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A characterization of the Poisson process using forward recurrence times

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1. *Introduction.* Consider a renewal process on the nonnegative real line with non-arithmetic distribution function $F(x)$. Denote by $V(x;t)$ the distribution function of the forward recurrence time from t, $t \geqslant 0$. If t is chosen at random with distribution function $\Phi(t)$, the corresponding unconditional forward recurrence time has distribution function

$$V(x) = \int_0^\infty d\Phi(t)\, V(x;t) = E\{V(x;T)\}.$$

This paper proves that if, for *some* Φ not concentrated at zero, $V(x) = F(x)$, $x \geqslant 0$, then the renewal process is a Poisson process, while the converse is true for *any* Φ. There is an analogous result for arithmetic F. A more formal presentation is given in section 2.

A number of characterization results for the Poisson process based on forward recurrence times already exist in the literature. These are, essentially, of two kinds; results assuming that $V(x;t)$ is independent of t and results assuming that

$$V(x;t) = F(x).$$

An important example in the first category, due to Smith (7), is that $V(x;t)$ is independent of t for all $t \geqslant \tau \geqslant 0$ if and only if the renewal process is quasi-Poisson, that is, the renewal function is linear on $[\tau, \infty)$. (Contrast this with our result in the case Φ concentrated at a single point. The apparently weaker assumption is possible because we specify precisely the form of t-independent function $V(x;t)$ is to be. Loosely speaking, we are making a Markov-type assumption whereas Smith effectively assumes only stationarity. This is the distinction between the two categories of results mentioned above.) More recently, Çinlar and Jagers (3) showed that the mean forward recurrence time being independent of t for all t is sufficient to characterize the Poisson process.

In the second category, Chung (2) demonstrated that if $V(x;t) = F(x)$ for either a sequence of t-values tending to infinity or else all t in $[0, t_0]$ for some $t_0 > 0$ then the renewal process is Poisson. The result proved here is a stronger one since we may take Φ to be concentrated at a single point. Chung's technique in the former case was to show that F must equal the equilibrium forward recurrence time distribution function, which implies a differential equation for $1 - F$ the only solution of which is exponential. In this guise the result is well known; the only stationary Markov renewal (indeed,

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point) process is the Poisson process. Another piece of folklore in this area is that to make a point process on the nonnegative half-line stationary we merely relocate the origin at a point uniformly distributed on $[-u, 0]$ and let $u \to \infty$; a rigorous proof is given in Slivnyak (6). Thus the recent result of Kotz and Johnson (5), in which they choose the t from which to measure the forward recurrence time according to a uniform 'prior' on $[0, \infty)$ and then show that if the forward recurrence time distribution equals F the process is Poisson, is not surprising. Our result is that this conclusion is valid for any proper prior distribution Φ, in all except some very special cases which are mentioned separately. The Kotz–Johnson result can then be derived by taking Φ uniform on $[0, u]$ and letting $u \to \infty$. This answers a question posed to one of us by Professor Kotz.

2. *Notation and statement of result.* We define a renewal process as usual. Let F be a (possibly defective) right-continuous distribution function on $[0, \infty)$ with $F(0) = 0$ and $0 < F(\infty) \leqslant 1$, and with mean μ which may be infinite. Let the class of all such F be $\mathscr{F}$. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with distribution F and define $\xi_0 = 0$, $\xi_n = X_1 + \ldots + X_n$ $(n = 1, 2, \ldots)$ and call the number of renewals in $(0, x]$ $N(x)$, that is $N(x) = \sup\{n: \xi_n \leqslant x\}$ for all $x > 0$. Further, $H(x) = E\{N(x)\}$ is the renewal function which, of course, uniquely determines the process. The forward recurrence time from t is $V_t = \xi_{N(t)+1} - t$, that is, the residual lifetime at t, its distribution function being $V(x; t)$. It is clear that H is right-continuous and $H(0) = 0$. Also, since $F(\infty) > 0$, the possibility that $H(x) \equiv 0$ for all x is excluded; this is necessary because, if $X_1 = \infty$ a.s. then $V(x) = F(x)$ trivially.

We choose the starting point t for the forward recurrence time randomly with arbitrary distribution function $\Phi(t)$ not concentrated at zero. For if $\Phi(0+) = 1$ then $V(x) = F(x)$ for any renewal process.

For completeness we include the following definition (see for instance Feller (4), p. 138):

A distribution F on $[0, \infty)$ is *arithmetic* if it is concentrated on the point set $B = \{0, \lambda, 2\lambda, \ldots\}$. The largest λ with this property is called the span of F.

The Poisson process is, of course, the renewal process for which F is the exponential distribution. Its arithmetic equivalent, when F is the distribution of a multiple of a geometric variable, we call a binomial process. In each case H is a linear function. The object of this note is to prove the following

PROPOSITION

1. *A renewal process with nonarithmetic distribution $F \in \mathscr{F}$ is a Poisson process with $\mu < \infty$ if $V(x) = F(x)$, $x \geqslant 0$, for some Φ not concentrated at zero. Conversely, if the process is Poisson with $\mu < \infty$ then $V(x) = F(x)$, $x \geqslant 0$, for any Φ.*

2. *A renewal process with arithmetic distribution $F \in \mathscr{F}$ is a binomial process with $\mu < \infty$ if $V(x) = F(x)$, $x \geqslant 0$, for some Φ not concentrated at zero with support in B. Conversely, if the process is binomial with $\mu < \infty$ then $V(x) = F(x)$, $x \geqslant 0$, for any Φ with support in B.*

3. *Proof.* Let T be a random variable with distribution Φ, independent of $\{X_i\}$; if F is arithmetic we assume T takes values only in B. Then the hypothesis $V(x) = F(x)$

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implies that the forward recurrence time from T has the same distribution as a typical interval, hence that $N(T+x)-N(T)$ and $N(x)$ are identically distributed. Taking expectations of these variables and using Fubini's theorem, we find

$$E\{H(T+x)-H(T)\} = H(x) \quad x \geqslant 0. \tag{1}$$

Thus it suffices to prove that the only admissible solutions of (1) are linear. It is possible to give a direct though lengthy analytical proof; however, we shall proceed as follows.

Let $\{T_i\}$ be a sequence of independent random variables, identically distributed as T, independent of $\{X_i\}$, with $S_n = \sum_{i=1}^{n} T_i$. It is easy to see from (1) that

$$E\{H(T+x+y)-H(T+y)\} = H(x+y)-H(y) \quad x, y \geqslant 0. \tag{2}$$

By successively setting y equal to $S_{n-1}, S_{n-2}, \ldots, S_0 = 0$, T equal to $T_n, T_{n-1}, \ldots, T_1$ and then taking expectations in (2), we deduce that, for any $x \geqslant 0$,

$$\begin{aligned} E\{H(S_n+x)-H(S_n)\} &= E\{H(S_{n-1}+x)-H(S_{n-1})\} \\ &= \ldots = H(x). \end{aligned} \tag{3}$$

Now $S_n \to \infty$ almost surely, since T is nonnegative and not identically zero, and by Blackwell's theorem (Feller (4), p. 360)

$$H(x+y)-H(y) \to \frac{x}{\mu} \quad \text{as} \quad y \to \infty, x \geqslant 0;$$

in the arithmetic case x must be a multiple of the span. So from a simple variant of Slutsky's theorem

$$H(S_n+x)-H(S_n) \to \frac{x}{\mu} \quad \text{a.s.} \quad \text{as} \quad n \to \infty, \quad x \geqslant 0. \tag{4}$$

Finally, it is well known (Blackwell (1)) that

$$H(x+y)-H(y) \leqslant 1+H(x) \quad x, y \geqslant 0.$$

So for any fixed x, the left side of (4) is bounded and application of the dominated convergence theorem to (3) gives

$$H(x) = \frac{x}{\mu}.$$

Since it was assumed that $H(x) \not\equiv 0$, $\mu < \infty$ which proves the proposition, since the converse is trivially true.

Note that the above technique affords a rather simple proof of Theorem 3 in Smith (7).

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REFERENCES

(1) BLACKWELL, D. A renewal theorem. *Duke Math. J.* **15** (1948), 145–150.
(2) CHUNG, K. L. The Poisson process as a renewal process. *Periodica Math. Hung.* **2** (1972), 41–48.
(3) ÇINLAR, E. and JAGERS, P. Two mean values which characterise the Poisson process. *J. Appl. Probability* **10** (1973), 678–681.
(4) FELLER, W. *An Introduction to Probability Theory and its Applications*, vol. II (2nd ed.), Wiley (1971).
(5) KOTZ, S. and JOHNSON, N. L. A characterization of the exponential distribution by a waiting time property. *Communications in Statistics* **3** (1974), 257–258.
(6) SLIVNYAK, I. M. Some properties of stationary flows of homogeneous random events. *Theor. Probability Appl.* **7** (1962), 336–341.
(7) SMITH, W. L. On renewal theory, counter problems and quasi-Poisson processes. *Proc. Cambridge Philos. Soc.* **53** (1957), 175–193.

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