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A geometric path from zero Lyapunov exponents to rotation cocycles

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Abstract. We consider cocycles of isometries on spaces of non-positive curvature H. We show that the supremum of the drift over all invariant ergodic probability measures equals the infimum of the displacements of continuous sections under the cocycle dynamics. In particular, if a cocycle has uniform sublinear drift, then there are almost invariant sections, that is, sections that move arbitrarily little under the cocycle dynamics. If, in addition, H is a symmetric space, then we show that almost invariant sections can be made invariant by perturbing the cocycle.

1. *Introduction*

1.1. *Basic setting and results.*

Let $F : \Omega \to \Omega$ be a continuous mapping of a compact Hausdorff topological space Ω. A *cocycle* over the dynamics F is a continuous function

$$A : \Omega \to G, \tag{1.1}$$

where G is a topological group. We write $A^{(0)}(\omega) := \mathrm{id}$ and, for $n \in \mathbb{Z}_+$,

$$A^{(n)}(\omega) := A(F^{n-1}\omega) \cdots A(F\omega)A(\omega). \tag{1.2}$$

Notice the *cocycle relation*

$$A^{(n+m)}(\omega) = A^{(m)}(F^n\omega)A^{(n)}(\omega). \tag{1.3}$$

Two cocycles A and B over F are said to be *cohomologous* whenever there exists a continuous map $U : \Omega \to G$ such that

$$A(\omega) = U(F\omega)B(\omega)U(\omega)^{-1} \quad \text{for all } \omega \in \Omega. \tag{1.4}$$

In most of this paper, G will be the group $\mathrm{Isom}(H)$ of isometries of a metric space (H, d); then A is called a *cocycle of isometries*. We will assume at least that H is a

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Busemann space (i.e. a separable complete geodesic space of non-positive curvature in the sense of Busemann). The group $\mathrm{Isom}(H)$ is endowed with the bounded-open topology. (Definitions are given in §2.2.1.)

The *maximal drift* of a cocycle of isometries is defined as

$$\mathrm{drift}(F, A) := \lim_{n\to\infty} \frac{1}{n} \sup_{\omega\in\Omega} d(A^{(n)}(\omega)p_0, p_0). \tag{1.5}$$

Notice that the limit exists by subadditivity, and is independent of the choice of $p_0 \in H$.

Remark 1.1. It follows from Kingman's subadditive ergodic theorem that for every ergodic probability measure μ for $F : \Omega \to \Omega$, the limit

$$\lim_{n\to\infty} \frac{1}{n} d(A^{(n)}(\omega)p_0, p_0)$$

exists for μ-almost every $\omega \in \Omega$, and is a constant (also independent of $p_0 \in H$). This is the *drift* of the cocycle of isometries with respect to the measure μ; let us denote it by $\mathrm{drift}(F, A, \mu)$†. The following 'variational principle' holds‡:

$$\mathrm{drift}(F, A) = \sup_{\mu} \mathrm{drift}(F, A, \mu), \tag{1.6}$$

where μ runs over all invariant ergodic probabilities for F.

We say that the cocycle has *uniform sublinear drift* if $\mathrm{drift}(F, A) = 0$. By the remark above, this happens if and only if A has zero drift with respect to every F-invariant probability measure.

The *displacement* of a (continuous) section $\varphi : \Omega \to H$ is defined by

$$\mathrm{displ}(\varphi) := \sup_{\omega\in\Omega} d(A(\omega)\varphi(\omega), \varphi(F\omega)). \tag{1.7}$$

(When necessary, we use the more precise notation $\mathrm{displ}_{F,A}(\varphi)$.)

Notice that $\mathrm{displ}(\varphi) = 0$ if and only if the section φ is *invariant*, that is, $A(\omega)\varphi(\omega) = \varphi(F\omega)$ holds for every $\omega \in \Omega$.

It is not hard to show (see §2.1) that the displacement of any continuous section $\varphi : \Omega \to H$ is at least the drift of the cocycle:

$$\mathrm{displ}(\varphi) \geq \mathrm{drift}(F, A). \tag{1.8}$$

Our first main result is a converse of this fact.

THEOREM A. (Existence of sections of nearly minimal displacement; discrete time) *Assume that H is a Busemann space. Given a cocycle $A : \Omega \to \mathrm{Isom}(H)$ over $F : \Omega \to \Omega$, for each $\varepsilon > 0$ there exists a continuous section $\varphi : \Omega \to H$ such that* $\mathrm{displ}(\varphi) \leq \mathrm{drift}(F, A) + \varepsilon$.

Together with (1.8), this theorem implies that the maximal drift is the infimum of the displacements of the continuous sections φ.

For the next result, we need extra assumptions on the space H, the most important being that H is a symmetric space (see §2.4.3).

† The results of [**KM**] give important information in the case where $\mathrm{drift}(F, A, \mu) > 0$.

‡ This follows from [**Schr**, Theorem 1] or [**SS**, Theorem 1.7]. Although these references assume Ω to be compact metrizable, the proofs also work for compact Hausdorff Ω. (See also [**AB**, proof of Proposition 1].)

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THEOREM B. (Creating invariant sections; discrete time) *Assume that H is:*

- *either a proper† Busemann space;*
- *or a space of bounded non-positive curvature in the sense of Alexandrov‡.*

Also assume that H is symmetric. Let A be a cocycle of isometries of H with uniform sublinear drift. Then there exists a cocycle $\tilde{A}$, arbitrarily close to A, that has a continuous invariant section, and so is cohomologous to a cocycle taking values in the stabilizer in $\mathrm{Isom}(H)$ *of a point $p_0 \in H$.*

Here, the approximation is meant in the following sense: there is a sequence of cocycles $\tilde{A}_N$ satisfying the conclusions of the theorem, and such that for every bounded set B of H, the sequence $\tilde{A}_N(\omega)p$ converges to $A(\omega)p$ uniformly with respect to $(\omega,\ p) \in \Omega \times B$.

To prove Theorem A, we explicitly construct sections φ that almost realize the drift. The main construction uses an appropriate concept of barycenter. This construction is suitable for extensions to flows and nilpotent group actions, as we will see. We also give an alternative argument (based on the referee's comments); however, this argument seems to be less suitable for generalizations.

The proof of Theorem B is also explicit: we use the symmetries of H to construct the required perturbation. Some care is needed to ensure that the perturbation is small. While the task is easy in the locally compact case, the general case requires finer geometric arguments, making full use of the assumptions on curvature. In any case, what we ultimately show is that the space H has a certain *uniform homogeneity* property, which may be of independent interest.

For another interpretation of Theorem B, see Remark 2.4. For some extensions of the results above, see Remarks 2.3 and 2.13–2.15.

1.2. *Examples and applications.* The simplest space H to which our results apply is the real line. If we restrict ourselves to orientation-preserving isometries, then Theorems A and B become results on $\mathbb{R}$-valued cocycles. They appear repeatedly in the literature; see [**MOP1**], [**Kat**, Proposition 2.13], [**CNP**, Proposition 6]. We call this the *classical* setting.

Another natural situation is when H is the Euclidean space $\mathbb{R}^n$. For an interesting class of examples of cocycles of isometries of $\mathbb{R}^2$ that have uniform sublinear drift, see [**CNP**, §2.3].

Other situations where our results can be applied are naturally related to *matrix cocycles*, as we now explain.

We say that a matrix cocycle $A : \Omega \to \mathrm{GL}(d, \mathbb{R})$ has *uniform subexponential growth* if

$$\lim_{n\to+\infty} \frac{1}{n} \log \|(A^{(n)}(\omega))^{\pm 1}\| = 0 \quad \text{uniformly over } \omega \in \Omega,$$

for some (and hence any) matrix norm $\|\cdot\|$.

† A metric space is called *proper* if bounded closed sets are compact.
‡ See §2.4.5 for the definitions.

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We will show the following theorem.

THEOREM 1.2. *Let G be an algebraic subgroup of* $\mathrm{GL}(d, \mathbb{R})$ *that is closed under matrix transposition, and let K be its intersection with the orthogonal group $O(n)$. Let $A : \Omega \to G$ be a cocycle with uniform subexponential growth. Then there exists a cocycle $\tilde{A} : \Omega \to G$ arbitrarily close to A that is cohomologous to a cocycle taking values in K.*

For an elementary proof of this result in the case where $G = \mathrm{GL}(d, \mathbb{R})$ (assuming F invertible), and for further applications, see the companion paper [**BN**].

Other examples of groups G where Theorem 1.2 applies are the complex general linear group $\mathrm{GL}(n, \mathbb{C})$ (embedded in $\mathrm{GL}(2n, \mathbb{R})$ in the usual way) and the symplectic group $\mathrm{Sp}(2n)$; in both examples, K is the unitary group $\mathrm{U}(n)$.

Theorem 1.2 essentially follows from Theorem B applied to the space $H := G/K$. We can also obtain similar results for infinite-dimensional Lie groups. See §2.6 for details.

1.3. *Continuous-time versions.* We now assume that H is a Cartan–Hadamard manifold, that is, a complete simply connected Riemannian manifold (possibly of infinite dimension, modeled on a Hilbert space) of non-positive sectional curvature. (See §4.1.2 for details.)

A *semiflow* $\{F^t\}$ on Ω is a continuous-time dynamical system, that is, a continuous map $(\omega, t) \in \Omega \times \mathbb{R}_+ \mapsto F^t\omega \in \Omega$ such that $F^0 = \mathrm{id}$ and $F^{t+s} = F^t \circ F^s$ for all s, t in $\mathbb{R}$.

A *cocycle of isometries* (of H) over $\{F^t\}$ is a one-parameter family of maps $A^{(t)} : \Omega \to \mathrm{Isom}(H)$ (where $t \in \mathbb{R}_+$) satisfying

$$A^{(0)}(\omega) = \mathrm{id}, \quad A^{(s+t)}(\omega) = A^{(s)}(F^t\omega)A^{(t)}(\omega) \tag{1.9}$$

and such that

$$\begin{aligned} &A^{(t)}(\omega)p \text{ is continuous with respect to } (t, \omega, p) \\ &\quad \text{and continuously differentiable with respect to } t. \end{aligned} \tag{1.10}$$

Given the cocycle $\{A^{(t)}\}$, we can associate the vector field $\mathfrak{a}(\omega)$ on H defined by

$$\mathfrak{a}(\omega)(p) = \frac{\partial}{\partial t} A^{(t)}(\omega)p \bigg|_{t=0}. \tag{1.11}$$

This defines a continuous map $\mathfrak{a} : \Omega \to \mathrm{Kill}(H)$, where $\mathrm{Kill}(H)$ is the set of Killing fields on H. (See §4.1.2 for the topology on $\mathrm{Kill}(H)$.) Conversely, given any continuous map $\mathfrak{a} : \Omega \to \mathrm{Kill}(H)$, there is a unique cocycle of isometries $(F^t, A^{(t)})$ satisfying the ODE

$$\frac{\partial}{\partial t} A^{(t)}(\omega)p = \mathfrak{a}(F^t\omega)(A^{(t)}(\omega)p). \tag{1.12}$$

The map $\mathfrak{a}$ is called the *infinitesimal generator* of the cocycle.

We define the *maximal drift* of a continuous-time cocycle of isometries in the same way as the discrete-time situation:

$$\mathrm{drift}(F, A) := \lim_{t\to\infty} \frac{1}{t} \sup_{\omega\in\Omega} d(A^{(t)}(\omega)p_0, p_0), \quad p_0 \in H \text{ arbitrary.} \tag{1.13}$$

Again, we say that the cocycle has *uniform sublinear drift* if $\mathrm{drift}(F, A) = 0$.

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A (continuous) section $\varphi : \Omega \to H$ is said to be *differentiable with respect to the semiflow*† if for every $\omega \in \Omega$, the derivative

$$\varphi'(\omega) := \frac{\partial}{\partial t}\varphi(F^t\omega)\bigg|_{t=0} \tag{1.14}$$

exists and defines a continuous map $\varphi' : \Omega \to TH$. If φ is such a section then its *speed* with respect to $\{A^t\}$ is defined by

$$\operatorname{speed}(\varphi) := \sup_{\omega\in\Omega} \|\mathfrak{a}(\omega)(\varphi(\omega)) - \varphi'(\omega)\|, \tag{1.15}$$

where $\mathfrak{a}$ is the infinitesimal generator of the cocycle. (A more precise notation is $\operatorname{speed}_{F,\mathfrak{a}}(\varphi)$.) The speed is the continuous-time analogue of the displacement (1.7). Notice that $\operatorname{speed}(\varphi) = 0$ holds if and only if the section φ is *invariant*, that is, $A^{(t)}(\omega)\varphi(\omega) = \varphi(F^t\omega)$.

Analogously to (1.8), we have

$$\operatorname{speed}(\varphi) \geq \operatorname{drift}(F, A). \tag{1.16}$$

The continuous-time versions of Theorems A and B are given below.

THEOREM C. (Existence of sections of nearly minimal speed; continuous time) *Assume that H is a Cartan–Hadamard manifold. Given a continuous-time cocycle of isometries $\{A^{(t)}\}$ of H over a semiflow $\{F^t\}$ on Ω, for each $\varepsilon > 0$ there exists a continuous section $\varphi : \Omega \to H$ that is differentiable with respect to the semiflow $\{F^t\}$ and such that* $\operatorname{speed}(\varphi) \leq \operatorname{drift}(F, A) + \varepsilon$.

THEOREM D. (Creating invariant sections; continuous time) *In the context of the preceding theorem, assume moreover that H is a symmetric space and that the cocycle $\{A^{(t)}\}$ has uniform sublinear drift. Let $\mathfrak{a}$ be the infinitesimal generator of $\{A^{(t)}\}$. Then there exists $\tilde{\mathfrak{a}} : \Omega \to \operatorname{Kill}(H)$, arbitrarily close to $\mathfrak{a}$ such that the associated cocycle $\{\tilde{A}^{(t)}\}$ has an invariant continuous section (and hence is cohomologous to a cocycle taking values in the stabilizer of a point $p_0 \in H$).*

Here, the convergence of a sequence $\tilde{\mathfrak{a}}_N : \Omega \to \operatorname{Kill}(H)$ to $\mathfrak{a}$ is in the following sense: for each bounded subset B of H, the sequence $\tilde{\mathfrak{a}}_N(\omega)(p)$ converges to $\mathfrak{a}(\omega)(p)$ uniformly with respect to $(\omega, p) \in \Omega \times B$. (See §4.1.2 for a more precise explanation of the topologies.)

Although the proofs of these theorems follow the same ideas as the discrete-time versions, the technical details are of a different nature. Thus we prove the two kinds of results in nearly independent ways.

Like its discrete-time analogue, the proof of Theorem C uses barycenters, but we also need to concern ourselves with differentiability with respect to the flow.

The proof of Theorem D uses an infinitesimal uniform homogeneity property of the space H. The proof of this property, like that of its macroscopic version, is simpler in the locally compact case but uses finer geometrical arguments in the general case.

† A similar definition appears in [**Schw**].

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1.4. *Other group actions.* It is natural to ask whether the previous theorems extend to cocycles of isometries over actions of semigroups more complicated than $\mathbb{Z}_+$ or $\mathbb{R}_+$. We will concentrate on discrete groups, leaving the generalizations to continuous groups as a task for the reader.

Given a group Γ acting on the left by homeomorphisms of a compact Hausdorff topological space Ω and a topological group G, a cocycle over the Γ-action with values in G is a continuous map

$$\begin{aligned} A : \Gamma \times \Omega &\to G \\ (g, \omega) &\mapsto A^{(g)}(\omega) \end{aligned}$$

such that

$$A^{(gh)}(\omega) = A^{(g)}(h\omega)A^{(h)}(\omega) \quad \text{for all } g, h \text{ in } \Gamma \text{ and all } \omega \in \Omega.$$

In the classical case (that is, when $G = \mathbb{R}$), the analogue of Theorem A holds for nilpotent group actions, but it does not hold for solvable (in particular, amenable) group actions; see [**MOP1, MOP2**]. The next result establishes that the analogue of Theorem A remains true for cocycles of isometries over abelian group actions. Generalizations for virtually nilpotent group actions will be discussed in §3.

Assume that A is a cocycle of isometries of a space (H, d). We say that A has *sublinear drift along cyclic subgroups* if, for each fixed $p_0 \in H$ and all $g \in \Gamma$, the limit

$$\lim_{n\to\infty} \frac{1}{n} d(A^{(g^n)}(\omega)p_0,\ p_0)$$

equals zero uniformly on Ω. We say that A *admits almost invariant sections* if there exists a sequence of continuous functions $\varphi_N : \Omega \to H$ such that, for all $g \in \Gamma$,

$$\lim_{N\to\infty} d(A^{(g)}(\omega)\varphi_N(\omega),\ \varphi_N(g\omega)) = 0$$

uniformly on Ω.

THEOREM E. (Existence of almost invariant sections; abelian groups) *Let Γ be a finitely generated abelian group acting by homeomorphisms of a compact Hausdorff metric space Ω. Let A be a cocycle over this group action with values in the group of isometries of a Busemann space H. If A has sublinear drift along cyclic subgroups, then A admits almost invariant sections.*

As is easy to see, in order to check the condition on drift along all cyclic subgroups above, it suffices to check it only for those associated to the generators of the group Γ.

We actually provide an extension of Theorem E to virtually nilpotent groups Γ; see §3. This is somewhat related to results obtained by de Cornulier *et al* [**CTV**]; in their considerations, Ω is a point and H is a Hilbert space.

1.5. *Further questions.* We next mention a few other questions that are suggested by our results.

The first question is whether our results can be extended to cocycles of *semicontractions*. Notice that the basic fact (1.8) as well as the related theorem from [**KM**] still hold in this case.

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Fix the dynamics $F : \Omega \to \Omega$, and let $A : \Omega \to \mathbb{R}$ be a real function. As mentioned above (§1.2), we can regard this as a cocycle of orientation-preserving isometries of the line. It follows from the 'variational principle' (1.6) that $\mathrm{drift}(F, A) = \sup_\mu |\int A \, d\mu|$. The study of the probability measures which realize this supremum is an interesting problem with rich ramifications in *ergodic optimization*; see [**Je**]. In this spirit, it could also be interesting to study *drift-maximizing measures*, that is, those which realize the supremum in (1.6).

A perhaps related problem is to find out when a displacement-minimizing section exists in Theorem A (or, in the language of Remark 2.4, when Γ is a semisimple isometry.)

All our results concern approximations in the C^0 topology. It is natural to ask whether the differentiability class can be improved. However, this is non-trivial already in the classical case (i.e. with $G = \mathbb{R}$), where it is closely related to the existence of invariant distributions for the base dynamics (see [**AK, Kat, NT**]). We do not know whether such a relation can be extended to the cocycles considered in this work.

Finally, a natural problem raised by this work concerns the case of diffeomorphisms: in the situation of Theorem 1.2, if a cocycle is given by the derivative of a diffeomorphism, then it is natural to require that the perturbed cocycle is also a derivative. More precisely, we pose the following question: given a diffeomorphism f of a compact manifold all of whose Lyapunov exponents are zero, under what circumstances is f close to a diffeomorphism that is conjugate to an isometry? Our methods fail in dealing with this problem, as it involves a simultaneous (and coherent) perturbation of the base dynamics and the cocycle. Let us point out, however, that the answer (in C^1 regularity) is known in the one-dimensional case; see [**BGu, Na2**].

1.6. *Organization of the paper.* In §2 we prove the discrete-time Theorems A and B; we also explain how to obtain Theorem 1.2.

In §3 we deal with other (still discrete) group actions, thus proving Theorem E and an extension thereof. That section is shorter and uses the tools explained in the previous one.

In §4 we deal with continuous-time cocycles, thus proving Theorems C and D. Although the main ideas of the proofs are similar to those of the discrete-time results, the technical details are somewhat different, and this section is actually nearly independent of the previous ones.

Along the way, we explain the geometrical tools and properties that are required for the proofs. Some of these properties become simpler to prove if the space H is proper (i.e. finite-dimensional in the case of manifolds). Thus in §§2 and 4 we give proofs for the proper case, and leave for Appendices A and B the geometrical arguments which allow us to extend the proofs to the general case. A technical property which is needed for the proof of Theorem D is proved in Appendix C.

2. *Discrete-time cocycles*

In this section we prove Theorems A and B. We also explain how to deduce Theorem 1.2.

2.1. *The easy inequality.* Let us prove that the displacement of any section is an upper bound for the drift of the cocycle, as asserted in (1.8). Here, no assumptions on the geometry of H are needed.

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Proof of (1.8). We have

$$\begin{aligned} d(A^{(n)}(\omega)\varphi(\omega),\, \varphi(F^n\omega)) &\le \sum_{j=0}^{n-1} d(A^{(n-j)}(F^j\omega)\varphi(F^j\omega),\, A^{(n-j-1)}(F^{j+1}\omega)\varphi(F^{j+1}\omega)) \\ &= \sum_{j=0}^{n-1} d(A(F^j\omega)\varphi(F^j\omega),\, \varphi(F^{j+1}\omega)) \\ &\le n\ \mathrm{displ}(\varphi). \end{aligned}$$

Dividing by n and passing to the limit, we obtain $\mathrm{drift}(F,\, A) \le \mathrm{displ}(\varphi)$, as desired. □

2.2. *Preliminaries.* The proof of Theorem A requires the preliminaries below.

2.2.1. *Busemann spaces.* Let (H, d) be a separable metric space. We say that H is a *geodesic space* if it is complete and every pair of points p, q in H can be joined by a *geodesic*, that is, a curve $\gamma : [0, 1] \to H$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $d(\gamma(t), \gamma(s)) = |t - s|d(p, q)$ for all s, t in $[0, 1]$. If these curves are unique for arbitrarily prescribed p, q, then we say that H is *uniquely geodesic*.

The space H has *non-positive curvature in the sense of Busemann* (it is a *Busemann space* for short) if it is geodesic and the distance function along geodesics is convex. Equivalently, given any two pairs of points p, q and p', q', their corresponding midpoints $m := \mathrm{mid}(p, q)$ and $m' := \mathrm{mid}(p', q')$ satisfy

$$d(m, m') \le \frac{d(p,\, p')}{2} + \frac{d(q, q')}{2}. \tag{2.1}$$

The family of such spaces obviously includes all strictly convex Banach spaces. (General Banach spaces may also be included in this category when considering only segments of lines as geodesics.) It also includes complete simply connected Riemannian manifolds of non-positive curvature (such as those that will appear in the proof of Theorem 1.2). For infinite-dimensional examples, see the remarks in §2.6.

2.2.2. *Barycenter maps.* Given a metric space (H, d), we denote by $\mathcal{P}^1(H)$ the space of probability measures on H with finite first moment, that is, such that for some (equivalently, for every) $p_0 \in H$,

$$\int_H d(p_0,\, p)\, d\mu(p) < \infty.$$

We endow this space with the 1-Wasserstein metric W_1 defined as

$$W_1(\mu,\, \nu) := \inf_{P \in (\mu \mid \nu)} \int_{H \times H} d(p, q)\, dP(p, q), \tag{2.2}$$

where $(\mu \mid \nu)$ denotes the set of all *couplings* of μ and ν, that is, all probability measures P on $H \times H$ whose projection along the first (respectively, second) coordinate coincides with μ (respectively, ν).

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Example 2.1. If $\mu, \nu_1, \nu_2 \in \mathcal{P}^1(H)$, then, for every $\lambda \in [0, 1]$,

$$W_1((1-\lambda)\mu + \lambda\nu_1, (1-\lambda)\mu + \lambda\nu_2) \leq \lambda \cdot \sup\{d(p_1, p_2) : p_i \in \operatorname{supp} \nu_i\}.$$

Indeed, this follows directly from the definitions by considering the coupling $(1-\lambda)i_*\mu + \lambda\, \nu_1 \times \nu_2$, where $i(p) := (p, p)$.

For much more on Wasserstein metrics, see, for example, [**Vi**].

Let δ_p denote Dirac measure on p.

THEOREM 2.2. *If H is a Busemann space, then there exists a map* $\operatorname{bar} : \mathcal{P}^1(H) \to H$ *that satisfies* $\operatorname{bar}(\delta_p) = p$ *for each p, is equivariant with respect to the action of the isometries, and is* 1*-Lipschitz for the* 1*-Wasserstein metric on* $\mathcal{P}^1(H)$.

In particular, in the situation of Example 2.1, we have

$$d(\operatorname{bar}((1-\lambda)\mu + \lambda\nu_1), \operatorname{bar}((1-\lambda)\mu + \lambda\nu_2)) \leq \lambda \cdot \sup\{d(p_1, p_2) : p_i \in \operatorname{supp} \nu_i\}. \tag{2.3}$$

We also denote

$$\operatorname{bar}(p_1, \ldots, p_n) := \operatorname{bar} \frac{1}{n}(\delta_{p_1} + \cdots + \delta_{p_n}).$$

It follows from (2.3) that

$$d(\operatorname{bar}(p_1, \ldots, p_{n-1}, p_n), \operatorname{bar}(p_1, \ldots, p_{n-1}, p_n')) \leq \frac{1}{n} d(p_n, p_n'). \tag{2.4}$$

Henceforth, we will not need to know what precisely the map bar above is, although its geometrical flavor should be intuitively transparent. For instance, $\operatorname{bar}(p)$ coincides with p, while $\operatorname{bar}(p_1, p_2)$ is the midpoint between p_1 and p_2. The definition of $\operatorname{bar}(p_1, p_2, p_3)$ is, however, quite involved.

In its full generality, Theorem 2.2 above was proved by Navas in [**Na1**] by elaborating on an idea introduced by Es-Sahib and Heinich in [**EH**]. Nevertheless, for compactly supported measures μ on CAT(0) spaces, a much more classical notion of barycenter due to Cartan (see [**Ca**, Note III, Part IV]) is enough for our purposes. (See §2.4.5 for the definition of CAT(0) spaces; see also [**AL, BK, Jo**].) The Cartan barycenter of μ as above is defined as the unique point that minimizes the function

$$f_\mu(p) = \int_H d^2(q, p)\, d\mu(q). \tag{2.5}$$

The fact that Cartan's barycenter is 1-Lipschitz with respect to the 1-Wasserstein metric is proved in [**St**].

2.3. *Existence of sections of nearly minimal displacement: proof of Theorem A.*

First proof of Theorem A. Given a cocycle A of isometries of a Busemann space H over $F : \Omega \to \Omega$, we fix any $p_0 \in H$. Let

$$\varphi_N(\omega) := \operatorname{bar}(p_0, A(\omega)^{-1} p_0, [A^{(2)}(\omega)]^{-1} p_0, \ldots, [A^{(N-1)}(\omega)]^{-1} p_0), \tag{2.6}$$

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where bar stands for the barycenter introduced in Theorem 2.2. Then, by equivariance of the barycenter,

$$A(\omega)\varphi_N(\omega) = \mathrm{bar}(A(\omega)p_0,\ p_0,\ A(F\omega)^{-1}p_0,\ \ldots,\ [A^{(N-2)}(F\omega)]^{-1}p_0),$$
$$\varphi_N(F\omega) = \mathrm{bar}(p_0,\ A(F\omega)^{-1}p_0,\ \ldots,\ [A^{(N-2)}(F\omega)]^{-1}p_0,\ [A^{(N-1)}(F\omega)]^{-1}p_0).$$

Using (2.4), we obtain

$$\begin{aligned} d(A(\omega)\varphi_N(\omega),\ \varphi_N(F\omega)) &\le \frac{1}{N} d(A(\omega)p_0,\ [A^{(N-1)}(F\omega)]^{-1}p_0) \\ &= \frac{1}{N} d(A^{(N)}(\omega)p_0,\ p_0). \end{aligned}$$

In particular,

$$\mathrm{displ}(\varphi_N) \le \frac{1}{N} \sup_{\omega\in\Omega} d(A^{(N)}(\omega)p_0,\ p_0).$$

The theorem then follows by taking $\varphi = \varphi_N$ with sufficiently large N (depending on ε). □

Remark 2.3. We can extend the family $\{\varphi_N\}_{N\in\mathbb{Z}_+}$ to a family $\{\varphi_t\}_{t\in\mathbb{R}}$ by letting

$$\varphi_t(\omega) := \mathrm{bar}((1-t+N)\delta_{\varphi_N(\omega)} + (t-N)\delta_{\varphi_{N+1}(\omega)}) \quad \text{where } N = \lfloor t \rfloor.$$

Then $\lim_{t\to\infty} \mathrm{displ}(\varphi_t) = \mathrm{drift}(F,\ A)$; moreover, φ_t depends continuously on t and also on A. (Such parameterized sections play an important role for the particular case considered in [**ABD2**].)

As pointed out by the referee, Theorem A was already known in the case where Ω is a point; see [**BGS**, Lemma 6.6]. In fact, it is possible to adapt the argument therein to give another proof of Theorem A.

Second proof of Theorem A. Let $\varphi_0\colon \Omega \to H$ be any section (e.g., constant equal to the base point p_0). Then

$$\mathrm{drift}(F,\ A) = \lim_{n\to\infty} \frac{1}{n} \mathrm{displ}_{F^n, A^{(n)}}(\varphi_0).$$

Take n of the form 2^k such that $(1/n)\,\mathrm{displ}_{F^n,A^{(n)}}(\varphi_0) < \mathrm{drift}(F,\ A) + \varepsilon$. If $k = 0$, then we are done, so assume that $k \ge 1$.

Define a new section by

$$\varphi_1(\omega) := \mathrm{mid}[A^{(n/2)}(F^{-n/2}\omega)\varphi_0(F^{-n/2}\omega),\ \varphi_0(\omega)],$$

where mid stands for the midpoint of a segment. Then

$$A^{(n/2)}(\omega)\varphi_1(\omega) = \mathrm{mid}[A^{(n)}(F^{-n/2}\omega)\varphi_0(F^{-n/2}\omega),\ A^{(n/2)}(\omega)\varphi_0(\omega)],$$
$$\varphi_1(F^{n/2}\omega) = \mathrm{mid}[A^{(n/2)}(\omega)\varphi_0(\omega),\ \varphi_0(F^{n/2}\omega)].$$

Thus, by the Busemann property,

$$d(A^{(n/2)}(\omega)\varphi_1(\omega),\ \varphi_1(F^{n/2}\omega)) \le d(A^{(n)}(F^{-n/2}\omega)\varphi_0(F^{-n/2}\omega),\ \varphi_0(F^{n/2}\omega));$$

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in particular,

$$\operatorname{displ}_{F^{n/2},A^{(n/2)}}(\varphi_1) \le \tfrac{1}{2}\operatorname{displ}_{F^n,A^{(n)}}(\varphi_0).$$

Repeating this construction, we recursively find sections $\varphi_2, \dots, \varphi_k$ such that

$$\operatorname{displ}_{F^{n/2^j},A^{(n/2^j)}}(\varphi_j) \le \tfrac{1}{2}\operatorname{displ}_{F^{n/2^{j-1}},A^{(n/2^{j-1})}}(\varphi_{j-1}) \quad \text{for } 1 \le j \le k.$$

Therefore,

$$\operatorname{displ}_{F,A}(\varphi_k) \le \frac{1}{2^k}\operatorname{displ}_{F^n,A^{(n)}}(\varphi_0) < \operatorname{drift}(F, A) + \varepsilon,$$

that is, $\varphi := \varphi_k$ has the required properties. □

The proof above does not require general barycenters. On the other hand, while it seems feasible to adapt this proof to the continuous case, it is unclear whether or not it can be adapted to more complicate group actions.

2.4. *More preliminaries.* The proof of Theorem B requires additional preliminaries.

2.4.1. *Topologies.* On the space $\operatorname{Isom}(H)$, we consider the first-countable topology for which the convergence of sequences is uniform convergence on bounded subsets. This is called the *bounded-open topology*†.

We also endow the set $C(\Omega, \operatorname{Isom}(H))$ of continuous functions from Ω into $\operatorname{Isom}(H)$ with the compact-open topology. A sequence (A_n) converges to A in this topology if and only if for every bounded subset B of H, the sequence $(A_n(\omega)(p))$ converges to $A(\omega)(p)$ uniformly with respect to $(\omega, p) \in \Omega \times B$.

2.4.2. *Translation length.* Recall that the *displacement function* of $J \in \operatorname{Isom}(H)$ is the function

$$p \in H \mapsto d(J(p), p). \tag{2.7}$$

Since H is a Busemann space, this function is convex. The infimum of the displacement function is called the *translation length* of J.

Remark 2.4. Let $A : \Omega \to \operatorname{Isom}(H)$ be a cocycle of isometries over a *homeomorphism* $F : \Omega \to \Omega$. Let us explain how its drift can be seen as a translation length of a certain isometry.

Let $C(\Omega, H)$ be the set of sections, endowed with the distance $d(\varphi_1, \varphi_2) = \sup_\omega d(\varphi_1(\omega), \varphi_2(\omega))$. (This is a geodesic space, but not a uniquely geodesic one.) Let $\Gamma = \Gamma_{F,A} : C(\Omega, H) \to C(\Omega, H)$ be the *graph transform* defined by $(\Gamma\varphi)(\omega) := A(\omega)\varphi(F^{-1}\omega)$. Then Γ is an isometry of $C(\Omega, H)$, and $\operatorname{displ}(\varphi) = d(\Gamma\varphi, \varphi)$. Therefore, Theorem A states that *the drift of a cocycle of isometries equals the translation length of the associated graph transform.*

† If H is proper then this coincides with the compact-open topology, which is the usual topology on $\operatorname{Isom}(H)$; see, for example, [**He**].

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2.4.3. *Symmetric geodesic spaces.* We say that a uniquely geodesic space H is *geodesically complete* if the maximal interval of definition of all geodesics is $\mathbb{R}$. For such a space, the *symmetry* at a point $p_0 \in H$ is the map $\sigma_{p_0} : H \to H$ that sends p to the unique point p' such that p_0 is the midpoint between p and p'. So σ_{p_0} is an involution. We say that H is a *symmetric geodesic space* if σ_{p_0} is an isometry for every $p_0 \in H$, and the map $(p_0, p) \mapsto \sigma_{p_0}(p)$ is continuous.

2.4.4. *Transvections and a displacement estimate.* Assume that H is a symmetric geodesic space. Following Cartan, we call an isometry of the form $J = \sigma_{p_2} \circ \sigma_{p_1}$ a *transvection*. If $\gamma : \mathbb{R} \to H$ is an isometric (unit-speed) parametrization of the geodesic passing through p_1 and p_2, say with $\gamma^{-1}(p_1) < \gamma^{-1}(p_2)$, then $J(\gamma(t)) = \gamma(t + b)$ holds for all $t \in \mathbb{R}$, where $b = 2d(p_1, p_2)$. We say that J translates the geodesic γ by length b.

We remark that if, in addition, H is a Busemann space, then $d(J(q), q) \geq b$ for all $q \in H$; see [**Pa**]. So the translation length of J is precisely b.

LEMMA 2.5. *Assume that H is a symmetric geodesic space. Also assume that H is proper. Let J be a transvection that translates a geodesic γ by length b. Then*

$$d(J(q), q) \leq f(b, d(q, \gamma)) \quad \text{for every } q \in H, \tag{2.8}$$

where $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function that depends only on the space H, and is monotonically increasing with respect to each variable.

Proof. Define a function $\tilde{f} : H \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{f}(p_0, b, \ell) := \sup\{d(J(q), q) : J = \sigma_{p_1} \circ \sigma_{p_0}, d(p_1, p_0) \leq b/2, d(q, p_0) \leq \ell\}.$$

The supremum is finite by the properness of H and continuity. Since the group of isometries acts transitively on H, the value $\tilde{f}(p_0, b, \ell)$ actually does not depend on p_0; call it $f(b, \ell)$. Then (2.8) holds. □

Remark 2.6. It follows that under the assumptions of Lemma 2.5, the map $p_0 \in H \mapsto \sigma_{p_0} \in \mathrm{Isom}(H)$ is continuous (where $\mathrm{Isom}(H)$ is endowed with the bounded-open topology, as explained in §2.4.1).

2.4.5. *Curvature bounds in the sense of Alexandrov, and a displacement estimate.* In the case where H is infinite-dimensional, the proof of Lemma 2.5 given above obviously does not work. Nevertheless, the lemma holds if properness is replaced by some curvature hypotheses, as we next explain. Readers who are not interested in infinite-dimensional applications can skip this section.

Given $\kappa \leq 0$, the *model space* (M_κ, d_κ) is the two-dimensional space of constant curvature κ†.

Let H be a uniquely geodesic space. A *triangle* $\triangle(p_1, p_2, p_3)$ in H consists of three points p_1, p_2, p_3 and three geodesic segments joining them. Suppose that $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ is a triangle in the model space M_κ such that $d(p_i, p_j) = d_\kappa(\tilde{p}_i, \tilde{p}_j)$ for all i, j in $\{1, 2, 3\}$. Then we say that $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ is an *SSS-comparison triangle*‡ for $\triangle(p_1, p_2, p_3)$.

† We will not consider $\kappa > 0$ in order to avoid unnecessary complications.
‡ SSS stands for *side–side–side*.

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We say that H has *curvature $\leq\kappa$ ($\geq\kappa$) in the sense of Alexandrov* if for every triangle $\triangle(p_1, p_2, p_3)$ in H and every SSS-comparison triangle $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ in the model space M_κ, the following inequality holds for all $t \in [0, 1]$:

$$d(p_3, (1-t)p_1 + tp_2) \leq (\geq)\ d_\kappa(\tilde{p}_3, (1-t)\tilde{p}_1 + t\tilde{p}_2),$$

where $t \mapsto (1-t)p_1 + tp_2$ is shorthand for the geodesic segment joining p_1 and p_2.

Remark 2.7. Actually the usual definition requires only local comparisons. However, (in the cases that we consider here) this turns out to be equivalent to our (global) definition; see [**BBI**, § 4.6.2] and references therein.

Spaces of curvature ≤ 0 are also called CAT(0) spaces. It is a standard fact that every complete CAT(0) space is a Busemann space (see [**BH**, p. 176] or [**St**, Corollary 2.5]).

We will say that a uniquely geodesic space H has *bounded non-positive curvature in the sense of Alexandrov* if it has curvature ≤ 0 and $\geq\kappa$ for some $\kappa \leq 0$.

We now have the following version of Lemma 2.5.

LEMMA 2.8. *Assume that H is a symmetric space of bounded non-positive curvature in the sense of Alexandrov. Let J be a transvection that translates a geodesic γ by length b. Then*

$$d(J(q), q) \leq f(b, d(q, \gamma)) \quad \text{for every } q \in H, \tag{2.9}$$

where $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function that depends only on the space H, and is monotonically increasing with respect to each variable.

We leave the proof of this lemma to Appendix A.

Remark 2.9. Similarly to Remark 2.6, we conclude that under the assumptions of Lemma 2.8, the map $p_0 \in H \mapsto \sigma_{p_0} \in \mathrm{Isom}(H)$ is continuous.

2.4.6. *Macroscopic uniform homogeneity.* Every symmetric space is homogeneous (in the sense that the group of isometries acts transitively). We will, however, need a stronger property, given by the following lemma.

LEMMA 2.10. (Macroscopic uniform homogeneity) *Assume that H is:*

- *either a proper Busemann space;*
- *or a space of bounded non-positive curvature in the sense of Alexandrov.*

Also assume that H is symmetric. Then there exists a continuous map $J : H \times H \to \mathrm{Isom}(H)$ with the following properties:

(a) $J(p, q)p = q$ *for all p, q in H.*

(b) $J(p, q)$ *converges to the identity as the distance between p and q converges to zero.*

More explicitly, assertion (b) means that, for every $\varepsilon > 0$ and each bounded subset $B \subset H$, there exists $\delta > 0$ such that $d(J(p, q)r, r) < \varepsilon$ holds for all $r \in B$ whenever $d(p, q) < \delta$. (Notice that p and q are not restricted to a bounded set.)

Proof. Fix some $p_0 \in H$, and consider the transvection (see Figure 1)

$$J(p, q) := \sigma_m \circ \sigma_{p_0} \quad \text{where } m \text{ is the midpoint between } \sigma_{p_0}(p) \text{ and } q. \tag{2.10}$$

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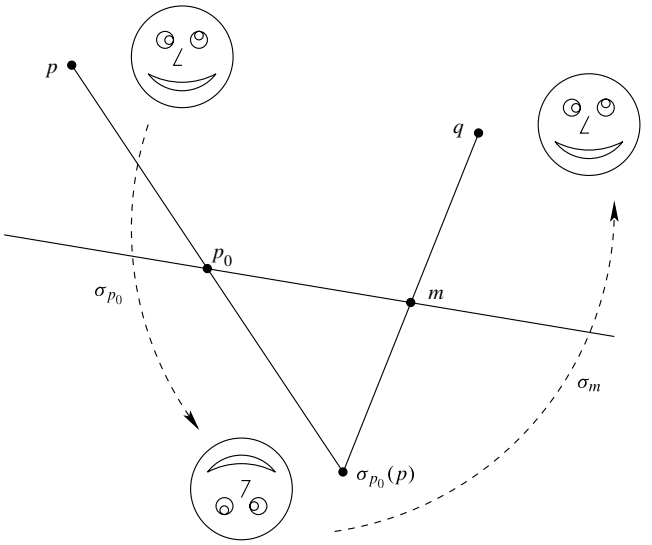


FIGURE 1. m is the midpoint between q and $\sigma_{p_0}(p)$. The isometry $J(p, q) := \sigma_m \circ \sigma_{p_0}$ sends p to q and translates the geodesic joining p_0 and m by length $2d(p_0, m) \le d(p, q)$.

Applying the Busemann inequality (2.1) to the points $\sigma_{p_0}(p)$, p and q, we obtain $d(p_0, m) \le \frac{1}{2}d(p, q)$. Therefore, the length by which the transvection $J(p, q)$ translates γ is at most $d(p, q)$. So assertion (b) follows directly from Lemma 2.5 or Lemma 2.8, depending on the case. □

Remark 2.11. In the case where H is the hyperbolic plane, Lemma 2.10 follows from [**ABD1**, Lemma 5]. Although the construction presented therein is specific to the hyperbolic plane, it actually produces the same isometries as our formula (2.10) in this particular case.

Remark 2.12. Despite the fact that the perturbative argument that appears in [**BN**] is elementary and does not allude to any geometry, it is actually the construction above specialized to $H = \mathrm{GL}(d, \mathbb{R})/\mathrm{O}(d)$.

2.5. *Creating invariant sections: proof of Theorem B.*

Proof of Theorem B. By Theorem A, there exists a sequence of sections φ_N such that $\lim_{N\to\infty} \mathrm{displ}(\varphi_N) = 0$. Define

$$\tilde{A}_N(\omega) = J(A(\omega)\varphi_N(\omega), \varphi_N(F\omega)) \circ A(\omega), \tag{2.11}$$

where J is given by Lemma 2.10. Then:

- $\tilde{A}_N(\omega)\varphi_N(\omega) = \varphi_N(F\omega)$, that is, φ_N is $\tilde{A}_N$-invariant.
- for each bounded subset B of H, the sequence $\tilde{A}_N(\omega)p$ converges to $A(\omega)p$ uniformly with respect to $(\omega, p) \in \Omega \times B$.

This shows the theorem except for the claim concerning the cohomologous cocycle. To prove this last issue, take any point $p_0 \in H$ and consider the cocycle $B_N(\omega) = U(F\omega)^{-1} \circ$

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$A_N(\omega) \circ U(\omega)$, where $U(\omega) = J(p_0, \varphi_N(\omega))$. Then B_N is cohomologous to A_N and takes values in the stabilizer of p_0, as desired. □

Remark 2.13. A 'non-perturbative' version of Theorem B goes as follows: *Under the hypotheses of Theorem B, it follows that A is cohomologous to cocycles arbitrarily close to cocycles taking values in stabilizers of points.*

Indeed, by Theorem A, there exists a sequence of sections φ_N such that $\lim_{N\to\infty} \mathrm{displ}(\varphi_N) = 0$. Take any point $p_0 \in H$ and consider the cocycle $B_N(\omega) = U(F\omega)^{-1} \circ A(\omega) \circ U(\omega)$, where $U(\omega) = J(p_0, \varphi_N(\omega))$ and J is provided by Lemma 2.10. If N is large then

$$\begin{aligned} d(B_N(\omega)p_0,\, p_0) &= d(A(\omega) \circ U(\omega)p_0,\, U(F(\omega))p_0) \\ &= d(A(\omega)\varphi_N(\omega),\, \varphi_N(\omega)) \le \mathrm{displ}(\varphi_N) \end{aligned}$$

is small. Thus B_N is a cocycle cohomologous to A close to a cocycle taking values in the stabilizer of p_0.

Remark 2.14. Let us see how to obtain certain 'accessibility' properties, which play an important role for the particular case treated in [**ABD2**]. First, by Remark 2.3 we can find a continuous family $\{\varphi_t\}_{t\in\mathbb{R}}$ of sections such that $\mathrm{displ}_{F,A}(\varphi_t) \to 0$ as $t \to +\infty$. Repeating the construction of the proof of Theorem B, we conclude the following: *for every cocycle A with uniform sublinear drift, there exists a continuous family of cocycles* $\{A_t\}_{t\in[0,\infty]}$, *satisfying* $A_\infty = A$ *and such that for each* $t < \infty$, A_t *has a continuous invariant section* φ_t *(that also depends continuously on t). Moreover, such correspondence is continuous: given a continuous family* $A(s)$ *of cocycles (s in an arbitrary topological space), the resulting* $A_t(s)$ *and* $\varphi_t(s)$ *are jointly continuous.*

Remark 2.15. Replace $\Omega \times H$ by a fiber bundle Σ with base space Ω, fiber H, and structural group $\mathrm{Isom}(H)$. Then the mappings $\Sigma \to \Sigma$ that preserve the bundle structure and project over F play the role of the cocycles of isometries. Fibered versions of Theorems A and B actually hold. The proofs are basically the same, replacing the base point p_0 that appears (explicitly or implicitly) in the fundamental formulas (1.5), (2.6), (2.10), and (2.11) by any continuous section $\Omega \to H$.

2.6. *Application to matrix cocycles.* In the proof below, we use some geometrical facts that can be found in [**BH**, Ch. II.10 (especially pp. 328–329)]; see also [**Lang**, Ch. XII].

Proof of Theorem 1.2. Let G be an algebraic subgroup of $\mathrm{GL}(d, \mathbb{R})$ that is closed under matrix transposition, and let $K = G \cap \mathrm{O}(n)$. Consider the action of G on the space $H := G/K$ of left cosets. Then we can metrize H so that the action becomes isometric, and moreover H becomes a symmetric Busemann space. Actually, for each $g \in G$, the distance between the cosets gK and K is $(\sum(\log \sigma_i)^2)^{1/2}$, where $\sigma_1, \ldots, \sigma_d$ are the singular values of the matrix g.

Now let $A : \Omega \to G$ be a cocycle, and let $[A] : \Omega \to \mathrm{Isom}(H)$ be the induced cocycle of isometries. Assume that A has uniform subexponential growth. It follows from the distance formula above that $[A]$ has uniform sublinear drift.

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By Theorem B, there is a perturbation of $[A]$ that has an invariant section $\varphi : \Omega \to H$. Actually, this perturbation is obtained by composition with transvections (recall (2.11) and (2.10)), which are induced by elements of G (see [**KN**, Lemma 1, p. 235]). So the perturbed cocycle of isometries is induced by a perturbation $\tilde{A}$ of the original G-cocycle.

Choose (e.g., using Lemma 2.10) a continuous map $U : \Omega \to G$ such that for each $\omega \in \Omega$, the coset containing $U(\omega)$ is precisely $\varphi(\omega)$. Then U is a conjugacy between $\tilde{A}$ and a K-valued cocycle, as desired. □

Remark 2.16. It is actually possible to state Theorem 1.2 in a Lie group setting, and prove it using, for example, [**KN**, Theorem 8.6(2), p. 256]. We preferred, however, to keep the statements simpler, relying only on more elementary results such as those from [**BH**] or [**Lang**].

Remark 2.17. Let $\mathcal{H}$ be an infinite-dimensional separable real Hilbert space. Let $\mathrm{GL}^2(\infty, \mathbb{R})$ be the group of all invertible operators on $\mathcal{H}$ that may be written in the form $\mathrm{Id} + L$, where L is a Hilbert–Schmidt operator. Let $\mathrm{O}^2(\infty)$ be the orthogonal subgroup of $\mathrm{GL}^2(\infty, \mathbb{R})$. Then $H := \mathrm{GL}^2(\infty, \mathbb{R})/\mathrm{O}^2(\infty)$ can be given the structure of a symmetric Cartan–Hadamard manifold on which $\mathrm{GL}^2(\infty, \mathbb{R})$ acts by isometries; see [**Lar**]. In particular, H has bounded non-positive curvature in the sense of Alexandrov (see [**KM**, §7] for more on this; see also Remarks B.1 and 4.1). Hence Theorems A and B apply to this space. In particular, Theorem 1.2 extends to cocycles of this kind of operators.

Remark 2.18. Consider now the space $H = \mathrm{GL}(\infty, \mathbb{R})/\mathrm{O}(\infty)$, where $\mathrm{GL}(\infty, \mathbb{R})$ is the group of all bounded invertible operators on $\mathcal{H}$, and $\mathrm{O}(\infty)$ is the orthogonal subgroup. It is possible (see [**CPR, LL**]) to metrize H so it becomes a Busemann space, besides being a symmetric space and a Banach manifold; however, the resulting space is not CAT(0). Therefore, Theorem A applies to the space H. However, we do not know whether Theorem B applies to this space, or whether Theorem 1.2 applies to $\mathrm{GL}(\infty, \mathbb{R})$ cocycles.

3. *Cocycles over other group actions*

We now consider other (still discrete) group actions. Before going into the proofs of our results, let us make an observation about the construction of almost invariant sections.

Let Γ be a (not necessarily abelian) group acting by homeomorphisms of a compact Hausdorff metric space Ω. Let A be a cocycle over this group action with values in the group of isometries of a Busemann space H.

Suppose that C_N is a sequence of finite subsets of Γ. Fix $p_0 \in \Omega$ and define a sequence of sections $\varphi_N : \Omega \to H$ by

$$\varphi_N(\omega) := \mathrm{bar}((A^{(h)}(\omega))^{-1} p_0 : h \in C_N), \tag{3.1}$$

where bar is given by Theorem 2.2 (compare with (2.6)). Now fix any $g \in \Gamma$. By equivariance of the barycenter,

$$A^{(g)}(\omega)\varphi_N(\omega) := \mathrm{bar}((A^{(h)}(g\omega))^{-1} p_0 : h \in C_N \cdot g^{-1}). \tag{3.2}$$

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By property (2.4) of the barycenter map,

$$d(A^{(g)}(\omega)\varphi_N(\omega), \varphi_N(g\omega)) \le \frac{n}{|C_N|} \max_j d((A^{(h_j)}(g\omega))^{-1} p_0, (A^{(h'_j)}(g\omega))^{-1} p_0)$$
$$\le \frac{n}{|C_N|} \max_j d(A^{(h'_j h_j^{-1})}(h_j g\omega) p_0, p_0), \tag{3.3}$$

where n is the cardinality of the union of $(C_N \cdot g^{-1}) \setminus C_N$ and $C_N \setminus (C_N \cdot g^{-1})$, which are enumerated as $\{h_1, \dots, h_n\}$ and $\{h'_1, \dots, h'_n\}$, respectively.

3.1. *Proof of Theorem E.* We first consider the case where $\Gamma = \mathbb{Z}^d$. Let $A\colon \Omega \times \mathbb{Z}^d \to \mathrm{Isom}(H)$ be a cocycle of isometries of a Busemann space H. Assume that A has uniform sublinear growth along cyclic subgroups. We need to exhibit a sequence of continuous maps $\varphi_N : \Omega \to H$ such that for all $i = 1, \dots, d$,

$$\lim_{N\to\infty} d(A^{(e_i)}(\omega)\varphi_N(\omega), \varphi_N(e_i\omega)) = 0 \quad \text{uniformly on } \omega \in \Omega, \tag{3.4}$$

where $e_i := (0, \dots, 0, 1, 0, \dots, 0)$ is the ith canonical generator of $\mathbb{Z}^d$.

To do this, consider the sequence of 'cubes'

$$C_N := \{(m_1, \dots, m_d) \in \mathbb{Z}^d : 0 \le m_j < N\},$$

and define φ_N by (3.1). The sets $(C_N - e_i) \setminus C_N = \{h_j\}$ and $C_N \setminus (C_N - e_i)$ have cardinality $n = N^{d-1} = |C_N|/N$. Moreover, they can be enumerated respectively as $\{h_1, \dots, h_n\}$ and $\{h'_1, \dots, h'_n\}$ in a way such that $h'_j - h_j = Ne_i$. Then (3.3) gives

$$d(A^{(e_i)}(\omega)\varphi_N(\omega), \varphi_N(e_i + \omega)) \le \frac{2}{N} \max_j d(A^{(Ne_i)}((h_j + e_i)\omega) p_0, p_0).$$

Since A has sublinear drift along the cyclic subgroup generated by e_i, (3.4) follows. This proves the theorem in the case $\Gamma = \mathbb{Z}^d$.

Now consider the general case where Γ is finitely generated and abelian. Let $\Gamma = \Gamma_0 \oplus \mathbb{Z}^d$ be the torsion decomposition, where Γ_0 is the torsion subgroup. Consider the sequence of sets

$$C_N := \Gamma_0 \oplus \{(m_1, \dots, m_d) \in \mathbb{Z}^d : 0 \le m_j < N\},$$

and define φ_N by (3.1). If $g \in \Gamma_0$ then $C_N - g = C_N$, and thus (3.2) gives $A^{(g)}(\omega)\varphi_N(\omega) = \varphi_N(g\omega)$. On the other hand, if $g = e_i$ then we can estimate as before $d(A^{(e_i)}(\omega)\varphi_N(\omega), \varphi_N(e_i\omega)) = o(N)$. So φ_N is a sequence of almost invariant sections, as desired. □

3.2. *Generalization to virtually nilpotent group actions.* We close this section with a further generalization of Theorem E for cocycles over virtually nilpotent group actions.

Let Γ be a finitely generated group acting on a compact space Ω, and let A be a cocycle of isometries of a space (H, d) over this action. We say that A has *uniform sublinear drift* if for each fixed $p_0 \in H$,

$$\sup_{\omega\in\Omega} d(A^{(g)}(\omega) p_0, p_0) = o(\ell(g)),$$

where ℓ denotes word length with respect to some finite system of generators.

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THEOREM 3.1. *Let Γ be a finitely generated virtually nilpotent group acting by homeomorphisms of a compact Hausdorff metric space Ω. Let A be a cocycle over this group action with values in the group of isometries of a Busemann space H. If A has uniform sublinear drift then A admits almost invariant sections.*

Proof. We follow an argument of [**CTV**]. Since Γ is virtually nilpotent, it has polynomial growth (with respect to any finite system of generators). Denoting by $B(n)$ the ball of radius n in Γ, we claim that there exist $D > 0$ and an increasing sequence of integers k_N such that for all N,

$$\frac{|B(k_N + 1) \setminus B(k_N)|}{|B(k_N)|} \leq \frac{D}{k_N}. \tag{3.5}$$

Otherwise, for each $D > 0$ there would exist positive constants C, C', C'' such that

$$|B(k)| \geq C \prod_{j=1}^{k-1} \left(1 + \frac{D}{j}\right) \geq C' \exp\left(\sum_{j=1}^{k-1} \frac{D}{j}\right) \geq C'' k^D,$$

thus contradicting polynomial growth.

Now fix $p_0 \in \Omega$, let $C_N := B(k_N)$, and define $\varphi_N : \Omega \to H$ by (3.1). Let g be a generator of Γ. Then, by (3.5),

$$n := |(C_N \cdot g^{-1}) \setminus C_N| \leq |B(k_N + 1) \setminus B(k_N)| \leq \frac{D}{k_N} |C_N|.$$

Therefore, (3.3) gives

$$\sup_{\omega \in \Omega} d(A^{(g)}(\omega)\varphi_N(\omega), \varphi_N(g\omega)) \leq \frac{D}{k_N} \sup_{\omega \in \Omega} \max_{h \in B(2k_N+1)} d(A^{(h)}(\omega)p_0, p_0),$$

which converges to 0 as $N \to \infty$. We conclude that φ_N is a sequence of almost invariant sections. □

Together with Theorem 3.1, the next general proposition shows that Theorem E extends to virtually nilpotent groups.

PROPOSITION 3.2. *Let A be a cocycle of isometries of a space H over a group action by homeomorhisms on a space Ω. If Γ is virtually nilpotent, then A has zero drift along cyclic subgroups if and only if it has uniform sublinear drift.*

Proof. Without loss of generality, we may assume that Γ is torsion-free and nilpotent. As is well known, such a group is boundedly generated in a strong form: there exist a generating system $\mathcal{G} = \{h_1, \ldots, h_k\}$ and a constant C such that every element $h \in \Gamma$ can be written as $h = h_{i_1}^{n_1} \cdots h_{i_m}^{n_m}$, where each h_{i_j} belongs to $\mathcal{G}$, $m \leq C$ and $|i_j| \leq C\ell(h)$. (In the torsion-free case, this follows, for instance, from [**BGr**, Appendix B].) Using this fact, the direct implication follows easily. The converse is straightforward and we leave it to the reader. □

Remark 3.3. For the case where H is the real line and the cocycle is by translations, this yields an alternative (and simpler) proof of [**MOP2**, Théorème 2].

Remark 3.4. We do not know whether Theorem B may also be extended to (finitely generated) abelian or virtually nilpotent group actions. The difficulty in adapting the proof

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is that the group relations must be preserved. Of course, if we consider Γ as a quotient of the free group $\mathbb{F}_k$, where k is the number of generators, and the action is lifted to $\mathbb{F}_k$, then the cocycle can be perturbed (as a cocycle above the $\mathbb{F}_k$-action) so that it has a continuous invariant section.

4. *Continuous-time cocycles*

In this section we prove the continuous-time Theorems C and D.

4.1. *Preliminaries.*

4.1.1. *Cartan–Hadamard manifolds.* Assume that H is a Hilbert manifold, that is, a separable C^∞ manifold modeled on a separable real Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Fix a Riemannian metric on H. (See [**Lang**] for the precise definition.)

If H is complete, simply connected, and has non-positive sectional curvature, then H is called a *Cartan–Hadamard manifold*. In this case, the Cartan–Hadamard–McAlpin theorem (see [**Lang**, §IX.3]) states that for each point $p \in H$, the exponential map $\exp_p : T_pH \to H$ is a diffeomorphism.

Remark 4.1. Let $\kappa \le 0$. If H is complete, simply connected, and has sectional curvature $\le\kappa$ (respectively, $\ge\kappa$) everywhere, then H has curvature $\le\kappa$ (respectively, $\ge\kappa$) in the sense of Alexandrov; see [**BBI**, Ch. 6].

4.1.2. *Killing fields and symmetric Cartan–Hadamard manifolds.* Here we recall some general facts about symmetric manifolds and Killing fields; more information can be found in [**Lang**, Ch. XIII].

If H is a Hilbert manifold, a *Killing field* is a vector field that generates a (globally defined) flow of isometries. Then the flow also preserves the Riemannian connection. On the space $\mathrm{Kill}(H)$ of these fields, we consider the first-countable topology for which the convergence of sequences is uniform convergence on bounded subsets.

We endow $C(\Omega, \mathrm{Kill}(H))$ with the compact-open topology. Then a sequence $(\mathfrak{a}_n)$ in $C(\Omega, \mathrm{Kill}(H))$ converges to $\mathfrak{a}$ if and only if for every bounded set $B \subset H$, $\|\mathfrak{a}_n(\omega)(p) - \mathfrak{a}(\omega)(p)\|$ converges to 0 uniformly with respect to $(\omega,\ p) \in \Omega \times B$.

Now let H be a Cartan–Hadamard manifold. Suppose that it is symmetric in the sense of §2.4.3†. If $v_0 \in T_{p_0}H$ is a non-zero vector, let $\alpha : \mathbb{R} \to H$ be the geodesic such that $\alpha(0) = p_0$, $\alpha'(0) = v_0$. Consider the transvection

$$\tau_{\alpha,s} := \sigma_{\alpha(s/2)} \circ \sigma_{\alpha(0)}.$$

Then $\tau_{\alpha,s}$ is a flow of isometries called the *translation along* α. More precisely, $\tau_{\alpha,s}(\alpha(t)) = \alpha(t+s)$. Moreover, the derivative

$$T_{\alpha(t)}\tau_{\alpha,s} : T_{\alpha(t)}H \to T_{\alpha(t+s)}H$$

is the parallel transport along the geodesic α. Let ξ_{v_0} denote the Killing field that generates the flow $\tau_{\alpha,s}$. (For $v_0 = 0$, we define $\xi_{v_0} \equiv 0$.) Then the map $v_0 \in TH \mapsto \xi_{v_0} \in \mathrm{Kill}(H)$ is continuous.

† It is easy to check that this agrees with the definition from [**Lang**, p. 359].

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As is customary, we denote by $\mathfrak{m}_{p_0}$ the set of Killing fields ξ_{v_0}, where $v_0 \in T_{p_0}H$. This is a vector space, and it can also be expressed as

$$\mathfrak{m}_{p_0} = \{\xi \in \mathrm{Kill}(H) : \nabla_\zeta \xi(p) = 0 \text{ for all vector fields } \zeta\},$$

where ∇ denotes the covariant derivative.

4.1.3. *Infinitesimal displacement estimates.* The following lemma is the infinitesimal counterpart of Lemmas 2.5 and 2.8.

LEMMA 4.2. *Let H be a symmetric Cartan–Hadamard manifold. There is a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 1$ such that*

$$\|v_0\| \overset{\text{(I)}}{\leq} \|\xi_{v_0}(p)\| \overset{\text{(II)}}{\leq} f(d(p,\, p_0))\|v_0\|$$

for all $p_0,\ p \in H,\ v_0 \in T_{p_0}H$.

Inequality (I) above is related to non-positive curvature; let us prove it first.

Proof of part (I) in Lemma 4.2. Fix $p_0,\ p$ in H, $v_0 \in T_{p_0}H$. Assume that $v_0 \neq 0$, otherwise there is nothing to prove. Let β be a unit-speed geodesic joining p_0 and p. Let $\eta(t) := \xi_{v_0}(\beta(t))$. Then (see [**Lang**, Proposition 2.2, Ch. XIII]) η is a Jacobi field over the geodesic β. By [**Lang**, Proposition 5.6, Ch. XIII], we have $\nabla_{\beta'}\eta(0) = 0$. Let $g(t) := \|\eta(t)\|^2$. By non-positive curvature, this function is convex; see [**Lang**, Lemma 1.1, Ch. X]. The same lemma also says that $g' = 2\langle \nabla_{\beta'}\eta,\, \eta\rangle$, which vanishes at $t = 0$. It follows that $g(t) \geq g(0)$ for all $t \in \mathbb{R}$. In particular, $\|\xi_{v_0}(p)\|^2 \geq \|\xi_{v_0}(p_0)\|^2 = \|v_0\|^2$, thus completing the proof of inequality (I). □

If H is finite-dimensional then the existence of a function with property (II) in Lemma 4.2 is nearly trivial, and does not rely on non-positive curvature:

Proof of part (II) in Lemma 4.2 assuming $\dim H < \infty$. Consider

$$\tilde{f}(p_0,\, \ell) := \sup\{\|\xi_{v_0}(p)\| : p \in H \text{ with } d(p,\, p_0) \leq \ell,\ v_0 \in T_{p_0}H \text{ with } \|v_0\| = 1\},$$

which is finite by compactness. Since $\mathrm{Isom}(H)$ acts transitively on H, the value $\tilde{f}(p_0,\, \ell)$ actually does not depend on p_0, and so defines a function $f(p_0)$ with the required properties. □

The proof of (II) in the infinite-dimensional case requires geometric arguments and is given in Appendix B.

4.1.4. *Infinitesimal uniform homogeneity.* The following is an infinitesimal version of the macroscopic uniform homogeneity (Lemma 2.10). It basically says that we can move any point p in any desired direction w by an infinitesimal isometry (Killing field), and these fields can be chosen so that they converge uniformly (with respect to p) in bounded sets to zero as $\|w\| \to 0$.

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LEMMA 4.3. (Infinitesimal uniform homogeneity) *Let $p_0 \in H$. There is a continuous map*

$$K : TH \to \mathrm{Kill}(H)$$
$$w \mapsto K_w$$

with the following properties.

(a) *The vector field K_w extends w, that is, if $p = \pi(w) \in H$ is the base point of w, then $K_w(p) = w$.*

(b) *For any $q \in H$,*

$$\|K_w(q)\| \le f(d(q, p_0))\|w\|,$$

where f is given by Lemma 4.2.

Proof. Let $p_0 \in H$ be fixed. For $p \in H$, consider the map

$$L_p : T_{p_0}H \to T_pH$$
$$v_0 \mapsto \xi_{v_0}(p).$$

We list below some properties of L_p:

- it is linear; see [**Lang**, p. 363];
- it is continuous; see part (II) of Lemma 4.2;
- it is one-to-one, and the inverse (on the image) is continuous; see part (I) of Lemma 4.2;
- it is onto; see Lemma B.2 in Appendix B. (In finite dimension, this part would of course be a trivial consequence of the others.)

Given $w \in T_pH$, define $K_w := \xi_{(L_p)^{-1}(w)}$. Using parts (II) and (I) of Lemma 4.2, we have

$$\|K_w(q)\| \le f(d(q, p_0))\|K_w(p_0)\| \le f(d(q, p_0))\|w\|. \qquad \square$$

Remark 4.4. Although there is no apparent advantage in doing so, it is possible to give an alternative proof of Lemma 2.10 using Lemma 4.3: given two points p and q, join them by a geodesic $\gamma : [0, \ell] \to H$, and integrate the time-varying Killing field $K_{\gamma'(t)}$ (where K is given by Lemma 4.3) to get the map $J(p, q)$.

4.2. *Existence of sections of nearly minimal speed: proof of Theorem C.* For each $p \in H$, we let δ_p or $\delta(p)$ denote the Dirac measure at the point p. If $\gamma : [a, b] \to H$ is a curve, we denote by $\int_a^b \delta(\gamma(t))\, dt$ the measure on H obtained by pushing forward by γ the Lebesgue measure on $[a, b]$.

In the proof of Theorem C, we will need the following technical result, whose proof is given in Appendix C.

LEMMA 4.5. (Differentiability of the Cartan barycenter) *Let $I \subset \mathbb{R}$ be an open interval and let $h : I \times [0, T] \to H$ be a continuous mapping that is continuously differentiable with respect to the first variable. Then the map $\bar{h} : I \to H$ defined by*

$$\bar{h}(t) = \mathrm{bar}\left(\frac{1}{T}\int_0^T \delta_{h(t,s)}\, ds\right) \tag{4.1}$$

(where bar *denotes the Cartan barycenter) is continuously differentiable.*

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Proof of Theorem C. Fix any $p_0 \in H$. For $T > 0$, let

$$\varphi_T(\omega) = \mathrm{bar}\left(\frac{1}{T}\int_0^T \delta([A^{(t)}(\omega)]^{-1} p_0)\, dt\right). \tag{4.2}$$

(Compare with (2.6).)

CLAIM. *The function $\varphi_T : \Omega \to H$ is differentiable with respect to the semiflow.*

Proof of the claim. We have

$$\varphi_T(F^t\omega) = \mathrm{bar}\left(\frac{1}{T}\int_0^T \delta\left([A^{(s)}(F^t\omega)]^{-1} p_0\right)\, ds\right).$$

By Lemma 4.5, to show that the map $\varphi_T(F^t\omega)$ is continuously differentiable with respect to t, it suffices to check that the map $(s, t) \mapsto [A^{(s)}(F^t\omega)]^{-1} p_0 \in H$ is continuous and continuously differentiable with respect to t. But these properties follow from the cocycle identity (1.9) and the regularity assumptions (1.10). □

Next, we want to estimate the distance

$$d(\varphi_T(F^t\omega),\, A^{(t)}(\omega)\varphi_T(\omega)) = d(\underbrace{[A^{(t)}(\omega)]^{-1}(\varphi_T(F^t\omega))}_{(\star)},\, \varphi_T(\omega)). \tag{4.3}$$

Assuming $t \in (0, T)$, we have

$$\begin{aligned}(\star) &= \mathrm{bar}\,\frac{1}{T}\int_0^T \delta([A^{(t)}(\omega)]^{-1}[A^{(s)}(F^t\omega)]^{-1} p_0)\, ds \\ &= \mathrm{bar}\,\frac{1}{T}\int_0^T \delta([A^{(s+t)}(\omega)]^{-1} p_0)\, ds \\ &= \mathrm{bar}\,\frac{1}{T}\int_t^{T+t} \delta([A^{(s)}(\omega)]^{-1} p_0)\, ds.\end{aligned}$$

Using the barycenter property (2.3), we obtain that the distance (4.3) is at most

$$\frac{t}{T}\sup\{d([A^{(s)}(\omega)]^{-1} p_0,\, [A^{(u)}(\omega)]^{-1} p_0) : s \in [0,\, t],\, u \in [T,\, T + t]\}.$$

Dividing by t and letting $t \to 0$, we obtain

$$\|\varphi_T'(\omega) - \mathfrak{a}(\omega)(\varphi_T(\omega))\| \le \frac{1}{T} d(A^{(T)}(\omega) p_0,\, p_0).$$

In particular,

$$\mathrm{speed}(\varphi_T) \le \sup_{\omega\in\Omega} \frac{1}{T} d(A^{(T)}(\omega) p_0,\, p_0).$$

The theorem follows by taking $\varphi = \varphi_N$ with sufficiently large N. □

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4.3. *Creating invariant sections: proof of Theorem D.*

Proof of Theorem D. By Theorem C, there exists a family of sections φ_T such that $\lim_{T\to\infty} \mathrm{speed}(\varphi_T) = 0$. Let $K(w) = K_w$ be the map given by Lemma 4.3. Define

$$\tilde{\mathfrak{a}}_T(\omega) = \mathfrak{a}(\omega) + K(\varphi_T'(\omega) - \mathfrak{a}(\omega)(\varphi_T(\omega))). \tag{4.4}$$

(Compare with (2.11).) Then:

- $\tilde{\mathfrak{a}}_T(\omega)(\varphi_T(\omega)) = \varphi_T'(\omega)$, that is, φ_T is an invariant section for the cocycle generated by $\tilde{\mathfrak{a}}_T$;
- for each bounded subset B of H, the sequence $\tilde{\mathfrak{a}}_T(\omega)(p)$ converges to $\mathfrak{a}(\omega)(p)$ uniformly with respect to $(\omega,\ p) \in \Omega \times B$.

Thus the theorem is proved. □

Remark 4.6. Similarly to Remark 2.15, it should be possible to state and prove fibered versions of Theorems C and D, but we have not checked this. It seems to be necessary to use a connection on the bundle in order to define the speed of a section.

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A. *Appendix. The displacement estimate for non-proper spaces*

In this appendix we prove Lemma 2.8. We will actually obtain an explicit formula for the function f. Some preliminaries are needed.

A.1. *Angles and more comparisons.*

PROPOSITION A.1. *Let H be a space of curvature $\le\kappa$ (respectively, $\ge\kappa$), with $\kappa \le 0$. Let γ_1, γ_2 be two geodesics such that $\gamma_1(0) = \gamma_2(0) = p_0$. For each $t > 0$, $s > 0$, consider the triangle $\triangle(p_0, \gamma_1(t), \gamma_2(s))$, and let $\triangle(\tilde{p}_0, \tilde{p}_{1,t}, \tilde{p}_{2,s})$ be an SSS-comparison triangle in M_κ. Let $\theta_\kappa(t, s)$ be the angle at the vertex $\tilde{p}_0$. Then the function $\theta_\kappa(t, s)$ is monotonically non-decreasing (respectively, non-increasing) with respect to each variable.*

Proof. See [**ABN**]. □

In particular, the limit of $\theta_\kappa(t, s)$ as $(t, s) \to (0, 0)$ exists. (Notice that the limit actually does not depend on κ.) It is called the *angle* between γ_1 and γ_2 at p_0.

An immediate consequence of Proposition A.1 is that if H has curvature $\le\kappa$ (respectively, $\ge\kappa$), with $\kappa \le 0$, then the angles of an SSS-comparison triangle in M_κ are smaller (respectively, greater) than or equal to the corresponding angles for the triangle in H.

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If $\triangle(p_1, p_2, p_3)$ is a triangle in H and $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ is a triangle in M_κ such that the angles at the vertices p_1 and $\tilde{p}_1$ are equal and the corresponding sides at these vertices have equal lengths (i.e., $d_\kappa(\tilde{p}_1, \tilde{p}_j) = d(p_1, p_j)$ for $j = 2, 3$), then we say that $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ is an *SAS-comparison triangle*† for $\triangle(p_1, p_2, p_3)$.

LEMMA A.2. *Let H be a uniquely geodesic space of curvature $\geq\kappa$ (where $\kappa \leq 0$). Let $\triangle(p_1, p_2, p_3)$ be a triangle in H, and let $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ be an SAS-comparison triangle in M_κ (with equal angles at vertices p_1 and $\tilde{p}_1$). Then:*

(a) *the side p_2p_3 is shorter than or has the same length as the side $\tilde{p}_2\tilde{p}_3$;*

(b) *if the angles at p_2 and p_3 are less than $\pi/2$, then the angle at p_2 (respectively, p_3) is larger than or equal to the angle at $\tilde{p}_2$ (respectively, $\tilde{p}_3$).*

Proof. Consider a triangle $\triangle(p_1, p_2, p_3)$ in H (which we assume has curvature $\geq\kappa$). In M_κ, we take an SAS-comparison triangle $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ (so that the angles at p_1 and $\tilde{p}_1$ are equal) and an SSS-comparison triangle $\triangle(q_1, q_2, q_3)$. Then the angle at q_i is less than the angle at p_i. Thus, to complete the proof of the lemma, we need the following facts about plane hyperbolic geometry, whose proof we will leave as an exercise.

CLAIM. *Suppose $\triangle(q_1, q_2, q_3)$ and $\triangle(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ are triangles in M_κ (where $\kappa \leq 0$) so that the angle at vertex $\tilde{p}_1$ is bigger than the angle at vertex q_1, and the adjacent sides are equal. Then:*

(a) *the side $\tilde{p}_2\tilde{p}_3$ is longer than the side q_2q_3;*

(b) *if the angles at $\tilde{p}_2$ and $\tilde{p}_3$ are both less than $\pi/2$, then they are smaller than the angles at q_2 and q_3, respectively.* □

A.2. *The displacement estimate.* The following is a more precise version of Lemma 2.8.

LEMMA A.3. *Assume that H is a geodesic symmetric space of bounded non-positive curvature in the sense of Alexandrov. Let $\kappa < 0$ be a lower bound for the curvature. Let J be a transvection that translates a geodesic γ by length b. Given $q \in H$, let $s = d(J(q), q)$, $\ell = d(q, \gamma)$, and $\lambda = \sqrt{-\kappa}$. Then*

$$\cosh \lambda s \leq \cosh^2 \frac{\lambda b}{2} \cosh^2 2\lambda\ell + \sinh^2 \frac{\lambda b}{2} \cosh 2\lambda\ell - \cosh \frac{\lambda b}{2} \sinh^2 2\lambda\ell. \tag{A.1}$$

Proof. Multiplying the metric by a constant, we can assume that $\kappa = -1$, that is, $\lambda = 1$.

Let J be a transvection that translates a geodesic γ by length b, and let $q \in H$. Let $s = d(J(q), q)$ and $\ell = d(q, \gamma)$. We can assume that $b > 0$ (because for $b = 0$ formula (A.1) means $s = 0$) and $\ell > 0$ (because for $\ell = 0$ formula (A.1) means $s \leq b$).

Let p_0 be the point in γ which is closest to q. Let p_1 be the midpoint of p_0 and $J(p_0)$; then $J = \sigma_{p_1} \circ \sigma_{p_0}$. Consider the triangle with vertices p_0, p_1, and $\sigma_{p_0}(q)$. The angle at vertex p_0 is $\pi/2$; let α and β the angles at vertices p_1 and $\sigma_{p_0}(q)$, respectively. See Figure A.1.

Now consider an SAS-comparison triangle in the hyperbolic plane M_{-1}, more precisely a triangle in M_{-1} with two sides $b/2$ and ℓ and angle $\pi/2$ between them. Let $\tilde{\alpha}$, $\tilde{\beta}$ be the

† SAS stands for side–angle–side.

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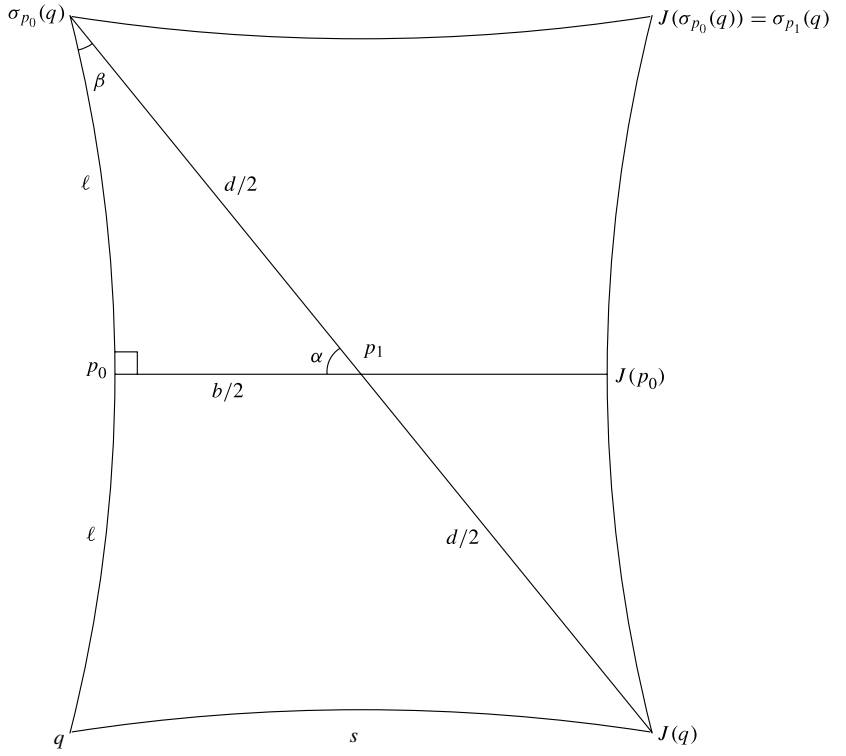


FIGURE A.1. Proof of Lemma A.3.

respective angles, and let $\tilde{d}/2$ be the third side. By Lemma A.2,

$$d \le \tilde{d}, \quad \alpha \ge \tilde{\alpha}, \quad \beta \ge \tilde{\beta}.$$

By the law of cosines in M_{-1},

$$\cosh \frac{\tilde{d}}{2} = \cosh \frac{b}{2} \cosh \ell. \tag{A.2}$$

By the law of sines in M_{-1},

$$\sinh \frac{\tilde{d}}{2} = \frac{\sinh \ell}{\sin \tilde{\alpha}}. \tag{A.3}$$

By the law of cosines in H (an inequality which comes automatically from the curvature lower bound),

$$\begin{aligned}
\cosh s &\le \cosh d \cosh 2\ell - \sinh d \sinh 2\ell \cos \beta \quad \text{(law of cosines)} \\
&\le \cosh d \cosh 2\ell - \sinh d \sinh 2\ell \sin \alpha \quad (\text{since } \alpha + \beta \le \pi/2) \\
&\le \cosh d \cosh 2\ell - \sinh d \sinh 2\ell \sin \tilde{\alpha} \quad (\text{since } \tilde{\alpha} \le \alpha \le \pi/2) \\
&\le \cosh \tilde{d} \cosh 2\ell - \sinh \tilde{d} \sinh 2\ell \sin \tilde{\alpha} \quad (\text{since } \tilde{d} \ge d \ge 2\ell) \\
&= \left(2 \cosh^2 \frac{\tilde{d}}{2} - 1\right) \cosh 2\ell - 2 \sinh \frac{\tilde{d}}{2} \cosh \frac{\tilde{d}}{2} \sinh 2\ell \sin \tilde{\alpha}.
\end{aligned}$$

Substituting (A.2) and (A.3) and manipulating, we obtain (A.1). □

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Remark A.4. Figure A.1 is not necessarily contained in a 'two-dimensional' totally geodesic subspace. If this were the case, it is possible to show that the following improved version of (A.1) holds: $\cosh \lambda s \leq \cosh \lambda b \cosh^2 \lambda \ell - \sinh^2 \lambda \ell$. Moreover, if $H = M_{-\lambda^2}$, then this becomes an equality, expressing the summit s of a Saccheri quadrilateral as a function of the legs ℓ and the base b; see [**BK**, p. 104].

B. *Appendix. Some lemmas on Killing fields*

In this appendix, we complete the proofs of Lemmas 4.2 and 4.3, which were proven in §4.1 only in the finite-dimensional case.

Remark B.1. The sectional curvatures of a symmetric Cartan–Hadamard manifold are bounded from below. Indeed, since isometries act transitively, it is sufficient to show that sectional curvatures are bounded at each point; but this follows directly from the boundedness of the Riemann tensor.

Proof of part (II) of Lemma 4.2. Let κ be the infimum of the sectional curvature of H, which is finite by the previous remark. We will show that (II) holds with

$$f(\ell) := \cosh(2\sqrt{-\kappa} \cdot \ell).$$

Fix p_0, p in H, $v_0 \in T_{p_0}H$. Assume $v_0 \neq 0$, otherwise there is nothing to prove. Let α be the geodesic passing through p_0 with velocity v_0. Let β be a unit-speed geodesic joining p_0 and p. Let $\eta(t) = \xi_{v_0}(\beta(t))$. Then (see [**Lang**, Proposition 2.2, Ch. XIII]) η is a Jacobi field over the geodesic β.

In view of the Rauch–Berger comparison theorem (see [**Bi, CE**]), in order to show that

$$\|\eta(t)\| \leq f(d(\beta(t), \alpha))\|v_0\| \leq f(t)\|v_0\|, \tag{B.1}$$

we need only consider the case where H is the hyperbolic plane of constant curvature κ. But then it is a simple calculation; actually, in this case, the first inequality in (B.1) becomes an equality. □

LEMMA B.2. *Let H be a symmetric Cartan–Hadamard manifold. For any p_0, p in H, $v \in T_pH$, there exists $\chi \in \mathfrak{m}_{p_0}$ such that $\chi(p) = v$.*

Proof. Let $\ell = d(p, p_0)$. Assume $\ell > 0$, otherwise the claim is trivial. Let $\beta : \mathbb{R} \to H$ be the geodesic such that $\beta(0) = p_0$ and $\beta(\ell) = p$. Let $q = \beta(-\ell)$. There is a Jacobi vector field η over β such that

$$\eta(\ell) = v, \quad \eta(-\ell) = 0.$$

(The existence of η follows from the Cartan–Hadamard–McAlpin theorem; see [**Lang**, §IX.3 and Theorem IX.3.1].) We claim that $\xi_{2\eta(0)}$ is the sought-after Killing field.

Let $\sigma = \sigma_{p_0}$ be the symmetry at p_0; then $\sigma(\beta(t)) = \beta(-t)$. Let ζ be the Jacobi field over β obtained by pushing forward η by σ, that is,

$$\zeta(t) = T\sigma(\beta(-t)) \cdot \eta(-t).$$

Consider the Jacobi field $\chi = \eta - \zeta$. (See Figure B.1.)

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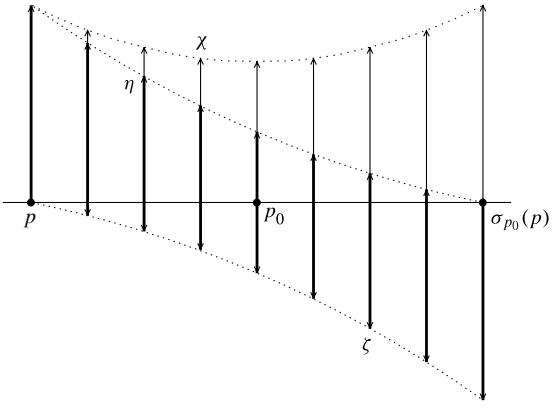


FIGURE B.1. The Jacobi field η, its reflection ζ, and the field $\chi = \eta - \zeta$.

The isometry σ preserves covariant derivatives, hence

$$\nabla_{\beta'}\zeta(t) = -T\sigma(\beta(-t)) \cdot \nabla_{\beta'}\eta(-t).$$

Taking $t = 0$, we get $\nabla_{\beta'}\chi(0) = 0$. As a consequence (use [**Lang**, Proposition XIII.5.6]), the Jacobi field χ over β can be extended to a Killing field $\chi \in \mathfrak{m}_{p_0}$. This is exactly $\xi_{\chi(0)} = \xi_{2\eta(0)}$. □

C. *Appendix. Differentiability of the Cartan barycenter*

In this appendix, we give the proof of Lemma 4.5.

Assume H is a Cartan–Hadamard manifold. Let μ be a probability measure of bounded support on H, and let $f = f_\mu$ be given by (2.5).

LEMMA C.1. *The gradient vector field*† *of f is given by*

$$\operatorname{grad} f(p) = -\int_H \exp_p^{-1}(q)\, d\mu(q)\,.$$

Proof. See [**BK**, p. 132]. □

LEMMA C.2. *For each p, the linear map*

$$L : v \in T_pM \mapsto \nabla_v \operatorname{grad} f(p) \in T_pM$$

(given by the covariant derivative) is bounded and has a bounded inverse.

Proof. By the previous lemma, $\operatorname{grad} f = \int_H \xi_q\, d\mu(q)$, where ξ_q is the vector field $\xi_q(p) = -\exp_p^{-1}(q)$. For each q and p, the linear map $v \in T_pM \mapsto \nabla_v \xi_q(p) \in T_pM$ is symmetric and $\geq \mathrm{Id}$; see [**Kar**, pp. 172, 188]. Integrating over q, we conclude that L is self-adjoint and $\geq \mathrm{Id}$.

We now need the following claim. Once it is proved, the lemma follows.

† Defined by the relation $Tf_p(v) = \langle \operatorname{grad} f(p), v\rangle$.

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CLAIM. *Let L be a bounded self-adjoint operator on a real Hilbert space such that $L \geq \mathrm{Id}$. Then L is invertible, with a bounded inverse.*

Proof of the claim. Let $c > 0$ and estimate

$$\|(\mathrm{Id} - cL)v\|^2 = \langle v, v\rangle - 2c\langle Lv, v\rangle + c^2\langle Lv, Lv\rangle \leq (1 - 2c + c^2\|L\|^2)\|v\|^2.$$

Thus if c is small enough, then $\|\mathrm{Id} - cL\| < 1$. In particular, cL is invertible, and so is L. □

Lemma C.2 follows. □

Proof of Lemma 4.5. Define a one-parameter family of vector fields ξ_t ($t \in I$) on H by

$$\xi_t(p) = -\frac{1}{T}\int_0^T \exp_p^{-1}(h(t, s))\, ds.$$

Then $\bar{h}(t)$ is the unique solution of $\xi_t(\bar{h}(t)) = 0$. To deduce differentiability of $\bar{h}$ from the implicit function theorem, it is sufficient to check that for each $t \in H$ and $p \in H$, the linear map $v \in T_pM \mapsto \nabla_v\xi_t(p) \in T_pM$ (given by the covariant derivative) is invertible (with a bounded inverse)—see [**BK**, p. 143] for details. But this was proved in Lemma C.2 above. □

Remark C.3. Another approach to the differentiability of the barycenter is to consider barycenters in the tangent bundle TH; see [**AL**].

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