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On the dynamics of generalized McMullen maps

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Abstract. In this paper, we study the dynamics of the two-parameter family of rational maps

$$F_{a,b}(z) = z^n + \frac{a}{z^n} + b.$$

We give the topological description of Julia sets and Fatou components of $F_{a,b}$ according to the dynamical behavior of the orbits of its free critical points.

1. *Introduction*

In this paper, let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ denote a rational map on the Riemann sphere $\overline{\mathbb{C}}$. The Julia set $J(R)$ of R is defined to be the set of points at which the family of iterates of R fails to be a normal family in the sense of Montel. Equivalently, the Julia set is the closure of repelling periodic points of R. The complement of the Julia set is called the Fatou set. A connected component of the Fatou set is called a Fatou component. According to Sullivan's eventual periodicity theorem, every Fatou component is eventually periodic and there are only five kinds of periodic Fatou components which are attracting domains, superattracting domains, parabolic domains, Siegel disks and Herman rings. The critical point of R is defined to be the point at which R is not univalent in any neighborhood. For properties of the Julia set and general background in complex dynamics, we refer to [**1, 17**].

An interesting and important problem in complex dynamics is to describe the topology of the Julia sets of rational maps. For a polynomial, it is well known that its Julia set is connected if and only if the orbits of its finite critical points are bounded. Moreover, Qiu and Yin obtained a sufficient and necessary condition for the Julia set of a polynomial to be a Cantor set in [**20**]. For a genuine rational map, this is a difficult problem if the rational map is not essentially topologically conjugate to a polynomial. Usually we can discuss the topology of Julia sets for some special families of rational maps. For example,

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Devaney *et al* studied the so-called McMullen family of rational maps $F_\lambda(z) = z^n + \lambda/z^d$ in a series of papers [**2, 7, 9–11, 13**]; see also [**21–23**]. In [**9**], it was proved that if the orbits of critical points of F_λ tend to ∞, then the Julia set of F_λ is either a Cantor set, or a Sierpiński curve, or a Cantor set of simple closed curves. In [**24**], Steinmetz discussed the dynamics of the rational family $R_t(z) = t(1 + (4/47)z^3/(1 - z))$. He proved that, for the escape parameter t, the Julia set of R_t is either a Cantor set, or a Sierpiński curve, or a curve with one or else infinitely many cut-points. The same problem has been studied in a recent paper [**15**] for the family of rational maps $f_t(z) = -t(z^2 - 2)^2/4(z^2 - 1)$ by Steinmetz and his student. They proved that for any escape parameter t, the boundary of the basin of infinity of f_t is either a Cantor set, a curve with infinitely many complementary components, or else a Jordan curve. In the latter case, the Julia set of f_t is a Sierpiński curve.

Our goal in this paper is to describe Julia sets of rational maps in the family given by

$$F_{a,b}(z) = z^n + \frac{a}{z^n} + b,$$

where $n \in \mathbb{Z}^+$ and $n \geq 2$. When $b = 0$, these maps reduce to the McMullen maps, and when $a = 0$ they reduce to the well-studied polynomials $z \mapsto z^n + b$. Noting that the McMullen maps can be regarded as the singular perturbation of $z \to z^n$, $F_{a,b}$ are the singular perturbation of the polynomials $z \mapsto z^n + b$, so we call $F_{a,b}$ the generalized McMullen maps. Different from all the above mentioned families, for which there is essentially only one free critical orbit, the generalized McMullen maps have two free critical orbits. So the dynamics of $F_{a,b}$ will be more complicated. The generalized McMullen maps $F_{a,b}$ were first studied by Devaney and his coauthors; see [**3, 4, 8**]. In [**4**], it was shown that when b is the center of a hyperbolic component of the multibrot set of degree n and a is sufficiently small, then the Julia set of $F_{a,b}$ consists of a countable collection of closed curves together with an uncountable collection of point components. In [**8**], it was also shown that if $n = 2$ and b is the center of a hyperbolic component in the Mandelbrot set, then the Julia sets of $F_{a,b}$ converge to the filled Julia sets of $F_{0,b}$ as a tends to 0. In this paper, we will give a quite complete description of the Julia sets of the generalized McMullen maps $F_{a,b}$ for arbitrary parameters a, b.

The paper is organized as follows. In §2, some basic properties of $F_{a,b}$ are presented. In §3, we describe the Julia set of $F_{a,b}$ for the case that both critical orbits escape to infinity. In §4, we discuss the case that one critical orbit tends to infinity while the other is bounded. In §5, we deal with the case that both critical orbits are bounded.

2. *Preliminaries*

Let

$$F_{a,b}(z) = z^n + \frac{a}{z^n} + b,$$

where $a, b \in \mathbb{C}\backslash\{0\}$, $n \in \mathbb{Z}^+$ and $n \geq 2$.

The point at ∞ is a superattracting fixed point of $F_{a,b}$. Let $A(\infty)$ denote the attracting basin of ∞ and B denote the immediately attracting basin of ∞; that is, B is the Fatou component containing ∞. Unlike for polynomials, the attracting basin $A(\infty)$ may consist of infinitely many components. Let T denote the component of $A(\infty)$ which contains

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the pole 0. Then T may be disjoint from B. If this is the case, both $F_{a,b}: T \to B$ and $F_{a,b}: B \to B$ are n-to-1 branched coverings since both 0 and ∞ have local degree n, and T is the unique component of $f^{-1}(B)$. So every component of $A(\infty)$, except T and B themselves, must be an iterative preimage of T, that is, a component of $F_{a,b}^{-k}(T)$ for some integer $k \geq 1$.

The points ∞ and 0 are two critical points of $F_{a,b}$ at which $F_{a,b}$ has local degree n. These two critical points have simple critical orbits. There are also $2n$ simple critical points of $F_{a,b}$ which have the form $c_k = \omega^k \sqrt[2n]{a}$, where $\omega = e^{i\pi/n}$ and $k = 0, \ldots, 2n-1$. However, there are only two critical values corresponding to these critical points, which are

$$F_{a,b}(c_k) = \begin{cases} v_+ = b + 2\sqrt{a} & \text{when } k \text{ is even,} \\ v_- = b - 2\sqrt{a} & \text{when } k \text{ is odd.} \end{cases} \tag{1}$$

The behavior of critical orbits of c_k will determine the dynamics of $F_{a,b}$. We call c_k ($k = 0, 1, \ldots, 2n-1$) the free critical points.

Set $\Omega_+ = \{\omega : \omega^n = 1\}$, $\Omega_- = \{\omega : \omega^n = -1\}$ and $\Omega = \Omega_+ \cup \Omega_-$. Set $\mathrm{C}_+ = \{\omega \sqrt[2n]{a} : \omega \in \Omega_+\}$, $\mathrm{C}_- = \{\omega \sqrt[2n]{a} : \omega \in \Omega_-\}$ and $\mathrm{C} = \mathrm{C}_+ \cup \mathrm{C}_-$. C is the set of free critical points. Since $F_{a,b}$ has local degree two at every critical point $c \in \mathrm{C}$ and $F_{a,b}$ is a degree-$2n$ rational map, we have $F_{a,b}^{-1}(v_+) = \mathrm{C}_+$ and $F_{a,b}^{-1}(v_-) = \mathrm{C}_-$. All free critical points lie on the critical circle $\Gamma_a = \{z \in \mathbb{C} : |z| = \sqrt[2n]{|a|}\}$. Let $H(z) = \sqrt[n]{a}/z$. Then $H(\Gamma_a) = \Gamma_a$ and H interchanges the two complementary components of Γ_a.

The set $K_{a,b} = \mathbb{C} \backslash A(\infty)$ is called the filled Julia set of $F_{a,b}$. A component of $K_{a,b}$ is called critical if it contains at least one critical point. We denote the component of $K_{a,b}$ containing z by $K(z)$.

The following results are straightforward.

LEMMA 2.1. *Let $\omega \in \Omega_+$. Then the following hold.*

(1) $F_{a,b}(\omega z) = F_{a,b}(z)$ *and* $F_{a,b}(H(z)) = F_{a,b}(z)$.
(2) *For any Fatou component U, both ωU and $H(U)$ are also Fatou components, where $\omega U = \{\omega z : z \in U\}$. In particular, $\omega B = B$, $\omega T = T$ and $H(B) = T$, $H(T) = B$.*
(3) *Let W_0 be the component of $\overline{\mathbb{C}} \backslash \overline{B}$ which contains T. Then $\omega W_0 = W_0$.*

That is, both B and T are n-fold symmetric. For a Fatou component U different from B and T, either U is n-fold symmetric, or ωU are disjoint Fatou components for all $\omega \in \Omega_+$. To prove this, we first show the following lemma.

LEMMA 2.2. *Suppose $\gamma \subset \mathbb{C}$ is a closed curve which separates 0 from ∞. Then $\sigma\gamma \cap \gamma \neq \emptyset$ for all $\sigma = e^{2\pi i\theta}$, $\theta \in [0, 1)$.*

Proof. We take two points z_1 and z_2 on γ such that

$$d(z_1, 0) = \inf_{z \in \gamma} d(z, 0) \quad \text{and} \quad d(z_2, 0) = \sup_{z \in \gamma} d(z, 0),$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{C}$. Then γ is contained in $A(r_1, r_2) = \{z, r_1 \leq |z| \leq r_2\}$ where $r_1 = d(z_1, 0)$, $r_2 = d(z_2, 0)$ and γ connects two boundary circles of $A(r_1, r_2)$. For any $\sigma = e^{2\pi i\theta}$ with $\theta \in [0, 1)$, we have $\sigma z_1 \in \{|z| = r_1\}$ and $\sigma z_2 \in \{|z| = r_2\}$. If none of them lie on the curve γ, then they must lie in different components

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of $\overline{\mathbb{C}}\backslash\gamma$. Since γ is a closed curve in $A(r_1, r_2)$ surrounding 0, any curve that connects the boundary circles $\{|z| = r_1\}$ and $\{|z| = r_2\}$ must intersect γ. Hence, the arc $\alpha \subset \sigma\gamma$ that connects σz_1 and σz_2 must intersect γ. This implies $\sigma\gamma \cap \gamma \neq \emptyset$. □

LEMMA 2.3. *Let U be a Fatou component of $F_{a,b}$. Suppose that both z_0 and $\omega^j z_0$ belong to U, where $\omega = \exp(2\pi i/n)$ and $j \in \{1, 2, \ldots, n-1\}$. Then $\omega^k z_0$ belongs to U for all $k \in \{0, 1, \ldots, n-1\}$. As a consequence, U is n-fold symmetric.*

Proof. Suppose that both z_0 and $\omega^j z_0$ lie in U but $\omega^k z_0$ does not lie in U for some $k \notin \{0, j\}$. Let γ_1 be a curve in U that connects z_0 and $\omega^j z_0$. Then $\gamma_2 = \omega^j\gamma_1$ is a curve in $\omega^j U$, which is also a Fatou component by Lemma 2.1. Since

$$\omega^j z_0 \in \gamma_1 \cap \gamma_2 \subset U \cap \omega^j U,$$

we obtain $U = \omega^j U$. Hence, γ_2 is a curve in U which connects $\omega^j z_0$ and $\omega^{2j} z_0$. Consequently, U contains $\gamma_l = \omega^{lj}\gamma_1$ for all $l \in \mathbb{N}$.

Now, there must exist a smallest positive integer m such that $\omega^{mj} = 1$. Let $\gamma = \bigcup_{l=1}^{m} \gamma_l$. Then $\gamma \subset U$ is a closed curve, 0 and ∞ lie in different connected components of $\mathbb{C}\backslash\gamma$.

Let $\beta = \omega^k\gamma$. Then $\beta \subset \omega^k U$. From Lemma 2.2, we obtain

$$\emptyset \neq \beta \cap \gamma \subset U \cap \omega^k U.$$

Since $\omega^k U$ also is a Fatou component, we have $U = \omega^k U$. This implies that $\omega^k z_0 \in \beta \subset U$. □

In the following, we prove that the rational map $F_{a,b}$ has no Herman ring in its Fatou set.

THEOREM 2.4. *The rational map $F_{a,b}$ has no Herman ring.*

Proof. Suppose $F_{a,b}$ has a cycle of Herman rings $\{D_0, D_1, \ldots, D_p = D_0\}$. Then $F_{a,b}^p$ is conjugate to the irrational rotation $z \mapsto \lambda z$ on D_0, where $\lambda = \exp(2\pi i\alpha)$ and α is an irrational number. For each $0 \leq k \leq p-1$, $F_{a,b} : D_k \to D_{k+1}$ is a conformal mapping. Take a Jordan curve L in D_0 invariant under $F_{a,b}^p$ which separates two connected components of ∂D_0. Let U be the bounded connected component of $\mathbb{C}\backslash L$. Then $U \cap J_{a,b} \neq \emptyset$. Since $F_{a,b}^m(L) \subset D_k$ where $k = m \bmod p$ and D_k is disjoint from B for all k, we have that $\{F_{a,b}^m(L)\}$ are uniformly bounded.

We claim $U \cap \{z : \exists m \geq 0,\ F_{a,b}^m(z) = 0\} = \emptyset$. Otherwise, if there exists a point $z_0 \in U$ and a smallest positive integer $m \geq 0$ such that $F_{a,b}^m(z_0) = 0$, then $F_{a,b}^m(L)$ is a closed curve surrounding 0 in D_k, where $k = m \bmod p$. For any $\omega^l \in \Omega_+$, $\omega^l F_{a,b}^m(L)$ is also a closed curve surrounding 0. From Lemma 2.2, we obtain $\omega^l F_{a,b}^m(L) \cap F_{a,b}^m(L) \neq \emptyset$. However, $\omega^l F_{a,b}^m(L) \subset \omega^l D_k$, which implies that $\omega^l D_k = D_k$. This shows that $F_{a,b}$ is not a conformal map on D_k, which is a contraction. Now the fact $U \cap \{z : \exists m \geq 0,\ F_{a,b}^m(z) = 0\} = \emptyset$ implies that $F_{a,b}^m$ is holomorphic on the domain U for all m. Since $\{F_{a,b}^m(L)\}$ are uniformly bounded, by the maximal modulus principle, $F_{a,b}^m(U)$ is uniformly bounded. So $\{F_{a,b}^m\}$ is normal on U, which contradicts $J_{a,b} \cap U \neq \emptyset$. □

LEMMA 2.5. *Suppose U is a bounded simply connected domain and contains exactly one free critical value. Then there are exactly n connected components of $F_{a,b}^{-1}(U)$ and*

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every connected component of $F_{a,b}^{-1}(U)$ is simply connected. Moreover, if every connected component of $F_{a,b}^{-m}(U)$ contains at most one free critical value for every $m \in \mathbb{N}$, then they are all simply connected.

Proof. Let V denote a connected component of $F_{a,b}^{-1}(U)$. By the Riemann–Hurwitz formula we obtain

$$\chi(V) + \delta(V) = \deg(F_{a,b}|V)\chi(U),$$

where $\chi(V)$ denotes the Euler characteristic of V and $\delta(V)$ denotes the number of critical points (counted with multiplicity) in V. By the assumption, $\chi(U) = 1$. Suppose that U contains the free critical value v_+. Since $F_{a,b}^{-1}(v_+) = \mathrm{C}_+$ and the local degree of $F_{a,b}$ at every free critical point is two, we get $\deg(F_{a,b}|V) = 2\delta(V)$. Thus $\chi(V) = \deg(F_{a,b}|V)\chi(U) - \delta(V) = \delta(V) \geq 1$. However, $V \subsetneq \mathbb{C}$ is a domain, $\chi(V) \leq 1$. So we have $\chi(V) = \delta(V) = 1$. This shows that V is simply connected and V contains only one critical point in C_+. The number of components of $F_{a,b}^{-1}(U)$ is equal to the number of critical points in C_+ which is n.

If U contains no critical values, then for every component V of $F_{a,b}^{-1}(U)$, $F_{a,b} : V \to U$ is a homeomorphism. V is certainly simply connected. Under the assumption of the second part of the lemma, an inductive argument gives that every component of $F_{a,b}^{-m}(U)$ is simply connected. □

LEMMA 2.6.

(1) *If a Fatou component D contains both free critical values v_-, v_+, then $F_{a,b}^{-1}(D)$ has only one component.*
(2) *If a Fatou component $U \nsubseteq A(\infty)$ contains only one free critical value, then U is simply connected.*
(3) *If v_-, $v_+ \notin B$, then $B \neq T$ and both B and T are simply connected.*
(4) *If at least one of v_-, v_+ is in B, then $B = T$ is completely invariant and B is infinitely connected.*

Proof. (1) Suppose that V is the Fatou component of $F_{a,b}^{-1}(D)$ containing the free critical point $\sqrt[2n]{a}$. Since v_-, $v_+ \in D$, there must exist a free critical point $\omega_- \sqrt[2n]{a} \in V$ for some $\omega_- \in \Omega_-$.

Notice that

$$H(\sqrt[2n]{a}) = \sqrt[2n]{a}, \quad H(\omega_- \sqrt[2n]{a}) = \omega_-^{-1} \sqrt[2n]{a},$$

so $\sqrt[2n]{a} \in H(V) \cap V$. This shows that $H(V) = V$ and $\omega_-^{-1} \sqrt[2n]{a}$, $\omega_- \sqrt[2n]{a} \in V$.

If $\omega_-^{-1} \sqrt[2n]{a} \neq \omega_- \sqrt[2n]{a}$, then, by Lemma 2.3, we obtain $\mathrm{C}_- \subset V$ and the degree $F_{a,b}$ on V is $2n$, and so V is the unique component of $F_{a,b}^{-1}(D)$.

If $\omega_-^{-1} \sqrt[2n]{a} = \omega_- \sqrt[2n]{a}$, then $\omega_-^2 = 1$. Noting that $\omega_- \in \Omega_-$, that is $\omega_-^n = -1$, we have $\omega_- = -1$. This shows that $-\sqrt[2n]{a}$, $\sqrt[2n]{a} \in V$. Take a curve γ disjoint from 0 connecting $-\sqrt[2n]{a}$ and $\sqrt[2n]{a}$ in V. Then $H(\gamma)$ is also a curve connecting $-\sqrt[2n]{a}$ and $\sqrt[2n]{a}$ in V. Obviously $\beta = \gamma \cup H(\gamma) \subset V$ is a closed curve separating 0 and ∞. From Lemma 2.2 we obtain $\omega\beta \cap \beta \neq \emptyset$ for every $\omega \in \Omega_+$. Thus $\omega V \cap V \neq \emptyset$, which implies $\omega V = V$. This shows that V contains all critical points in C_+. Again, V is the unique component of $F_{a,b}^{-1}(D)$.

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(2) Suppose $v_+ \in U$. According to Sullivan's eventual periodicity theorem, there exists a positive integer m such that $D = F_{a,b}^m(U)$ is a periodic component of period p. If $D = U$, D certainly contains the only free critical value v_+. If $D \neq U$, then D contains at most a free critical value v_-. From Theorem 2.4, D cannot be the Herman ring: D is either a (super)attracting domain, or a parabolic domain, or a Siegel disk.

Case 1. Suppose that D is a (super)attracting domain with a (super)attracting periodic point $z_0 \in D$. Let $z_k = F_{a,b}^k(z_0) \in D_k = F_{a,b}^k(D)$ for $0 \leq k \leq p-1$. Take a small disk $\Delta_0 = \{z : |z - z_0| < r\} \subset D$ such that $\partial\Delta_0$ is disjoint from the orbits of all free critical points. Let Δ_m denote the connected component of $F_{a,b}^{-m}(\Delta_0)$ containing z_k for $k = m \bmod p$. Then we obtain $\Delta_{lp+k} \subset \Delta_{p(l+1)+k} \subset \cdots \subset D_k$, and $D_k = \bigcup_{l\geq 0} \Delta_{lp+k}$ for $0 \leq k \leq p-1$. Noting that Δ_0 is simply connected and Δ_m contains at most one free critical value for every $m \geq 0$, from Lemma 2.5 we obtain that every Δ_m is simply connected. Thus $D = D_0$ is simply connected. Applying Lemma 2.5 again, we get that U is simply connected.

Case 2. Suppose that D is a parabolic domain with a parabolic period point $z_0 \in \partial D$. The argument is the same as in Case 1. The only difference is to take a petal at the parabolic point z_0 instead of the small disk Δ_0 at the attracting point.

Case 3. Suppose that D is a Siegel disk. Then D is, by definition, simply connected. In this case, the Julia set $J_{a,b}$ contains at least one free critical value, so every component of $F_{a,b}^{-m}(D)$ contains at most one free critical value for every $m \in \mathbb{N}$. Applying Lemma 2.5 gives that U is simply connected as well.

(3) Note that in this case there is no critical value in B other than the superattracting fixed point ∞. According to the Böttcher theorem, there exists a conformal map

$$\phi : B \to \overline{\mathbb{C}}\backslash\Delta,$$

where $\Delta = \{z : |z| < 1\}$, which conjugates $F_{a,b}|B$ to $z \mapsto z^n$. So B is simply connected and $\deg(F_{a,b}|B) = n$. Thus $0 \notin B$ and $T \neq B$. Since 0 is the only preimage of ∞ in T and the local degree of $F_{a,b}$ at 0 is n, we get $\deg(F_{a,b}|T) = n$ and $\delta(T) = n - 1$. From the Riemann–Hurwitz formula, $\chi(T) = \deg(F_{a,b}|T) - \delta(T) = n - (n-1) = 1$; T is also simply connected.

(4) Suppose $v_+ \in B$, without loss of generality. Then B contains at least one preimage of v_+ which is a critical point in C_+. However, B is n-fold symmetric. This implies that all critical points in C_+ are in B. Note that $\mathrm{C}_+ = F_{a,b}^{-1}(v_+)$. We get that B is completely invariant, which follows from the fact that $T = B$. Now, B contains at least $3n-2$ critical points, say 0, ∞ (both with multiplicity $n-1$) and all critical points in C_+. Suppose B is m-connected with $1 \leq m < \infty$. Then by the Riemann–Hurwitz formula, we have

$$\deg(F_{a,b}|B)\chi(B) = \chi(B) + \delta(B) \geq \chi(B) + 3n - 2.$$

Since $\deg(F_{a,b}|B) = 2n$ and $\chi(B) = 2 - m$, we get $(2n-1)(2-m) \geq 3n-2$, a contradiction. So B is infinitely connected. □

3. *The escape case*

In this section, we discuss the case that the orbits of both free critical values are unbounded; that is, $v_+, v_- \in A(\infty)$. By Sullivan's classification theorem, the Fatou set of $F_{a,b}$ is

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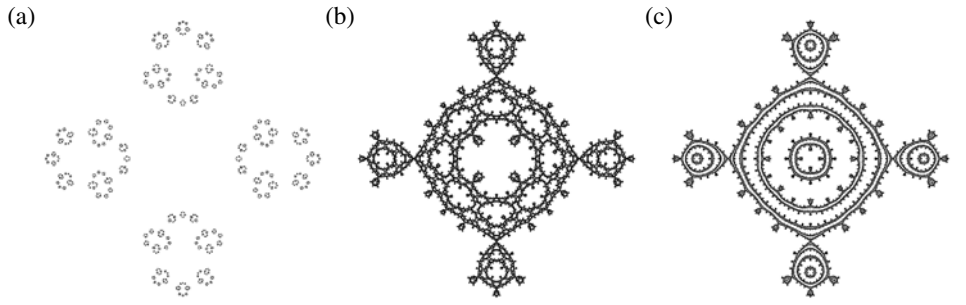


FIGURE 1. Julia sets of $F_{a,b}(z) = z^4 + a/z^4 + b$ for the cases of Theorem 3.1. (a) Case (1), Cantor Julia set ($a = 0.1$, $b = -1$). (b) Case (2.1), connected Julia set ($a = -0.002$, $b = -1$). (c) Case (2.2), disconnected Julia set but not Cantor ($a = 0.00004$, $b = -1$).

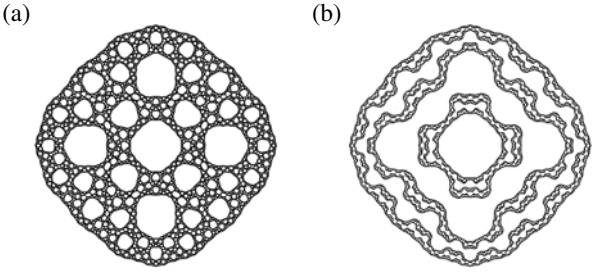


FIGURE 2. Special Julia sets of $F_{a,b} = z^4 + a/z^4 + b$ in the cases of Theorem 3.1(2). (a) Case (2.1) Julia set of Sierpiński curve ($a = -0.002$, $b = -0.2$). (b) Case (2.2) Julia set of Cantor circles ($a = -0.005$, $b = -0.2$).

equal to $A(\infty) = \bigcup_{m=0}^{\infty} F_{a,b}^{-m}(B)$ and the Julia set $J_{a,b} = \overline{\mathbb{C}} - \bigcup_{m=0}^{+\infty} F_{a,b}^{-m}(B)$. If $B \neq T$, we use W_0 to denote the connected component of $\overline{\mathbb{C}} \backslash \overline{B}$ containing T. In the following, we use $\mathrm{Card}(A)$ to denote the cardinality of a set A. Our goal is to prove the following theorem.

THEOREM 3.1. *Suppose the orbits of both free critical values of $F_{a,b}$ tend to ∞, that is, $v_+, v_- \in A(\infty)$.*

(1) *If $\mathrm{Card}(B \cap \{v_+, v_-\}) \geq 1$, then B is completely invariant and the Julia set $J_{a,b} = \partial B$ is a Cantor set.*

(2) *If $\mathrm{Card}(B \cap \{v_+, v_-\}) = 0$, then $B \neq T$. There are two subcases.*

(2.1) *If v_+, v_- lie in the different Fatou components, then every Fatou component is simply connected and the Julia set $J_{a,b}$ is connected and locally connected. In particular, if one of the free critical values lies in W_0, then $J_{a,b}$ is a Sierpiński curve.*

(2.2) *If v_+, v_- lie in the same Fatou component, then every Fatou component is either simply connected or doubly connected. The Julia set $J_{a,b}$ is disconnected and has infinitely many components. In particular, if both $v_+, v_- \in T$ and $n > 2$, then $J_{a,b}$ is a Cantor set of circles.*

Remark 3.1. By definition, a Sierpiński curve is a planar set homeomorphic to the well-known Sierpiński carpet fractal. From Whyburn [**25**], it is known that any planar set which

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is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpiński curve. For the family $F_{a,0}$, Devaney *et al* [**9**] showed that the Julia set is a Sierpiński curve when one of the free critical values lies in a iterative preimage of T.

Remark 3.2. A planar set is a Cantor set of circles if it is homeomorphic to $\Lambda \times S^1$, where Λ is the standard Cantor set and S^1 is the unit circle. In [**16**], McMullen found that the Julia set is a Cantor set of circles for the map $F_{a,0}(z) = z^n + a/z^d$ when $|a|$ is sufficiently small and $1/n + 1/d < 1$. Devaney *et al* [**9**] proved that if one of the critical values lies in T and $1/n + 1/d < 1$, then the Julia set of $F_{a,0}$ is a Cantor set of circles.

Remark 3.3. The structure of the Julia set of $F_{a,b}$ for the case (2.2) in Theorem 3.1 was deeply studied in [**4**], when b is the center of a hyperbolic component of the corresponding multibrot set and a is sufficiently small.

A rational map R is said to be hyperbolic if the orbit of every critical point of R accumulates at a (super)attracting cycle of R. Thus, in this section, $F_{a,b}$ is a hyperbolic rational map. A rational map R is called geometrically finite if the accumulation set of the orbits of all critical points of R is a finite set.

The following two lemmas are elementary, and can be found, for example, in [**1, 17**].

LEMMA 3.2. *Let R be a rational map of degree d, where $d \geq 2$. Let ζ be a (super)attracting fixed point of R. If all of the critical points of R lie in the immediate attracting basin of ζ, the Julia set of R is a Cantor set.*

LEMMA 3.3.

(1) *If the Julia set of a hyperbolic rational map is connected, then it is locally connected.*
(2) *If U is a simply connected Fatou component of a hyperbolic rational map, then the boundary ∂U is locally connected.*

LEMMA 3.4. *If $F_{a,b}$ is hyperbolic and $v_+, v_- \notin B$, then $\partial B \cap \partial T = \emptyset$. Moreover, the boundaries of components of $A(\infty) = \bigcup_{m=0}^{\infty} F_{a,b}^{-m}(B)$ are pairwise disjoint.*

Proof. From Lemma 2.6(3), $B \neq T$ and both of them are simply connected. By the Böttcher theorem, there exists a unique conformal map

$$\phi : B \to \overline{C} \backslash \overline{\Delta}$$

tangent at ∞ such that

$$\phi \circ F_{a,b}(z) = (\phi(z))^n,$$

where $\Delta = \{z : |z| < 1\}$. For a given angle $\theta \in [0, 1)$, by definition, the external ray in B with angle θ is $\gamma(\theta) = \{z \in B : \phi(z) = r \exp(2\pi i\theta), r > 1\}$. Obviously, $F_{a,b}(\gamma(\theta)) = \gamma(n\theta)$. If $\gamma(\theta) \to z \in \partial B$ as $r \to 1$, we say that the external ray $\gamma(\theta)$ lands at z, and z is a landing point of $\gamma(\theta)$. Since $F_{a,b}$ is hyperbolic, ∂B is locally connected by Lemmas 2.6(3) and 3.3(2). From the Carathéodory theorem, all external rays are landing and every point on ∂B is a landing point of an external ray.

If D is a component of $F_{a,b}^{-m}(B)$, a component μ of $F_{a,b}^{-m}(\gamma(\theta))$ in D is called an internal ray in D. If $\gamma(\theta)$ is landing at $z \in \partial B$, then μ is also landing on ∂D, which lands at a point (or some points) in $F_{a,b}^{-m}(z)$.

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Suppose that $\partial B \cap \partial T \neq \emptyset$, say $z \in \partial B \cap \partial T$. Then there is an external ray $\gamma(\theta)$ in B and an internal ray μ in T which both land at z and are mapped by $F_{a,b}$ to the external ray $\gamma(n\theta)$ landing at $F_{a,b}(z) \in B$. This implies that z is a critical point, which contradicts the assumption that $F_{a,b}$ is hyperbolic. So ∂B and ∂T are disjoint.

Now suppose that there are two components D_1 and D_2 of $A(\infty)$ such that $\partial D_1 \cap \partial D_2 \neq \emptyset$. Then there are smallest integers $m, k \geq 0$ such that $F_{a,b}^m(D_1) = F_{a,b}^k(D_2) = B$. If $m \neq k$, say $m < k$, then $F_{a,b}^{k-1}(D_1) = B$ and $F_{a,b}^{k-1}(D_2) = T$, which implies that $\partial B \cap \partial T \neq \emptyset$. This is a contradiction. If $n = m$, then $F_{a,b}^m(D_1) = F_{a,b}^m(D_2) = B$. There are internal rays μ_1 and μ_2 in D_1 and D_2, respectively, such that both of them land at a point $z \in \partial D_1 \cap \partial D_2$, and $\gamma = F_{a,b}^m(\mu_1) = F_{a,b}^m(\mu_2)$ is an external ray in B landing at $F_{a,b}^m(z)$. Again, z is a critical point of $F_{a,b}^m$, which is a contradiction. So we have $\partial D_1 \cap \partial D_2 = \emptyset$. □

The following lemma was proved by Morosawa for sub-hyperbolic rational maps in [**19**]. However, his proof is valid for geometrically finite rational maps.

LEMMA 3.5. [**19**] *Let R be a geometrically finite rational map with connected Julia set. Let U be a forward invariant (super)attracting Fatou component. If there exists a complementary component W of $\overline{U}$ and a Fatou component D such that $D \cup R^{-1}(D) \subset W$, then the boundary of U is a Jordan curve.*

By applying the above lemma, we can prove the following lemma.

LEMMA 3.6. *If $v_+, v_- \in A(\infty)$ but $v_+, v_- \notin B$, then v_+, v_- must belong to the same connected component of $\overline{\mathbb{C}}\backslash\overline{B}$. In particular, if $v_+, v_- \in W_0$ but they are in different Fatou components, then ∂B is a Jordan curve and $\overline{\mathbb{C}}\backslash\overline{B} = W_0$ is connected.*

Proof. Suppose that v_+, v_- belong to different connected components of $\overline{\mathbb{C}}\backslash\overline{B}$. Then there is a connected component U of $\overline{\mathbb{C}}\backslash\overline{B}$ different from W_0 which contains one of the free critical values v_+ and v_-, say $v_+ \in U$. Note that U is simply connected, since $\overline{B}$ is connected. By Lemma 2.5, $F_{a,b}^{-1}(U)$ has exactly n connected components and every component of $F_{a,b}^{-1}(U)$ is simply connected. Note that every component of $F_{a,b}^{-1}(U)$ contains exactly one critical point in C_+, since $v_+ \in U$. Thus, if V is a component of $F_{a,b}^{-1}(U)$, then $V, \omega V, \ldots, \omega^{n-1}V$ are all n different components of $F_{a,b}^{-1}(U)$, where $\omega = e^{2\pi i/n} \in \Omega_+$. Since $\partial U \subset \partial B$ and $F_{a,b}^{-1}(\partial B) = \partial B \cup \partial T$, we have $\partial V \subset \partial B \cup \partial T$. However, ∂V is connected, since V is simply connected and $\partial B \cap \partial T = \emptyset$ from Lemma 3.4; we get that either $\partial V \subset \partial B$ or $\partial V \subset \partial T$. Assume $\partial V \subset \partial B$, without loss of generality. Then by the symmetry of ∂B, we have $\partial(\omega^j V) \subset \partial B$ for all $j = 0, 1, \ldots, n-1$. Now $H(\omega^j V)$, $j = 0, 1, \ldots, n-1$, are also components of $F_{a,b}^{-1}(U)$. Noting that $H(\partial B) = \partial T$, we have $\partial H(\omega^j V) \subset \partial T$. This implies that $H(V), H(\omega V), \ldots, H(\omega^{n-1}V)$ are connected components of $F_{a,b}^{-1}(U)$ other than $V, \omega V, \ldots, \omega^{n-1}V$, which is a contradiction.

Now suppose $v_+, v_- \in W_0$ but that they are in different Fatou components. From Lemmas 2.5 and 2.6, all Fatou components are simply connected. Let $D(v_+)$ be the Fatou component containing v_+. Then $F_{a,b}^{-1}(D(v_+))$ consists of exactly n components and every one contains a critical point in C_+. As above, if D is a component of $F_{a,b}^{-1}(D(v_+))$, then $D, \omega D, \ldots, \omega^{n-1}D$ are all n components of $F_{a,b}^{-1}(D(v_+))$. Obviously, $D(v_+) \subset W_0$. We claim that $F_{a,b}^{-1}(D(v_+)) \subset W_0$. Suppose that there is a component of $F_{a,b}^{-1}(D(v_+))$, say

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D, disjoint from W_0. Then by the symmetry of W_0, $\omega^j D$, $j = 0, 1, \ldots, n-1$, are all disjoint from W_0. Let V_j be the component of $\overline{\mathbb{C}}\backslash\overline{B}$ containing $\omega^j D$, $j = 0, 1, \ldots, n-1$. Then $\partial V_j \subset \partial B$ and V_j is also disjoint from W_0 for every $j = 0, 1, \ldots, n-1$. Noting that $T \subset W_0$, we have $\partial T \subset \overline{W}_0$. However, $\partial W_0 \subset \partial B$ and $\partial B \cap \partial T = \emptyset$; we get $\partial T \subset W_0$. Since $\partial H(V_j) \subset \partial T$ and both $H(V_j)$ and W_0 are simply connected, we get $H(V_j) \subset W_0$. Hence, $H(\omega^j D) \subset H(V_j) \subset W_0$. Thus $H(D)$, $H(\omega D)$, $\ldots$, $H(\omega^{n-1} D)$ are components of $F_{a,b}^{-1}(D(v_+))$ different from $D, \omega D, \ldots, \omega^{n-1} D$, which is, again, a contradiction. Now $D(v_+) \cup F_{a,b}^{-1}(D(v_+)) \subset W_0$. By Lemma 3.5, ∂B is a Jordan curve. Consequently, $\overline{\mathbb{C}}\backslash\overline{B} = W_0$ is connected. □

Now we are in the position to prove Theorem 3.1.

Proof of Theorem 3.1. (1) Suppose $\mathrm{Card}(B \cap \{v_+, v_-\}) \geq 1$, say $v_+ \in B$. By Lemma 2.6, $B = T$ and B is completely invariant. Hence $v_- \in A(\infty) = B$. This implies that all critical points are contained in B. From Lemma 3.2, the Julia set $J_{a,b}$ is a Cantor set.

(2) Suppose $\mathrm{Card}(B \cap \{v_+, v_-\}) = 0$. By Lemma 2.6, $B \neq T$ and both B and T are simply connected.

(2.1) If v_+, v_- are in different Fatou components, then, by Lemma 2.5, all Fatou components are simply connected. This implies that the Julia set $J_{a,b}$ is connected. By Lemma 3.3, $J_{a,b}$ is also locally connected. It is well known that $J_{a,b}$ is compact and nowhere dense in $\overline{\mathbb{C}}$. Suppose that one of the free critical values lies in W_0. Then, by Lemma 3.6, both critical values are in W_0, and ∂B is a Jordan curve. Consequently, the boundary of every Fatou component is a Jordan curve. From Lemma 3.4, boundaries of Fatou components are pairwise disjoint. According to Whyburn [**25**], $J_{a,b}$ is a Sierpiński curve.

(2.2) Let V denote the Fatou component in $A(\infty)$ that contains both free critical values v_+, v_-. There is a unique integer $m \geq 0$ such that $F_{a,b}^m : V \to T$ is a conformal map, since there is no critical point in $F_{a,b}^k(V)$ for $k = 0, 1, \ldots, m-1$. So $F_{a,b}^k(V)$, $k = 0, 1, \ldots, m-1$, including V itself, are simply connected components. By Lemma 2.6, we obtain that $F_{a,b}^{-1}(V)$ has only one component U. Thus U contains all free critical points, which gives $\delta(U) = 2n$ and $\deg(F_{a,b}|U) = 2n$. By the Riemann–Hurwitz formula, $\chi(U) = 0$ which implies that U is doubly connected. Let U_m be a connected component of $F_{a,b}^{-m}(U)$ for some integer $m > 0$. Since U_m does not contain any critical point, we have that $F_{a,b}^m : U_m \to U$ is a covering map. It follows that U_m is also doubly connected. Hence, all components of $\bigcup_{m=1}^{\infty} F_{a,b}^{-m}(V)$ are doubly connected. Any other Fatou component will be conformally mapped to one of $F_{a,b}^k(V)$ for $k = 1, 2, \ldots, m$. Hence, they are simply connected. Thus we have shown that the Julia set is disconnected and consists of infinitely many components and that every Fatou component is either simply connected or doubly connected.

If $V = T$, that is both $v_+, v_- \in T$, then $F_{a,b}^{-1}(T)$ has only one component that is doubly connected. This doubly connected component contains all critical points in C and separates 0 and ∞. Thus all Fatou components except B and T are doubly connected. The argument is completely the same as that given by Devaney *et al* [**9**], and it follows that $J_{a,b}$ is a

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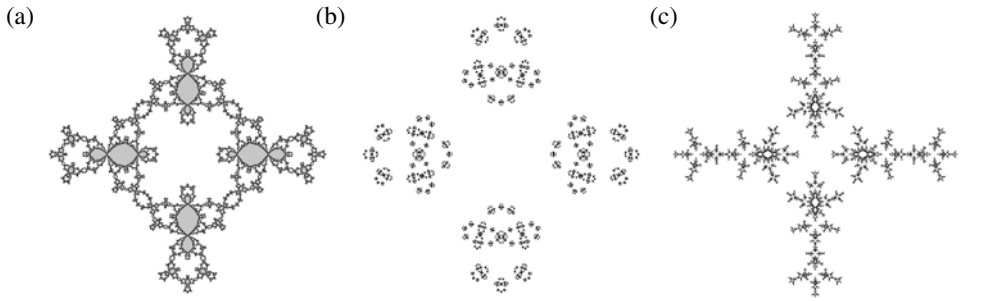


FIGURE 3. Julia sets of $F_{a,b}(z) = z^4 + a/z^4 + b$ in the cases of Theorem 4.1. (a) Case (1), connected Julia set ($a = 0.01$, $b = -1$). (b) Case (2.1), Cantor Julia set ($a \approx 0.074$, $b = -1$). (c) Case (2.2), disconnected Julia set but not Cantor ($a = 0.0009$, $b = -1.2$).

Cantor set of circles. As shown in [**16**], a modulus consideration shows that we must have $n > 2$ in order that T contains both free critical values. □

4. *The semi-escape case*

In this section, we discuss the case that the orbit of one of the free critical values tends to infinity while the orbit of the other one is bounded. Without loss of generality, we assume that $v_- \in A(\infty)$ and $v_+ \in K_{a,b}$ in this section. Our goal is to prove the following theorem.

THEOREM 4.1. *Suppose $v_- \in A(\infty)$ and $v_+ \in K_{a,b}$.*

(1) *If $v_- \notin B$, then all Fatou components of $F_{a,b}$ are simply connected. Consequently, the Julia set $J_{a,b}$ is connected.*

(2) *If $v_- \in B$, then the Julia set $J_{a,b}$ is disconnected. Moreover, the following hold.*

(2.1) *If every critical component $K(c)$ of $K_{a,b}$ is not periodic, then $J_{a,b}$ is a Cantor set.*

(2.2) *If there is a critical component $K(c)$ of $K_{a,b}$ such that $K(c)$ is periodic, then $K(c)$ is homeomorphic to the filled Julia set of a quadratic polynomial. The Julia set $J_{a,b}$ is the union of countably many homeomorphic images of the Julia set of a quadratic polynomial and uncountably many points.*

Proof of Theorem 4.1(1). Since $v_\pm \notin B$, by Lemmas 2.5 and 2.6, we know that all connected components of $A(\infty) = \bigcup_{m=0}^{\infty} F_{a,b}^{-m}(B)$ are simply connected.

If U is a Fatou component which is not contained in $A(\infty)$, then $U \subset K_{a,b}$. The component U will be mapped by $F_{a,b}^m$ for some integer $m \geq 0$ to a periodic Fatou component D. Since $F_{a,b}$ has no Herman ring, D is either a (super)attracting component, a parabolic component, or a Siegel disk. If D is a Siegel disk, D is certainly simply connected. If D is a (super)attracting component or a parabolic component, we can assume that D contains the only critical value v_+. By Lemma 2.6, D is also simply connected. In both cases, every component of $F_{a,b}^{-m}(D)$ is simply connected by Lemma 2.5. So U is simply connected. This shows that all Fatou components are simply connected. Thus $J_{a,b}$ is connected. □

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The proof of Theorem 4.1(2) is very complicated. Firstly, we will use the polynomial-like mapping theory introduced by Douady and Hubbard in [**14**] and the Branner–Hubbard–Yoccoz puzzle technique [**5**]; see also [**18**].

Recall that a polynomial-like mapping of degree d is a triple (U, V, g), where U and V are simply connected planar domains with $\overline{V} \subset U$ and $g : V \to U$ is a holomorphic proper mapping of degree d. The filled Julia set K_g of the polynomial-like mapping g is defined as

$$K_g = \{z \in V : g^k(z) \in U \text{ for all } k \geq 0\}.$$

Two polynomial-like mappings (U_1, V_1, g_1) and (U_2, V_2, g_2) of degree d are said to be hybrid equivalent if there exists a quasiconformal homeomorphism h from a neighborhood of K_{g_1} onto a neighborhood of K_{g_2} such that h is a conjugation between g_1 and g_2 and satisfies $\overline{\partial} h = 0$ on K_{g_1}. The following theorem was proved by Douady and Hubbard in [**14**].

THEOREM 4.2. (The straightening theorem)

(1) *Every polynomial-like mapping (U, V, g) of degree d is hybrid equivalent to a polynomial of degree d.*

(2) *If K_g is connected, then the polynomial is uniquely determined up to conjugation by an affine map.*

In the following, we introduce the Branner–Hubbard–Yoccoz puzzle technique. We refer to [**18**] for basic concepts of this technique.

Let $v_- \in B$. We construct the Branner–Hubbard–Yoccoz puzzle for $F_{a,b}$ as follows. Take a small neighborhood D_0 of ∞ such that the orbit of every critical point in C_- is disjoint from ∂D_0. Let D_k denote the component of $F_{a,b}^{-k}(D_0)$ containing D_0. Then $D_0 \subset D_1 \subset \cdots \subset D_k \subset \cdots$ and $B = \bigcup_{k\geq 0} D_k$. If $v_- \in B$, there exists a positive integer n_0 such that $v_- \in D_{n_0} \setminus D_{n_0-1}$, and then both D_{n_0} and $U = \overline{\mathbb{C}} \setminus \overline{D}_{n_0}$ are simply connected. Obviously $v_+ \in U$, $F_{a,b}^{-1}(U) \subset U$ and the filled Julia set is $K_{a,b} = \bigcap_{k\geq 0} F_{a,b}^{-k}(U)$. From Lemma 2.5, every component of $F_{a,b}^{-(k+1)}(U)$, for $k \geq 0$, is simply connected, and we call it a puzzle piece of depth k. For two puzzle pieces, either they are disjoint from each other or one contains the other. Every puzzle piece of depth k contains finitely many puzzle pieces of depth $k+1$, and every puzzle piece of depth $k+1$ is compactly contained in a puzzle piece of depth k. For every point $z \in K_{a,b}$, there is a unique puzzle piece of depth k containing z. Such a puzzle piece is denoted by $P_k(z)$. Thus every point $z \in K_{a,b}$ determines a nested sequence of puzzle pieces

$$\cdots \subset P_k(z) \subset P_{k-1}(z) \subset \cdots \subset P_0(z),$$

and $K(z) = \bigcap_{k\geq 0} P_k(z)$ is exactly the component of $K_{a,b}$ containing z. The puzzle piece containing a critical point is called a critical puzzle piece.

LEMMA 4.3. *There are exactly n disjoint puzzle pieces of depth 0 for $F_{a,b}$ and every of them contains a free critical point in C_+. Let $c_j = \omega^j \sqrt[2n]{a} \in \mathrm{C}_+$, $j = 0, 1, \ldots, n-1$, $\omega = e^{2\pi i/n}$. Then these puzzle pieces are $P_0(c_0)$, $P_0(c_1)$, $\ldots$, $P_0(c_{n-1})$. Furthermore,*

$$P_0(c_j) = \omega^j P_0(c_0).$$

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Proof. Since U is simply connected and contains only a free critical value v_+, according to Lemma 2.5 there are exactly n connected components of $F_{a,b}^{-1}(U)$ and every component contains a free critical point $c_j \in \mathrm{C}_+$. So there are n puzzle pieces of depth 0 for $F_{a,b}$ and all of them are $P_0(c_0)$, $P_0(c_1)$, $\ldots$, $P_0(c_{n-1})$. Since $\omega^j P_0(c_0)$ is also a component of $F_{a,b}^{-1}(U)$, which means that it is a puzzle piece containing $\omega^j c_0 = c_j$, we get $\omega^j P_0(c_0) = P_0(c_j)$. □

Similarly, we have, for all $k \in \mathbb{N}$ and $j = 0, 1, \ldots, n-1$,

$$P_k(c_j) = \omega^j P_k(c_0).$$

We call $P_k(c_j)$ the j-critical puzzle piece of depth k. Obviously, c_j is the only free critical point in $P_k(c_j)$.

For any $z \in K_{a,b}$, $k \in \mathbb{N}$, we call $A_k(z) = P_k(z) \backslash \overline{P_{k+1}(z)}$ the annulus of depth k at z.

An annulus $A_k(z) = P_k(z) \backslash \overline{P_{k+1}(z)}$ is called off-critical if $\mathrm{C}_+ \cap P_k(z) = \emptyset$. In this case, $F_{a,b} : A_k(z) \to A_{k-1}(F_{a,b}(z))$ is a conformal map. It follows that

$$\mathrm{mod}(A_k(z)) = \mathrm{mod}(A_{k-1}(F_{a,b}(z))),$$

where $\mathrm{mod}(A)$ denotes the conformal modulus of an annulus A.

For given $j = 0, 1, \ldots, n-1$, $A_k(z)$ is called j-critical if $c_j \in P_{k+1}(z)$. In this case, $F_{a,b} : A_k(z) \to A_{k-1}(F_{a,b}(z))$ is a degree-two covering map. This implies

$$\mathrm{mod}(A_k(z)) = \tfrac{1}{2}\mathrm{mod}(A_{k-1}(F_{a,b}(z))).$$

For given $j = 0, 1, \ldots, n-1$, $A_k(z)$ is called j-semi-critical if $c_j \in A_k(z)$. In this case, we have

$$\mathrm{mod}(A_k(z)) > \tfrac{1}{2}\mathrm{mod}(A_{k-1}(F_{a,b}(z))).$$

We call the annulus $A_k(z)$ is critical or semi-critical if it is j-critical or j-semi-critical for some j, respectively. By symmetry,

$$\mathrm{mod}(A_k(c_j)) = \mathrm{mod}(A_k(c_0))$$

for all $j = 0, 1, \ldots, n-1$.

For each $z \in K_{a,b}$, the tableau $T(z)$ is the two dimensional array $\{A_{kl}(z)\}_{k,l \geq 0}$, where $A_{kl}(z) = A_k(F_{a,b}^l(z))$ and $A_{k0}(z) = A_k(z)$. The tableau $T(c_j)$ of a critical point $c_j \in \mathrm{C}_+$ is called a j-critical tableau. We use $\mathrm{Row}_z(k)$ and $\mathrm{Col}_z(l)$ to denote the kth row and the lth column of the tableau $T(z)$, respectively.

As shown in [**18**], the tableau $T(z)$ satisfies the following three rules.

TABLEAU RULES.

(T1) *For a given $l \geq 0$, either the annulus $A_{kl}(z)$ for every $k \geq 0$ is j-critical for some j (in this case, we say that the column $\mathrm{Col}_z(l)$ is fully j-critical) or there exists a $k_0 \geq 0$ such that $A_{k_0 l}(z)$ is j-semi-critical for some j and $A_{kl}(z)$ is j-critical when $k < k_0$ while $A_{kl}(z)$ is off-critical when $k > k_0$.*

(T2) *If $A_{kl}(z)$ is j-critical, that is $A_{kl}(z) = A_{k0}(c_j)$, then*

$$A_{(k-d)(l+m)}(z) = A_{(k-d)m}(c_j)$$

for all $0 \leq m \leq d \leq k$.

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(T3) *If $A_{kl}(z)$ is j-semi-critical, $A_{(k-m)m}(c_j)$ is off-critical when $0 < m < d$ and $A_{(k-d)d}(c_j)$ is i-critical for some i, then $A_{(k-m)(l+m)}(z)$ is off-critical when $0 < m < d$ and $A_{(k-d)(l+d)}(z)$ is i-semi-critical.*

The following lemma is a direct consequence of the Grötzsch inequality, see [**5**].

LEMMA 4.4. *If $\sum_{k=0}^{\infty} \operatorname{mod}(A_k(z)) = \infty$, then $K(z) = \bigcap_{k=0}^{\infty} P_k(z) = \{z\}$.*

LEMMA 4.5. *If there exists a $k_0 \geq 0$ such that the annulus $A_{kl}(z)$ in the tableau $T(z)$ is off-critical when $k > k_0$ and $l > 0$, then*

$$\sum_{k=0}^{\infty} \operatorname{mod}(A_k(z)) = \infty.$$

Proof. See the proof in [**18**, Lemma 2.3]. □

LEMMA 4.6. *If $\sum_{k=0}^{\infty} \operatorname{mod}(A_k(c_0)) = \infty$, then $K(z) = \bigcap_{k=0}^{\infty} P_k(z) = \{z\}$ for all $z \in K_{a,b}$.*

Proof. If there exists $k_0 \geq 0$ such that $A_{kl}(z)$ are off-critical for all $k > k_0$ and $l > 0$, then, by Lemma 4.5, we have $\sum_{k=0}^{\infty} \operatorname{mod}(A_k(z)) = \infty$.

Otherwise, there exists a positive minimum integer $l = l_k$ for every $k > 0$ such that $A_{kl_k}(z)$ is j-critical for some j. Then, by rule (T1), we obtain that $A_{(k+d)(l_k-d)}(z)$ is off-critical when $0 < d < l_k$. Thus

$$F_{a,b}^{l_k-1} : A_{(k+l_k-1)1}(z) \to A_{kl_k}(z) = A_k(c_j)$$

is a conformal map. We get

$$\operatorname{mod}(A_{k+l_k}(z)) \geq \tfrac{1}{2}\operatorname{mod}(A_{(k+l_k-1)1}(z)) = \tfrac{1}{2}\operatorname{mod}(A_k(c_j)) = \tfrac{1}{2}\operatorname{mod}(A_k(c_0)).$$

It follows that

$$\sum_{k=0}^{\infty} \operatorname{mod}(A_k(z)) \geq \sum_{k=0}^{\infty} \operatorname{mod}(A_{k+l_k}(z)) \geq \frac{1}{2} \sum_{k=0}^{\infty} \operatorname{mod}(A_k(c_0)) = \infty.$$

By Lemma 4.4, $K(z) = \bigcap_{k=0}^{\infty} P_k(z) = \{z\}$. □

Since $F_{a,b}^{l}(c_j) = F_{a,b}^{l}(c_0) = F_{a,b}^{l-1}(v_+)$ when $l > 0$ and $j = 1, 2, \ldots, n-1$, we have $A_{kl}(c_j) = A_{kl}(c_0)$ for all $k \geq 0$, $l > 0$. This implies that $\operatorname{Col}_{c_j}(l) = \operatorname{Col}_{c_0}(l)$ for all $l > 0$. The tableau rules (T2) and (T3) can be rewritten as follows.

MODIFIED TABLEAU RULES.

(T2′) *If $A_{kl}(z)$ is critical, that is, $A_{kl}(z) = A_{k0}(c_j)$ for some j, then*

$$A_{(k-d)(l+m)}(z) = A_{(k-d)m}(c_0)$$

for all $0 < m \leq d \leq k$.

(T3′) *If $A_{kl}(z)$ is semi-critical, $A_{(k-m)m}(c_0)$ is off-critical when $0 < m < d$ and $A_{(k-d)d}(c_0)$ is critical, then $A_{(k-m)(l+m)}(z)$ is off-critical when $0 < m < d$ and $A_{(k-d)(l+d)}(z)$ is semi-critical.*

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Now, we consider the critical tableau $T(c)$ for $c \in \mathrm{C}_+$.

Definition. Let $c \in \mathrm{C}_+$ and $T(c)$ be a critical tableau.

(1) If $A_{(k+l-m)m}(c)$ is off-critical when $0 < m < l$ but $A_{kl}(c)$ is critical, we say that the $(k+l)$th row $\mathrm{Row}_c(k+l)$ is a child of the kth row $\mathrm{Row}_c(k)$. In this case, $F^l_{a,b} : A_{k+l}(c) \to A_{kl}(c)$ is a degree-two covering map.

(2) If there is no semi-critical annulus in $\mathrm{Row}_c(k)$, we say that $\mathrm{Row}_c(k)$ is excellent.

(3) If there exists $k_0 \geq 0$ such that $A_{kl}(c)$ is off-critical when $k > k_0$ and $l > 0$, we say that $T(c)$ is non-recurrent. Otherwise, we say that $T(c)$ is recurrent.

(4) If there exists a column $\mathrm{Col}_c(l)$ in $T(c)$ for some $l > 0$ such that it is fully j-critical for some j, we say that $T(c)$ is pre-periodic. The smallest such l is the period of $T(c)$. Moreover, if $c_j = c$, we say that $T(c)$ is periodic.

LEMMA 4.7. *Suppose that the critical tableau $T(c_0)$ is recurrent but not pre-periodic. Then for every $c \in \mathrm{C}_+$, we have the following.*

(1) $\mathrm{Row}_c(k)$ *has at least one child for every* $k \geq 0$.

(2) *If* $\mathrm{Row}_c(k)$ *is excellent, then* $\mathrm{Row}_c(k)$ *has at least two children.*

(3) *If* $\mathrm{Row}_c(k)$ *is excellent and* $\mathrm{Row}_c(k+l)$ *is its child, then* $\mathrm{Row}_c(k+l)$ *is also excellent.*

(4) *If* $\mathrm{Row}_c(k)$ *has only one child, say* $\mathrm{Row}_c(k+l)$, *then* $\mathrm{Row}_c(k+l)$ *is excellent.*

Proof. Applying $A_{kl}(c) = A_{kl}(c_0)$ for every $c \in \mathrm{C}_+$ and every $l > 0$, the proof is the same as the proof of [**18**, Lemma 1.3]. □

Now we are in the position to prove Theorem 4.1(2).

Proof of Theorem 4.1(2). By Lemma 2.6, $B = T$ and B is completely invariant and infinitely connected. Hence $J_{a,b} = \partial B$ is disconnected.

(2.1) Under the assumption, the tableau $T(c_0)$ is not periodic. If $T(c_0)$ is non-recurrent, then every connected component of $K_{a,b}$ is a single point by Lemmas 4.5 and 4.6. That is, $J_{a,b} = K_{a,b}$ is a Cantor set. In the following, we assume that $T(c_0)$ is recurrent. Then by Lemma 4.7, there exists an integer $k_0 \geq 0$ such that $\mathrm{Row}_{c_0}(k_0)$ has the following properties.

(i) $\mathrm{Row}_{c_0}(k_0)$ has at least two children. We call them the first generation children of $\mathrm{Row}_{c_0}(k_0)$.

(ii) Every child in the dth generation of $\mathrm{Row}_{c_0}(k_0)$ has at least two children, which are $(d+1)$th generation children.

Let $\mathcal{C}(d)$ denote the collection of the dth generation descendants of $\mathrm{Row}_{c_0}(k_0)$. Then $\mathrm{Row}_{c_0}(k_0)$ has at least 2^d descendants in the dth generation for every d; that is, $\mathrm{Card}(\mathcal{C}(d)) \geq 2^d$. By definition, if $\mathrm{Row}_{c_0}(k+l)$ is a child of $\mathrm{Row}_{c_0}(k)$, then

$$\mathrm{mod}(A_{k+l}(c_0)) = \tfrac{1}{2}\mathrm{mod}(A_k(c_0)).$$

It follows that, for every $k \in \mathcal{C}(d)$,

$$\mathrm{mod}(A_k(c_0)) = \tfrac{1}{2^d}\mathrm{mod}(A_{k_0}(c_0)).$$

Thus

$$\sum_{k \in \mathcal{C}(d)} \mathrm{mod}(A_k(c_0)) \geq \mathrm{Card}(\mathcal{C}(d))\frac{1}{2^d}\mathrm{mod}(A_{k_0}(c_0)) \geq \mathrm{mod}(A_{k_0}(c_0)).$$

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So

$$\sum_{k=0}^{\infty} \mathrm{mod}(A_k(c_0)) \geq \sum_{d=0}^{\infty} \sum_{k\in\mathcal{C}(d)} \mathrm{mod}(A_k(c_0)) \geq \sum_{d=0}^{\infty} \mathrm{mod}(A_{k_0}(c_0)) = \infty.$$

By Lemma 4.6, we obtain $K_{a,b}(z) = \{z\}$ for every $z \in K_{a,b}$. This shows that $J_{a,b} = K_{a,b}$ and that $J_{a,b}$ is a Cantor set.

(2.2) There exists a critical point $c \in \mathrm{C}_+$ such that the component $K(c)$ of $K_{a,b}$ is periodic with period p. Without loss of generality, we assume that $K(c_0)$ is periodic. Then $T(c_0)$ is also periodic and the column $\mathrm{Col}_{c_0}(p)$ is fully 0-critical. We say that p is the period of $T(c_0)$. In fact, if there is $0 < l < p$ such that the column $\mathrm{Col}_{c_0}(l)$ is also fully j-critical for some j, then $F_{a,b}^{p-l}(K(c_j)) = K(c_0)$. However, $F_{a,b}(K(c_j)) = F_{a,b}(K(c_0))$, which implies that $K(c_0)$ has period at most $p - l$; this is a contradiction.

Now, every column $\mathrm{Col}_{c_0}(l)$ is not fully critical for $0 < l < p$. Let k_0 denote the largest semi-critical depth between the 0th column and the pth column in $T(c_0)$. Then

$$F \equiv F_{a,b}^{p} : P_{k_0+p}(c_0) \to P_{k_0}(c_0)$$

is a branched covering of degree two with the only critical point c_0. Thus $(P_{k_0}(c_0), P_{k_0+p}(c_0), F)$ is a polynomial-like map of degree two. The filled Julia set of F is exactly $K(c_0)$ and the orbit of c_0 under the iteration of F is bounded. According to Theorem 4.2, F is hybrid equivalent to a polynomial P of degree two. So $K(c_0)$ is homeomorphic to the filled Julia set of P. Then the corresponding component of $J_{a,b}$ is homeomorphic to the Julia set of P.

For every $c \in \mathrm{C}_+$, $K(c) = \omega K(c_0)$ for some $\omega \in \Omega_+$. $K(c)$ is certainly homeomorphic to the filled Julia set of P. For any point $z \in K_{a,b}$, if $K(z)$ is eventually mapped to some $K(c)$ by some iteration of $F_{a,b}$, then there is a smallest integer $l > 0$ such that $F_{a,b}^{l} : K(z) \to K(c)$ is conformal. So $K(z)$ is also homeomorphic to the filled Julia set of P. Obviously, there are only countably many such components.

For $z \in K_{a,b}$ such that $K(z)$ is not eventually mapped to $K(c)$, the corresponding tableau $T(z)$ has no fully critical column. If there is a $k_0 \geq 0$ such that $A_{kl}(z)$ are off-critical for all $k > k_0$ and $l > 0$, then, by Lemma 4.5, $K(z)$ is a single point. Otherwise, there are infinitely many rows $\mathrm{Row}_z(k_i)$, $i = 1, 2, \ldots$, where $k_{i+1} > k_i$ such that, for every k_i, there is an $l_i > 0$ such that $A_{k_i l_i}(z)$ is semi-critical but $A_{k_i(l_i-l)}(z)$ is off-critical for every $0 < l \leq l_i$. By the tableau rule (T1), $A_{(k_i+m)(l_i-m)}(z)$ is also off-critical when $0 < m \leq l_i$. Note that, for every $c \in \mathrm{C}_+$, $T(c)$ is pre-periodic with period p. The columns $\mathrm{Col}_c(mp)$ are all fully critical for $m > 0$, but $\mathrm{Col}_c(l)$ is not if $l \neq mp$. Let k_0 denote the largest semi-critical depth of columns $\mathrm{Col}_c(l)$ for $0 < l < p$. Then, by the tableau rules (T3) and (T3′), the annulus $A_{(k_i-m)(l_i+m)}(z)$ is semi-critical or off-critical. More precisely, when $0 < m < k_i - k_0$ and $l_i + m = 0 \bmod p$, the annulus $A_{(k_i-m)(l_i+m)}(z)$ is semi-critical; otherwise, it is off-critical. It follows that the annuli $A_{(k_i-m+1)(l_i+m)}(z)$ are all off-critical for $-l_i \leq m < k_i - k_0$ by the rule (T1). This means that

$$F_{a,b}^{k_i+l_i-k_0} : A_{k_i+l_i+1}(z) \to A_{(k_0+1)(k_i+l_i-k_0)}(z)$$

is a conformal map. So we have

$$\mathrm{mod}(A_{k_i+l_i+1}(z)) = \mathrm{mod}(A_{(k_0+1)(k_i+l_i-k_0)}(z)).$$

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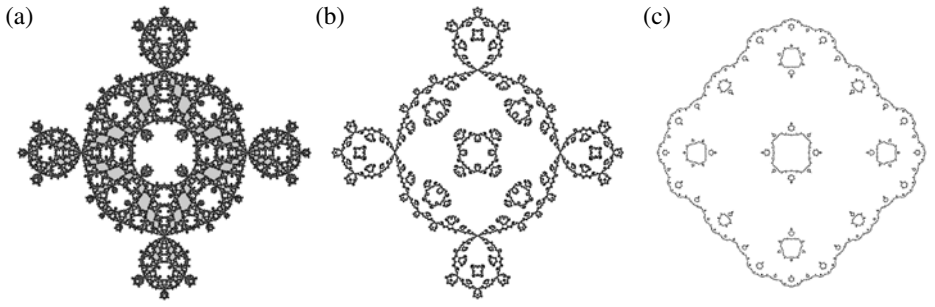


FIGURE 4. Julia sets of $F_{a,b}(z) = z^4 + a/z^4 + b$ for the cases of Theorem 5.1. (a) Case (1), connected Julia set ($a = -0.001$, $b = -0.89$). (b) Case (2), disconnected Julia set ($a = 0.0001$, $b = -0.9$). (c) Special case of case (2), Julia set of Cantor bubbles ($a = 0.0006$, $b = -0.4$).

Note that there are only finitely many annuli in the $(k_0 + 1)$th row. The moduli of these annuli have a positive low bound $a > 0$. We get $\mathrm{mod}(A_{k_i+l_i+1}(z)) \geq a > 0$. By definition, we must have $l_{i+1} > l_i$. So $k_{i+1} + l_{i+1} + 1 > k_i + l_i + 1$; that is, $A_{k_i+l_i+1}(z)$ are different annuli for $i = 1, 2, \ldots$. We then obtain that

$$\sum_{k=0}^{\infty} \mathrm{mod}(A_k(z)) \geq \sum_{i=1}^{\infty} \mathrm{mod}(A_{k_i+l_i+1}(z)) = \infty.$$

It follows from Lemma 4.6 that $K(z)$ is a single point. This completes the proof of (2.2).□

5. *The non-escape case*

In this section, we consider the case that the orbits of both free critical values are bounded.

THEOREM 5.1. *Suppose v_-, $v_+ \in K_{a,b}$. Then we have the following.*

(1) *If each Fatou component contains at most one free critical value, then every Fatou component is simply connected and the Julia $J_{a,b}$ set is connected.*

(2) *If there exists a Fatou component D that contains both free critical values, then the following hold.*

(2.1) *The component D is periodic and infinitely connected. Every Fatou component which is eventually mapped to D is infinitely connected, and every other Fatou component is simply connected.*

(2.2) *The Julia set $J_{a,b}$ is disconnected and has infinitely many components. Moreover, If D is invariant, then $J_{a,b}$ is the union of countably many Jordan curves and uncountably many points (Cantor bubbles).*

Remark 5.1. The Cantor bubbles structure of the Julia sets was also found by Devaney and Marotta in [**12**] for the rational maps $z \mapsto z^n + c/(z - a)^d$ when $|a| \neq 0, 1$, c is sufficiently small, and $1/n + 1/d < 1$.

Proof of Theorem 5.1. (1) By Lemma 2.6(3), both B and T are simply connected. Since every component of $A(\infty)$ is eventually mapped to T, it follows from Lemma 2.5 that all Fatou components in $A(\infty)$ are simply connected. Let U be a Fatou component disjoint from $A(\infty)$. If U contains only one free critical value, then, by Lemma 2.6(2), it is simply

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connected. Now every Fatou component U which is disjoint from $A(\infty)$ will be eventually mapped to a periodic component which is either a (super)attracting domain, a parabolic domain, or a Siegel disk, since $F_{a,b}$ has no Herman ring, by Theorem 2.4. However, such a (super)attracting or parabolic domain contains only one critical value, which implies it is simply connected, and a Siegel disk is simply connected by definition. So U is also simply connected by Lemma 2.5. This shows that all Fatou components are simply connected, which implies that the Julia set $J_{a,b}$ is connected.

(2) Suppose that the Fatou component D contains both free critical values. Then D will be eventually mapped to a periodic Fatou component U. The periodic component U cannot be a Siegel disk, since there is no critical point in $J_{a,b}$, so it is a (super)attracting or parabolic domain which we can assume contains a critical value. Since the only possible critical values are v_+ and v_-, which belong to D, it follows that D itself is a periodic component.

Let D have the period p and

$$D_{p-1} \to \cdots \to D_1 \to D_p = D$$

be the cycle of the periodic component D. According to Lemma 2.6, D_1 is the only preimage of D which contains all $2n$ free critical points and $F_{a,b} : D_1 \to D$ is a degree-$2n$ branched covering map. Suppose that D is m-connected with $1 \le m < \infty$, then, by the Riemann–Hurwitz formula, we have

$$\chi(D_1) + \delta(D_1) = \deg(F_{a,b}|D_1)\chi(D).$$

It follows that

$$\chi(D_1) = 2n(1-m) < 2 - m = \chi(D) \quad \text{and} \quad \chi(D_1) \le 0.$$

Since $D_j (j = 2, \ldots, p)$ contains no critical point, $F_{a,b} : D_j \to D_{j-1}$ is a covering map. We have $\chi(D_j) \le \chi(D_{j-1}) \le 0$. Thus

$$\chi(D) = \chi(D_p) \le \cdots \le \chi(D_1) < \chi(D).$$

This is a contradiction. So D is infinitely connected. Since every Fatou component which is disjoint from $A(\infty)$ will eventually map to D, we get that such a component is also infinitely connected. As pointed out in case (1), every component in $A(\infty)$ is simply connected. We have proved case (2.1).

Obviously, from case (2.1), $J_{a,b}$ is disconnected and has infinitely many components. In the following, we assume that D is invariant. Then D contains all free critical points and hence D is completely invariant. Thus the Fatou set of $F_{a,b}$ is equal to $D \cup A(\infty)$ and the Julia set $J_{a,b} = \partial D = \partial A(\infty)$. Furthermore, D is n-fold symmetric and contains a closed curve surrounding the origin 0 which separates ∂B and ∂T. So D, and hence $\{v_+, v_-\}$, is contained in W_0, the component of $\mathbb{C}\backslash\overline{B}$ which contains 0. By Lemma 3.6, ∂B is a Jordan curve. So ∂T and the boundaries of components of $A(\infty)$ are all Jordan curves. By Lemma 3.4, the boundaries of components of $A(\infty)$ are pairwise disjoint. Let $\tilde{K} = \overline{\mathbb{C}}\backslash D = \overline{A(\infty)}$. Then $J_{a,b} = \partial\tilde{K}$. For every $z \in \tilde{K}$, let $\tilde{K}(z)$ denote the component of $\tilde{K}$ which contains z. We will show that either $\tilde{K}(z)$ is a single point or $\tilde{K}(z)$ is the closure of some component of $A(\infty)$ and hence $\partial\tilde{K}(z)$ is a Jordan curve.

Suppose that D is a (super)attracting component. By the Böttcher theorem or the Schröder theorem (refer to [**17**]), there is a neighborhood U of the fixed point z_0 in D

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and a conformal map $\varphi : U \to D_r = \{z : |z| < r\}$, $z_0 \mapsto 0$, for some $0 < r < 1$, such that $F_{a,b}$ conjugates to the function $z \mapsto z^2$ or $z \mapsto \lambda z$ for some λ satisfying $0 < |\lambda| < 1$. There exists $k_0 > 0$ such that $F_{a,b}^{-k_0}(U)$ is connected and contains all free critical points. Then we construct the puzzle pieces of depth 0 to be the components of $\overline{\mathbb{C}}\backslash F_{a,b}^{-k_0}(U)$. Taking k_0 large enough, we can assume that B and T are contained in different puzzle pieces of depth 0. The puzzle pieces of depth k are the components of $\overline{\mathbb{C}}\backslash F_{a,b}^{-(k_0+k)}(U)$ for $k = 1, 2, \ldots$. We denote by $\tilde{P}_k(z)$ the puzzle piece of depth k which contains the point z. For every $z \in \tilde{K}$, there is a nested sequence of puzzle pieces

$$\cdots \subset \tilde{P}_k(z) \subset \cdots \subset \tilde{P}_1(z) \subset \tilde{P}_0(z).$$

We have $\bigcap_{k=0}^{\infty} \tilde{P}_k(z) = \tilde{K}(z)$. Note that there are only two critical points, ∞ and 0, in $\tilde{K}$. $\tilde{K}(\infty)$ is fixed and $F_{a,b}(\tilde{K}(0)) = \tilde{K}(\infty)$. Since there is a closed curve in D surrounding 0 which separates 0 and ∞, we know that $\tilde{K}(0)$ and $\tilde{K}(\infty)$ are two different components of $\tilde{K}$. Similar to the proof of Theorem 4.1(2.2), we get that $\tilde{K}(\infty)$ is quasiconformally homeomorphic to the filled Julia set K_g of a polynomial g of degree n. Obviously, $\tilde{K}(\infty)$ contains B which corresponds to a bounded Fatou component $V_g \subset K_g$ of g. We claim that there is no other Fatou component of $F_{a,b}$ contained in $\tilde{K}(\infty)$. Otherwise, if there is a Fatou component $U \subset \tilde{K}(\infty)$, then $U \subset A(\infty)$ and there is a $k \geq 0$ such that $F_{a,b}^k(U) = T$. It follows that $\tilde{K}(\infty) = F_{a,b}^k(\tilde{K}(\infty)) \supset T \ni 0$, a contradiction. So we get that V_g is the unique Fatou component contained in K_g which implies that it is completely invariant. Hence $K_g = \overline{V}$. Correspondingly, $\tilde{K}(\infty) = \overline{B}$, or, equivalently, $\partial\tilde{K}(\infty) = \partial B$ is a Jordan curve. Consequently, every component $\tilde{K}(z)$ which is eventually mapped to $\tilde{K}(\infty)$ is a closure of a component of $A(\infty)$ which is also a Jordan curve. Again, as the proof of Theorem 4.1(2.2), every component $\tilde{K}(z)$ which is not eventually mapped to $\tilde{K}(\infty)$ is a single point.

Suppose that D is a parabolic component. Then there is a parabolic fixed point on ∂D and it is the only parabolic periodic point. All other periodic points on $J_{a,b}$ are repelling. We can perturb $F_{a,b}$ to $F_{a_\varepsilon,b_\varepsilon}$ such that the latter has an attracting fixed point and $F_{a,b}$ is conjugate to $F_{a_\varepsilon,b_\varepsilon}$ on their Julia sets (see, for example, [**6**]). So $J_{a,b}$ has the same topology as $J_{a_\varepsilon,b_\varepsilon}$, which consists of Jordan curves and single points. This completes the proof of (2.2). □

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