# HYPERKÄHLER STRUCTURES AND GROUP ACTIONS

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## 1. *Introduction*

A $4n$-dimensional Riemannian manifold $(M, g)$ is hyperkähler if it possesses three anti-commuting complex structures $I$, $J$, $K$ such that the metric $g$ is Kähler with respect to each of them. The reduced holonomy group of such a manifold is necessarily a subgroup of $\mathrm{Sp}(n)$ so the Ricci tensor of $g$ vanishes and $(M, g)$ can be regarded as a positive definite solution to Einstein's equations in vacuum.

Considerable amount of effort has been devoted to establishing the existence of hyperkähler manifolds. One particularly fruitful method of constructing such manifolds is the hyperkähler generalization of the Marsden–Weinstein quotient given in [**13**]. Briefly, if $G$ is a compact Lie group acting freely on a hyperkähler manifold $(M, g, I, J, K)$ and preserving the hyperkähler structure, one obtains three moment maps $\mu_1, \mu_2, \mu_3 : M \to \mathfrak{g}^*$ corresponding to the three Kähler forms

$$\omega_1 = g(I, \ ), \quad \omega_2 = g(J, \ ), \quad \omega_3 = g(K, \ )$$

from $M$ into the dual of the Lie algebra of $G$. Then the induced metric on $(\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0))/G$ is hyperkählerian.

Examples of hyperkähler manifolds arising in this way include multi-Eguchi–Hanson instantons [**20**], coadjoint orbits of complex Lie groups [**21**, **22**, **4**, **17**, **16**], different monopole moduli spaces [**1**, **6**] and moduli spaces of instantons [**23**].

If we forget about the Riemannian structure of a hyperkähler manifold and think of it as a complex manifold with respect to a chosen complex structure, say $I$, we obtain more than a complex manifold: the form $\omega = \omega_2 + \mathrm{i}\omega_3$ is closed, non-degenerate, and holomorphic with respect to $I$, so we obtain a *complex-symplectic* manifold. The relation between the complex-symplectic structure and the hyperkähler one is very close. For example, Beauville showed [**2**], using Yau's theorem, that a compact Kählerian manifold on which a holomorphic symplectic form exists possesses a hyperkähler structure.

In this work we shall be dealing also with an action of a compact Lie group $G$ on a hyperkähler manifold $M$ preserving the hyperkähler structure. We shall show that whenever such an action (not necessarily free) extends to a holomorphic (with respect to $I$) action of the complexified group $G^c$, one obtains a class of hyperkähler manifolds. These manifolds can be realized as complex (in fact complex-symplectic) submanifolds of $M$ with respect to the chosen structure $I$. They can be thought of as transverse slices to the action of $G^c$ on $M$.

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Let us be more precise. Let $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ be a homomorphism of Lie algebras which we extend to a homomorphism of $\mathfrak{sl}(2)$ into $\mathfrak{g} \otimes \mathbb{C}$. Let

$$h = \rho\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x = \rho\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

Define, after Slodowy [**26**], an affine subspace of $\mathfrak{g} \otimes \mathbb{C}$ by

$$S(\rho) = y + Z(x), \tag{1.2}$$

where $Z(x)$ is the centralizer of $x$ in $\mathfrak{g} \otimes \mathbb{C}$. Thus $S(\rho)$ is the transverse slice to the adjoint action of $G^c$ and so it is transverse to any adjoint orbit it meets.

Now consider the complex moment map

$$\mu = \mu_2 + i\mu_3 : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{C} \simeq \mathfrak{g} \otimes \mathbb{C}. \tag{1.3}$$

This is the moment map for the action of $G^c$ with respect to the complex symplectic form $\omega$. Here and in what follows we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the $\mathrm{Ad}(G)$-invariant inner product.

We are going to show that the subsets of $M$ of the form

$$\mu^{-1}(S(\rho)) \tag{1.4}$$

are submanifolds carrying a natural hyperkähler structure. The motivation for seeking such a result comes from the work of Kronheimer. In [**22**] he defines a hyperkähler structure on $\mathcal{O} \cap S(\rho)$, where $\mathcal{O}$ is an adjoint (alias coadjoint) nilpotent orbit in the Lie algebra of a complex semisimple Lie group $G^c$. In particular, when $\rho = 0$ is the trivial representation, we have a hyperkähler structure on $\mathcal{O}$ itself. The complex-symplectic form of $\mathcal{O}$ is given by the complex Kirillov–Konstant form [**9**] and so the moment map for the action of $G^c$ on $\mathcal{O}$ is the inclusion $\mathcal{O} \to \mathfrak{g} \otimes \mathbb{C}$. Hence $\mathcal{O} \cap S(\rho)$ is of the form (1.4).

Furthermore, in [**20**] Kronheimer constructs hyperkähler ALE (asymptotically locally euclidean) metrics on minimal resolutions of simple singularities. It is well known (see [**26**]) that the underlying complex manifolds occur naturally as $\mathcal{O} \cap S(\rho)$, where $\mathcal{O}$ is a regular semisimple adjoint orbit in a simple Lie algebra. On the other hand Kronheimer also constructed [**21**] a hyperkähler metric on such an orbit $\mathcal{O}$. It is natural to ask whether these metrics are related. It will follow from our result that this is indeed the case and that the ALE spaces arise naturally from the regular semisimple orbits.

Let us say a few words about the construction of a hyperkähler structure on (1.4). We shall do it first in the special case $M = T^*G^c$, the cotangent bundle of the group. The existence of a hyperkähler structure on $T^*G^c$ has been shown by Kronheimer [**19**]. The left and right translations on $G^c$ induce two actions of $G$ on $T^*G^c$ and these actions preserve the hyperkähler structure of $T^*G^c$. If we identify $T^*G^c$ with $G^c \times \mathfrak{g}^c$ by means of right-invariant forms of $G^c$, then the complex moment map (1.3) for the left action of $G^c$ becomes simply the projection $G^c \times \mathfrak{g}^c \to \mathfrak{g}^c$. Hence the set $\mu^{-1}(S(\rho))$ is in this case

$$G^c \times S(\rho). \tag{1.5}$$

Our first step will be to show that these manifolds carry a hyperkähler structure whose underlying complex-symplectic structure is induced from $T^*G^c$. The hyperkähler structure of manifolds (1.5) is of some interest: for $G = U(m)$ they are different moduli spaces of magnetic monopoles with minimal symmetry breaking (see [**6**]).

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There is a free hyperkähler action of $G$ on $G^c \times S(\rho)$ given by right translations on the first factor. Therefore, for a hyperkähler manifold $M$ with a $G$-action preserving the hyperkähler structure, we obtain a free $G$-action on the product manifold $M \times G^c \times S(\rho)$. This action preserves the (product) hyperkähler structure and so we can consider the hyperkähler quotient of $M \times G^c \times S(\rho)$ by $G$. Let us state the main result of this paper.

THEOREM 1. *Let $(M, g, I, J, K)$ be a hyperkähler manifold with an action of a connected compact semisimple Lie group $G$ which preserves the hyperkähler structure and extends to an action of $G^c$ holomorphic with respect to the complex structure $I$. Let $\mu = \mu_2 + i\mu_3$ be the moment map for the complex form $\omega$ and let $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ be a homomorphism of Lie algebras. Then,*

(i) *the manifold $G^c \times S(\rho)$ carries a natural hyperkähler structure which is preserved by the action of $G$ given by right translations on $G^c$;*

(ii) *the hyperkähler quotient of $M \times G^c \times S(\rho)$ by $G$ is isomorphic, as a complex-symplectic manifold, to an open subset $U_\rho$ of $\mu^{-1}(S(\rho))$;*

(iii) *if $M$ has a $G$-invariant Kähler potential for $\omega_1$, bounded from below on each $G^c$-orbit, then $U_\rho = \mu^{-1}(S(\rho))$.*

Recall that a Kähler potential for $\omega_1$ is a real function $K$ such that $\omega_1 = 2i\,\partial_1 \bar{\partial}_1 K$, where $\partial_1$ is the Dolbeaut operator associated to $I_1$.

REMARKS. (i) The set $\mu^{-1}(S(\rho))$ can obviously be empty (for example, for the trivial group action). However we do not have to assume that the action is free; this will allow for some interesting examples.

(ii) In general the submanifolds of the form $\mu^{-1}(S(\rho))$ will have a complicated topology, even for simple manifolds $M$. For example, we shall show that the space of based rational maps of degree $m$ from $\mathbb{C}P^1$ to itself is such a submanifold of the cotangent bundle $T^*\mathrm{GL}(m)$. Nevertheless the complex moment map is usually not difficult to write down, nor are the transverse slices $S(\rho)$, so one does obtain a fairly explicit description of these submanifolds. The question of identifying the hyperkähler metric is, of course, much more difficult.

(iii) It is reasonable to expect that the metric on $G^c \times S(\rho)$ is complete (this is the case of $\rho = 0$, that is, $T^* G^c$, see [**19**]). If this were the case, then the metric on $\mu^{-1}(S(\rho))$ would be complete whenever the metric on $M$ was.

(iv) The idea of defining a hyperkähler structure on a manifold by taking the hyperkähler quotient of a certain product manifold was also used by Kronheimer [**21**] to obtain the hyperkähler quotient 'with momentum'. The similarity extends to the fact that in both cases the manifolds which are used to define the product action of the group (in our case the manifolds $S(\rho) \times G^c$) are certain moduli spaces of solutions to Nahm's equations.

(v) The hyperkähler quotient of $M \times G^c \times S(\rho)$ by $G$ has a particularly simple interpretation when the hyperkähler structure on $M$ is given by Nahm's equations. The resulting metric on $\mu^{-1}(S(\rho))$ will be also given by Nahm's equations. In particular, the ALE spaces will be realized as moduli spaces of solutions to Nahm's equations.

(vi) If $M$ admits an isometric action of SO(3) or SU(2) rotating the complex structures, then so does $U_\rho$.

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(vii) Part (i) holds for any connected compact Lie group. Parts (ii) and (iii) will hold if the assumption of (iii) holds and the complex moment map for the action of $G^c$ on $M$ exists (the assumption of (iii) guarantees that $\mu_1$ exists).

## 2. *$T^*G^c$ and Nahm's equations*

In this section we shall describe, after Kronheimer [**19**], the hyperkähler structure on $T^*G^c$ for a compact group $G$ and then calculate the complex symplectic form and the moment map for the action of $G^c \times G^c$ (see also [**7**]).

The hyperkähler structure on $T^*G^c$ is given by Kronheimer in terms of the Nahm equations. These are ordinary differential equations

$$\dot{T}_i + [T_0, T_i] + \tfrac{1}{2} \sum_{j,k=1,2,3} \varepsilon_{ijk}[T_j, T_k] = 0 \quad \text{for } i = 1, 2, 3 \tag{2.1}$$

for Lie algebra valued functions $T_j\colon [0,1] \to \mathfrak{g}$ with $j = 0, 1, 2, 3$. These equations occur in many examples of hyperkähler geometry: moduli spaces of magnetic monopoles [**12**, **8**, **15**, **6**], coadjoint orbits of a semisimple complex group [**21**, **22**, **4**, **17**]. In these examples different boundary conditions are imposed and one studies the moduli space of solutions.

For $T^*G^c$ the situation is particularly simple: we consider all analytic solutions to (2.1) on [0, 1]. One can think of $(T_0(t), T_1(t), T_2(t), T_3(t))$ as components of an anti-self-dual $\mathbb{R}^3$-invariant connection on $\mathbb{R}^4$. It is then clear that the gauge group of smooth functions $g\colon [0,1] \to G$ with $g(0) = g(1) = 1$, acts as follows:

$$T_0 \longmapsto \mathrm{Ad}(g)\, T_0 - \dot{g}g^{-1}, \quad T_i \longmapsto \mathrm{Ad}(g)\, T_i, \quad \text{for } i = 1, 2, 3.$$

Kronheimer considers then the space $N$ of smooth solutions to (2.1) by this gauge group and shows that it is diffeomorphic to $T^*G^c$.

The existence of a hyperkähler structure on moduli spaces of solutions to Nahm's equations is a general phenomenon and can be explained by the infinite-dimensional hyperkähler quotient as follows. Consider the Banach space $\mathscr{A}$ of all $\mathfrak{g}$-valued functions $T_0$, $T_1$, $T_2$, $T_3$ analytic on [0, 1] with the norm

$$\|(T_0, T_1, T_2, T_3)\|^2 = 2\int_0^1 \langle T_0, T_0\rangle + \langle T_1, T_1\rangle + \langle T_2, T_2\rangle + \langle T_3, T_3\rangle,$$

where $\langle\,,\rangle$ is a (positive-definite) invariant inner product on $\mathfrak{g}$, and we can see the three complex structures $I$, $J$, $K$ by writing an element as

$$T_0 + \mathrm{i}T_1 + \mathrm{j}T_2 + \mathrm{k}T_3,$$

where i, j, k is the basis of quaternions. The gauge group $\mathscr{G}$ of gauge transformations $g\colon [0,1] \to G$ which are 1 at the endpoints acts freely on $\mathscr{A}$ preserving the hyperkähler structure, and the three moment maps $\mu_1$, $\mu_2$, $\mu_3$ for this action are the three components of Nahm's equations. Hence $N$ is the hyperkähler quotient of $\mathscr{A}$ by $\mathscr{G}$.

REMARK. Kronheimer showed [**19**] that the above metric on $T^*G^c$ is complete.

Let us now exhibit the complex structure of the space $N$. Following Donaldson [**8**], choose an isomorphism (compatible with the usual metrics) $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ which allows

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us to introduce complex coordinates $(t+ip_1, p_2+ip_3)$ and to write the matrices $T_i$ in a similar fashion: if the direction chosen is, say, the positive $x_1$-axis (which corresponds to the complex structure $I$), we put

$$\alpha := \tfrac{1}{2}(T_0+iT_1), \quad \beta := \tfrac{1}{2}(T_2+iT_3). \tag{2.2}$$

Having chosen $\alpha$ and $\beta$, we can write Nahm's equations as

$$\frac{d}{dt}(\alpha+\alpha^*)+2[\alpha,\alpha^*]+2[\beta,\beta^*]=0 \tag{2.3}$$

and

$$\frac{d}{dt}\beta = 2[\beta,\alpha]. \tag{2.4}$$

The action of the gauge group $\mathcal{G}$ now becomes:

$$B \longmapsto \mathrm{Ad}(g)\,\beta, \quad \alpha \longmapsto \mathrm{Ad}(g)\,\alpha - \tfrac{1}{2}\dot{g}g^{-1}. \tag{2.5}$$

The complex equation (2.4) is preserved by the complexified gauge group $\mathcal{G}^c$, that is, by the smooth maps $g\colon [0,1]\to G^c$, acting as above, that are 1 at 0, 1. This is an example of the equivalence, under certain hypotheses, of hyperkähler and complex-symplectic quotients: we start with the infinite-dimensional manifold $\mathcal{A}$ written as a complex manifold of pairs $(\alpha,\beta)$. The group $\mathcal{G}^c$ is the complexification of the group $\mathcal{G}$ and performing the complex-symplectic quotient with respect to the form $\omega=\omega_2+i\omega_3$ gives us $N$. To show this we must argue (see [**13**]) that the Kähler quotient of $(\mu_2+i\mu_3)^{-1}(0)$ by $\mathcal{G}$ can be identified with the ordinary quotient of $(\mu_2+i\mu_3)^{-1}(0)$ by $\mathcal{G}^c$. From [**9**, Section 5.6.2], or from [**13**, Section 3(E)], we can infer the following.

PROPOSITION 2.1. *Let $(X,I,\rho)$ be a Kähler manifold admitting an isometric and holomorphic free action of a compact semisimple Lie group $G$, which extends to a holomorphic action of $G^c$. Let $\phi$ be the moment map for this action and let $K$ be a globally defined $G$-invariant Kähler potential for $\rho$. Then, for any $G^c$-orbit $\mathcal{O}$, we have*

(i) *$K$ is a strictly convex on $G^c/G$, where we identify $\mathcal{O}$ with $G^c$;*

(ii) *the critical points of the restriction of $K$ to $\mathcal{O}$ and of the restriction of $\|\phi\|^2$ to $\mathcal{O}$ coincide.*

Hence, if the $G^c$-orbits are closed and $K$ is proper on each such orbit, then the Kähler quotient of $X$ by $G$ is biholomorphic to $X/G^c$.

This proposition can be also used in our infinite-dimensional setting, where we take $K=\int_0^1 \|T_1\|^2+\|T_3\|^2$ and observe that for each $\mathcal{G}^c$-orbit, $\mathcal{G}^c/\mathcal{G}$ is a Hilbert space and $K$ is a weakly coercive convex functional on it. This is essentially the argument used by Donaldson in [**8**]. In other words $N$ can be described as the (complex) manifold of smooth solutions to the complex Nahm equation modulo the gauge group $\mathcal{G}^c$. Let us remark that instead considering *analytic* solutions to the complex equation modulo *analytic* gauge transformations, we can equivalently consider $C^1$ solutions modulo $C^2$ gauge transformations (see [**8**]).

It is now easy to see that the holomorphic symplectic form $\omega$ is given by

$$\omega((a,b),(\hat{a},\hat{b})) = 2\int_0^1 \langle a,\hat{b}\rangle - \langle \hat{a},b\rangle, \tag{2.6}$$

where $\langle\,,\rangle$ is an invariant complex inner product on $\mathfrak{g}\otimes\mathbb{C}$.

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The diffeomorphism between $N$ and $T^*G^c$ can be seen as follows [**19**]: for each $(\alpha, \beta)$ we can find a unique solution to the equation $\dot{p}p^{-1} = -2\alpha$, where $p: [0,1] \to G$ satisfies $p(0) = 1$. Then $g = p^{-1}$ is a gauge transformation such that, under the action (2.3), $\alpha$ becomes identically 0 and therefore (as the complex equation remains satisfied) $\beta$ becomes constant equal to $\beta(0)$. The map

$$(\alpha, \beta) \longmapsto (g(1), \beta(0)) \tag{2.7}$$

is well defined and it is a map into $T^*G^c$ if we identify $T^*G^c \simeq G^c \times \mathfrak{g}^c$. It is not difficult to see that this map is a holomorphic diffeomorphism. Let us see what happens to the complex-symplectic form (2.4) under this biholomorphism. Under the action of $g$ we have

$$d\alpha \longmapsto a_g = 0, \quad d\beta \longmapsto b_g = \mathrm{Ad}(g)\, d\beta + [dg\, g^{-1}, \beta(0)].$$

Let $(a, b)$, $(\hat{a}, \hat{b})$ be two tangent vectors. Let us put $\rho = dg\, g^{-1}$. Then we have

$$\begin{aligned}\omega((a,b),(\hat{a},\hat{b})) &= 2\int_0^1 \langle a, \hat{b}\rangle - \langle \hat{a}, b\rangle \\ &= 2\int_0^1 \langle a_g, \hat{b}_g\rangle - \langle \hat{a}_g, b_g\rangle + (\langle \rho, \hat{b}_g\rangle - \langle \hat{\rho}, b_g\rangle - \langle \beta(0), [\rho, \hat{\rho}]\rangle)|_0^1 \\ &= (\langle \rho, \hat{b}_g\rangle - \langle \hat{\rho}, b_g\rangle - \langle \beta(0), [\rho, \hat{\rho}]\rangle)|_0^1.\end{aligned}$$

Therefore, since $\rho(0) = \hat{\rho}(0) = 0$, it follows that $N$ is symplectomorphic to $T^*G^c \simeq G^c \times \mathfrak{g}^c$ with the symplectic form at $(\beta, g)$ satisfying

$$\omega((\rho, b), (\hat{\rho}, \hat{b})) = \langle \rho, \hat{b}\rangle - \langle \hat{\rho}, b\rangle - \langle \beta, [\rho, \hat{\rho}]\rangle, \tag{2.8}$$

where $\rho, \hat{\rho}$ are right invariant 1-forms on $G^c$.

Next we discuss the group actions on $T^*G^c$. There is a left and a right action of $G^c$ on $T^*G^c$ and if we identify $T^* \simeq G^c \times \mathfrak{g}^c$, we can write these as an action of $G^c \times G^c$ as follows:

$$(p, q) \cdot (g, \beta) = (pgq^{-1}, \mathrm{Ad}(p)\beta). \tag{2.9}$$

Notice that while both left and right actions are free, the action of the product group is not.

We also have a hyperkähler action of $G \times G$ on $N$ defined by allowing gauge transformations with arbitrary values at the endpoints; $(p, q) \in G \times G$ are the values of the gauge transformation at $\{0, 1\}$. This action preserves the hyperkähler structure. Similarly, if we consider $N$ as a complex manifold with the complex-symplectic form $\omega$ and allow complex gauge transformations with arbitrary values at the endpoints, we obtain an action of $G^c \times G^c$ preserving the symplectic form. It is not difficult to see that under the above identification of $N$ with $T^*G^c$ this action agrees with (2.9).

Let us now calculate the complex moment map for this action of $G^c \times G^c$. We shall think of a $(p, q) \in G \times G$ as a complex gauge transformation $g(t)$ such that $g(0) = p$, $g(1) = q$. Let $(a, b)$ be a tangent vector to $N$ at $(\alpha, \beta)$. An element $\rho \in \mathfrak{g}^c$ gives us a vector field on $N$ and at $(\alpha, \beta)$ it is $[\rho, \beta]$, $[\rho, \alpha] - \frac{1}{2}\dot{\rho}$. Contracting this field with $\omega$ gives us

$$\begin{aligned}(d\mu)(a,b) &= 2\int_0^1 \langle [\rho, \alpha], b\rangle - \tfrac{1}{2}\langle \dot{\rho}, b\rangle - \langle a, [\rho, \beta]\rangle \\ &= -\int_0^1 2\langle [\beta, \alpha], \rho\rangle + 2\langle [b, \alpha], \rho\rangle + \langle b, \dot{\rho}\rangle = -d\int_0^1 \langle \dot{\beta}, \rho\rangle + \langle \beta, \dot{\rho}\rangle,\end{aligned}$$

which means that $\mu(\alpha, \beta)(\rho) = -\langle \beta, \rho\rangle|_0^1$. Hence

$$\mu(\alpha, \beta) = (\beta(0), -\beta(1)). \tag{2.10}$$

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Similarly one sees that in terms of the action (2.9) on $T^*G^c$ the moment map is

$$\mu(g,\beta) = (\beta, -\mathrm{Ad}(g^{-1})\beta). \tag{2.11}$$

## 3. *Hyperkähler structure on* $G^c \times S(\rho)$

In this section we shall prove Theorem 1 in case of $T^*G^c$. As shown in the previous section there is a hyperkähler action of $G \times G$ on $T^*G^c \simeq N$ and so we consider Lie algebra homomorphisms $\rho: \mathfrak{su}(2) \to \mathfrak{g} \oplus \mathfrak{g}$, that is, pairs of homomorphisms $\rho_-, \rho_+: \mathfrak{su}(2) \to \mathfrak{g}$. We want to show that $\mu^{-1}(S(\rho)) = \mu^{-1}(S(\rho_-, \rho_+))$ carries a natural hyperkähler structure. Let us make a few remarks about the complex submanifolds $\mu^{-1}(S(\rho))$.

REMARKS. 1. From (2.10) it follows that $\mu^{-1}(S(\rho)) = \mu^{-1}(S(\rho_-, \rho_+))$ is the submanifold of $N$ consisting of equivalence classes of solutions $(\alpha, \beta)$ such that $\beta(0) \in S(\rho_-)$, $-\beta(1) \in S(\rho_+)$.

2. If $\rho_+ = 0$, then, by (2.11), $\mu^{-1}(S(\rho))$ is $G^c \times S(\rho_-)$ as a complex-symplectic submanifold of $T^*G^c \simeq G^c \times \mathfrak{g}^c$.

3. Since $S(\rho)$ is an affine subspace of $\mathfrak{g} \times \mathbb{C}$, it follows that, as a complex manifold, $G^c \times S(\rho)$ is simply $G^c \times \mathbb{C}^{\dim Z(y)}$. By writing $G^c \times S(\rho)$ we want to stress that it is a *complex-symplectic* manifold with the symplectic form given by (2.8).

In fact the case when $\rho_+ = 0$ suffices to prove Theorem 1 in full generality (from which the case of arbitrary $\rho_+$ then follows) and so we shall concentrate on constructing a hyperkähler structure on the complex-symplectic manifolds $G^c \times S(\rho)$, where $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ is an arbitrary Lie algebra homomorphism. The hyperkähler structure on $G^c \times S(\rho)$ will be also defined using Nahm's equations. We shall consider solutions to (3.2) on $(0, 1)$ with first-order poles at 0 and residues

$$\mathrm{res}_{t=0}\, T_i = \tfrac{1}{2}\rho(\sigma_i), \tag{3.1}$$

for $i = 1, 2, 3$, where the $\sigma_i$ are the Pauli matrices. The fact that the singular part of the solutions is fixed guarantees that an infinitesimal deformation (a tangent vector) is regular and so the $L^2$-metric is well defined. However, we have to be careful about the question of analycity of the solutions, since, as is easily seen, if

$$\mathrm{ad}\,\rho: \mathfrak{sl}(2) \to \mathrm{End}(\mathfrak{g} \otimes \mathbb{C})$$

has odd weights, then half-powers of the interval parameter $t$ will occur. Therefore we shall assume that the regular part of the solutions $T_i$ is analytic on $(0, 1]$ and has an expansion in powers of $t^{1/2}$ near $t = 0$. Then the group $\mathcal{G}$ of gauge transformations $g: [0, 1] \to G$ that are analytic on $(0, 1]$, $C^1$ at $t = 0$ and satisfy $g(0) = g(1) = 1$, acts as in the previous section and preserves the singular part of the solutions.

We shall denote the corresponding moduli space by $N(\rho)$. Our aim is to show that $N(\rho)$ is diffeomorphic to $G^c \times S(\rho)$.

As in the previous section the existence of a hyperkähler structure on $N(\rho)$ follows by means of an infinite hyperkähler quotient. This time we consider the flat hyperkähler Banach manifold $\mathcal{M}(\rho)$ consisting of all $\mathfrak{g}$-valued functions $T_0$, $T_1$, $T_2$, $T_3$ with simple poles at 0 and residues (3.1) and appropriate analytic behaviour. The space $N(\rho)$ is then the hyperkähler quotient of $\mathcal{M}(\rho)$ by $\mathcal{G}$.

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As in the previous section we define the complex structures by considering solutions $(\alpha, \beta)$ to (2.3) and (2.4). The arguments used there still work to show that $(N(\rho), I_1)$ is biholomorphic to the complex-symplectic quotient of $\mathscr{M}(\rho)$ by $\mathscr{G}^c$. The complex symplectic form $\omega$ on $N(\rho)$ is still given by (2.6).

The remainder of the section will be devoted to proving the following.

PROPOSITION 3.1. *For each homomorphism* $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ *the manifold* $N(\rho)$ *is isomorphic, as a complex-symplectic manifold, to* $G^c \times S(\rho)$ *with complex symplectic form* (2.8).

The proof will be achieved by showing that both objects are isomorphic to a third, namely a complex-symplectic quotient of $N \simeq G^c \times \mathfrak{g}^c$ by a certain nilpotent subgroup of $G^c$.

Let us define, for any homomorphism $\rho: \mathfrak{su}(2) \to \mathfrak{g}$, the Lie subalgebra $\mathfrak{h}(\rho)$ as the sum of negative eigenspaces of $\operatorname{ad} h$. This is the nilradical of the parabolic subalgebra determined by $\rho$ and therefore $H(\rho) = \exp \mathfrak{h}(\rho)$ is a unipotent subgroup of $G^c$. Let $\mathfrak{h}'(\rho)$ denote the sum of eigenspaces of $\operatorname{ad} h$ corresponding to eigenvalues less than or equal to $-2$, and let $H'(\rho) = \exp \mathfrak{h}'(\rho)$. If we identify $\mathfrak{h}(\rho)^*$ with the sum of positive eigenspaces of $\operatorname{ad} h$, then $H'(\rho)$ is the subgroup of $H(\rho)$ stabilizing $y$ as an element of $\mathfrak{h}(\rho)^*$. Let

$$P(\rho) = \left\{ y + m; m \in \bigoplus_{\lambda \leqslant 0} V_\lambda \right\},$$

where the $V_\lambda$ are the weight spaces of $\operatorname{ad} \rho$. Let us finally define $N'(\rho)$ as the space of solutions $(\alpha, \beta)$ to the equation (2.4) such that $\beta(0) \in P(\rho)$ modulo the group $\mathscr{G}_H$ of $G^c$-valued gauge transformations $g$ such that $g(0) \in H'(\rho)$. From (2.10) it follows that $N'(\rho)$ is the complex symplectic quotient of $N$ by $H(\rho)$, that is, $N'(\rho) = \bar{\mu}^{-1}(y)/\operatorname{Stab}(y)$ where $\bar{\mu}$ is the complex moment map for the action of $H(\rho)$ (that is, $\mu$ followed by the projection $\mathfrak{g}^* \otimes \mathbb{C} \to \mathfrak{h}(\rho)^*$).

LEMMA 3.2. *For each homomorphism* $\rho: \mathfrak{su}(2) \to \mathfrak{g}$, *the manifold* $G^c \times S(\rho)$ *is isomorphic, as a complex-symplectic manifold, to* $N'(\rho)$.

*Proof.* In view of the remarks at the beginning of this section it is enough to show that for every element $y + m$ of $P(\rho)$ there is a unique $g \in H'(\rho)$ such that $\operatorname{Ad}(g)(y + m) \in S(\rho)$. Since $H'(\rho)$ is nilpotent, the exponential mapping is surjective and, moreover, we can write

$$\operatorname{Ad}(e^n)(y + m) = y + m + [n, y] + [n, m] + \tfrac{1}{2}[n, [n, y]] + \ldots.$$

To find an $n$ such that $\operatorname{Ad}(e^n)(y + m) \in S(\rho)$ we decompose $\mathfrak{g} \otimes \mathbb{C}$ into irreducible $\mathfrak{sl}(2)$ modules $V^i$ and, further, each $V^i$ into weight spaces $V^i_\lambda$. Now in each $V^i$ we look for an appropriate $n^i = \sum n^i_\lambda \in \bigoplus_{\lambda < -1}$ by induction on $-\lambda$. To see that it is possible to find $n$ this way we notice that

$$[x, [y, n^i_\lambda]] = \tfrac{1}{4}(\lambda_i + \lambda + 2)(\lambda_i - \lambda) \cdot n^i_\lambda,$$

where $\lambda_i = \max\{\lambda; V^i_\lambda \neq 0\}$. Since $\lambda_i \geqslant 0$ and $-\lambda_i \leqslant \lambda < 0$, the coefficient on the right-hand side is never 0 and so we can find a solution to $[x, \operatorname{Ad}(e^n)(y + m) - y] = 0$. The uniqueness follows from the same argument.

To finish the proof of Proposition 3.1 we have to show the following.

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LEMMA 3.3. *For each homomorphism* $\rho\colon \mathfrak{su}(2)\to\mathfrak{g}$ *the manifold* $N'(\rho)$ *is isomorphic, as a complex-symplectic manifold, to* $N(\rho)$.

*Proof.* Let us consider an $(\alpha,\beta)\in N'(\rho)$. We can find a complex gauge transformation $g$ such that $g(0)=g(1)=1$ and $g$ makes $\alpha$ equal to 0 near $t=0$. This means that $\beta$ is constant and in $P(\rho)$ near $t=0$. Now we can gauge $(\alpha,\beta)$ by

$$D(t)=t^{-h/2}. \tag{3.2}$$

This gives us a solution to the complex equation with residues of an element of $N(\rho)$ (and analytic in $t^{1/2}$). Moreover if $(\alpha,\beta)$, $(\alpha',\beta')$ are two solutions differing by a gauge transformation $g\in\mathscr{G}_H$ (that is, $g(0)\in H'(\rho)$), then $D(\alpha,\beta)$, $D(\alpha',\beta')$ differ by a gauge transformation that is in $\mathscr{G}^c$. Therefore we have a well-defined map

$$D\colon N'(\rho)\longrightarrow N(\rho).$$

Let us show that the inverse map exists. Consider an $(\alpha,\beta)\in N(\rho)$.

We shall use the $\mathscr{G}^c$-gauge freedom to make $\alpha$ of a particularly simple form. Let us put $f(t)=\exp(f_1 t+f_{3/2}t^{3/2}+\ldots)$, where the number of terms will be finite, and try and solve the equation $\mathrm{Ad}(f)\alpha-\frac{1}{2}\dot{f}f^{-1}=0$ up to some order of $t^{1/2}$. We can see that, after decomposing $\mathfrak{g}\times\mathbb{C}$ into eigenspaces $V_\lambda$ of $\mathrm{ad}\,h$, we can remove all non-negative powers of $t$ except the $t^{(-\lambda-2)/2}$-term of the $\lambda$-component of $\alpha$. Therefore we can assume that $\alpha'=\mathrm{Ad}(D^{-1})\alpha+\frac{1}{2}\dot{D}D^{-1}$ is of the form $a/t+O(t^m)$, where $a\in\bigoplus_{\lambda<-1}V_\lambda$ and $m$ is as large as we need. Now we can use the complex equation and show by induction on weights that if $a\neq 0$ or $\beta'=\mathrm{Ad}(D^{-1})\beta$ is not regular, then $d\beta'/dt$ has a first order pole, which is impossible. Hence $\beta'$ is regular and $\alpha'(t)=O(t^m)$ with $m$ as large as we need; and so we can assume that $\alpha'$, $\beta'$ are $C^1$. Moreover, because of $D$, we have that $\beta'(0)\in P_\rho$. Therefore to show that the inverse map $N(\rho)\to N'(\rho)$ is well defined, we have to prove only that if $(\alpha,\beta)$, $(\hat{\alpha},\hat{\beta})\in N(\rho)$ differ by a gauge transformation $g\in\mathscr{G}^c$, then $(\alpha',\beta')$ and $(\hat{\alpha}',\hat{\beta}')$ differ by a $g'\in\mathscr{G}_H$. We can find gauge transformations $p,\hat{p}$ such $p(\alpha')$, $\hat{p}(\hat{\alpha}')$ are 0 near $t=0$. Moreover we can assume that $p$, $\hat{p}=1+O(t^{m+1})$. Recall that the mapping $\alpha\mapsto\alpha'$ is given by the composition of two gauge transformations: $f\in\mathscr{G}^c$ and $D^{-1}$. Similarly we have an $\hat{f}\in\mathscr{G}^c$ for $\hat{\alpha}$. Now $\hat{p}D^{-1}\hat{f}gf^{-1}Dp^{-1}$ must be constant near $\pm 1$ and so $D^{-1}\hat{f}gf^{-1}D$ is at least $C^2$. Since $\hat{f}gf^{-1}$ is regular and equal to 1 at 0 and, since $D^{-1}$ makes anything in a positive eigenspace of $\mathrm{ad}\,h$ equal to 0 at $t=0$ and it does not change the 0-eigenspace, we conclude that $D^{-1}\hat{f}gf^{-1}D(0)\in H(\rho)$. To see that it is actually in $H'(\rho)$ notice that both $\beta'(0)$ and $\hat{\beta}'(0)$ are in $P_\rho$.

This proves that the two spaces are diffeomorphic. As the diffeomorphism is defined by a gauge transformation $D$ not depending on $(\alpha,\beta)$, it preserves the complex-symplectic form.

## 4. *Proof of Theorem* 1

Suppose now that we have an action of a connected compact semisimple group $G$ on a hyperkähler manifold $M$ preserving the hyperkähler structure and extending to an action of $G^c$. Since the group is semisimple, equivariant moment maps $\mu_1$, $\mu_2$, $\mu_3$ corresponding to the three Kähler forms exist [**10**] and so does the moment map $\mu=\mu_2+\mathrm{i}\mu_3$ corresponding to the complex-symplectic form $\omega=\omega_2+\mathrm{i}\omega_3$. Let $\rho\colon\mathfrak{su}(2)\to\mathfrak{g}$ be a homomorphism. We consider the hyperkähler manifold $N(\rho)$ defined in the previous section. As a complex-symplectic manifold it is $G^c\times S(\rho)$. From the discussion in Section 2, it follows that $N(\rho)$ admits a free hyperkähler action of

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$G$ defined by allowing gauge transformations with arbitrary values at $t = 1$. From (2.9) the action of the complexified group $G^c$ on $G^c \times S(\rho)$ is given by right multiplication on the first factor, and from (2.11) the complex moment map for this action is

$$(g, \beta) \longmapsto -\mathrm{Ad}\,(g^{-1})\,\beta. \tag{4.1}$$

Therefore we have a free hyperkähler action of $G$ on the product manifold $M \times N(\rho)$ and so its hyperkähler quotient is a manifold. Part (ii) of Theorem 1 will be proved once we show that this hyperkähler quotient is isomorphic, as a complex-symplectic manifold, to an open subset of $\mu^{-1}(S(\rho))$. For this it will be enough to show that $\mu^{-1}(S(\rho))$ is isomorphic to the complex-symplectic quotient of

$$M \times N(\rho) \simeq M \times (G^c \times S(\rho))$$

by $G^c$. The moment map for this action is

$$\hat{\mu}(m, g, \beta) = \mu(m) - \mathrm{Ad}\,(g^{-1})\,\beta.$$

Consider $\hat{\mu}^{-1}(0)$. It is acted upon by $G^c$ and it is easy to see that the assignment $(m, \beta, g) \mapsto (g \cdot m, \beta)$ maps the $G^c$-orbits of $\hat{\mu}^{-1}(0)$ bijectively onto the set

$$\{(m, \beta) \in M \times S(\rho)\,;\, \mu(m) - \beta = 0\}.$$

This set is simply $\mu^{-1}(S(\rho))$. The above argument shows also that the complex-symplectic forms agree. This proves Part (ii).

For Part (iii), we have to show, in view of Proposition 2.1, that $M \times N(\rho)$ has a Kähler potential for $\omega_1$ which has a critical point on each $G^c$-orbit ($G^c$-orbits on $M \times G^c \times S(\rho)$ are closed). We claim that for this it is enough to show that $N(\rho)$ has a ($G$-invariant) Kähler potential $K_o$ for $\omega_1$ which is proper on each $G^c$-orbit. Indeed, let $K$ be the $G$-invariant Kähler potential for $(M, \omega_1)$, which is, by assumption, bounded from below on each $G^c$-orbit. Then the Kähler potential on $M \times N(\rho)$ is $K + K_o$ and, since the sum of a proper bounded from below function and a function bounded from below is proper, $K + K_o$ has a minimum on each $G^c$-orbit. Therefore we have to show that $N(\rho)$ has a $G$-invariant Kähler potential $K_o$ for $\omega_1$ which is proper on each $G^c$-orbit (Proposition 2.1(i) implies then that $K_o$ is bounded from below on each $G^c$-orbit). The existence of a $G$-invariant Kähler potential follows [**13**] from the fact that $N(\rho)$ admits an isometric SO(3)-action rotating the complex structures. We claim that this potential $K_o$ will be proper on each $G^c$-orbit if the square $\|\mu_1\|^2$ of the norm of the moment map for $\omega_1$ is proper on each $G^c$-orbit. Indeed, Proposition 2.1(ii) implies then that $K_o$ has a critical point on each $G^c$-orbit, and, as a strictly convex function has a critical point if and only if it is proper, Proposition 2.1(i) shows that $K_o$ is proper on each $G^c$-orbit. Let us then finally show that $\|\mu_1\|^2$ is proper on each $G^c$-orbit. The moment map $\mu_1$ at $(T_0, T_1, T_2, T_3)$ is given by $-T(1)$ (see (2.10)) and so $\|\mu_1\|^2$ at $(\alpha, \beta)$ is given by $\|\alpha + \alpha^*\|^2$. Hence, if we consider the $G^c$-orbit of $(\alpha, \beta)$, then the restriction of $\|\mu_1\|^2$ as a function of $h = g^* g \in G^c/G$ with $g \in G^c$, is given by

$$\|\mathrm{Ad}\,(h)\,\alpha(1) + \alpha^*(1) + \tfrac{1}{2}\dot{h}(1)\,h^{-1}\|^2, \tag{4.2}$$

where $h(t) = g^*(t)\,g(t)$ and $g(t)$ is a (unique) gauge transformation with $g(0) = 1$, $g(1) = g$ and $g(\alpha, \beta)$ still satisfying (2.3), (2.4). Now we proceed as in [**8**] or [**14**]. Fix a positive Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{g}$. Then, using the Cartan decomposition of $G^c$, we have that $h(t) = k(t)\,e^{i\lambda(t)}\,k(t)^{-1}$ for each $t$, where $k(t) \in G$ and $\lambda(t) \in \overline{\mathfrak{a}^+}$. Formula (4.2) can be now rewritten, as in [**14**, Proposition 2.20], as

(*quadratic polynomial in* $\dot{\lambda}$ + *positive function in* $\lambda$ + *constant*).

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Lemma 2.10 in [**8**] implies that $\|\lambda(t)\|$ is a convex function. Therefore, if $h = h(1)$ tends to infinity in $G^c/G$, the same is true of $\dot{\lambda}(1)$ and so of $\|\mu_1\|^2$. This shows that $\|\mu_1\|^2$ is proper. This finishes the proof of Theorem 1.

REMARKS. 1. Observe that the submanifold $\mu^{-1}(S(\rho))$ can also be identified with the complex-symplectic quotient of $M$ by the unipotent subgroup $H(\rho)$ defined in the previous section. This remains true in the non-Riemannian case, that is, when $M$ is a complex-symplectic manifold acted upon by a complex reductive group.

2. The above construction of a hyperkähler metric on $\mu^{-1}(S(\rho))$ has a simple interpretation when $M$ itself is a moduli space of smooth solutions $T_0, T_1, T_2, T_3: [1, a) \to \mathfrak{g}$ to Nahm's equations. Let $\hat{T}_0, \ldots, \hat{T}_3: (0, 1] \to \mathfrak{g}$ be an element of $N(\rho)$. The point $(\{T_i\}, \{\hat{T}_i\}) \in M \times N(\rho)$ is in the intersection of the 0-sets of the three moment maps if and only if $T_i(1) = \hat{T}_i(1)$ for $i = 1, 2, 3$. Since $T_0, \hat{T}_0$ can be made identically equal to 0 near $t = 1$ by a gauge transformation, we obtain a smooth solution to Nahm's equations on $(0, a)$. The quotienting by $G$ is now equivalent to allowing gauge transformations with an arbitrary value at $t = 1$. Hence in this case our construction produces the moduli space of solutions to Nahm's equations on $(0, a)$ with boundary behaviour of an element of $N(\rho)$ at $t = 0$ and of an element of $M$ at $t = a$.

We can apply the last remark to the manifold $N(\rho)$ with the $G$-action. Let $\rho': \mathfrak{su}(2) \to \mathfrak{g}$ be a Lie algebra homomorphism. By the discussion at the beginning of the previous section $\mu^{-1}(S(\rho'))$ can be identified with the submanifold $\mu^{-1}(S(\rho, \rho'))$ of $T^* G^c$ and from the remark above (and using the fact, proven above, that $N(\rho)$ satisfies the assumption of Theorem 1(iii)) we have the following.

COROLLARY 4.1. *The submanifold $\mu^{-1}(S(\rho, \rho'))$ of $T^* G^c$ can be identified with the moduli space of solutions $T_0, T_1, T_2, T_3$ to Nahm's equations analytic on $(0, 2)$ and having simple poles at $0, 2$ with $\operatorname{res}_{t=0} T_i = \frac{1}{2}\rho(\sigma_i)$, $\operatorname{res}_{t=2} T_i = \frac{1}{2}\rho'(\sigma_i)$, where $\sigma_i$ for $i = 1, 2, 3$, are the Pauli matrices.*

REMARK. Observe that in the case of $\rho, \rho'$ being regular (that is, $\dim_{\mathbb{C}} S(\rho)$, $\dim_{\mathbb{C}} S(\rho') = \operatorname{rank} \mathfrak{g}$) one can think of $\mu^{-1}(S(\rho, \rho'))$ as a desingularisation of $(T \times \mathfrak{h})/W$, where $T \subset G^c$ is a maximal torus, $\mathfrak{h} \subset \mathfrak{g}^c$ its Lie algebra, and $W$ is the Weyl group.

## 5. *Adjoint orbits and* ALE *spaces*

If $\Gamma$ is a finite subgroup of SU(2), then the quotient $\mathbb{C}^2/\Gamma$ has a simple singularity at 0 and it has been shown by Kronheimer [**20**] that the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ admits a family of hyperkähler ALE metrics. As mentioned in the Introduction ALE stands for *asymptotically locally euclidean* and means that the metrics approximate the Euclidean metric up to $O(r^{-4})$. For the cyclic subgroups these are the multi-Eguchi–Hanson instantons as described by Hitchin [**11**].

THEOREM (Kronheimer). *Let three 2-cohomology classes $\alpha_1, \alpha_2, \alpha_3$ on $\widetilde{\mathbb{C}^2/\Gamma}$ be given such that on each homology 2-sphere with self-intersection equal to $-2$, at least one of the $\alpha_i$ is non-zero. Then there is an* ALE *hyperkähler structure on $\widetilde{\mathbb{C}^2/\Gamma}$ for which the cohomology classes of the three Kähler forms are the given $\alpha_i$.*

Kronheimer constructs these hyperkähler structures as hyperkähler quotients of certain flat spaces. On the other hand the simple singularities $\mathbb{C}^2/\Gamma$ are an example of

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the A-D-E pattern as shown in the celebrated theorem of Brieskorn [**5**], which states that the intersection of the nilpotent variety of a simple Lie group with the transverse slice $S(\rho)$ to the subregular orbit (that is, the unique nilpotent orbit such that $\dim_{\mathbb{C}} Z(y) = \operatorname{rank} \mathfrak{g} + 2$ for any of its elements $y$) is biholomorphic to $C^2/\Gamma$.

The group $\Gamma$ is determined by the Dynkin diagram of $\mathfrak{g}^c$ (see [**26**]); for Lie algebras of type $A$ these are cyclic groups, for type $D$ dihedral, for $E_6$, $E_7$, $E_8$ they are tetrahedral, octahedral and icosahedral, respectively.

On the other hand, the intersection of the transverse slide $S(\rho)$ (to the subregular orbit) with a regular semisimple orbit is biholomorphic to the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ [**26**]. It is also a result of Kronheimer [**21**] that a regular semisimple orbit possesses a family of hyperkähler structures invariant under the adjoint action of $G$. Moreover the associated complex symplectic form of these is the Kirillov–Konstant–Souriau form, so the complex moment map for the adjoint action of $G^c$ on an orbit $\mathcal{O}$ is just the embedding $\mathcal{O} \hookrightarrow \mathfrak{g}^c$. Moreover it is simple to verify (see below) that the assumption of Theorem 1(iii) is satisfied and so we obtain hyperkähler structures on $\mathcal{O} \cap S(\rho) \simeq \widetilde{\mathbb{C}^2/\Gamma}$. These will be the ALE spaces.

In addition, since the hyperkähler structures on adjoint orbits are given by Nahm's equations, it follows from Remark 2 in Section 4 that the same will be true for the hyperkähler structures on $\mathcal{O} \cap S(\rho)$.

Let us recall that Kronheimer [**21**] shows that the space $G^c/T^c$, where $T^c$ is a maximal torus in $G^c$, carries a family of hyperkähler structures parameterized by triples $\tau_1, \tau_2, \tau_3$ in a fixed Cartan algebra $\mathfrak{h}$ in $\mathfrak{g}$ such that the centralizer of the triple is $\mathfrak{h}$. The corresponding hyperkähler manifold is defined as the moduli space of smooth ($\mathfrak{g}$-valued) solutions $(T_0, T_1, T_2, T_3)$ to Nahm's equations on $(0, \infty)$ such that $\|T_0(t)\|, \|\dot{T}_0(t)\|, \|T_i(t) - \tau_i\|, \|\dot{T}_i(t)\|$ are all $O(e^{-\eta t})$ for $\eta > 0$, $i = 1, 2, 3$. The relevant gauge group consists of gauge transformations $g(t)$ such that $g(0) = 1$ and $g(t)$ preserves the above decay condition. Let us notice that the Kähler potential for $\omega_1$ is $\int_0^\infty (\|T_1\|^2 + \|T_3\|^2 - \|\tau_1\|^2 - \|\tau_3\|^2)$ and so it is bounded from below.

We can now phrase a result which shows that the ALE spaces arise naturally from the hyperkähler metrics on adjoint orbits.

THEOREM 5.1. *Let $G$ be a simple group, $S(\rho)$ the transverse slice to the subregular nilpotent adjoint orbit defined by* (1.2), *and $\tau_1, \tau_2, \tau_3$ three elements of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that the centralizer of the triple is $\mathfrak{h}$. Then,*

(i) *the hyperkähler quotient $L$ of the product manifold $M(\tau_1, \tau_2, \tau_3) \times N(\rho)$ by the group $G$ is an* ALE *space;*

(ii) *$L$ can be identified with the moduli space of solutions $(T_0, T_1, T_2, T_3)$ to Nahm's equations, smooth on $(0, \infty)$ with simple poles of residues* (3.1) *at $t = 0$ and such that $\|T_0(t)\|, \|\dot{T}_0(t)\|, \|T_i(t) - \tau_i\|, \|\dot{T}_i(t)\|$ are all $O(e^{-\eta t})$ for $\eta > 0$, $i = 1, 2, 3$;*

(iii) *the cohomology classes of the three Kähler forms on $L$ are $2\pi\tau_1, 2\pi\tau_2, 2\pi\tau_3$;*

(iv) *if $\tau_2 + i\tau_3$ is a regular element of $\mathfrak{g} \oplus \mathbb{C}$, then $(L, \omega_2 + i\omega_3)$ is isomorphic, as a complex-symplectic manifold, to $\mathcal{O}(\tau_2 + i\tau_3) \cap S(\rho)$.*

The only parts that require arguing are (i) and (iii). While the proof is not difficult, it is not quite along the lines of this paper. It will be given in [**3**]; let us, however, remark here that the ALE property of the metric is a consequence of the action of $\mathbb{R}^+$ given by $r \cdot T_i(t) = rT_i(rt)$ and mapping $M(\tau_1, \tau_2, \tau_3)$ to $M(r\tau_1, r\tau_2, r\tau_3)$.

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## 6. *Magnetic monopoles*

The SU($n$) monopoles constitute another example of a situation where hyperkähler manifolds arise by the construction of Theorem 1. Without going into details, let us recall [**12**, **15**, **6**] that there is a well-known relation between moduli spaces of (framed) SU($n$) monopoles and moduli spaces of solutions to Nahm's equations (the functions $T_i$ take values in a unitary Lie algebra). In particular the moduli spaces of monopoles with minimal symmetry breaking [**24**] are related to the moduli spaces of solutions to Nahm's equations described in Corollary 4.1 (for $G = U(m)$), that is, to submanifolds of $T^*\mathrm{GL}(m)$ of the form $\mu^{-1}(S(\rho, \rho'))$. If $\rho, \rho'$ are irreducible as representations of $\mathfrak{su}(2)$ on $\mathbb{C}^m$, we obtain the moduli space of framed SU(2) monopoles of charge $m$ (cf. [**8**]). It is well known that in this case the two moduli spaces (monopoles and solutions to Nahm's equations) are diffeomorphic [**12**] and even isometric [**25**]. The complex-symplectic structure in this case is also known [**8**]: the moduli space of framed SU(2) monopoles of charge $m$ is isomorphic to the space $\mathrm{Rat}_m(\mathbb{C}P^1, \mathbb{C}P^1)$ of based rational maps of degree $m$ from $\mathbb{C}P^1$ to itself (with the complex-symplectic structure given by poles and residues; see [**1**]). Combining this information with the formula (2.11) for the complex moment map, we obtain the following.

COROLLARY 6.1. *The space* $\mathrm{Rat}_m(\mathbb{C}P^1, \mathbb{C}P^1)$ *is isomorphic, as a complex-symplectic manifold, to the intersection* $V_1 \cap V_2$ *of two copies of* $\mathrm{GL}(m) \times S(\rho)$ *inside* $\mathrm{GL}(m) \times \mathfrak{gl}(m)$, *where*

$$V_1 = \mathrm{GL}(m) \times S(\rho) \quad \textit{and} \quad V_2 = \{(g, \beta) \in \mathrm{GL}(m) \times \mathfrak{gl}(m);\ -g\beta g^{-1} \in S(\rho)\}.$$

*Here* $\rho\colon \mathfrak{su}(2) \to \mathfrak{u}(m)$ *is an irreducible m-dimensional representation of* $\mathfrak{su}(2)$, *that is, y is a regular nilpotent element of* $\mathfrak{gl}(m)$.

As mentioned above, all this can be extended to monopoles with non-minimal symmetry breaking. In such a case the relevant group $G$ will be a product of unitary groups and the manifold $M$ will be a product of $T^*G$ and Euclidean spaces (cf. [**14**, **15**]). It is somewhat interesting that moduli spaces of monopoles with *maximal* symmetry breaking arise from moduli spaces of monopoles with *minimal* symmetry breaking (as hyperkähler quotients of products of the latter ones).

It is more interesting to notice that the moduli space of charge 2 SU(2)-monopoles can be obtained as a slice to a coadjoint orbit. For this consider the minimal nilpotent orbit $\mathcal{O}$ in $\mathrm{Sl}(3, \mathbb{C})$ which carries a hyperkähler structure described in [**22**, **16**]. This hyperkähler structure is invariant under the adjoint action of SU(3) and we consider the action of the subgroup $\mathrm{SU}(2)/\mathbb{Z}_2 \simeq \mathrm{SO}(3)$, given by the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$. A simple calculation shows that $\mu^{-1}(S(\rho))$, where $\rho$ is the identity map on $\mathfrak{su}(2)$, is biholomorphic to the surface with equation $x^2 - yz^2 = 1$, which is the double cover of the Atiyah–Hitchin manifold [**1**]. By virtue of Remark (vi) of the Introduction the hyperkähler metric on $\mu^{-1}(S(\rho))$ admits an action of SO(3) rotating the complex structures and whose orbits can be seen to be generically 3-dimensional. This implies [**1**] that the hyperkähler metric on $\mu^{-1}(S(\rho))$ is the Atiyah–Hitchin metric.

If, instead of a nilpotent orbit, we start with a minimal semisimple orbit, we obtain the deformations of the Atiyah–Hitchin metric constructed by Dancer in [**6**]. It would be interesting to know what happens in the case of the action of SU($k$) on minimal orbits in $\mathfrak{sl}(k(k+1)/2, \mathbb{C})$, where $\mathfrak{sl}(k, \mathbb{C})$ is embedded into $\mathfrak{sl}(k(k+1)/2, \mathbb{C})$ via the symmetric square.

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## 7. *Flat hyperkähler metrics*

The obvious hyperkähler structure on $\mathbb{H}^k \simeq \mathbb{R}^{4k}$ is preserved under the natural action of $\mathrm{Sp}(k) \subset \mathrm{SO}(4k, \mathbb{R})$. The complex moment map $\mu\colon \mathbb{C}^k \times \mathbb{C}^k \to \mathfrak{sp}(k, \mathbb{C})$ (1.3) is easily seen to be

$$\mu(u, w) = \begin{pmatrix} uw^T & uu^T \\ ww^T & -wu^T \end{pmatrix}.$$

In addition the Kähler potential for $\omega_1$ is $\|u\|^2 + \|w\|^2$, and so it has a global lower bound. The same is true for any open subset of $\mathbb{H}^k$.

If we consider a subgroup $G$ of $\mathrm{Sp}(k)$, then the complex moment map for the action of $G^c$ is simply the above $\mu$ followed by projection $\mathfrak{sp}(k, \mathbb{C}) \to \mathfrak{g}^c$. For each such a $G$ we obtain hyperkähler metrics on complex submanifolds $\mu^{-1}(S(\rho))$ of $\mathbb{C}^{2k}$. Let us give two non-trivial examples.

(1) *Flat metric on* $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$.

Consider the diagonal adjoint action of a compact semisimple Lie group $G$ on the flat hyperkähler space $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. The moment map for the complexified action of $G^c$ on $\mathfrak{g}^c \oplus \mathfrak{g}^c$ is easily seen to be $(\alpha, \beta) \mapsto [\alpha, \beta]$. Therefore for any homomorphism $\rho\colon +\mathfrak{su}(2) \to \mathfrak{g}$ we obtain a hyperkähler structure on

$$\{(\alpha, \beta) \in \mathfrak{g}^c \oplus \mathfrak{g}^c ; [\alpha, \beta] \in S(\rho)\}.$$

In the simplest case $\mathfrak{g} = \mathfrak{su}(2)$ the arguments at the end of the previous section imply that the hyperkähler metric obtained is related to the metric on the moduli space of charge 2 SU(2) monopoles. Our guess is that

$$\{(\alpha, \beta) \in \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) ; [\alpha, \beta] \in S(\rho)\}$$

is biholomorphic to a branched covering of $\mathrm{Rat}_2(\mathbb{C}P^1, \mathbb{C}P^1)$.

(2) *Monads*.

Consider the set $V$ of matrices

$$(\alpha, \beta, a, b) \in M_{k+k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{k\times r}(\mathbb{C}) \times M_{r\times k}(\mathbb{C})$$

satisfying

$$\begin{pmatrix} \alpha + \lambda \\ \beta + \lambda \\ a \end{pmatrix} \quad \text{injects as map } \mathbb{C}^k \longrightarrow \mathbb{C}^{2k+r} \quad \text{for all } \lambda \in \mathbb{C},$$

$$(-\beta - \mu \quad \alpha + \mu \quad b) \quad \text{surjects as a map } \mathbb{C}^{2k+r} \longrightarrow \mathbb{C}^k \quad \text{for all } \mu \in \mathbb{C},$$

with the natural flat hyperkähler metric. The group $U(k)$ acts freely preserving the hyperkähler structure, and the hyperkähler quotient construction produces the space of *monads* which was shown by Maciocia [**23**] to be isometric to the space of framed instantons over $\mathbb{R}^4$. By taking any homomorphism $\rho\colon \mathfrak{su}(2) \to \mathfrak{u}(k)$, we obtain from Theorem 1 a hyperkähler structure on a complex submanifold of $V$ defined as

$$\{(\alpha, \beta, a, b) \in V ; [\alpha, \beta] + ab \in S(\rho)\}.$$

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