$ satisfying the following three properties.

(1) $i(\mathbf{T}^m)$ is invariant for the Euler–Lagrange flow (from now on E–L flow) of $\mathcal{L}$.
(2) i conjugates the E–L flow on $i(\mathbf{T}^m)$ with a translation of frequency ω on $\mathbf{T}^m$.
(3) $i(\mathbf{T}^m)$ is 'close' to the torus $\{\dot{Q} = \omega\}$.

If $h \in C^\infty$, then the frequencies ω for which the theorem holds are those satisfying the following condition

$$\exists C, \gamma > 0 : \quad \forall k \in \mathbf{Z}^m \setminus \{0\} \quad |\langle \omega, k\rangle| \geq \frac{C}{|k|^\gamma} \tag{$*$}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^m$. A proof of the sufficiency of ($*$) can be found, for instance, in [**14**]; as a corollary of [**9**] it follows that, for twist maps, ($*$) is also necessary. With the additional condition that $i(\mathbf{T}^m)$ projects diffeomorphically on $\mathbf{T}^m$ it is simple to show, using only the most elementary ideas of [**9**], that ($*$) is necessary in any dimension.

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If h is analytic, the situation is more intricate. It is known (see for instance [**6**]) that in this case the KAM theorem holds under a weaker condition than $(*)$, the Bruno condition which we presently explain. Let

$$\alpha_r = \min_{0<|k|\le r} |\langle k, \omega\rangle|.$$

The Bruno numbers are those for which

$$B(\omega) := -\sum_{i=0}^{n} \frac{1}{2^{j-1}} \log_2 \alpha_{2^j} < \infty. \qquad (**)$$

It is easy to see that $(**)$ is implied by $(*)$, while the converse is not true. As for necessity, there is numerical evidence (see for instance [**8**]) that the smallest h destroying the invariant torus is roughly of norm $e^{-B(\omega)}$; if $B(\omega) = \infty$, then the torus can be destroyed by arbitrarily small analytic perturbations.

The rigorous results are those of [**5**] which deals exclusively with twist maps. As a corollary of [**5**] we have that the KAM theorem does not hold for all ω for which the general term of the series in $(**)$ does not tend to 0. In other words, the invariant circles of an integrable twist map of rotation number $\omega \in \mathbf{R}$ satisfying

$$\exists C, \gamma > 0, \quad \exists (p_n, q_n) \to \infty : |q_n\omega - p_n| \le \frac{C}{\exp(\gamma(|p_n| + |q_n|))}$$

do not survive to arbitrarily small analytic perturbations.

Our aim is to generalize this part of [**5**] to higher dimension. We begin by introducing some notation. We call A_σ the set of all real analytic functions defined in a complex strip of radius σ around $\mathbf{T}^m$; for $h \in A_\sigma$ we consider the norm

$$\|h\|_\sigma = \sup\{|h(z)| : |\mathrm{Im}(z)| \le \sigma\}.$$

We will prove the following theorem.

THEOREM 1. *Let $m \ge 2$ and $\sigma > 0$. Let $\bar\omega \in \mathbf{R}^m$, $|\bar\omega| \in (1, 2)$ and let it be rationally independent (i.e. $\{\bar\omega t\}_{t\in\mathbf{R}}$ is dense on $\mathbf{T}^m$). Moreover, let it satisfy*

$$\exists \{k_n\} \subset \mathbf{Z}^m, |k_n| \to \infty : 0 < \langle \bar\omega, k_n\rangle \le e^{-D|k_n|}. \qquad (***)$$

Then if $D > \sigma$ $\forall \epsilon \in (0, 1]$ there is $h \in A_\sigma$ with $\|h\|_\sigma \le \epsilon$ such that $\mathcal{L}(Q, \dot Q) = \frac{1}{2}|\dot Q|^2 + h(Q)$ does not have any invariant torus $\mathcal{T}$ satisfying

(i) *$\mathcal{T}$ projects diffeomorphically on $\mathbf{T}^m$.*

(ii) *There is a C^1 diffeomorphism $i : \mathbf{T}^m \to \mathcal{T}$ which conjugates the motion on $\mathcal{T}$ to a translation of frequency $\bar\omega$.*

For twist maps, it is a theorem of Birkhoff (see [**12**] for a proof) that any homotopically non-trivial invariant circle projects diffeomorphically on $\mathbf{T}^1$; we do not know whether in arbitrary dimension invariant tori of rotation number $\bar\omega$ always project diffeomorphically on $\mathbf{T}^m$. However, if a torus $\mathcal{T}$ not satisfying (i) exists, then its Kolmogoroff normal form cannot be positive definite; this is because of the following theorem of [**10**]. If close to $\mathcal{T}$ $\mathcal{L}$ behaves like $\mathcal{L}(Q, \dot Q) = \langle \dot Q, \bar\omega\rangle + \frac{1}{2}\langle A\dot Q, \dot Q\rangle +$ h.o.t. with A positive definite, then the

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torus supports a minimal measure and this implies that $\mathcal{T}$ projects diffeomorphically on $\mathbf{T}^m$. This makes it very unlikely that such a $\mathcal{T}$ can result from any of the usual proofs of the KAM theorem; indeed these proofs provide positive definite KAM tori whenever the unperturbed torus $\{\dot{Q} = \bar{\omega}\}$ is, as in our case, positive definite.

The proof of Theorem 1 is contained in the next section and is based on an observation of [**13**] and [**7**]: the orbits lying on a KAM torus are minimizers of the action functional. But the projection on $\mathbf{T}^m$ of a KAM torus of frequency $\bar{\omega}$ is surjective: thus through each point of $\mathbf{T}^m$ passes a minimal orbit of 'mean frequency' $\bar{\omega}$. Thus our counterexample reduces to finding a quasi-integrable Lagrangian whose minimal orbits of mean frequency $\bar{\omega}$ avoid some points of $\mathbf{T}^m$. This Lagrangian is the same as in [**1**]; this is not surprising since [**1**] is practically the Nekhorocheff normal form near a first-order resonance and, by (∗∗∗), we are very close to a first-order resonance.

Proof of Theorem 1

We begin recalling the definition of [**10**] of action minimizing orbit; actually, we will give a slightly weaker definition, but sufficient for our purposes. Let us consider the Lagrangian $\mathcal{L}(Q, \dot{Q}) = \frac{1}{2}|\dot{Q}|^2 + h(Q)$. We say that the orbit $Q \in$ A.C.$(\mathbf{R}, \mathbf{T}^m)$ (i.e. absolutely continuous) is minimal for $\mathcal{L}$ if

$$\forall t_1 < t_2, \quad \int_{t_1}^{t_2} \mathcal{L}(Q, \dot{Q})\, dt \leq \int_{t_1}^{t_2} \mathcal{L}(\bar{Q}, \dot{\bar{Q}})\, dt \tag{1}$$

for every $\bar{Q} \in$ A.C. such that

(a) $Q(t_i) = \bar{Q}(t_i)$, $i = 1, 2$.

(b) Q and $\bar{Q}$ restricted to $[t_1, t_2]$ are in the same homotopy class or, equivalently, their lifts to $\mathbf{R}^m$ connect the same points.

If (1) holds for a particular couple of $t_1 < t_2$, we say that Q is minimal for $\mathcal{L}$ on $[t_1, t_2]$.

In the above definition we take minima in the very wide class of A.C. functions; however, since in our case minima are always analytic, it is the same if we restrict ourselves to a smaller class. In particular, in the proof of Lemma 2 it will be useful to consider our minima as taken in $H^1([t_1, t_2], \mathbf{T}^m)$, the set of A.C. functions with square-integrable derivative.

We now observe that an invariant torus satisfying point (ii) of Theorem 1 is Lagrangian. Indeed, let Ω be the canonical 2-form on $\mathbf{T}^m \times \mathbf{R}^m$ and let us suppose by contradiction that $\Omega|_{\mathcal{T}} \neq 0$. Then there is a point $x_0 \in \mathbf{T}^m$ and two vectors v and w such that $i^*(\Omega)(x_0)(v, w) > 0$, where i^* denotes the pullback. By the invariance of $i^*(\Omega)$ under translations of frequency $\bar{\omega}$ we get that $i^*(\Omega)(x)(v, w) > 0$ for any $x \in \mathbf{T}^m$. Moreover, slightly changing v and w, we can suppose that the plane π which contains them is, quotiented by $2\pi\mathbf{Z}^m$, a two-dimensional torus. If we call $\tilde{\pi}$ this torus we get that $\int_{\tilde{\pi}} i^*(\Omega) > 0$, which is a contradiction since $i^*(\Omega)$ is exact.

Thus we have that $\mathcal{T}$ is Lagrangian and projects diffeomorphically on $\mathbf{T}^m$; by Appendix 2 of [**10**], all orbits Q with $(Q(t), \dot{Q}(t)) \in \mathcal{T}\ \forall t$ are minimal for $\mathcal{L}$. Moreover, since $\mathcal{T}$ is invariant, any point on $\mathcal{T}$ is the initial condition of an orbit lying on $\mathcal{T}$. We conclude that, given any $Q_0 \in \mathbf{T}^m$, we can find $Q(t)$ such that

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(1) Q is minimal for $\mathcal{L}$ and $(Q, \dot{Q})$ lies on the invariant torus.
(2) $Q(0) = Q_0$.

Thus the proof will go along the following lines: we will define a perturbation h on whose minimal orbits we have much information, then we will suppose by contradiction that an invariant torus exists and show that for some Q_0, no orbits satisfying (1) and (2) exist; this contradiction will prove the theorem.

We note that we can always find a vector k_n' which is orthogonal to k_n and satisfies

$$\forall n \quad \left\langle \bar{\omega}, \frac{k_n'}{|k_n'|} \right\rangle \geq a > 0, \quad |k_n'| \leq |k_n| \tag{2}$$

for some a not depending on n. We choose our perturbation as in [**1**]

$$\tilde{\mathcal{L}}^n_{\epsilon,\mu} = \tfrac{1}{2}|\dot{Q}|^2 + h^n_{\epsilon,\mu}(Q) = \tfrac{1}{2}|\dot{Q}|^2 + \epsilon e^{-\sigma|k_n|}[1 - \cos\langle k_n, Q\rangle] + \epsilon\mu e^{-\sigma|k_n|}[1 - \cos\langle k_n, Q\rangle]e^{-\sigma|k_n'|}\cos\langle k_n', Q\rangle.$$

We note that $\|h^n_{\epsilon,\mu}\|_\sigma \leq 4\epsilon$ if $\mu \leq 1$. To make $\tilde{\mathcal{L}}^n_{\epsilon,\mu}$ more wieldy, we make a change of coordinates. We define P and $\bar{P}$ to be the orthogonal projections on k_n and on $k_n^\perp$, respectively, and we set

$$q = |k_n| P Q, \quad w = \bar{P} Q.$$

In the new coordinates we have that $\tilde{\mathcal{L}}^n_{\epsilon,\mu}(Q, \dot{Q}) = \mathcal{L}^n_{\epsilon,\mu}(w, q, \dot{w}, \dot{q})$ with

$$\mathcal{L}^n_{\epsilon,\mu}(w, q, \dot{w}, \dot{q}) = \frac{1}{2}|\dot{w}|^2 + \frac{1}{|k_n|^2}\frac{1}{2}|\dot{q}|^2 + \epsilon e^{-\sigma|k_n|}[1 - \cos(q)] + \epsilon\mu e^{-\sigma|k_n|}[1 - \cos(q)]e^{-\sigma|k_n'|}\cos\langle k_n', w\rangle.$$

Thus $\mathcal{L}^n_{\epsilon,\mu}$ consists in a rotator w, a pendulum q of Lagrangian

$$A^n_\epsilon(q, \dot{q}) = \frac{1}{|k_n|^2}\frac{1}{2}|\dot{q}|^2 + \epsilon e^{-\sigma|k_n|}[1 - \cos(q)]$$

and a coupling between the two systems, small with μ. We write below the homoclinic q_ϵ of A^n_ϵ such that $\dot{q}_\epsilon > 0$, $q_\epsilon(0) = \pi$:

$$q_\epsilon(t) = 4\,\mathrm{arctg}(e^{\sqrt{\epsilon\delta_n}t}), \quad \dot{q}_\epsilon(t) = \frac{2\sqrt{\epsilon\delta_n}}{\mathrm{Ch}(\sqrt{\epsilon\delta_n}t)}$$

where

$$\delta_n = |k_n|^2 e^{-\sigma|k_n|}$$

and arctg takes values between $-\pi/2$ and $\pi/2$. We set

$$\beta_n = e^{-\sigma|k_n|}, \quad \alpha_n = e^{-\sigma|k_n|}e^{-\sigma|k_n'|}$$

and define the Melnikoff function as

$$M_\epsilon(\omega, w_0, t_0) = \epsilon\alpha_n \int_{\mathbf{R}} [1 - \cos(q_\epsilon(t - t_0))]\cos\langle k_n', \omega(t - t_0) + w_0\rangle\, dt.$$

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In other words, M_ϵ is the integral of the perturbation along a homoclinc of $\mathcal{L}^n_{\epsilon,0}$. It does not depend on t_0 and on the projection of w_0 on $(k'_n)^\perp$ and is $2\pi/|k'_n|$-periodic in the projection of w_0 on k'_n. It can be explicitly calculated

$$M_\epsilon(\omega, w_0, t_0) = \frac{1}{|k_n|^2} e^{-\sigma|k'_n|} 2\pi \frac{\langle \omega, k'_n\rangle}{\mathrm{Sh}((\pi/2)(\langle \omega, k'_n\rangle/\sqrt{\epsilon\delta_n}))} \cos\langle w_0, k'_n\rangle.$$

If we specialize the above formula to $\bar\omega$ and recall equation (2), with a simple calculation we can get the next lemma.

LEMMA 1. *Let $\epsilon > 0$, $n \in \mathbf{N}$ and let $\langle w_0, k'_n\rangle \in 2\pi\mathbf{Z}$, $\langle w_1, k'_n\rangle \in \pi + 2\pi\mathbf{Z}$. Then*

$$M_\epsilon(\bar\omega, w_0, t_0) - M_\epsilon(\bar\omega, w_1, t_0) \geq e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}.$$

It is easy to see that the transformation: $Q \to (w, q)$ preserves the minimality of the orbits; thus, if $(Q, \dot Q)$ lies on the invariant torus of frequency $\bar\omega$, we have that (w, q), the image of Q, is minimal for $\mathcal{L}^n_{\epsilon,\mu}$. We will show that this implies that for all $\epsilon \in (0, 1)$ there is n big enough such that

$$q(t) \in \pi + 2\pi\mathbf{Z} \implies \langle w(t), k'_n\rangle \notin 2\pi\mathbf{Z} \tag{3}$$

whenever $(Q, \dot Q)$ belongs to a torus $\mathcal{T}$ enjoying the properties of Theorem 1. But this is tantamount to showing that the properties (1) and (2), listed at the beginning of the proof, do not hold. Thus the proof of Theorem 1 reduces to the proof of equation (3), which contradicts the existence of an invariant torus.

We remark that equation (3) says that minimal orbits do not pass through the maxima of the Melnikoff function; this is the same fact underlying the variational proof of [**1**] (see for instance [**3**].) Also as a side remark, the Melnikoff function is closely connected to the 'barrier' of [**11**]; the Lagrangian of [**1**] is practically the only case where this barrier can be estimated explicitly (see for instance [**4**].)

We now prove a lemma which says that, for μ small enough, M_ϵ approximates the integral of the perturbation along the minimal orbits of $\mathcal{L}^n_{\epsilon,\mu}$. From now on we will consider our orbits as valued in $\mathbf{R}^m$, the universal cover of $\mathbf{T}^m$; C_i will always denote a constant not depending on ϵ and n.

LEMMA 2. *Let $M > 16$. There are $C_0, C_1 > 0$ and $l \in \mathbf{N}$, $l \geq 2$ with the following property. Let*

$$\epsilon \in (0, 1], \quad \mu = e^{-C_0|k'_n|/\sqrt{\epsilon\delta_n}} \tag{4}$$

$$T_3 \geq T_1 + \frac{lC_1|k'_n|}{\epsilon\delta_n}. \tag{5}$$

Let (w, q) be minimal for $\mathcal{L}^n_{\epsilon,\mu}$ on $[T_1, T_3]$ and let it satisfy

$$\frac{w(T_3) - w(T_1)}{T_3 - T_1} = \omega, \quad |\omega| \in [1, 2] \tag{6}$$

$$q(T_1) = 0, \quad q(T_3) = 2\pi. \tag{7}$$

Then for any $T_2 \in (T_1, T_3)$ such that $q(T_2) = \pi$ we have that

$$\left|\epsilon\alpha_n \int_{T_1}^{T_3} [1 - \cos(q(t))] \cos\langle k'_n, w(t)\rangle\, dt - M_\epsilon(\omega, w(T_2), T_2)\right| \leq \frac{1}{M} e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}. \tag{8}$$

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Proof. The proof follows the lines of Lemma 1.1 of [**3**]; here we will only sketch the general argument and point out those details which are different. First of all we define

$$w_\epsilon(t) = w(T_1) + \omega(t - T_1).$$

As in [**3**], (8) is implied by the following four inequalities:

$$\epsilon\delta_n \int_{T_2 - C_1|k_n'|/\epsilon\delta_n}^{T_2 + C_1|k_n'|/\epsilon\delta_n} |\cos\langle k_n', w\rangle - \cos\langle k_n', w_\epsilon + w(T_2) - w_\epsilon(T_2)\rangle|\, dt \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \tag{9}$$

$$\epsilon\delta_n \int_{T_2 - C_1|k_n'|/\epsilon\delta_n}^{T_2 + C_1|k_n'|/\epsilon\delta_n} |\cos q(t) - \cos q_\epsilon(t - T_2)|\, dt \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \tag{10}$$

$$\epsilon\delta_n \int_{[T_1, T_3]\setminus[T_2 - C_1|k_n'|/\epsilon\delta_n, T_2 + C_1|k_n'|/\epsilon\delta_n]} |1 - \cos q|\, dt \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \tag{11}$$

$$\epsilon\delta_n \int_{\mathbf{R}\setminus[T_2 - C_1|k_n'|/\epsilon\delta_n, T_2 + C_1|k_n'|/\epsilon\delta_n]} |1 - \cos q_\epsilon(t - T_2)|\, dt \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}. \tag{12}$$

In the course of the proof we will also see that

$$T_1 \leq T_2 - \frac{C_1|k_n'|}{\epsilon\delta_n} \quad \text{and} \quad T_3 \geq T_2 + \frac{C_1|k_n'|}{\epsilon\delta_n}$$

so that (11) makes sense. We will always assume that $C_1, C_0 > 1$.

First of all, by direct computation it is possible to see that, taking C_1 big enough, we have

$$\epsilon\delta_n \int_{\mathbf{R}\setminus[T_2 - C_1|k_n'|/\epsilon\delta_n, T_2 + C_1|k_n'|/\epsilon\delta_n]} |1 - \cos q_\epsilon(t - T_2)|\, dt \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}.$$

This proves (12) and sets the value of C_1. Now we consider ϕ, a C^∞ cutoff

$$\phi(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t \geq 1 \end{cases}$$

and define

$$\bar{q}(t) = \begin{cases} q_\epsilon\left(t - \dfrac{T_1 + T_3}{2}\right) \phi\left(t - \left(\dfrac{T_1 + T_3}{2} - \dfrac{C_1|k_n'|}{\epsilon\delta_n}\right)\right), & t \leq \dfrac{T_1 + T_3}{2} \\ \left(q_\epsilon\left(t - \dfrac{T_1 + T_3}{2}\right) - 2\pi\right) \phi\left(-t + \left(\dfrac{T_1 + T_3}{2} + \dfrac{C_1|k_n'|}{\epsilon\delta_n}\right)\right) + 2\pi, & \dfrac{T_1 + T_3}{2} \leq t. \end{cases}$$

We note that, by (5) and (6), $(w_\epsilon, \bar{q})(T_i) = (w, q)(T_i)$, $i = 1, 3$. Since (w, q) is minimal we have that

$$\int_{T_1}^{T_3} \mathcal{L}^n_{\epsilon,\mu}(w_\epsilon, \bar{q})\, dt \geq \int_{T_1}^{T_3} \mathcal{L}^n_{\epsilon,\mu}(w, q)\, dt.$$

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As in [**3**], from this fact we deduce that, for C_0 big enough,

$$\int_{T_1}^{T_3} \left\{ \frac{1}{|k_n|^2} \frac{1}{2} |\dot{q}|^2 + \epsilon\beta_n [1 - \cos(q)] \right\} dt \leq C_2 \sqrt{\epsilon\beta_n}. \tag{13}$$

By the E–L equation we get that

$$|\ddot{w}| \leq \mu |k_n'| e^{-\sigma |k_n'|} \epsilon\beta_n [1 - \cos(q)].$$

Thus

$$\forall t_0 < t_1 \in [T_1, T_3], \quad |\dot{w}(t_1) - \dot{w}(t_0)| \leq \mu |k_n'| e^{-\sigma |k_n'|} \int_{t_0}^{t_1} \epsilon\beta_n [1 - \cos(q)]\, dt$$

which by (13) and the fact that $|k_n'| e^{-\sigma |k_n'|}$ is bounded yields

$$\forall t_0 < t_1 \in [T_1, T_3], \quad |\dot{w}(t_1) - \dot{w}(t_0)| \leq \mu C_3 \sqrt{\epsilon\beta_n}$$

implying by (6)

$$\forall t \in [T_1, T_3], \quad |\dot{w}(t) - \omega| \leq \mu C_3 \sqrt{\epsilon\beta_n}.$$

If we now integrate $\dot{w}$ from T_2 to t and use the above formula we get

$$\sup_{[T_2 - C_1 |k_n'|/\epsilon\delta_n, T_2 + C_1 |k_n'|/\epsilon\delta_n]} |w(t) - [w_\epsilon(t) + w(T_2) - w_\epsilon(T_2)]|$$
$$\leq \mu C_3 \sqrt{\epsilon\beta_n} \frac{C_1 |k_n'|}{\epsilon\delta_n} \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma |k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \frac{1}{2C_1 |k_n'|}.$$

The last inequality follows by (4) taking C_0 big enough. Inequality (9) now follows from the above formula and the Lipschitz continuity of the cosine.

We are now going to prove (10). By (5) we have that either

$$T_2 - T_1 \geq \frac{l}{2} \frac{C_1 |k_n'|}{\epsilon\delta_n} \quad \text{or} \quad T_3 - T_2 \geq \frac{l}{2} \frac{C_1 |k_n'|}{\epsilon\delta_n}.$$

To fix ideas we suppose that the first case occurs and we consider, on $[T_1, T_3]$, the solution $\bar{q}$ of

$$\begin{cases} \ddot{\bar{q}} = \epsilon\delta_n \sin(\bar{q}) \\ \bar{q}(T_1) = q(T_1) = 0 \\ \bar{q}(T_2) = q(T_2) = \pi. \end{cases}$$

Since $T_2 - T_1 \geq (l/2)(C_1 |k_n'|/\epsilon\delta_n)$, it is easy to see that h, the energy of $\bar{q}$, satisfies

$$0 < h \leq \epsilon\beta_n e^{-C_4 C_1 l |k_n'|/\sqrt{\epsilon\delta_n}}. \tag{14}$$

We set

$$h(t) = \frac{1}{|k_n|^2} \frac{1}{2} |\dot{q}|^2 - \epsilon\beta_n [1 - \cos(q)].$$

As in [**3**], it is possible to show that there is $t_0 \in [T_1, T_2]$ such that $h(t_0) = h$; moreover, using the E–L equation we can estimate $\dot{h}(t)$; integrating from t_0 to t we get

$$\sup_{t \in [T_1, T_3]} |h(t) - h| \leq C_5 \mu \sqrt{\epsilon\beta_n}. \tag{15}$$

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We set

$$\gamma = C_5 \mu \sqrt{\epsilon \beta_n}.$$

As in [**3**], the minimality of (w, q) implies that $\dot{q}(T_2) > 0$; clearly, $\dot{\bar{q}}(T_2) > 0$. If $(\bar{T}, T_2]$ is the maximal interval to the left of T_2 where $\dot{q} > 0$ (i.e. where q is invertible) denoting by $t(q)$ the inverse function we have that, by (15), for $q < \pi$,

$$T_2 + \frac{1}{|k_n|} \int_\pi^q \frac{ds}{\sqrt{2(h - \gamma + \epsilon \beta_n [1 - \cos(s)])}} \leq t(q) \leq T_2 + \frac{1}{|k_n|} \int_\pi^q \frac{ds}{\sqrt{2(h + \gamma + \epsilon \beta_n [1 - \cos(s)])}}.$$

On the other side, $\bar{t}(q)$, the inverse function of $\bar{q}(t)$, satisfies

$$\bar{t}(q) = T_2 + \frac{1}{|k_n|} \int_\pi^q \frac{ds}{\sqrt{2(h + \epsilon \beta_n [1 - \cos(s)])}}.$$

From the last two formulae we get that

$$|t(q) - \bar{t}(q)| \leq C_6 \frac{1}{|k_n|} \frac{\gamma}{\{\epsilon \beta_n [1 - \cos(q)]\}^{3/2}}$$

if

$$\gamma \leq \tfrac{1}{4} \epsilon \beta_n [1 - \cos(q)]. \tag{16}$$

By (14) and (15) it is easy to see that, for l and C_0 big enough,

$$\|\dot{q}\| + \|\dot{\bar{q}}\| \leq C_7 |k_n| \sqrt{\epsilon \beta_n} = C_7 \sqrt{\epsilon \delta_n}.$$

The above formula and (16) imply that

$$|q(t) - \bar{q}(t)| \leq C_6 C_7 \frac{\gamma \sqrt{\epsilon \delta_n}}{(\epsilon \beta_n [1 - \cos(\bar{q}(t))])^{3/2}} \tag{17}$$

as long as

$$\gamma \leq \tfrac{1}{4} \epsilon \beta_n [1 - \cos(\bar{q}(t))]. \tag{18}$$

If $l > 1$, we have the estimate

$$\bar{q}\left(T_2 - \frac{C_1 |k_n'|}{\epsilon \delta_n}\right) \geq e^{-C_8 |k_n'| / \sqrt{\epsilon \delta_n}}.$$

If we choose C_0 big enough we have by the above formula that the maximal interval on which (18) holds contains $[T_2 - C_1 |k_n'| / \epsilon \delta_n, T_2]$. Moreover, it is easily seen by (15) and (14) that on this interval $\dot{q} > 0$, i.e. $[\bar{T}, T_2] \subset [T_2 - C_1 |k_n'| / \epsilon \delta_n, T_2]$. We now re-apply the same estimate to the maximal interval on the right of T_2 on which $\dot{q}(t) > 0$; we see that (17) continues to hold and thus

$$|q(t) - \bar{q}(t)| \leq C_6 C_7 \frac{\gamma |k_n|^3}{\epsilon \delta_n [1 - \cos(\bar{q}(t))]^{3/2}} \leq C_6 C_7 \frac{4 \gamma |k_n|^3}{\epsilon \delta_n e^{-3 C_8 |k_n'| / \sqrt{\epsilon \delta_n}}}$$
$$\forall t \in \left[T_2 - \frac{C_1 |k_n'|}{\epsilon \delta_n}, T_2 + \frac{C_1 |k_n'|}{\epsilon \delta_n}\right].$$

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For l big enough, $\bar{q}$ and $q_\epsilon(\cdot - T_2)$ are close; indeed, an estimate similar to the above one yields

$$|q_\epsilon(t - T_2) - \bar{q}(t)| \leq C_9 \frac{h|k_n|^3}{\epsilon\delta_n e^{-3C_8|k_n'|/\sqrt{\epsilon\delta_n}}}, \quad \forall t \in \left[T_2 - \frac{C_1|k_n'|}{\epsilon\delta_n}, T_2 + \frac{C_1|k_n'|}{\epsilon\delta_n}\right].$$

If we take l and C_0 big enough, from the above two formulae and (14) we get

$$|q_\epsilon(t - T_2) - q(t)| \leq \frac{1}{8M} \frac{|k_n|^2}{e^{-\sigma|k_n'|}} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \frac{1}{2C_1|k_n'|}.$$

Formula (10) now follows by the Lipschitz continuity of the cosine.

The proof of (11) is exactly the same as that of the corresponding inequality in [**3**]. □

We remark that the above lemma continues to hold if we substitute in (4) any constant C bigger that C_0, since this means to reduce further the size of the perturbation. Moreover, with the same proof it is possible to get that

$$\begin{aligned} \left|\epsilon\alpha_n \int_{T_1}^{T_2} [1 - \cos(q(t))] \cos\langle k_n', w(t)\rangle \, dt - M_\epsilon^-(\omega, w(T_2), T_2)\right| &\leq \frac{1}{M} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \\ \left|\epsilon\alpha_n \int_{T_2}^{T_3} [1 - \cos(q(t))] \cos\langle k_n', w(t)\rangle \, dt - M_\epsilon^+(\omega, w(T_2), T_2)\right| &\leq \frac{1}{M} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \end{aligned} \tag{19}$$

where

$$\begin{aligned} M_\epsilon^-(\omega, w_0, t_0) &= \epsilon\alpha_n \int_{-\infty}^{t_0} [1 - \cos(q_\epsilon(t - t_0))] \cos\langle k_n', \omega(t - t_0) + w_0\rangle \, dt \\ M_\epsilon^+(\omega, w_0, t_0) &= \epsilon\alpha_n \int_{t_0}^{\infty} [1 - \cos(q_\epsilon(t - t_0))] \cos\langle k_n', \omega(t - t_0) + w_0\rangle \, dt. \end{aligned}$$

We now note that, if we consider the motion of $q(t)$ as known, the motion of w is given by the non-autonomous Lagrangian $\frac{1}{2}|\dot{w}|^2 + \epsilon\mu\alpha_n[1 - \cos(q(t))]\cos\langle k_n', w\rangle$ whose first derivative in w is bounded by $2\epsilon\mu\alpha_n|k_n'|$. Moreover, this Lagrangian is periodic in w; this is all we need to apply Lemma 2 of [**2**], from which we get the following estimate.

LEMMA 3. *Let (w, q) be an action minimizing orbit. Then we have that, $\forall T_1 < T_2$,*

$$\forall t \in [T_1, T_2], \quad \left|\dot{w}(t) - \frac{w(T_2) - w(T_1)}{T_2 - T_1}\right| \leq C_{10}\sqrt{\mu}\sqrt{\epsilon\delta_n}. \tag{20}$$

The next lemma contains an estimate on the action functional of the unperturbed pendulum which we will need in the following. We will use it twice: in Lemma 5 we will use point (1) to show that a half-turn around the homoclinic (or whisker, as in [**1**]) takes a very long time to be done; in the end of the proof of Theorem 1 point (2) will tell us that a small translation in time of a homoclinic does not increase its action functional too much. Throughout the rest of the paper, μ and C_0 are defined as in (4).

LEMMA 4. *Let $M > 16$ be big enough. Then for all C_0 big enough there is $\Delta = \Delta(C_0) > 0$ such that the following holds.*

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Let $T_2, \bar{T}_2 \in [T_1, T_3]$. Let q be a solution of A^n_ϵ on (T_1, T_2) and on (T_2, T_3) with boundary conditions

$$q(T_1) = 0, \quad q(T_2) = \pi, \quad q(T_3) = 2\pi$$

and let $\bar{q}$ be a solution of A^n_ϵ on $(T_1, \bar{T}_2)$ and on $(\bar{T}_2, T_3)$ with boundary conditions

$$\bar{q}(T_1) = 0, \quad \bar{q}(\bar{T}_2) = \pi, \quad \bar{q}(T_3) = 2\pi.$$

(1) *If*

$$\begin{gathered} T_2 - T_1 \geq 4\Delta \\ T_3 - T_2 \leq \Delta \\ T_3 - T_1 \geq 5\Delta \\ \left| \bar{T}_2 - \frac{T_1 + T_3}{2} \right| \leq \frac{8\pi}{a} \end{gathered}$$

then we have

$$\int_{T_1}^{T_3} A^n_\epsilon(q, \dot{q})\, dt - \int_{\bar{T}_1}^{\bar{T}_3} A^n_\epsilon(\bar{q}, \dot{\bar{q}})\, dt > \frac{8\mu}{M} e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}.$$

(2) *If*

$$\begin{gathered} T_3 - T_1 \geq 2\Delta \\ \bar{T}_2 = \frac{T_1 + T_3}{2} \\ |T_2 - \bar{T}_2| \leq \frac{8\pi}{a} \end{gathered}$$

then we have

$$\int_{T_1}^{T_3} A^n_\epsilon(q, \dot{q})\, dt - \int_{\bar{T}_1}^{\bar{T}_3} A^n_\epsilon(\bar{q}, \dot{\bar{q}})\, dt \leq \frac{\mu}{8} e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}.$$

Moreover, we can take

$$\frac{C_0}{C_{11}} \frac{|k'_n|}{\epsilon\delta_n} \leq \Delta \leq C_{11}(C_0 + M) \frac{|k'_n|}{\epsilon\delta_n}. \tag{21}$$

Proof. Let us consider the situation of point (1). We define the function

$$\begin{aligned} L(T_2) &= \sum_{i=1}^{2} \int_{T_i}^{T_{i+1}} A^n_\epsilon(q, \dot{q})\, dt \\ &= \sum_{i=1}^{2} \left[\frac{1}{|k_n|} \int_{(i-1)\pi}^{i\pi} \sqrt{2(h(T_{i+1} - T_i) - V(s))}\, ds - h(T_{i+1} - T_i)(T_{i+1} - T_i) \right] \end{aligned}$$

where

$$V(s) = -\epsilon\beta_n[1 - \cos(s)]$$

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and $h(T)$ denotes the energy of the orbit of the pendulum doing one half turn in time T. A simple calculation yields

$$\frac{dL(T_2)}{dT_2} = (h(T_3 - T_2) - h(T_2 - T_1)). \tag{22}$$

Since $h(T)$ is decreasing in T we have that

if $T_2 \in [T_3 - \Delta - 1, T_3 - \Delta]$ then $h(T_3 - T_2) - h(T_2 - T_1) \geq h(\Delta + 1) - h(2\Delta)$.

If we now integrate (22) between $\bar{T}_2$ and T_2, we get that (1) holds if

$$h(\Delta + 1) - h(2\Delta) \geq \frac{8\mu}{M} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}. \tag{23}$$

Let us now consider the situation of (2). Integrating as before we get that (2) holds if

$$\frac{8\pi}{a} h\left(\Delta - \frac{8\pi}{a}\right) \leq \frac{\mu}{8} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}. \tag{24}$$

It is now easy to see (for instance estimating $(d/dT)h(T)$) that, for M big but independent of Δ,

$$h(\Delta + 1) - h(2\Delta) \geq \frac{64}{M} \frac{\pi}{a} h\left(\Delta - \frac{8\pi}{a}\right).$$

A simple calculation now shows

$$\epsilon\beta_n e^{-\bar{C}_{12} T\sqrt{\epsilon\delta_n}} \leq h(T) \leq \epsilon\beta_n e^{-C_{12} T\sqrt{\epsilon\delta_n}}.$$

We now choose Δ such that equality holds in (24); by the last two formulae, also (23) holds, yielding points (1) and (2). Inequality (21) is a consequence of our choice of Δ and of the last formula. □

In the following, we shall fix once and for all $M > 16$ such that Lemma 4 holds. As for C_0, we will choose it so big that both Lemmas 2 and 4 hold. Moreover, by (21) we can suppose C_0 so big that

$$\Delta \geq \frac{lC_1|k_n'|}{\epsilon\delta_n}. \tag{25}$$

We also require that

$$C_0 \geq 2lC_{12}C_1, \tag{26}$$

where C_{12} is the same that appears in the proof of the last lemma; this will come in handy in Lemma 5. By the explicit formula for M_ϵ we see that, for C_0 big enough,

$$|M_\epsilon(\omega, w_0, t_0) - M_\epsilon(\bar{\omega}, w_0, t_0)| \leq \frac{1}{M} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \quad \text{if } |\omega - \bar{\omega}| \leq C_{10}\sqrt{\mu}\sqrt{\epsilon\delta_n} \tag{27}$$

and the same is true for M_ϵ^+ and M_ϵ^-. From now on, C_0 is fixed and satisfies the above conditions.

The next lemma states that the turns around the homoclinics (or 'whiskers', as in [**1**]) are each spaced far enough from the other. It is rather similar to Lemma 2 of [**2**] but the fact that we are close to a homoclinic of the pendulum requires some special care. The constants D and σ appearing below are the same as in the statement of Theorem 1. We recall that at the beginning of the proof we have made a change of coordinates on $\mathbf{T}^m$ from Q to (w, q).

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LEMMA 5. *Let $D > \sigma$. Then for all $\epsilon \in (0,1)$ there is n big enough such that the following holds. Let (w,q) be minimal for $\mathcal{L}^n_{\epsilon,\mu}$, let Q be its pre-image and let $(Q,\dot{Q})$ lie on an invariant torus of frequency $\bar{\omega}$. Let us suppose that $t_0 < t_1$ and that $q(t_0) = 0$, $q(t_1) = \pi$ (or that $q(t_0) = \pi$, $q(t_1) = 2\pi$.) Then*

$$t_1 - t_0 \geq \Delta. \tag{28}$$

Proof. In the following, given a vector $v \in \mathbf{R}^m$, we will denote by v_1 its component orthogonal to k_n and k_n', and by v_2 its component along k_n'.

Let us consider t_0 and t_1 as in the hypotheses. We will prove the case $q(t_0) = 0$, $q(t_1) = \pi$, the other one being analogous. Since the motion on the invariant torus is conjugate to a rotation of frequency $\bar{\omega}$, we can find $\bar{T} > t_0$ such that, setting $Q = (\bar{P}Q, PQ)$,

$$\left| \left(\frac{w(\bar{T}) - w(t_0)}{\bar{T} - t_0}, \frac{q(\bar{T}) - q(t_0)}{|k_n|(\bar{T} - t_0)} \right) - \bar{\omega} \right| \leq C_{10}\sqrt{\mu}\sqrt{\epsilon\delta_n} \tag{29}$$

$$q(\bar{T}) = N\pi. \tag{30}$$

Moreover, since by $(***)$ the motion in the direction q has mean speed less than $|k_n|e^{-D|k_n|}$, we can suppose that

$$\bar{T} - t_0 \geq \frac{1}{2} N \frac{1}{|k_n|} e^{D|k_n|}.$$

By (21) and the fact that $D > \sigma$ we have that, for all n big enough,

$$\bar{T} - t_0 \geq 16N\Delta. \tag{31}$$

By (20) and (29) we get that, for C_0 big enough,

$$|\dot{w}_2(t) - \bar{\omega}_2| \leq 2C_{10}\sqrt{\mu}\sqrt{\epsilon\delta_n}, \quad \forall t \in [T_0, \bar{T}]. \tag{32}$$

We now choose a sequence of times $t_0 < t_1 < \cdots < t_N = \bar{T}$ such that $q(t_r) = \pi r$ for $r \in (0, 1, \ldots, N)$.

Let us suppose by contradiction that (28) does not hold. We distinguish two cases: in the first one

$$\Delta > t_1 - t_0 \geq lC_1 \frac{|k_n'|}{\epsilon\delta_n}. \tag{33}$$

We will compare (w,q) with $(\bar{w},\bar{q})$, an orbit which comes out of a surgery in the style of [**2**]. By (31) there is $j \in (1, \ldots, N-1)$ such that

$$t_{j+1} - t_j \geq 16\Delta. \tag{34}$$

In the new orbit $(\bar{w},\bar{q})$, $\bar{q}$ will take a longer time to go from 0 to π and a correspondingly shorter time to go from $j\pi$ to $(j+1)\pi$. Thus by Lemma 4 the action of A^n_ϵ will decrease; it will decrease enough to compensate the possible increase in the perturbation (actually, everything is chosen in such a way that the Melnikoff function remains the same). On the other side, $(\bar{w},\bar{q})$ will be the same as (w,q) (up to a translation) outside the intervals $[t_0, t_1]$ and $[t_j, t_{j+1}]$.

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We begin defining

$$T = \frac{t_1 - t_0 + t_{j+1} - t_j}{2}.$$

Let s_3 be the closest time to $t_{j+1} - T$ such that $w_2(s_3) - w_2(t_j) \in (2\pi/|k_n'|)\mathbf{Z}$; we set $s_2 = s_3 - (t_j - t_1)$. By (32) and (2), $|t_{j+1} - T - s_3| \leq 2\pi/a$. We define $\bar{w} \in$ A.C. in the following way

$$\bar{w}_1(t) = w_1(t)$$

and

$$\bar{w}_2(t) = \begin{cases} w_2(t_0) + \dfrac{w_2(t_1) - w_2(t_0) + m}{s_2 - t_0}(t - t_0) & t_0 \leq t \leq s_2 \\ w_2(t - s_2 + t_1) + m & s_2 \leq t \leq s_3 \\ \bar{w}_2(s_3) + \dfrac{w_2(t_{j+1}) - w_2(t_j) - m}{t_{j+1} - s_3}(t - s_3) & s_3 \leq t \leq t_{j+1} \\ w_2(t) & t_{j+1} \leq t \leq \bar{T} = t_N \end{cases}$$

where m is given by

$$m = w_2(s_3) - w_2(t_j) \in \frac{2\pi}{|k_n'|}\mathbf{Z}. \tag{35}$$

We will see that this choice of m keeps low the kinetic energy of $\bar{w}$ and minimizes the change in the Melnikoff function. We note that

$$\bar{w}_2(s_2) - w_2(t_1) = m \in \frac{2\pi}{|k_n'|}\mathbf{Z}, \quad \bar{w}_2(s_3) - w_2(t_j) = m \in \frac{2\pi}{|k_n'|}\mathbf{Z}. \tag{36}$$

We set

$$\bar{q}(t) = \begin{cases} q_1(t) & t_0 \leq t \leq s_2 \\ q(t - s_2 + t_1) & s_2 \leq t \leq s_3 \\ q_2(t) & s_3 \leq t \leq t_{j+1} \\ q(t) & t_{j+1} \leq t \leq \bar{T} = t_N \end{cases}$$

where q_1 is a solution of A_ϵ^n on (t_0, s_1) with boundary conditions

$$q_1(t_0) = q(t_0) = 0, \quad q_1(s_2) = q(t_0) = \pi$$

and q_2 is a solution of A_ϵ^n on (s_3, t_j) with boundary conditions

$$q_1(s_3) = q(t_j) = j\pi, \quad q_1(t_{j+1}) = q(t_{j+1}) = (j+1)\pi.$$

In order to deal with shorter formulae, we set

$$\text{pert}(w, q) = \epsilon e^{-\sigma|k_n|} e^{-\sigma|k_n'|}[1 - \cos(q)]\cos\langle k_n', w\rangle.$$

The minimality of (w, q) implies

$$\int_{t_0}^{\bar{T}} \mathcal{L}_{\epsilon,\mu}^n(\bar{w}, \bar{q})\, dt - \int_{t_0}^{\bar{T}} \mathcal{L}_{\epsilon,\mu}^n(w, q)\, dt \geq 0.$$

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After eliminating all the terms which are the same in the two integrals, we get that

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} \tfrac{1}{2}|\dot{\bar{w}}_2|^2\,dt + \int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A^n_\epsilon(\bar{q},\dot{\bar{q}})\,dt + \mu\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} \mathrm{pert}(\bar{w}_2,\bar{q})\,dt$$
$$- \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} \tfrac{1}{2}|\dot{w}_2|^2\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A^n_\epsilon(q,\dot{q})\,dt$$
$$- \mu\int_{[t_0,t_1]\cup[t_j,t_{j+1}]} \mathrm{pert}(w_2,q)\,dt \geq 0.$$

We assert that

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} \tfrac{1}{2}|\dot{\bar{w}}_2|^2\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} \tfrac{1}{2}|\dot{w}_2|^2\,dt \leq 0.$$

By (34) we have that $s_3 > t_j$ and, consequently, $s_2 > t_1$. By (35) and (36), on $[s_3, t_{j+1}]$ w_2 and $\bar{w}_2$ run the same distance (indeed, $\bar{w}_2(s_3) = w_2(s_3)$ and $\bar{w}_2(t_{j+1}) = w_2(t_{j+1})$) and thus on this interval the action functional of $\bar{w}_2$ is smaller than that of w_2. On the other side, since $\bar{w}_2(s_2) - \bar{w}_2(t_0) = w_2(t_1) - w_2(t_0) + w_2(s_3) - w_2(t_j)$, we have that the distance run by $\bar{w}_2$ on $[t_0, s_2]$ is less than the distance run by w_2 on $[t_0, t_1] \cup [t_j, s_3]$. Moreover, the measure of $[t_0, s_2]$ is the same as the measure of $[t_0, t_1] \cup [t_j, s_3]$. Using again the minimality of $\bar{w}$ for the action functional of kinetic energy, we get the above formula.

From the above two formulae we get

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A^n_\epsilon(\bar{q},\dot{\bar{q}})\,dt + \mu\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} \mathrm{pert}(\bar{w}_2,\bar{q})\,dt$$
$$- \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A^n_\epsilon(q,\dot{q})\,dt - \mu\int_{[t_0,t_1]\cup[t_j,t_{j+1}]} \mathrm{pert}(w_2,q)\,dt \geq 0. \tag{37}$$

To fix ideas, let us suppose that $q(t_j)$ is an odd multiple of π and $q(t_{j+1})$ is an even multiple. By (33) we can apply Lemma 2 and (19); by (32) we can apply (27); we thus get

$$\int_{[t_0,t_1]\cup[t_j,t_{j+1}]} \mathrm{pert}(w_2,q)\,dt \geq M^-_\epsilon(\bar{\omega}, w_2(t_1), \bar{t}_1) + M^+_\epsilon(\bar{\omega}, w_2(t_j), t_j) - \frac{4}{M}e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}.$$

Analogously, by applying Lemma 2 with $\mu = 0$, we get

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} \mathrm{pert}(\bar{w}_2,\bar{q})\,dt \leq M^-_\epsilon(\bar{\omega}, \bar{w}_2(s_2), s_2)$$
$$+ M^+_\epsilon(\bar{\omega}, \bar{w}_2(s_3), s_3) + \frac{4}{M}e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}.$$

From (36) and the periodicity of M^+_ϵ and M^-_ϵ we get that

$$M^-_\epsilon(\bar{\omega}, w_2(t_1), \bar{t}_1) + M^+_\epsilon(\bar{\omega}, w_2(t_j), t_j) = M^-_\epsilon(\bar{\omega}, \bar{w}_2(s_2), s_2) + M^+_\epsilon(\bar{\omega}, \bar{w}_2(s_3), s_3).$$

From the last four formulae we get

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A^n_\epsilon(\bar{q},\dot{\bar{q}})\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A^n_\epsilon(q,\dot{q})\,dt \geq -\frac{8}{M}\mu e^{-b|k'_n|/\sqrt{\epsilon\delta_n}}. \tag{38}$$

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Let now $\tilde{q}$ be a solution of A_ϵ^n on (t_0, t_1), (t_j, t_{j+1}) with the same boundary conditions as q in t_0, t_1, t_j, t_{j+1}. Clearly

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A_\epsilon^n(q,\dot{q})\,dt \le \int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A_\epsilon^n(\tilde{q},\dot{\tilde{q}})\,dt.$$

We now note that we are in the hypotheses of point (1) of Lemma 4: indeed, $t_1 - t_0 \le \Delta$, $t_{j+1} - t_j \ge 16\Delta$ while $|s_2 - t_0 - T| \le 2\pi/a$ and $|t_{j+1} - s_3 - T| \le 2\pi/a$. Moreover, the length of $[t_0, s_2] \cup [s_3, t_{j+1}]$ is the same as the length of $[t_0, t_1] \cup [t_j, t_{j+1}]$. Thus we can apply point (1) of Lemma 4 and get

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A_\epsilon^n(q,\dot{q})\,dt < -\frac{8\mu}{M} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}. \tag{39}$$

If we now put together (39) and (38), we get

$$-8\mu\frac{1}{M}e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} > -8\mu\frac{1}{M}e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}$$

a contradiction.

Now, only the other case remains,

$$t_1 - t_0 < l\frac{C_1|k_n'|}{\epsilon\delta_n}. \tag{40}$$

We build $(\bar{w}, \bar{q})$ with exactly the same surgery as before; the previous deductions continue to hold up to formula (37). From (37) and the definition of pert we get that

$$(1+\mu)\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt - (1-\mu)\int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A_\epsilon^n(q,\dot{q})\,dt \ge 0.$$

With an argument like that of Lemma 4 we get that

$$\int_{[t_0,s_2]\cup[s_3,t_{j+1}]} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt - \int_{[t_0,t_1]\cup[t_j,t_{j+1}]} A_\epsilon^n(q,\dot{q})\,dt \le -e^{-2C_{12}lC_1|k_n'|/\sqrt{\epsilon\delta_n}}.$$

Moreover, since $[t_0, s_2]$ is very big, we have that

$$\int_{t_0}^{s_2} A_\epsilon^n(\bar{q},\dot{\bar{q}})\,dt \le C_2\sqrt{\epsilon\delta_n}.$$

By (26), for n big enough the above three formulae are in contradiction. □

End of the proof of Theorem 1. As we remarked at the beginning of the proof, it suffices to prove (3). Given $\epsilon \in (0, 1)$ we fix n such that Lemma 5 holds. Let us suppose by contradiction that equation (3) does not hold, i.e. that there is $\bar{t}_0$ such that

$$q(\bar{t}_0) = \pi + 2l\pi, \quad w_2(\bar{t}_0) = \frac{2k\pi}{|k_n'|}, \quad l, k \in \mathbf{Z}$$

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where w_2 denotes, as in Lemma 5, the projection of w on k_n'. Simply by changing the lift of (w, q), we can suppose $l, k = 0$ so that

$$q(\bar{t}_0) = \pi, \quad w_2(\bar{t}_0) = 0. \tag{41}$$

We are going to show that (41) contradicts the minimality of (w, q). Indeed, let us consider $t_0 < \bar{t}_0$ and $t_1 > \bar{t}_0$ such that

$$q(t_0) = 0, \quad q(t_1) = 2\pi.$$

By Lemma 5 we have that

$$t_1 - t_0 \geq 2\Delta. \tag{42}$$

We now choose $\bar{t}$ as the last time to the left of $\bar{t}_0$ or as the first time to the right of $\bar{t}_0$ such that

$$w_2(\bar{t}) = \frac{\pi}{|k_n'|}.$$

By (32) and (2) we have that

$$|\bar{t} - t_0| \leq \frac{2\pi}{a}. \tag{43}$$

Thus there are two ways to choose $\bar{t}$: to the left or to the right of $\bar{t}_0$. We will explain in the following which is the suitable one. On $[t_0, t_1]$ we build an orbit in the following way: $\bar{q}$ is a solution of A_ϵ^n on $(t_0, \bar{t})$ and on $(\bar{t}, t_1)$ with boundary conditions

$$\bar{q}(t_0) = q(t_0) = 0, \quad \bar{q}(\bar{t}) = q(\bar{t}_0) = \pi, \quad \bar{q}(t_1) = q(t_1) = 2\pi.$$

By the minimality of (w, q) we have that

$$\int_{t_0}^{t_1} \mathcal{L}_{\epsilon,\mu}^n(w, \bar{q})\, dt - \int_{t_0}^{t_1} \mathcal{L}_{\epsilon,\mu}^n(w, q)\, dt \geq 0.$$

Eliminating all the terms which are the same in the two integrals we get

$$\int_{t_0}^{t_1} A_\epsilon^n(\bar{q}, \dot{\bar{q}})\, dt - \int_{t_0}^{t_1} A_\epsilon^n(q, \dot{q})\, dt + \mu \int_{t_0}^{t_1} \mathrm{pert}(w, \bar{q})\, dt - \mu \int_{t_0}^{t_1} \mathrm{pert}(w, q)\, dt \geq 0. \tag{44}$$

By Lemma 2 and the fact that $M > 16$ we get that

$$\mu \int_{t_0}^{t_1} \mathrm{pert}(w, \bar{q})\, dt - \mu \int_{t_0}^{t_1} \mathrm{pert}(w, q)\, dt \leq \mu \left[M_\epsilon\left(\omega, \frac{\pi}{|k_n'|}, \bar{t}\right) - M_\epsilon(\omega, 0, \bar{t}_0) + \frac{2}{16} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \right]$$

where $\omega = w(t_1) - w(t_0)/(t_1 - t_0)$. By Lemma 3 we have that $|\omega - \bar{\omega}| \leq C_{10}\sqrt{\mu}\sqrt{\epsilon\delta_n}$; if we recall (27) and Lemma 1 we get

$$\mu \int_{t_0}^{t_1} \mathrm{pert}(w, \bar{q})\, dt - \mu \int_{t_0}^{t_1} \mathrm{pert}(w, q)\, dt \leq -\tfrac{3}{4}\mu e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}. \tag{45}$$

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On the other hand, if $\tilde{q}$ is a solution of A_ϵ^n on $(t_0, \bar{t}_0)$ and on $(\bar{t}_0, t_1)$ with the same boundary conditions as q at $t_0, \bar{t}_0, t_1$ we have that

$$\int_{t_0}^{t_1} A_\epsilon^n(\bar{q}, \dot{\bar{q}})\, dt - \int_{t_0}^{t_1} A_\epsilon^n(q, \dot{q})\, dt \leq \int_{t_0}^{t_1} A_\epsilon^n(\bar{q}, \dot{\bar{q}})\, dt - \int_{t_0}^{t_1} A_\epsilon^n(\tilde{q}, \dot{\tilde{q}})\, dt.$$

Here it is important which $\bar{t}$ we choose. If $|\bar{t}_0 - (t_1 + t_0)/2| \leq 2\pi/a$, then we choose the $\bar{t}$ such that $|\bar{t} - (t_1 + t_0)/2| \leq 2\pi/a$; by (42) and (43) part (2) of Lemma 4 applies, yielding

$$\int_{t_0}^{t_1} A_\epsilon^n(\bar{q}, \dot{\bar{q}})\, dt - \int_{t_0}^{t_1} A_\epsilon^n(q, \dot{q})\, dt \leq \frac{\mu}{8} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}}.$$

If $|\bar{t}_0 - (t_1 + t_0)/2| > 2\pi/a$, we will choose the $\bar{t}$ closer to $(t_1 + t_0)/2$, thus making the motion of the pendulum more uniform; clearly, this implies that the above inequality holds with 0 on the right-hand side. From the last formula, (45) and (44) we get that

$$-\frac{5\mu}{8} e^{-b|k_n'|/\sqrt{\epsilon\delta_n}} \geq 0$$

a contradiction. □

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