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SQUARE-ROOT RULE OF TWO-DIMENSIONAL BANDWIDTH PROBLEM*

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Abstract. The bandwidth minimization problem is of significance in network communication and related areas. Let G be a graph of n vertices. The two-dimensional bandwidth $B_2(G)$ of G is the minimum value of the maximum distance between adjacent vertices when G is embedded into an $n \times n$ grid in the plane. As a discrete optimization problem, determining $B_2(G)$ is NP-hard in general. However, exact results for this parameter can be derived for some special classes of graphs. This paper studies the "square-root rule" of the two-dimensional bandwidth, which is useful in evaluating $B_2(G)$ for some typical graphs.

Mathematics Subject Classification. 05C78, 68R10.

1. Introduction

The bandwidth minimization problem of graphs and its variants have significant background in sparse matrix computation, circuit layout designs, and network communication. Especially, the *two-dimensional bandwidth problem* has close relations to the VLSI layout designs and parallel algorithm simulations [1,2,6,12].

Keywords and phrases. Network layout, two-dimensional bandwidth, lower and upper bounds, optimal embedding.

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Let us first formulate the problem as follows. Given a simple graph $G = (V(G), E(G))$ on n vertices, a bijection $f : V(G) \to \{1, 2, \cdots, n\}$ is called a *labelling* or *one-dimensional embedding* of G. This can be thought of as an embedding of G into a path P_n of n vertices. The bandwidth of f for G is $B(G, f) = \max\{|f(u)-f(v)| : uv \in E(G)\}$ and the bandwidth of G is $B(G) = \min\{B(G, f) : f \text{ is a labeling of } G\}$. The bandwidth problem in graph theory has been extensively studied in over 50 years (see surveys [4–6]). A generalization of the bandwidth problem is the *two-dimensional bandwidth problem* stated below.

First, $\{1, 2, \cdots, N\} \times \{1, 2, \cdots, N\}$ is called an $N \times N$ grid in the plane. The cartesian product $H = P_N \times P_N$ is a graph with vertex set $V(H) = \{1, 2, \cdots, N\} \times \{1, 2, \cdots, N\}$ and with vertical and horizontal edges joining adjacent lattice points, which is called a *grid graph* in the plane (see the next section for a formal definition). In circuit layout models, the wires are usually placed along the vertical and horizontal directions. So the distance in the grid graph usually means the rectilinear distance, namely the L_1-norm distance. In other words, the distance between two points $(i, j), (i', j') \in V(H)$ is

$$d_{L_1}((i, j), (i', j')) = |i - i'| + |j - j'|.$$

Moreover, for a graph $G = (V(G), E(G))$ on n vertices, an injection $f : V(G) \to V(H)$ is called a *two-dimensional embedding* of G. The two-dimensional bandwidth of an embedding f for G is

$$B_2(G, f) = \max_{uv \in E(G)} d_{L_1}(f(u), f(v)).$$

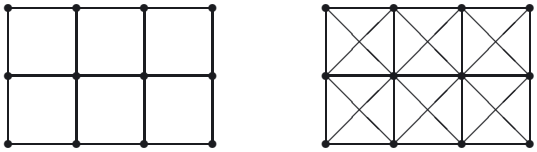
And the two-dimensional bandwidth of G is

$$B_2(G) = \min_f B_2(G, f),$$

where the minimum is taken over all embeddings f. An embedding f attaining the above minimum is called an *optimal embedding*. Here, the grid graph H is called the *host graph*, which may represent the parallel computer architecture in the parallel computation system and graph G is called the *guest graph*, which may represent the algorithm being performed in that system. In order that the grid graph has enough room to hold any embedding, we may assume that $N \geq |V(G)| = n$. However, it is sufficient to choose $N = n$.

In addition to the parallel computation, the two-dimensional bandwidth problem is mainly motivated by the VLSI layouts. However, there are some differences between these two concepts. In fact, the edge routings in VLSI layout are not allowed to overlap (edge-disjoint or with congestion 1) (see [1,12]). Instead, in the two-dimensional bandwidth problem mentioned above, we need not consider the edge routings (they are always along the shortest paths) and only the distances are taken into account. Therefore, the two-dimensional bandwidth problem is a relaxation of the "edge length" minimization problem in VLSI layout. So the two-dimensional bandwidth $B_2(G)$ is a lower bound of the minimum edge length of the VLSI layout.

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(a) $P_4 \times P_3$ (b) $P_4 \otimes P_3$

FIGURE 1. Grid graph and extended grid graph.

In the context of network layouts, the quality of an embedding is usually evaluated by two parameters, the *dilation* and the *congestion*. The dilation leads to the bandwidth problem (minimizing the maximum length of wires between two linked nodes) and the congestion gives rise to the cutwidth problem (minimizing the maximum number of wires passing through any edge of H in the embedding). When the host graph H is a path, we have the one-dimensional bandwidth and cutwidth problems. When H is a grid graph, we have the two-dimensional bandwidth and cutwidth problems. [5,6] presented surveys to these graph labelling and layout problems. Relatively, there are few results on the two-dimensional bandwidth problem. Some related work are as follows. [2] studied the dilation (bandwidth) problem and the congestion (cutwidth) problem for n-cubes into two-dimensional grids and presented lower and upper bounds. [3] solved the congestion problem for n-cubes into grids. [11] solved the wirelength (bandwidth-sum) problem for n-cubes into grids. Our previous paper [9] discussed two models of two-dimensional bandwidth problem. In this paper, we further investigate a basic property of two-dimensional bandwidth, called "square-root rule", by which more results for special graphs are derived.

The paper is organized as follows. In Section 2, we state some basic concepts and known results. In Section 3, we study the "square-root" relation between $B_2(G)$ and $B(G)$. Section 4 is concerned with graph products. Section 5 discusses a recursive algorithm for hypercubes. A short summary is given in Section 6.

2. PRELIMINARIES

We first recall the notion of graph products. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *cartesian product* $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent if and only if either $[u = u' \text{ and } vv' \in E_2]$ or $[v = v' \text{ and } uu' \in E_1]$. Moreover, the *strong product* $G_1 \otimes G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent if and only if $[u = u' \text{ and } vv' \in E_2]$ or $[v = v' \text{ and } uu' \in E_1]$ or $[uu' \in E_1 \text{ and } vv' \in E_2]$. For example, $H = P_m \times P_n$ is the *planar grid graph* (see Fig. 1a). Besides, we call $H' = P_m \otimes P_n$ the *extended grid graph* (following the version of [12]), which can be obtained from $P_m \times P_n$ by adding two diagonal edges in each mesh (see Fig. 1b).

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Moreover, we may consider another distance of L_∞-norm, for which the distance between two points $(i,j),(i',j') \in V(H)$ is

$$d_{L_\infty}((i,j),(i',j')) = \max\{|i-i'|,|j-j'|\}.$$

This distance does not mean the length of wire connecting points (i,j) and (i',j'), but it is meaningful in representing dilation in some interconnection network architectures (see, *e.g.*, [12]). Based on this type of distance in the plane, we have another model of two-dimensional bandwidth problem [9]: the bandwidth of labelling f for G is

$$B'_2(G,f) = \max_{uv\in E(G)} d_{L_\infty}(f(u),f(v))$$

and the bandwidth of G is

$$B'_2(G) = \min_f B'_2(G,f).$$

So we obtain two graph-theoretic invariants $B_2(G)$ and $B'_2(G)$. We will study them together. In fact, $B_2(G)$ is to consider embedding G into the grid graph $H = P_n \times P_n$ and $B'_2(G)$ is to consider embedding G into the extended grid graph $H' = P_n \otimes P_n$. Clearly, we have

Proposition 2.1. *$B_2(G) = 1$ if and only if G is isomorphic to a subgraph of $P_n \times P_n$; $B'_2(G) = 1$ if and only if G is isomorphic to a subgraph of $P_n \otimes P_n$.*

Proof. If $B_2(G) = 1$, then G can be embedded into $H = P_n \times P_n$ such that each edge has length 1. So each edge of G coincides with an edge of H. Hence the embedded graph of G is a subgraph of H. Conversely, if G is isomorphic to a subgraph of H, then it has an embedding of edge-length 1 in H and thus $B_2(G) = 1$. Likewise, we have the second conclusion. □

It is known that deciding whether $B_2(G) = 1$ even for a binary tree is NP-complete (see Thm. 5.8 of [5]). Besides, the following basic relation is implied by the definition:

Lemma 2.2. $B'_2(G) \le B_2(G) \le 2B'_2(G)$.

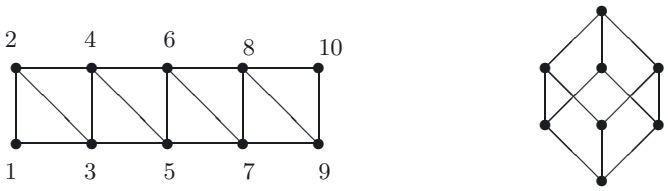
Proof. It follows from $d_{L_\infty}(x,y) \le d_{L_1}(x,y) \le 2d_{L_\infty}(x,y)$. □

It turns out that a result for $B'_2(G)$ is usually easier than that for $B_2(G)$ and the former is a useful lower bound of the latter. In fact, if an optimal embedding f of $B'_2(G)$ satisfies $B_2(G,f) = B'_2(G,f)$, then f is also an optimal embedding of $B_2(G)$ and thus $B_2(G) = B'_2(G)$. This fact may provide an efficient way for determining $B_2(G)$ in some cases. Moreover, if we can obtain a result of $B'_2(G)$, then it provides at least a 2-approximation of $B_2(G)$. This relation would have advantages in the study of asymptotic results.

The following lower bound, called density lower bound, plays an important role in the study of $B(G)$ [4,5]:

$$B(G) \ge \left\lceil \frac{n-1}{D(G)} \right\rceil,$$

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(a) Power graph P_{10}^2 (b) 3-cube Q_3

FIGURE 2. Subgraphs of $P_n \otimes P_n$.

where $D(G)$ is the diameter of G. For two-dimensional bandwidth, the density lower bound and the corresponding upper bound are as follows (obtained in [5,9]).

Lemma 2.3. *For any graph G with n vertices and diameter $D(G)$,*

$$\left\lceil \frac{\sqrt{n}-1}{D(G)} \right\rceil \le B_2'(G) \le \lceil \sqrt{n}-1 \rceil,$$

$$\left\lceil \frac{\delta(n)}{D(G)} \right\rceil \le B_2(G) \le \delta(n),$$

where

$$\delta(n) = \min\left\{ 2\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil, 2\left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1 \right\}.$$

In brief, we have $B_2(G) = O(\sqrt{n})$. This is the basic form of the square-root property.

3. Bandwidth-k graphs

We are concerned with the relation between $B(G)$ and $B_2'(G)$ or $B_2(G)$ in this section. A graph with $B(G) = k$ is called a bandwidth-k graph. A well-known characterization of the bandwidth is that $B(G)$ is the smallest integer k such that G can be embedded in P_n^k, where P_n^k is the kth power of path P_n (see [4]). Here is an easy observation:

Proposition 3.1. *For any bandwidth-2 graph G, $B_2'(G) = 1$.*

Proof. If $B(G) = 2$, then G can be embedded in P_n^2, which can be viewed as a subgraph of $P_n \otimes P_2$ (see Fig. 2a). By Proposition 2.1, the result follows. □

So this is a consequence of Proposition 2.1. Besides, there are more examples of this kind. The embedding of Figure 2b shows that $B_2'(Q_3) = 1$. Another example can be seen in Proposition 4.4 (Fig. 6).

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Proposition 3.2. $\lceil\sqrt{k+1}\rceil - 1 \le B'_2(P_n^k) \le \lceil\sqrt{k+1}\rceil$.

Proof. Let G' be a maximal complete subgraph of $G = P_n^k$. In other words, $V(G')$ is a maximal clique of G and $|V(G')| = k+1$. By Lemma 2.3, we have

$$B'_2(P_n^k) \ge B'_2(G') \ge \lceil\sqrt{k+1} - 1\rceil.$$

On the other hand, let $m = \lceil\sqrt{k+1}\rceil$ and $r = \lceil\frac{n}{m}\rceil$. Then we can embed G into the grid graph $H = P_r \otimes P_m$ as follows. Let $V(H) = \{(i,j) : 1 \le i \le m, 1 \le j \le r\}$ be the vertex set of the grid graph. Then we embed the n vertices of G (in the order of path P_n) into $V(H)$ column by column from left to right. That is to say, the n vertices of G are put into the first n positions of $(1,1),(2,1),\ldots,(m,1),(1,2),(2,2),\ldots,(m,2),\ldots,(1,r),(2,r),\ldots,(m,r)$. Note that in graph $G = P_n^k$, any $k+1$ consecutive vertices form a maximal clique, which has at most m^2 vertices. It follows that each of such cliques occupies at most $m+1$ columns of $V(H)$. Therefore the maximum L_∞-distance of adjacent vertices in G is $m = \lceil\sqrt{k+1}\rceil$, proving the upper bound. This completes the proof. □

Proposition 3.3. *For any bandwidth-k graph G, $B'_2(G) \le \lceil\sqrt{k+1}\rceil$.*

Proof. If $B(G) = k$, then G is a subgraph of P_n^k, thus $B'_2(G) \le B'_2(P_n^k) \le \lceil\sqrt{k+1}\rceil$ by Proposition 3.2, as required. □

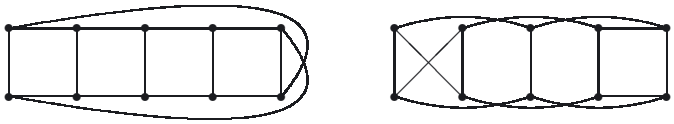
As a consequence, we have the following asymptotic result.

Proposition 3.4. $B_2(G) = O\left(\sqrt{B(G)}\right)$.

This is the second form of the square-root property. To explain in detail, let us see some interesting examples as follows. Notice that in determining the exact value of the bandwidth, we have two things to do: on the one hand, we derive a lower bound of the bandwidth; on the other hand, we construct an embedding of the graph attaining this lower bound (so that we obtain an upper bound that is equal to the lower bound).

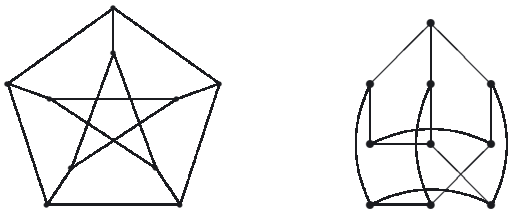
- First, for the complete graph K_n, $B(K_n) = n-1$ and $B'_2(K_n) = \lceil\sqrt{n} - 1\rceil$. In fact, both results can be easily seen by the density lower bounds (note that $D(G) = 1$).
- A typical example is $B(Q_3) = 4$ for the 3-dimensional cube Q_3 (see [4]). Now we have $B_2(Q_3) = 2 = \sqrt{4}$, exactly the value of square root. In fact, the lower bound is due to the fact that Q_3 is not isomorphic to any subgraph of $P_n \times P_n$ (since $Q_3 = C_4 \times K_2$ and if two disjoint C_4's are embedded into two meshes of the grid graph, then the distance between two corresponding vertices of the cycles is at least two); and the upper bound is due to the embedding of Figure 2b above.
- The Möbius ladder M_{2k} is a cycle of length $2k$ with all pairs of vertices at distance k apart in the cycle joined by an edge (see Fig. 3a). It is also well known that $B(M_{2k}) = 4$ (Thm. 4.4.11 of [4]). By the embedding of Figure 3b, we can see that $B_2(M_{2k}) = B'_2(M_{2k}) = 2 = \sqrt{4}$, also fitting the square-root property exactly. Note that the lower bound is also obtained by the fact that M_{2k} is not isomorphic to any subgraph of $P_n \times P_n$ (since M_{2k} is not a planar graph).

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(a) Möbius ladder (b) Embedding

FIGURE 3. Optimal embedding of Möbius ladders.



(a) Petersen graph (b) Embedding

FIGURE 4. Optimal embedding of Petersen graph.

- It is easy to show that the Petersen graph G has $B(G) = 5$ (the lower bound 5 is due to $D(G) = 2$ and the density lower bound). On the other hand, by the two-dimensional embedding of Figure 4b, we have $B_2(G) = B'_2(G) = 2$ for the Petersen graph. In fact, the lower bound is also because it is not a subgraph of $P_n \times P_n$ (since G is not a planar graph).
- Let $T_{2,k}$ be a complete binary tree with $n = 2^{k+1} - 1$ vertices. Paterson *et al.* (1981) proved that $B_2(T_{2,k}) = \Theta(\sqrt{n}/\log n)$ (see Thm. 5.5 of [5]). In fact, since $D(T_{2,k}) = 2k = O(\log n)$, the lower bound here is the same for $B'(T_{2,k})$ in Lemma 2.3. This is a meaningful example of asymptotic results for the square-root rule.

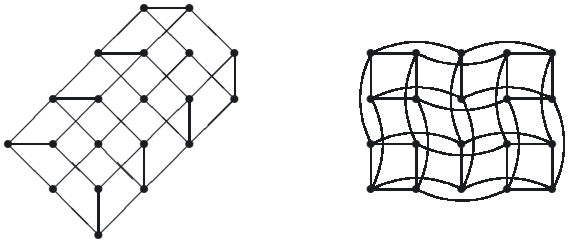
4. Graph products

Note that in the extended grid graph H', in addition to the horizontal and vertical lines, there are two families of parallel lines: one with slope $+1$ and one with slope -1. We call them *positive-slope lines* and *negative-slope lines* respectively.

Let P_n, C_n, and K_n denote the path, the cycle, and the complete graph on n vertices respectively. With respect to cartesian products of graphs, the basic results on the one-dimensional bandwidth were established in early years:

- for planar grids $P_m \times P_n$, $B(P_m \times P_n) = \min\{m, n\}$;
- for cylinder grids $P_m \times C_n$, $B(P_m \times C_n) = \min\{2m, n\}$;
- for torus grids $C_m \times C_n$, $B(C_m \times C_n) = 2\min\{m, n\} - \delta_{m,n}$.

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(a) $C_5 \times P_4$ (b) $C_5 \times C_4$

FIGURE 5. Cylinder and torus grids.

Here, the first two were presented by Chvátalová in 1975 (see [4]) and the third one was presented by Li *et al.* in 1981 [8].

For the two-dimensional bandwidth, the corresponding results are relatively easy. It is clear that $B_2(P_m \times P_n) = B_2'(P_m \times P_n) = 1$ by Proposition 2.1. Furthermore, we have

Proposition 4.1. *For the cartesian product of a path and a cycle $P_m \times C_n$ (the cylinder grid), $B_2'(P_m \times C_n) = 1$ and $B_2(P_m \times C_n) = 2$ ($m \geq 2, n \geq 3$).*

Proof. Let $G = P_m \times C_n$. For this cartesian product, the horizontal copies are paths P_m and vertical copies are cycles C_n.

First, for $B_2'(G) = 1$, it suffices to give an embedding of edge length 1 in H'. This can be done by putting the cycles C_n along two consecutive negative-slope lines and then translating the cycles along the positive-slope lines (the direction of P_m). An example is shown in Figure 5a.

Second, for $B_2(G) = 2$, we first show the lower bound. Suppose to the contrary that $B_2(G) = 1$, that is, G can be embedded into a subgraph of the grid graph H such that any two adjacent vertices have distance one. We take an edge e of the vertical cycle C_n and let S be the set of m vertical copies of e. Then $G - S$ is a grid graph $G' = P_m \times P_n$, which has a unique embedding in H (in the sense of symmetry). From this we can see that the edges in S must have length at least $n - 1 \geq 2$, which is a contradiction to $B_2(G) = 1$. On the other hand, the upper bound is obtained by the embedding of Figure 5a. This completes the proof. □

Proposition 4.2. *For the cartesian product of two cycles (the torus grid), $B_2(C_m \times C_n) = B_2'(C_m \times C_n) = 2$ ($m, n \geq 3$).*

Proof. Let $G = C_m \times C_n$. For this cartesian product, the horizontal copies are cycles C_m and vertical copies are cycles C_n.

We first show the lower bound that $B_2'(G) \geq 2$. Suppose not. Then $B_2'(G) = 1$ and by Proposition 2.1, G can be embedded into a subgraph of the extended grid H' such that any two adjacent vertices have distance one. We may call this a

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distance-1 embedding. We take an edge e of the vertical cycle C_n and let S be the set of m vertical copies of e. Then $G - S$ is a cylinder grid $G' = C_m \times P_n$. By the previous proposition, G' has a distance-1 embedding in H' (as shown in Fig. 5a). Furthermore, we can prove the following

Claim. This distance-1 embedding of cylinder grid $G' = C_m \times P_n$ is unique (up to symmetry).

In fact, let us consider two consecutive copies of cycle C_m in the cylinder grid, for which two corresponding vertices are joined by an edge. So there are m quadrilateral meshes (cycle C_4) between these two cycles. Note that in a distance-1 embedding, the opposite sides of a quadrilateral mesh are always parallel (no matter it is a square or a rhombus). Therefore, shifting a copy of cycle C_m to the next copy of C_m is a translation of distance one and these two copies are not overlapped. For this embedding of cylinder grid $G' = C_m \times P_n$, due to the non-overlapping translation, the only possible form of the embedding of cycle C_m is along two consecutive slope lines of one direction (say, negative-slope lines) and the translation moves along the slope lines of other direction (say, positive-slope lines). Thus the unique form of embedding is the one depicted in Figure 5a. Hence we show the claim.

Applying this claim to the embedding of torus G, we conclude that the edges in S must have length at least $n - 1 \geq 2$, contradicting $B'_2(G) = 1$. This proves the lower bound.

We next show the upper bound $B_2(G) \leq 2$. This can be done by constructing an embedding f with $B_2(G, f) = 2$. Recall the one-dimensional bandwidth embedding (labelling) of $B(C_n) = 2$ for cycle C_n. Suppose that the vertices of C_n are $1, 2, \cdots, n$ arranged clockwise. Then an optimal labelling (with bandwidth 2) is in the order $1, n, 2, n-1, \cdots, \frac{n}{2}, \frac{n}{2}+1$ (if n is even) or $1, n, 2, n-1, \cdots, \frac{n+1}{2}+1, \frac{n+1}{2}$ (if n is odd). Now let f_1 be an optimal embedding of C_m on the horizontal line and let f_2 be an optimal embedding of C_n on the vertical line. Then $f = (f_1, f_2)$ is a two-dimensional embedding that each horizontal copy C_m is embedded by f_1 while each vertical copy C_n is embedded by f_2. An example is shown in Figure 5b. So we have $B_2(G, f) = 2$ and thus $B_2(G) \leq 2$.

Finally, combining the lower and upper bounds gives $2 \leq B'_2(G) \leq B_2(G) \leq 2$. Therefore $B'_2(G) = B_2(G) = 2$. This completes the proof. □

The bandwidth of cartesian product of two complete graphs has been obtained by Mai and Luo [10]: $B(K_n \times K_n) = n(n+1)/2 - 1$. The following also reflects the square-root property.

Proposition 4.3. $\lceil (n-1)/2 \rceil \leq B'_2(K_n \times K_n) \leq B_2(K_n \times K_n) \leq n - 1$ *and the bounds are tight.*

Proof. Since the diameter of $G = K_n \times K_n$ is 2, the lower bound is due to Lemma 2.3. On the other hand, the upper bound can be obtained by the following embedding f: G is embedded in an $n \times n$ grid such that each horizontal or vertical line holds a copy of K_n in a given order. Then $B'_2(G, f) = B_2(G, f) = B(G) = n - 1$, giving the upper bound. For example, $B'_2(K_3 \times K_3) = B_2(K_3 \times K_3) = 2$

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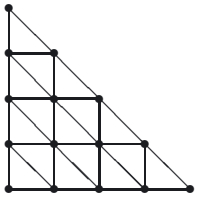


FIGURE 6. Embedding of T_n.

(by Prop. 4.2) attains the upper bound; $B_2'(K_4 \times K_4) = 2$ (by embedding G in a 4×4 grid with four copies of K_4 at four corners) attains the lower bound. □

The bandwidth of triangulated triangles T_n was a difficult problem (posed by D. West) in this direction. Here T_n is a graph with vertex set $\{(x, y, z) \in \mathbf{Z}^3 : x+y+z = n, x, y, z \geq 0\}$ and two vertices (x, y, z) and (x', y', z') are joined by an edge if $|x-x'|+|y-y'|+|z-z'| \leq 2$ (they agree in one coordinate and differ by 1 in the other two coordinates). Hochberg *et al.* [7] proved that $B(T_n) = n+1$ for the one-dimensional bandwidth. However, the result of two-dimensional bandwidth is rather trivial:

Proposition 4.4. *For the triangulated triangles T_n, $B_2'(T_n) = 1$ and $B_2(T_n) = 2$.*

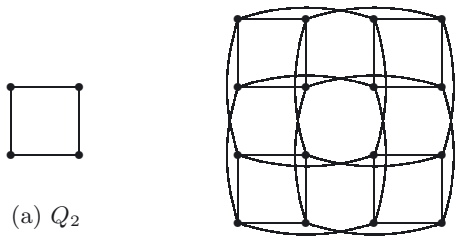
Proof. We may define an embedding of T_n into the extended grid graph H' by an injection $f : (x, y, z) \mapsto (x, y)$. Then the image of the vertex set $V(T_n) = \{(x, y, z) \in \mathbf{Z}^3 : x + y + z = n, x, y, z \geq 0\}$ in the plane is $\{(x, y) \in \mathbf{Z}^2 : x + y \leq n, x, y \geq 0\}$, and two vertices (x, y) and (x', y') are joined by an edge if $|x - x'| + |y - y'| \leq 1$ or $|x - x'| + |y - y'| = 2$ and $x + y = x' + y'$. So this embedding of T_n is a subgraph of H' (as shown in Fig. 6). Therefore $B_2'(T_n) = 1$. Moreover, this embedding implies an upper bound that $B_2(T_n) \leq B_2(T_n, f) = 2$. On the other hand, as a subgraph of T_n, a triangle C_3 has $B_2(C_3) = 2$. So we deduce a lower bound that $B_2(T_n) \geq B_2(C_3) = 2$. Combining the lower and upper bounds gives $B_2(T_n) = 2$. The proof is complete. □

5. HYPERCUBES

In this section we proceed to consider the hypercubes. By an n-cube Q_n, we mean the graph whose vertex set is the set of all n-tuples of 0's and 1's, where two vertices are adjacent if they differ in exactly one coordinate. Clearly, it has 2^n vertices and diameter n, and it is also a cartesian product $Q_n = K_2 \times K_2 \times \cdots \times K_2$. The one-dimensional bandwidth $B(Q_n)$ is a well-known result, due to L. Harper (see surveys [4–6]). The two-dimensional bandwidth (dilation) problem of embedding Q_n into grid graph H has been studied and lower and upper bounds are obtained in [2].

Now we consider embedding Q_n into extended grid graph H'.

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(a) Q_2 (b) Q_4

FIGURE 7. Embedding of Q_{2i}.

Proposition 5.1. *For the n-cube Q_n on 2^n vertices, we have*

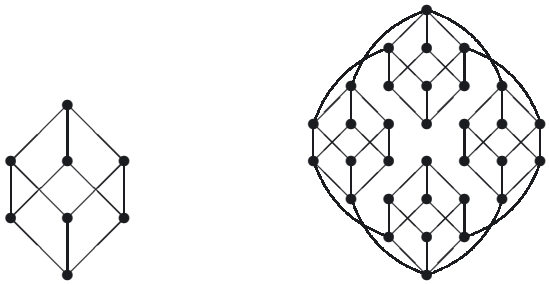
$$\lceil(\sqrt{2^n}-1)/n\rceil \le B_2'(Q_n) \le 2^{\lfloor\frac{n}{2}\rfloor-1}.$$

Proof. The lower bound is due to Lemma 2.3. To show the upper bound, it suffices to describe an embedding algorithm of Q_n with the required bandwidth. We distinguish two cases as follows.

Case 1: n is even. Clearly, Q_2 is a 4-cycle, which can be embedded in a 2×2 grid (see Fig. 7a) and so $B_2'(Q_2) = 1$. Further, $Q_4 = Q_2 \times K_2 \times K_2$ can be embedded in a 4×4 grid with four copies of Q_2 at the four corners (see Fig. 7b). Note that, by the definition of cartesian product, between two corresponding vertices of two Q_2's in the same vertical or horizontal line, there is an edge joining them. For convenience, we call the edges in each Q_2 the *inner edges*, and those between different Q_2's the *outer edges*. So, each inner edge has length 1 while each outer edge has length 2. Thus the maximum length of edges in this embedding is 2. In general, if Q_{2i} has been embedded in a $a_i \times a_i$ grid, then we embed Q_{2i+2} in a $a_{i+1} \times a_{i+1}$ grid, where $a_{i+1} = 2a_i$, with four copies of Q_{2i} at the four corners. By the recursive relation $a_{i+1} = 2a_i$ and $a_1 = 2$, we obtain $a_i = 2^i$. So the maximum length of outer edges in this embedding of Q_{2i+2} is $a_i = 2^i$ (where $i = n/2 - 1$).

Case 2: n is odd. We define a *k-diamond* to be a set of lattice points contained in the intersection of k consecutive positive-slope lines and k consecutive negative-slope lines. A 4-diamond is shown in Figure 8a. Now we state the embedding algorithm. First, Q_3 can be optimally embedded in a 4-diamond (see Fig. 8a) and so $B_2'(Q_3) = 1$. Similar to the previous case, if Q_{2i-1} has been embedded in a a_i-diamond, then we embed $Q_{2i+1} = Q_{2i-1} \times K_2 \times K_2$ in a a_{i+1}-diamond, where $a_{i+1} = 2a_i$, with four copies of Q_{2i-1} at the four corners. For example, Q_5 is embedded in a 8-diamond with maximum edge-length 2 (see Fig. 8b, some outer edges are omitted). By the recursive relation $a_{i+1} = 2a_i$ and $a_2 = 4$, we obtain $a_i = 2^i$. Consequently the length of each outer edge in this embedding of Q_{2i+1} is $a_i/2 = 2^{i-1}$. So the maximum length of edges in this embedding is $2^{\lfloor\frac{n}{2}\rfloor-1}$. This completes the proof. □

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(a) Q_3 (b) Q_5

FIGURE 8. Embedding of Q_{2i+1}.

For small n, the above embedding is optimal:

Corollary 5.2. *For $n \leq 5$, the upper bound in Proposition 5.1 gives exact results: $B_2'(Q_2) = B_2'(Q_3) = 1$, $B_2'(Q_4) = B_2'(Q_5) = 2$. Moreover, $B_2(Q_4) = 2$.*

Proof. The former result is obvious. We next see $B_2'(Q_4) = 2$. In fact, by using the method of Proposition 4.2, we can show that $B_2'(Q_4) = 1$ is impossible (since $Q_4 = Q_3 \times K_2$ and the distance-1 embedding of Q_3 is unique (see Fig. 8a), the distance-1 translation of two copies of Q_3 must be overlapped). So we have the lower bound $B_2'(Q_4) \geq 2$. On the other hand, the upper bound $B_2'(Q_4) \leq 2$ is obtained by the embedding of Figure 7b. This embedding also implies $B_2(Q_4) = 2$. Finally, the lower bound of $B_2'(Q_5)$ follows from $B_2'(Q_5) \geq B_2'(Q_4) = 2$ while the upper bound is obtained by the embedding of Figure 8b. This completes the proof. □

However, for large n the asymptotic behavior of Proposition 5.1 is not so good. For even n, [2] obtained a better asymptotic result $B_2(Q_n) = O(\sqrt{2^n/n})$. It seems that no better result is known for odd n yet.

6. CONCLUSION

In the foregoing sections, we study the two-dimensional bandwidth $B_2(G)$ and $B_2'(G)$ for several classes of special graphs. The core is the square-root rule $B_2(G) = O(\sqrt{B(G)})$, which describes a general picture of the topic. However, the exact relation of $B(G)$ and $B_2(G)$ is rather complicated. In fact, the characterization of $B(G) = 1$ is trivial while that of $B_2(G) = 1$ is hard. On the other hand, for the triangulated triangles T_n, the result of $B(T_n) = n + 1$ is quite involved (see [7]) while $B_2(T_n) = 2$ is almost trivial (see Prop. 4.4). Moreover, for P_n^k, the k-th power of path P_n, it is clear that $B(P_n^k) = k$. We have determined $B_2(P_n^k)$ in Proposition 3.2 (with two possible values). It is known that the result

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of $B(P_n^k) = k$ can be easily generalized to the proper interval graphs. The algorithm for determining $B_2(P_n^k)$ in Proposition 3.2 can also be generalized to the proper interval graphs. Furthermore, there has been polynomial-time algorithms of determining $B(G)$ for interval graphs. The problem of determining $B_2(G)$ for interval graphs should be worthwhile to study. Finally, more results for typical graphs are expected. Especially, the exact result for n-cubes should be settled for all n.

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