Proc. Camb. Phil. Soc. (1968), **64**, 895
PCPS 64–105
Printed in Great Britain

Rods, plates and shells

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(*Received* 27 *June* 1967)

Abstract. We discuss non-linear thermodynamical theories of rods and shells using the three-dimensional theory of classical continuum mechanics as a starting point. The three-dimensional theory is reduced to a two-dimensional theory for a shell, or plate, and a one-dimensional theory for a rod by employing an exact expansion for the displacement but an approximation for the temperature. For elastic rods and shells a method of approximation is suggested which brings the respective theories into correspondence with those of Green and Laws (1) and Green, Naghdi and Wainwright (2).

1. *Introduction.* The problem of rods and shells, mainly in the context of linear elasticity, has a long history. It is not our wish to discuss this history critically, nevertheless, we make some preliminary comments. One of the main difficulties in rod and shell theory arises in defining what we intuitively think of as a rod or shell. In fact it is not possible to give a unique definition to these concepts, even for a 'thin' rod or shell. In either case we are actually dealing with a three-dimensional continuum and we should, strictly speaking, attempt to solve the three-dimensional equations. Since there are few known solutions of the three-dimensional equations the aim of rod and shell theory is to provide simpler sets of equations. Ideally we wish to know the relation between the solution of the rod (or shell) equations and the full three-dimensional equations—under what circumstances do the rod equations supply an approximate solution of the three-dimensional equations and how 'close' is this approximate solution to the exact solution? This is a formidable problem and has not been solved even in the linear theory of elastic plates, and this problem is not solved here. We hope, however, that this paper may illuminate our understanding of existing theories of rods and shells.

It is sufficient for our purpose to consider two main approaches to these problems. In one approach a rod is defined to be a one-dimensional curve at every point of which two directors are attached and a shell is a two-dimensional surface at every point of which one director is attached. Using these ideas Green and Laws (1) and Green *et al.* (2) have, respectively, developed general non-linear thermodynamical theories of rods and shells valid for any type of material. References to earlier contributions along these lines are contained in these papers and further developments have been made by Laws (3), Green and Naghdi (4), (9) and Green, Laws and Naghdi (5). Other contributions,

https://doi.org/10.1017/S0305004100043565 Published online by Cambridge University Press

restricted to isothermal elastic shells or rods have been made by Cohen (6) and Cohen and De Silva (7).

In the second approach authors begin with three-dimensional equations of a continuum and usually restrict discussion to elastic materials, mostly using the linear isothermal theory. Subsequent developments usually contain additional assumptions, sometimes combined with variational procedures, and much effort in recent years has been given to attempts to reduce the number of additional assumptions to a minimum. In shell theory some success has also been had with asymptotic expansion methods but the corresponding work for rods is rather less clear. Here we mention a recent paper on rods by Antman and Warner (8) who attempt a general derivation of rod equations from the non-linear isothermal theory of three-dimensional elasticity. Green *et al.* (5) have made some criticisms of the approximation technique used in (8). However, in their paper Antman and Warner derive an energy balance equation for rods from their three-dimensional theory and this, together with a number of other ideas in their paper, provides a clue for the point of view adopted here for rods. We start with a three-dimensional energy equation, allowing for heat conduction and valid for any continuum. In section 6, using the same infinite series expansion for the displacement as that given by Antman and Warner, the energy equation is reduced to a one-dimensional form which is a generalization of that postulated by Green and Laws (1). This reduction is exact. We also discuss the entropy production inequality but here we have not yet succeeded in avoiding an approximation for the temperature distribution. Subsequently we can follow the procedure of Green and Laws (1) to obtain a one-dimensional theory in a systematic manner. Results for an elastic rod are stated in section 7. At this point we are in no better position than that attained by Antman and Warner, since we have an infinite number of equations for an infinite number of unknowns. In section 8 we propose a method of approximation which, first of all, yields a theory formally equivalent to that of Green and Laws (1). This retains the essential ingredients to deal with standard problems of deformations of rods and avoids the difficulties inherent in the method of approximation used by Antman and Warner (8).

A similar procedure is used for shells in section 3. Using a suitable expansion for the displacement of the shell the energy equation is reduced to a two-dimensional form which is a generalization of that postulated by Green *et al.* (2). We also discuss the entropy production inequality but, as in the problem of a rod, cannot avoid making an approximation for the temperature distribution. Subsequently we can adopt the procedure of Green *et al.* (2) to obtain a two-dimensional theory in a systematic manner. Results for an elastic shell are stated in section 4 and provide an infinite set of equations for an infinite number of unknowns. In section 5 we proceed to a method of approximation which, first of all, gives a theory formally equivalent to that of Green *et al.* (2). This retains the essential ingredients necessary to deal with standard problems of deformations of shells and plates.

2. *Notation and formulae.* Points of a three-dimensional continuum are defined by a general moving (convected) coordinate system θ^i. Covariant and contravariant base vectors at points of the continuum at time t are denoted by $\mathbf{g}_i$, $\mathbf{g}^i$ with corresponding

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metric tensors g_{ij}, g^{ij}, latin indices having values, 1, 2, 3. Thus

$$\left.\begin{aligned} g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j, \\ \mathbf{g}_i = \frac{\partial \mathbf{r}^*}{\partial \theta^i}, \quad \mathbf{r}^* = \mathbf{r}^*(\theta^1, \theta^2, \theta^3, t), \end{aligned}\right\} \tag{2·1}$$

where $\mathbf{r}^*$ is the position vector of a typical particle θ^i and δ^i_j is the Kronecker delta. The velocity vector $\mathbf{v}^*$ at time t is given by

$$\mathbf{v}^* = \frac{D\mathbf{r}^*}{Dt} = \dot{\mathbf{r}}^*, \tag{2·2}$$

where D/Dt or a superposed dot denotes the material time derivative (holding θ^i fixed).

The energy equation for an arbitrary (material) volume V bounded by a surface A at time t is

$$\frac{D}{Dt}\int_V (U^* + \tfrac{1}{2}\mathbf{v}^* \cdot \mathbf{v}^*)\rho^* dV = \int_V (r^* + \mathbf{f}^* \cdot \mathbf{v}^*)\rho^* dV + \int_A (\mathbf{t} \cdot \mathbf{v}^* - h^*) dA, \tag{2·3}$$

where ρ^* is density, U^* is internal energy per unit mass, $\mathbf{f}^*$ is the body force vector per unit mass, r^* is the heat supply function per unit mass per unit time, $\mathbf{t}$ is the stress vector and h^* the heat flux across A. We observe that an element of volume dV is given by

$$dV = \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3, \quad g = \det g_{ij}, \tag{2·4}$$

and that $\rho^*\sqrt{g}$ is independent of time because of mass conservation. The entropy production inequality is

$$\frac{D}{Dt}\int_V \rho^* S^* dV - \int_V \frac{\rho^* r^*}{T^*} dV + \int_A \frac{h^*}{T^*} dA \geqslant 0, \tag{2·5}$$

where S^* is the entropy per unit mass.

The stress vector $\mathbf{t}$ across a surface whose unit normal is $\mathbf{u}$ can be put in the form

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{\sqrt{g}}, \quad \mathbf{u} = u_i \mathbf{g}^i = u^i \mathbf{g}_i, \tag{2·6}$$

where

$$\mathbf{T}_i = \sqrt{(g)}\, \tau^{ij} \mathbf{g}_j, \tag{2·7}$$

and τ^{ij} is the symmetric contravariant stress tensor. When given surface forces $\mathbf{P}$ are applied on the boundary surface of the body then

$$\mathbf{t} = \mathbf{P} \tag{2·8}$$

on the boundary.

3. *Shells.* We adopt the convention that Greek indices take the values 1, 2 and write

$$\theta^3 = \xi. \tag{3·1}$$

The parametric equation $\xi = 0$ defines a surface σ in space at time t, which we assume to be smooth and non-intersecting, the position vector of any point of σ being given by

$$\mathbf{r} = \mathbf{r}(\theta^1, \theta^2, t) = \mathbf{r}^*(\theta^1, \theta^2, 0, t). \tag{3·2}$$

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We assume that the continuum is bounded by the surfaces

$$\xi = \alpha, \quad \xi = \beta \quad (\alpha < 0 < \beta), \tag{3·3}$$

which are non-intersecting with themselves, each other, or with $\mathcal{S}$, and which are such that $\mathcal{S}$ lies entirely between them. We also assume that the continuum is bounded by a surface

$$f(\theta^1, \theta^2) = 0, \tag{3·4}$$

which is such that $\xi =$ constant are closed smooth curves on this surface. We call such a continuum a shell†.

We assume that the position vector of any point of the shell at time t can be represented by the Maclaurin expansion

$$\mathbf{r}^* = \mathbf{r}(\theta^1, \theta^2, t) + \sum_{N=1}^{\infty} \xi^N \mathbf{d}_N, \tag{3·5}$$

where $\mathbf{d}_N$ are vector functions of θ^1, θ^2, t. We also assume that the series (3·5) may be differentiated as many times as required with respect to any of its variables.

From (2·2) and (3·5) we have

$$\mathbf{v}^* = \mathbf{v} + \sum_{N=1}^{\infty} \xi^N \mathbf{w}_N, \tag{3·6}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{w}_N = \dot{\mathbf{d}}_N. \tag{3·7}$$

For later convenience we put

$$\mathbf{d} = \mathbf{d}_1, \quad \mathbf{w} = \mathbf{w}_1. \tag{3·8}$$

We call the vectors $\mathbf{d}_N$ directors and $\mathbf{w}_N$ director velocities and observe that $\mathbf{d}_N$ are unchanged in length when the shell is subjected to superposed rigid body motions. From (3·5) and (2·1) we see that

$$\left.\begin{aligned} \mathbf{g}_\alpha &= \mathbf{a}_\alpha + \sum_{N=1}^{\infty} \xi^N \frac{\partial \mathbf{d}_N}{\partial \theta^\alpha}, \\ \mathbf{g}_3 &= \sum_{N=1}^{\infty} N \xi^{N-1} \mathbf{d}_N, \quad \mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}. \end{aligned}\right\} \tag{3·9}$$

We find it convenient to use the notation

$$\left.\begin{aligned} a_{\alpha\beta} &= \mathbf{a}_\alpha . \mathbf{a}_\beta, & a &= \det a_{\alpha\beta}, \\ \mathbf{a}^\alpha &= a^{\alpha\beta} \mathbf{a}_\beta, & a^{\alpha\lambda} a_{\lambda\beta} &= \delta^\alpha_\beta, \end{aligned}\right\} \tag{3·10}$$

and use the symbol $\mathbf{a}_3$ to denote a unit normal vector to the surface (3·2). This vector is a function of θ^1, θ^2, t. In the three-dimensional theory of continuum mechanics it is usual to assume that

$$[\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3] > 0$$

for all time, and all values of θ^i. In particular, it is valid for $\xi = 0$ so that, from (3·9) and (3·8), we have

$$[\mathbf{a}_1 \mathbf{a}_2 \mathbf{d}] > 0. \tag{3·11}$$

For the rest of this section we consider an arbitrary element of the shell bounded by the surfaces $\xi = \alpha$, $\xi = \beta$ and a surface of the form (3·4).

† At this stage there is nothing to distinguish our shell from a rod. The form of expansion (3·5) is, however, suggested by an intuitive idea of a shell.

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We define a mass per unit area of the surface $\mathcal{S}$ to be $\rho(\theta^1, \theta^2, t)$, given by the formula,

$$\rho\sqrt{a} = \int_\alpha^\beta \rho^* \sqrt{(g)}\, d\xi. \tag{3·12}$$

From the three-dimensional equation of mass conservation we know that $\rho^* \sqrt{g}$ is independent of time and, since θ^i are convected coordinates, it follows that $\rho\sqrt{a}$ is independent of time, although both ρ and a may depend upon t. Since the element of area of $\mathcal{S}$ is $d\sigma = \sqrt{(a)}\, d\theta^1 d\theta^2$, we have

$$\iiint \rho^* \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3 = \iint \rho \sqrt{(a)}\, d\theta^1 d\theta^2 = \int \rho\, d\sigma. \tag{3·13}$$

The ranges of integration in (3·13) are clear.

We fix the relation of the surface $\mathcal{S}$ to the bounding surfaces (3·3) by imposing the condition

$$\int_\alpha^\beta \rho^* \sqrt{(g)}\, \xi\, d\xi = 0. \tag{3·14}$$

This involves a minor loss of generality. The above condition is independent of time, so that once $\mathcal{S}$ is determined by such an equation (in, say, a reference state) it remains so determined.

The kinetic energy of the shell is

$$\frac{1}{2}\iiint \rho^* \mathbf{v}^* . \mathbf{v}^* \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3$$

and with the help of (3·6), (3·12) and (3·13) this becomes

$$\frac{1}{2}\int \rho \left(\mathbf{v} . \mathbf{v} + 2 \sum_{N=2}^{\infty} k^N \mathbf{v} . \mathbf{w}_N + \sum_{M,N=1}^{\infty} k^{MN} \mathbf{w}_M . \mathbf{w}_N \right) d\sigma, \tag{3·15}$$

where

$$\left.\begin{aligned} \rho k^N \sqrt{a} &= \int_\alpha^\beta \xi^N \rho^* \sqrt{(g)}\, d\xi \qquad (N \geqslant 2), \\ \rho k^{MN} \sqrt{a} &= \int_\alpha^\beta \xi^{M+N} \rho^* \sqrt{(g)}\, d\xi \quad (M \geqslant 1, N \geqslant 1). \end{aligned}\right\} \tag{3·16}$$

From (3·16) we note that k^N and k^{MN} are functions of θ^1, θ^2, but are independent of t.

We now need the fact that an element of area on any surface $\xi =$ constant is $\sqrt{(gg^{33})}\, d\theta^1 d\theta^2$. This helps us to define a heat supply function r, per unit mass of $\mathcal{S}$, per unit time by

$$\rho r \sqrt{a} = \int_\alpha^\beta \rho^* r^* \sqrt{(g)}\, d\xi - \{h^* \sqrt{(gg^{33})}\}_{\xi=\alpha} - \{h^* \sqrt{(gg^{33})}\}_{\xi=\beta}. \tag{3·17}$$

Let $\mathbf{P}$ stand for the prescribed surface forces on the bounding surfaces (3·3). Then we define the assigned force $\mathbf{F}$ and assigned director forces $\mathbf{L}^N (N = 1, 2, \ldots)$, per unit mass of $\mathcal{S}$ by the equations

$$\begin{aligned} \rho \mathbf{F} \sqrt{a} &= \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)}\, d\xi + \{\mathbf{P} \sqrt{(gg^{33})}\}_{\xi=\alpha} + \{\mathbf{P} \sqrt{(gg^{33})}\}_{\xi=\beta} \\ &= \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)}\, d\xi + \{\mathbf{t} \sqrt{(gg^{33})}\}_{\xi=\alpha} + \{\mathbf{t} \sqrt{(gg^{33})}\}_{\xi=\beta} \\ &= \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)}\, d\xi + [\mathbf{T}_3]_\alpha^\beta, \end{aligned} \tag{3·18}$$

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$$\rho \mathbf{L}^N \sqrt{a} = \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)} \xi^N d\xi + \{\mathbf{P}\xi^N \sqrt{(gg^{33})}\}_{\xi=\alpha} + \{\mathbf{P}\xi^N \sqrt{(gg^{33})}\}_{\xi=\beta}$$

$$= \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)}\, \xi^N d\xi + \{\mathbf{t}\xi^N \sqrt{(gg^{33})}\}_{\xi=\alpha} + \{\mathbf{t}\xi^N \sqrt{(gg^{33})}\}_{\xi=\beta}$$

$$= \int_\alpha^\beta \rho^* \mathbf{f}^* \sqrt{(g)}\, \xi^N d\xi + [\xi^N \mathbf{T}_3]_\alpha^\beta, \tag{3·19}$$

where

$$[\psi(\xi, t)]_\alpha^\beta = \psi(\beta, t) - \psi(\alpha, t). \tag{3·20}$$

We return to the energy equation (2·3) and take the volume V to be a portion of the shell bounded by the surfaces (3·3) and an edge surface whose equation is of the form (3·4). To simplify the calculations we first consider that part of the right-hand side of (2·3) composed of the volume integral and the surface integral over the surfaces $\xi = \alpha, \xi = \beta$. Use of the preceding formulae readily shows that this part of the right-hand side of (2·3) may be rewritten as

$$\int \rho \left(r + \mathbf{F}.\mathbf{v} + \sum_{N=1}^{\infty} \mathbf{L}^N.\mathbf{w}_N \right) d\sigma \tag{3·21}$$

and the integration is over that part of the surface σ which is within a surface of the form (3·4).

To obtain a corresponding expression for the remaining part of the right-hand side of (2·3), we first recall that if $\boldsymbol{\nu}$ is the unit outward normal in the surface σ, to a curve of the form $f(\theta^1, \theta^2) = 0$, $\xi = 0$ then

$$\boldsymbol{\nu} = \nu_\alpha \mathbf{a}^\alpha, \quad \nu_1 dc = \sqrt{(a)}\, d\theta^2, \quad \nu_2 dc = -\sqrt{(a)}\, d\theta^1, \tag{3·22}$$

where dc is an element of length of the curve. Also, if dA is an element of surface area in the surface (3·4) whose unit normal is $\mathbf{u}$ then

$$\left.\begin{aligned} u_1 dA &= \sqrt{(g)}\, d\theta^2 d\theta^3, \quad u_2 dA = -\sqrt{(g)}\, d\theta^1 d\theta^3, \\ dA &= (u^1 d\theta^2 - u^2 d\theta^1) \sqrt{(g)}\, d\theta^3, \end{aligned}\right\} \tag{3·23}$$

where

$$\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i.$$

We put

$$\mathbf{N}^\alpha \sqrt{a} = \int_\alpha^\beta \mathbf{T}_\alpha d\xi, \qquad \mathbf{N} = \mathbf{N}^\alpha \nu_\alpha, \tag{3·24}$$

$$\mathbf{M}^{N\alpha} \sqrt{a} = \int_\alpha^\beta \xi^N \mathbf{T}_\alpha d\xi, \quad \mathbf{M}^N = \mathbf{M}^{N\alpha} \nu_\alpha, \tag{3·25}$$

$$q^\alpha \sqrt{a} = \int_\alpha^\beta q^{*\alpha} \sqrt{(g)}\, d\xi, \quad h^* = u_i q^{*i}, \quad h = q^\alpha \nu_\alpha. \tag{3·26}$$

Now that part of the right-hand side of (2·3) which has not been incorporated in (3·21) is

$$\iint \mathbf{t}.\mathbf{v}^*\, dA - \iint h^*\, dA,$$

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the integration being over a surface of the form (3·4) between $\xi = \alpha$ and $\xi = \beta$. If we use (2·6), (3·6), and (3·22) to (3·26) this becomes

$$\begin{aligned}
&\iint \mathbf{t}.\left(\mathbf{v}+\sum_{N=1}^{\infty}\xi^N\mathbf{w}_N\right)dA-\iint h^*\,dA\\
&\quad=\iint(\mathbf{T}_1\,d\theta^2-\mathbf{T}_2 d\theta^1).\left(\mathbf{v}+\sum_{N=1}^{\infty}\xi^N\mathbf{w}_N\right)d\xi-\iint h^*(u^1\,d\theta^2-u^2 d\theta^1)\sqrt{(g)}\,d\xi\\
&\quad=\oint\left(\mathbf{N}^\alpha.\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{M}^{N\alpha}.\mathbf{w}_N\right)\nu_\alpha dc-\oint(q^1\,d\theta^2-q^2\,d\theta^1)\,\sqrt{a}\\
&\quad=\oint\left(\mathbf{N}.\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{M}^N.\mathbf{w}_N\right)dc-\oint q^\alpha\nu_\alpha dc\\
&\quad=\oint\left(\mathbf{N}.\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{M}^N.\mathbf{w}_N-h\right)dc,
\end{aligned}\tag{3·27}$$

the final integral being taken around a curve of the form $\xi = 0, f(\theta^1, \theta^2) = 0$.

To complete our definitions, the internal energy U, per unit mass of the surface $\mathcal{S}$, is given by

$$\rho U\sqrt{a}=\int_\alpha^\beta \rho^* U^*\sqrt{(g)}\,d\xi.\tag{3·28}$$

With the help of (3·15), (3·21), (3·27) and (3·28) the energy equation (2·3) reduces to

$$\begin{aligned}
&\frac{D}{Dt}\int\rho\left(U+\tfrac{1}{2}\mathbf{v}.\mathbf{v}+\sum_{N=2}^{\infty}k^N\mathbf{w}_N.\mathbf{v}+\frac{1}{2}\sum_{M,N=1}^{\infty}k^{MN}\mathbf{w}_M.\mathbf{w}_N\right)d\sigma\\
&\qquad=\int\rho\left(r+\mathbf{F}.\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{L}^N.\mathbf{w}_N\right)d\sigma+\oint\left(\mathbf{N}.\mathbf{v}+\sum_{N=1}^{\infty}\mathbf{M}^N.\mathbf{w}_N-h\right)dc.
\end{aligned}\tag{3·29}$$

The surface integral in (3·29) is over an arbitrary element of the surface $\mathcal{S}$ bounded by some closed curve $\mathcal{C}$. The line integrals are around $\mathcal{C}$.

To complete the general three-dimensional theory one also needs to postulate an entropy production inequality. The same is true for shells. On returning to the inequality (2·5) we see that we need to define an entropy S, per unit mass of the surface $\mathcal{S}$, and the temperature T of the shell. It is easy to define S by

$$\rho S\sqrt{a}=\int_\alpha^\beta \rho^* S^*\sqrt{(g)}\,d\xi,\tag{3·30}$$

but to define temperature necessitates an assumption. Following (3·5) we might expect to express T^* as a Maclaurin series in ξ but so far it has not proved possible to use such an expansion. Instead, we adopt the approximation

$$T^*(\theta^1,\theta^2,\theta^3,t)=T(\theta^1,\theta^2,t).\tag{3·31}$$

It then follows from (3·17), (3·26), (3·30) and (3·31) that the inequality (2·5) reduces to

$$\frac{D}{Dt}\int\rho S\,d\sigma-\int\frac{\rho r}{T}d\sigma+\oint\frac{h}{T}dc\geqslant 0.\tag{3·32}$$

In the three-dimensional context it is often more convenient to work with the Helmholtz free energy

$$A^*=U^*-T^*S^*.\tag{3·33}$$

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Accordingly, recalling (3·28) and (3·30), we define the Helmholtz function A, per unit mass of the surface $\mathscr{S}$, by

$$\rho A\sqrt{a} = \int_{\alpha}^{\beta} \rho^* A^* \sqrt{(g)}\, d\xi. \tag{3·34}$$

Then

$$A = U - TS. \tag{3·35}$$

The entropy production inequality (3·32) is the same as that postulated by Green *et al.* (2). The energy equation (3·29) is a generalization of that postulated by the same authors who used only one director. The development of the theory from (3·29) and (3·32) follows exactly the same method as that given by Green *et al.* (2). With the help of invariance conditions under superposed rigid body motions equations of motion are deduced and with their help the energy equation is simplified. The main results are quoted in the next section and the reader is referred to Green *et al.* (2) for details. These authors gave results both in vector and component form but only the component forms are given here.

4. *Shell equations*: *elastic shells*. We introduce further notation. Thus

$$\left.\begin{aligned} \mathbf{d}_N &= d_{Ni}\mathbf{a}^i = d_N{}^i\mathbf{a}_i, \qquad \mathbf{d}_{N,\alpha} = \lambda_{Ni\alpha}\mathbf{a}^i, \\ \mathbf{d}_1 &= \mathbf{d} = d_i\mathbf{a}^i = d^i\mathbf{a}_i, \qquad \mathbf{d}_{,\alpha} = \lambda_{i\alpha}\mathbf{a}^i, \end{aligned}\right\} \tag{4·1}$$

$$\left.\begin{aligned} &\lambda_{N\beta\alpha} = d_{N\beta|\alpha} - b_{\beta\alpha}d_{N3}, \quad \lambda_{N3\alpha} = d_{N3,\alpha} + b_\alpha^\beta d_{N\beta}, \\ &\lambda_N{}^\beta{}_{.\alpha} = a^{\beta\gamma}\lambda_{N\gamma\alpha}, \quad \lambda_N{}^3{}_{.\alpha} = \lambda_{N3\alpha}, \\ &\lambda_{1\beta\alpha} = \lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\beta\alpha}d_3, \quad \lambda_{13\alpha} = \lambda_{3\alpha} = d_{3,\alpha} + b_\alpha^\beta d_\beta, \\ &\lambda^\beta{}_{.\alpha} = a^{\beta\gamma}\lambda_{\gamma\alpha}, \quad \lambda^3{}_{.\alpha} = \lambda_{3\alpha}, \end{aligned}\right\} \tag{4·2}$$

$$\left.\begin{aligned} \mathbf{N}^\alpha &= N^{\gamma\alpha}\mathbf{a}_\gamma + N^{3\alpha}\mathbf{a}_3, \\ \mathbf{M}^{N\alpha} &= M^{Ni\alpha}\mathbf{a}_i, \quad \mathbf{M}^{1\alpha} = \mathbf{M}^\alpha = M^{i\alpha}\mathbf{a}_i, \end{aligned}\right\} \tag{4·3}$$

where a comma denotes partial differentiation and a vertical line denotes covariant differentiation with respect to $a_{\alpha\beta}$. Also

$$\left.\begin{aligned} \bar{\mathbf{F}} &= \mathbf{F} - \dot{\mathbf{v}} - \sum_{N=2}^{\infty} k^N \dot{\mathbf{w}}_N = \bar{F}^i\mathbf{a}_i, \\ \bar{\mathbf{L}}^N &= \mathbf{L}^N - k^N\dot{\mathbf{v}} - \sum_{M=1}^{\infty} k^{MN}\dot{\mathbf{w}}_M = \bar{L}^{Ni}\mathbf{a}_i \quad (N \geqslant 2), \\ \bar{\mathbf{L}} &= \bar{\mathbf{L}}^1 = \mathbf{L} - \sum_{M=1}^{\infty} k^{M1}\dot{\mathbf{w}}_M = \bar{L}^i\mathbf{a}_i. \end{aligned}\right\} \tag{4·4}$$

The equations of motion are

$$\left.\begin{aligned} N^{\beta\alpha}|_\alpha - b_\alpha^\beta N^{3\alpha} + \rho\bar{F}^\beta &= 0, \\ N^{3\alpha}|_\alpha + b_{\alpha\beta}N^{\beta\alpha} + \rho\bar{F}^3 &= 0, \end{aligned}\right\} \tag{4·5}$$

$$\left.\begin{aligned} M^{\beta\alpha}|_\alpha - b_\alpha^\beta M^{3\alpha} + \rho\bar{L}^\beta &= m^\beta = m^{1\beta}, \\ M^{3\alpha}|_\alpha + b_{\alpha\beta}M^{\beta\alpha} + \rho\bar{L}^3 &= m^3 = m^{13}, \end{aligned}\right\} \tag{4·6}$$

$$\left.\begin{aligned} M^{N\beta\alpha}|_\alpha - b_\alpha^\beta M^{N3\alpha} + \rho\bar{L}^{N\beta} &= m^{N\beta} \\ M^{N3\alpha}|_\alpha + b_{\alpha\beta}M^{N\beta\alpha} + \rho\bar{L}^{N3} &= m^{N3} \end{aligned}\right\} \quad (N \geqslant 2), \tag{4·7}$$

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$$N^{3\alpha}+\sum_{N=1}^{\infty}(m^{N3}d_N^{\alpha}-m^{N\alpha}d_N^{3})+\sum_{N=1}^{\infty}(M^{N3\gamma}\lambda_{N.\gamma}^{\alpha}-M^{N\alpha\gamma}\lambda_{N.\gamma}^{3})=0, \tag{4·8}$$

$$N'^{\beta\alpha}=N'^{\alpha\beta}=N^{\beta\alpha}-\sum_{N=1}^{\infty}(m^{N\alpha}d_N^{\beta}+M^{N\alpha\gamma}\lambda_{N.\gamma}^{\beta}). \tag{4·9}$$

The quantities m^{Ni} are defined by (4·6) and (4·7) and are specified by constitutive equations.

The energy equation reduces to

$$\rho r-q^{\alpha}|_{\alpha}-\rho(T\dot{S}+S\dot{T})-\rho\dot{A}+N'^{\beta\alpha}\eta_{\alpha\beta}+\sum_{N=1}^{\infty}m^{Ni}\dot{d}_{Ni}+\sum_{N=1}^{\infty}M^{Ni\alpha}\dot{\lambda}_{Ni\alpha}=0, \tag{4·10}$$

where

$$2\eta_{\alpha\beta}=\dot{a}_{\alpha\beta}, \tag{4·11}$$

and the entropy inequality is

$$-\rho(\dot{A}+S\dot{T})+N'^{\beta\alpha}\eta_{\alpha\beta}+\sum_{N=1}^{\infty}m^{Ni}\dot{d}_{Ni}+\sum_{N=1}^{\infty}M^{Ni\alpha}\dot{\lambda}_{Ni\alpha}-\frac{q^{\alpha}T_{,\alpha}}{T}\geqslant 0. \tag{4·12}$$

The above results are valid whatever the material properties of the shell. For an elastic shell we may follow the procedure of Green *et al.*(2) to produce constitutive equations

$$A=A(T,e_{\alpha\beta},\lambda_{Ni\alpha},d_{Ni},A_{\alpha\beta},\Lambda_{Ni\alpha},D_{Ni}), \tag{4·13}$$

$$S=-\frac{\partial A}{\partial T}, \tag{4·14}$$

$$N'^{\beta\alpha}=\rho\frac{\partial A}{\partial e_{\alpha\beta}},\quad m^{Ni}=\rho\frac{\partial A}{\partial d_{Ni}},\quad M^{Ni\alpha}=\rho\frac{\partial A}{\partial\lambda_{Ni\alpha}}, \tag{4·15}$$

where $A_{\alpha\beta}$, $\Lambda_{Ni\alpha}$, D_{Ni} are values of $a_{\alpha\beta}$, $\lambda_{Ni\alpha}$, d_{Ni} in the reference state and

$$2e_{\alpha\beta}=a_{\alpha\beta}-A_{\alpha\beta}. \tag{4·16}$$

The value of A in (4·13) is obtained from (3·12) and (3·3) by using the expansions (3·9) and the value of A^* expressed as a function of three-dimensional strains.

Also

$$-q^{\alpha}T_{,\alpha}\geqslant 0. \tag{4·17}$$

In (4·15), $\partial A/\partial e_{\alpha\beta}$ stands for $\frac{1}{2}(\partial A/\partial e_{\alpha\beta}+\partial A/\partial e_{\beta\alpha})$.

With the help of (4·14) to (4·16) the energy equation (4·10) reduces to

$$\rho r-q^{\alpha}|_{\alpha}-\rho T\dot{S}=0. \tag{4·18}$$

5. *Approximations for shells.* Given the basic postulates of Green *et al.*(2), the resulting theory of a Cosserat surface may be regarded as an exact theory of shells in which no attempt is made to relate the theory to three dimensions. Here we start with the three dimensional theory of section 4 and by systematic approximations, recover a theory of the same form as that given by Green *et al.* (2). In sections 3 and 4, an approximation has already been made for the temperature but the rest of the theory is exact. However, the theory consists of an infinite system of equations in an infinite number of unknowns. As a first step we assume that the base vectors in (3·9) may be written as

$$\left.\begin{aligned}\mathbf{g}_{\alpha}&=\mathbf{a}_{\alpha}+\xi\frac{\partial\mathbf{d}}{\partial\theta^{\alpha}},\\ \mathbf{g}_{3}&=\mathbf{d}+\sum_{N=2}^{\infty}N\xi^{N-1}\mathbf{d}_{N},\end{aligned}\right\} \tag{5·1}$$

where $\mathbf{g}_3$ is exact and $\mathbf{g}_{\alpha}$ is represented approximately by only the first two terms in

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the expansions (3·9). For an elastic body the Helmholtz function A^* is a function of G_{ij}, temperature and three dimensional strains γ^*_{ij} where

$$\gamma^*_{ij} = \tfrac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j), \tag{5·2}$$

where $\mathbf{G}_i, G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$ are the initial values of $\mathbf{g}_i, g_{ij}$. From (3·12), (3·34), (5·1) and the conservation of mass, it follows that A given by (4·13), reduces to

$$A = A(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_{Ni}, A_{\alpha\beta}, \Lambda_{i\alpha}, D_{Ni}). \tag{5·3}$$

Hence, from (4·15)

$$M^{i\alpha} = \rho \frac{\partial A}{\partial \lambda_{i\alpha}}, \quad M^{Ni\alpha} = 0 \quad (N \geqslant 2). \tag{5·4}$$

The system of equations of motion (4·7) then becomes

$$m^{N\beta} = \rho \bar{L}^{N\beta}, \quad m^{N3} = \rho \bar{L}^{N3} \quad (N \geqslant 2). \tag{5·5}$$

Inspection of (4·4), (3·16) and (3·19), shows that $\bar{L}^{Ni}$ involve moments of all orders through the thickness of the shell of body forces, applied surface forces on the faces of the shell, and inertia terms. In addition to (5·1) we now assume that†

$$\bar{L}^{Ni} = 0 \quad (N \geqslant 2), \tag{5·6}$$

so that, from (5·5)

$$m^{Ni} = 0 \quad (N \geqslant 2). \tag{5·7}$$

Equations $(4·15)_2$ for $N \geqslant 2$, are an infinite system of equations which we assume can be inverted to express $d_{Ni}\,(N \geqslant 2)$ as a function of $e_{\alpha\beta}, d_i, \lambda_{i\alpha}, T$ and $m^{Ni}\,(N \geqslant 2)$, together with certain reference state variables given in (5·3). Omitting reference state variables it follows that

$$A(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_i, d_{Ni}) = A'(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_i, m^{Ni}) \quad (N \geqslant 2), \tag{5·8}$$

and

$$\left.\begin{aligned} \rho \frac{\partial A'}{\partial T} &= \rho \frac{\partial A}{\partial T} + \sum_{N=2}^{\infty} m^{Ni} \frac{\partial d_{Ni}}{\partial T}, \\ \rho \frac{\partial A'}{\partial e_{\alpha\beta}} &= \rho \frac{\partial A}{\partial e_{\alpha\beta}} + \sum_{N=2}^{\infty} m^{Ni} \frac{\partial d_{Ni}}{\partial e_{\alpha\beta}}, \\ \rho \frac{\partial A'}{\partial \lambda_{i\alpha}} &= \rho \frac{\partial A}{\partial \lambda_{i\alpha}} + \sum_{N=2}^{\infty} m^{Nk} \frac{\partial d_{Nk}}{\partial \lambda_{i\alpha}}, \\ \rho \frac{\partial A'}{\partial d_k} &= \rho \frac{\partial A}{\partial d_k} + \sum_{N=2}^{\infty} m^{Ni} \frac{\partial d_{Ni}}{\partial d_k}. \end{aligned}\right\} \tag{5·9}$$

If the coefficients of m^{Ni} in (5·9) are finite when $m^{Ni} = 0\,(N \geqslant 2)$, it follows, on using (5·7), that

$$\left.\begin{aligned} S &= -\frac{\partial A'}{\partial T}, \quad N'^{\beta\alpha} = \rho \frac{\partial A'}{\partial e_{\alpha\beta}}, \quad M^{i\alpha} = \rho \frac{\partial A'}{\partial \lambda_{i\alpha}}, \\ A' &= A'(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_i), \quad m^i = \rho \frac{\partial A'}{\partial d_i} \end{aligned}\right\} \tag{5·10}$$

dependence of A' on reference state variables being understood. Moreover, in view of (5·4) and (5·7), equations (4·8) and (4·9) reduce to

$$N^{3\alpha} + m^3 d^\alpha - m^\alpha d^3 + M^{3\gamma} \lambda^\alpha_{\cdot\gamma} - M^{\alpha\gamma} \lambda^3_{\cdot\gamma} = 0, \tag{5·11}$$

$$N'^{\beta\alpha} = N'^{\alpha\beta} = N^{\beta\alpha} - m^\alpha d^\beta - M^{\alpha\gamma} \lambda^\beta_{\cdot\gamma}. \tag{5·12}$$

† If there are no body forces and the surfaces of the shell are free from applied stress then the moments of the body and surface forces are zero.

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Finally, in view of (5·6), we replace $\overline{\mathbf{F}}$ and $\overline{\mathbf{L}}$ in (4·4) by

$$\overline{\mathbf{F}} = \mathbf{F} - \dot{\mathbf{v}}, \quad \overline{\mathbf{L}} = \mathbf{L} - k^{11}\dot{\mathbf{w}}. \tag{5·13}$$

Equations (4·5), (4·6), (4·18), (5·10) to (5·13) are the equivalent to those derived by Green *et al.* (2). All the basic three-dimensional equations have been considered systematically, the only ones not used being

$$0 = m^{Ni} = \rho \frac{\partial A}{\partial d_{Ni}} \quad (N \geqslant 2), \tag{5·14}$$

but these merely serve to determine $d_{Ni} (N \geqslant 2)$.

Green and Naghdi (9) have shown how it is possible to simplify their Cosserat surface equations to correspond to shell theories derived by other means. If desired, further approximations to the complete set of three-dimensional equations may be obtained systematically by retaining more terms in $(3·9)_1$ and adopting a similar procedure to that described above. However, the theory of the Cosserat surface as given by Green *et al.* (2) reflects satisfactorily most of the basic properties of deformation of shells (and plates) and is sufficiently general for most purposes. We observe that in practice it will be impossible to compute A' in (5·10) from the three-dimensional Helmholtz function A^* and A' must be regarded as an arbitrary function of its arguments. Some of the elastic coefficients in A', at least in the linear theory of a plate, can, however, be identified by studying known exact solutions of the three-dimensional equations.

6. *Rods.* The notation in this and following sections must be regarded as separate from that in sections 3 to 5 so that no confusion should arise over different uses of the same symbols. Greek indices still take the values 1, 2.

The parametric equations

$$\theta^\alpha = 0 \tag{6·1}$$

define a curve c, in space, at time t. We assume that the curve c is sufficiently smooth and non-intersecting. The position vector $\mathbf{r}$ of any point of c is given by

$$\mathbf{r} = \mathbf{r}(\theta, t) = \mathbf{r}^*(0, 0, \theta^3, t) \quad (\theta^3 \equiv \theta). \tag{6·2}$$

We also assume that the region of space occupied by the continuum is some neighbourhood of c which is bounded by a surface

$$f(\theta^1, \theta^2) = 0, \tag{6·3}$$

such that $\theta =$ constant are curved sections of this continuum bounded by closed curves. We call such a continuum a rod.†

We represent the position vector of any point of the rod by the infinite series

$$\mathbf{r}^* = \mathbf{r}(\theta, t) + \sum_n \theta^{\alpha_1}\theta^{\alpha_2} \dots \theta^{\alpha_n} \mathbf{d}_{\alpha_1 \alpha_2 \dots \alpha_n}, \tag{6·4}$$

where $\mathbf{d}_{\alpha_1 \alpha_2 \dots \alpha_n}$ are vector functions of θ, t, completely symmetric in the suffices $\alpha \dots \alpha_n$, and the summation in (6·4) is over all values of $\alpha_1, \alpha_2, \dots, \alpha_n = 1, 2$, and $n = 1, 2, 3, \dots$. We assume that (6·4) may be differentiated as many times as required with respect to any of its variables.

† At this stage there is nothing to distinguish a rod from a shell. The form of expansion (6·4) is, however, suggested by an intuitive idea of a rod

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From (2·2) and (6·4) we have

$$\mathbf{v}^* = \mathbf{v} + \sum_n \theta^{\alpha_1}\theta^{\alpha_2} \dots \theta^{\alpha_n} \mathbf{w}_{\alpha_1 \alpha_2 \dots \alpha_n}, \tag{6·5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{w}_{\alpha_1 \alpha_2 \cdots \alpha_n} = \dot{\mathbf{d}}_{\alpha_1 \alpha_2 \cdots \alpha_n}. \tag{6·6}$$

For later convenience we put

$$\mathbf{d}_\alpha = \mathbf{a}_\alpha, \quad \mathbf{w}_\alpha = \dot{\mathbf{a}}_\alpha. \tag{6·7}$$

We call $\mathbf{d}_{\alpha_1 \dots \alpha_n}$ directors and $\mathbf{w}_{\alpha_1 \dots \alpha_n}$ director velocities and observe that $\mathbf{d}_{\alpha_1 \dots \alpha_n}$ are unchanged in length when the rod is subjected to superposed rigid body motions. We also use the notation

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \theta}, \quad a_{ij} = \mathbf{a}_i . \mathbf{a}_j, \tag{6·8}$$

where $\mathbf{a}_3$ is a tangent vector to the curve (6·2). From (6·4) and (2·1) we see that

$$\left.\begin{aligned} \mathbf{g}_\beta &= \mathbf{a}_\beta + \sum_{n=2}^{\infty} n\theta^{\alpha_2} \dots \theta^{\alpha_n} \mathbf{d}_{\beta \alpha_2 \dots \alpha_n}, \\ \mathbf{g}_3 &= \mathbf{a}_3 + \sum_{n=1}^{\infty} \theta^{\alpha_1} \dots \theta^{\alpha_n} \partial \mathbf{d}_{\alpha_1 \dots \alpha_n} / \partial \theta. \end{aligned}\right\} \tag{6·9}$$

With the displacement function (6·4) the restriction analogous to (3·11) is

$$[\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3] > 0. \tag{6·10}$$

For the rest of this section we consider an arbitrary element of the rod bounded by the surfaces $\theta = \alpha, \theta = \beta\ (\beta \geqslant \theta \geqslant \alpha)$, and a surface (6·3).

We denote a mass per unit length of the curve c by $\rho(\theta, t)$. To defins this we note that the total mass of the element of rod is

$$\iiint \rho^* \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3,$$

where the integration is over the relevant range of values of θ^i. The element of length of c is $\sqrt{(a_{33})}\, d\theta$ and so

$$\iiint \rho^* \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3 = \int \rho \sqrt{(a_{33})}\, d\theta.$$

Therefore ρ is defined by

$$\rho \sqrt{a_{33}} = \iint \rho^* \sqrt{(g)}\, d\theta^1 d\theta^2. \tag{6·11}$$

Since θ^i are convected coordinates and $\rho^* \sqrt{g}$ is independent of time, we observe that $\rho \sqrt{a_{33}}$ is independent of time.

For definiteness and with little loss of generality, we fix the relation of the line c to the boundary (6·3) of the rod by imposing the condition that

$$\iint \rho^* \sqrt{(g)}\, \theta^\alpha d\theta^1 d\theta^2 = 0 \quad (\alpha = 1, 2). \tag{6·12}$$

We observe that this condition is independent of time t, so that once c is determined by such an equation (in, say, a reference state) it is determined for all time.

The kinetic energy of the rod is

$$\frac{1}{2} \iiint \rho^* \mathbf{v}^* . \mathbf{v}^* \sqrt{(g)}\, d\theta^1 d\theta^2 d\theta^3,$$

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and with the help of (6·5), (6·11) and (6·12), this becomes

$$\frac{1}{2}\int\rho\left(\mathbf{v}.\mathbf{v}+2\sum_{n=2}^{\infty}k^{\alpha_1\dots\alpha_n}\mathbf{v}.\mathbf{w}_{\alpha_1\dots\alpha_n}+\sum_{m,n=1}^{\infty}k^{\alpha_1\dots\alpha_n:\,\beta_1\dots\beta_m}\mathbf{w}_{\alpha_1\dots\alpha_n}.\mathbf{w}_{\beta_1\dots\beta_m}\right)\sqrt{(a_{33})}\,d\theta, \tag{6·13}$$

where
$$\left.\begin{aligned}\rho k^{\alpha_1\dots\alpha_n}\sqrt{(a_{33})}&=\iint\rho^*\sqrt{(g)}\,\theta^{\alpha_1}\dots\theta^{\alpha_n}d\theta^1d\theta^2\quad(n\geqslant 2),\\ \rho k^{\alpha_1\dots\alpha_n:\,\beta_1\dots\beta_m}\sqrt{(a_{33})}&=\iint\rho^*\sqrt{(g)}\theta^{\alpha_1}\dots\theta^{\alpha_n}\theta^{\beta_1}\dots\theta^{\beta_m}d\theta^1d\theta^2.\end{aligned}\right\} \tag{6·14}$$

From (6·14) it follows that all the coefficients $k^{\alpha_1\dots\alpha_n}$, $k^{\alpha_1\dots\alpha_n:\,\beta_1\dots\beta_m}$ depend only upon θ.

We now need the result (3·23) which provides the formula for the element of surface area in the surface (6·3). This result helps us to define a heat supply function, r, per unit mass of the line c, per unit time, by

$$\rho r\sqrt{a_{33}}=\iint\rho^*r^*\sqrt{(g)}\,d\theta^1d\theta^2-\oint h^*(u^1d\theta^2-u^2d\theta^1)\sqrt{g}. \tag{6·15}$$

The first integral in (6·15) is over any section $\theta=$ constant of the rod ($\alpha<\theta<\beta$) bounded by the surface (6·3). The second integral is along the curve

$$\theta=\text{constant},\quad f(\theta^1,\theta^2)=0. \tag{6·16}$$

Let $\mathbf{P}$ denote the prescribed surface forces on the lateral surface (6·3). Then, if dA is the element of surface area in the surface (6·3) the resultant body force plus surface force on any section of the rod is

$$\iiint\rho^*\mathbf{f}^*\sqrt{(g)}\,d\theta^1d\theta^2d\theta^3+\iint\mathbf{P}\,dA.$$

From (2·6) and (2·8) this is equal to

$$\iiint\rho^*\mathbf{f}^*\sqrt{(g)}\,d\theta^1d\theta^2d\theta^3+\iint\frac{u_i\mathbf{T}_i}{\sqrt{g}}dA$$

and, with the result (3·23), this may be rewritten in the form

$$\iiint\rho^*\mathbf{f}^*\sqrt{(g)}\,d\theta^1d\theta^2d\theta^3+\iint(\mathbf{T}_1d\theta^2-\mathbf{T}_2d\theta^1)\,d\theta^3. \tag{6·17}$$

Accordingly, we define the assigned force, $\mathbf{f}$, per unit mass of the line c by

$$\rho\mathbf{f}\sqrt{a_{33}}=\iint\rho^*\mathbf{f}^*\sqrt{(g)}d\theta^1d\theta^2+\oint(\mathbf{T}_1d\theta^2-\mathbf{T}_2d\theta^1). \tag{6·18}$$

The range of integration in these integrals is the same as the range of the corresponding integrals in (6·15). A similar argument for the 'moments' of the body forces and surface forces leads to the definition of assigned director forces $\mathbf{l}^{\alpha_1\dots\alpha_n}$ per unit mass of the line c, through

$$\rho\mathbf{l}^{\alpha_1\dots\alpha_n}\sqrt{a_{33}}=\iint\rho^*\mathbf{f}^*\theta^{\alpha_1}\dots\theta^{\alpha_n}\sqrt{(g)}\,d\theta^1d\theta^2+\oint\theta^{\alpha_1}\dots\theta^{\alpha_n}(\mathbf{T}_1d\theta^2-\mathbf{T}_2d\theta^1), \tag{6·19}$$

for $n=1,2,3\dots$.

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The internal energy, U, of the rod, per unit mass of the rod, is given by

$$\rho U \sqrt{a_{33}} = \iint \rho^* U^* \sqrt{(g)}\, d\theta^1 d\theta^2. \tag{6·20}$$

To conclude our list of definitions we observe that the element of area in a surface $\theta =$ constant is $\sqrt{(gg^{33})}\, d\theta^1 d\theta^2$ and set

$$h = \iint h^* \sqrt{(gg^{33})}\, d\theta^1 d\theta^2, \tag{6·21}$$

$$\mathbf{n} = \iint \mathbf{T}_3\, d\theta^1 d\theta^2, \tag{6·22}$$

$$\mathbf{p}^{\alpha_1 \dots \alpha_n} = \iint \theta^{\alpha_1} \dots \theta^{\alpha_n} \mathbf{T}_3\, d\theta^1 d\theta^2, \tag{6·23}$$

these integrals again being over a surface $\theta =$ constant between $\theta = \alpha$, $\theta = \beta$ and bounded by the surface (6·3).

We turn our attention to the energy equation (2·3). When the volume V consists of an arbitrary segment of the rod, the range of the surface integral consists of three parts: one is part of the boundary surface (6·3), the other two being sections $\theta = \phi_1$, $\theta = \phi_2$ between $\theta = \alpha$, $\theta = \beta$. With the help of results of this section, a series of straightforward computations shows that equation (2·3) reduces to

$$\begin{aligned}\frac{D}{Dt}\int_{\phi_1}^{\phi_2} \rho \Big(U + \tfrac{1}{2}\mathbf{v}.\mathbf{v} + \sum_{n=2}^{\infty} k^{\alpha_1 \dots \alpha_n} \mathbf{v}.\mathbf{w}_{\alpha_1 \dots \alpha_n} \\ + \frac{1}{2} \sum_{m,n=1}^{\infty} k^{\alpha_1 \dots \alpha_n : \beta_1 \dots \beta_m} \mathbf{w}_{\alpha_1 \dots \alpha_n}.\mathbf{w}_{\beta_1 \dots \beta_m} \Big) \sqrt{(a_{33})}\, d\theta \\ = \int_{\phi_1}^{\phi_2} \rho \Big(r + \mathbf{f}.\mathbf{v} + \sum_{n=1}^{\infty} \mathbf{l}^{\alpha_1 \dots \alpha_n}.\mathbf{w}_{\alpha_1 \dots \alpha_n} \Big) \sqrt{(a_{33})}\, d\theta \\ + \Big[\mathbf{n}.\mathbf{v} + \sum_{n=1}^{\infty} \mathbf{p}^{\alpha_1 \dots \alpha_n}.\mathbf{w}_{\alpha_1 \dots \alpha_n} - h \Big]_{\phi_1}^{\phi_2}, \end{aligned} \tag{6·24}$$

where we have used the notation

$$[\psi(\theta, t)]_{\phi_1}^{\phi_2} = \psi(\phi_2, t) - \psi(\phi_1, t).$$

The discussion of the entropy production inequality follows closely that of section 3 on shells when an approximation for the temperature distribution is introduced. We define the entropy S, per unit mass of the rod by

$$\rho S \sqrt{a_{33}} = \iint \rho^* S^* \sqrt{(g)}\, d\theta^1 d\theta^2, \tag{6·25}$$

but for temperature we assume that

$$T^*(\theta^1, \theta^2, \theta^3, t) = T(\theta, t). \tag{6·26}$$

We then use (2·5), (6·15), (6·25) and (6·26) to obtain

$$\frac{D}{Dt}\int_{\phi_1}^{\phi_2} \rho S \sqrt{(a_{33})}\, d\theta - \int_{\phi_1}^{\phi_2} \frac{\rho r}{T} \sqrt{(a_{33})}\, d\theta + \left[\frac{h}{T}\right]_{\phi_1}^{\phi_2} \geqslant 0. \tag{6·27}$$

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In parallel with the discussion of section 3 we define the Helmholtz free energy A, per unit mass of the rod, by

$$\rho A \sqrt{a_{33}} = \iint \rho^* A^* \sqrt{(g)}\, d\theta^1 d\theta^2, \tag{6·28}$$

and then

$$A = U - TS. \tag{6·29}$$

The entropy production inequality (6·27) is the same as that postulated by Green and Laws (1). The energy equation (6·24) is a generalization of that postulated by the same authors who used only two directors. The development of the theory from (6·24) and (6·27) is similar to that given by Green and Laws. With the help of invariance conditions under superposed rigid body motions the equations of motion are deduced and with their help the energy equation is simplified. Green and Laws used the three vectors $\mathbf{a}_i$ as a reference basis with $\mathbf{a}_\alpha$ as two directors, so that some differences arise here in the analysis.

7. *Rod equations*: *elastic rods*. We wish to introduce further notation so that we may exhibit the field equations in component form. Thus, let

$$\left.\begin{aligned} \mathbf{a}^i . \mathbf{a}_j = \delta^i_j, \quad a^{ij} = \mathbf{a}^i . \mathbf{a}^j, \\ \frac{\partial \mathbf{a}_i}{\partial \theta} = \kappa_{is} \mathbf{a}^s = \kappa_i{}^s \mathbf{a}_s, \end{aligned}\right\} \tag{7·1}$$

and if $\mathbf{b}$ is a vector such that

$$\mathbf{b} = b^i \mathbf{a}_i = b_i \mathbf{a}^i, \tag{7·2}$$

then

$$\frac{\partial \mathbf{b}}{\partial \theta} = \frac{\delta b^i}{\delta \theta} \mathbf{a}_i = \frac{\delta b_i}{\delta \theta} \mathbf{a}^i, \tag{7·3}$$

where

$$\left.\begin{aligned} \frac{\delta b^i}{\delta \theta} = \frac{\partial b^i}{\partial \theta} + \kappa_r{}^i b^r, \\ \frac{\delta b_i}{\delta \theta} = \frac{\partial b_i}{\partial \theta} - \kappa_i{}^r b_r. \end{aligned}\right\} \tag{7·4}$$

Next†

$$\left.\begin{aligned} \mathbf{d}_{\alpha_1 \dots \alpha_n} = d_{\alpha_1 \dots \alpha_n i} \mathbf{a}^i = d_{\alpha_1 \dots \alpha_n}{}^i \mathbf{a}_i, \\ \frac{\partial \mathbf{d}_{\alpha_1 \dots \alpha_n}}{\partial \theta} = \lambda_{\alpha_1 \dots \alpha_n i} \mathbf{a}^i = \lambda_{\alpha_1 \dots \alpha_n}{}^i \mathbf{a}_i, \\ \lambda_{\alpha_1 \dots \alpha_n i} = \frac{\delta d_{\alpha_1 \dots \alpha_n i}}{\delta \theta}, \end{aligned}\right\} \tag{7·5}$$

and, in view of (6·7),

$$d_{\alpha i} = a_{\alpha i}, \quad \lambda_{\alpha i} = \kappa_{\alpha i}. \tag{7·6}$$

Also

$$\left.\begin{aligned} \bar{\mathbf{f}} = \mathbf{f} - \dot{\mathbf{v}} - \sum_{n=2}^{\infty} k^{\alpha_1 \dots \alpha_n} \dot{\mathbf{w}}_{\alpha_1 \dots \alpha_n}, \\ \mathbf{q}^{\alpha_1 \dots \alpha_n} = \mathbf{l}^{\alpha_1 \dots \alpha_n} - k^{\alpha_1 \dots \alpha_n} \dot{\mathbf{v}} - \sum_{m=1}^{\infty} k^{\alpha_1 \dots \alpha_n : \beta_1 \dots \beta_m} \dot{\mathbf{w}}_{\beta_1 \dots \beta_m} \quad (n \geqslant 2), \\ \mathbf{q}^{\alpha} = \mathbf{l}^{\alpha} - \sum_{m=1}^{\infty} k^{\alpha : \beta_1 \dots \beta_m} \dot{\mathbf{w}}_{\beta_1 \dots \beta_m}, \end{aligned}\right\} \tag{7·7}$$

$$\bar{\mathbf{f}} = \bar{f}^i \mathbf{a}_i, \quad \mathbf{q}^{\alpha_1 \dots \alpha_n} = q^{\alpha_1 \dots \alpha_n i} \mathbf{a}_i, \tag{7·8}$$

$$\mathbf{n} = n^i \mathbf{a}_i, \quad \mathbf{p}^{\alpha_1 \dots \alpha_n} = p^{\alpha_1 \dots \alpha_n i} \mathbf{a}_i. \tag{7·9}$$

† $d_{\alpha_1 \dots \alpha_n i}$ and $\lambda_{\alpha_1 \dots \alpha_n i}$ are completely symmetric in $\alpha_1 \dots \alpha_n$.

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The equations of motion are

$$\frac{1}{\sqrt{a_{33}}}\frac{\delta n^i}{\delta\theta}+\rho\bar{f}^i=0, \tag{7·10}$$

$$\pi^{\lambda\mu}-\pi^{\mu\lambda}-\frac{1}{\sqrt{a_{33}}}(p^{\gamma\lambda}\kappa_\gamma^{\cdot\mu}-p^{\gamma\mu}\kappa_\gamma^{\cdot\lambda})+\sum_{n=2}^{\infty}(\pi^{\alpha_1\dots\alpha_n\mu}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}-\pi^{\alpha_1\dots\alpha_n\lambda}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \mu})$$
$$+\frac{1}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}(p^{\alpha_1\dots\alpha_n\mu}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}-p^{\alpha_1\dots\alpha_n\lambda}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \mu})=0, \tag{7·11}$$

$$\pi^{\lambda 3}-\frac{1}{\sqrt{a_{33}}}(n^\lambda+p^{\gamma\lambda}\kappa_\gamma^{\cdot 3}-p^{\gamma 3}\kappa_\gamma^{\cdot\lambda})+\sum_{n=2}^{\infty}(\pi^{\alpha_1\dots\alpha_n 3}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}-\pi^{\alpha_1\dots\alpha_n\lambda}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ 3})$$
$$+\frac{1}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}(p^{\alpha_1\dots\alpha_n 3}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}-p^{\alpha_1\dots\alpha_n\lambda}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ 3})=0, \tag{7·12}$$

where we have put
$$\pi^{\alpha_1\dots\alpha_n i}=\frac{1}{\sqrt{a_{33}}}\frac{\delta p^{\alpha_1\dots\alpha_n i}}{\delta\theta}+\rho q^{\alpha_1\dots\alpha_n i}. \tag{7·13}$$

The energy equation reduces to

$$\rho r-\rho(\dot{A}+S\dot{T}+T\dot{S})+\bar{\pi}^{(\lambda\mu)}\eta_{\lambda\mu}+\bar{\pi}^\lambda\eta_{\lambda 3}+\bar{\pi}\eta_{33}+\frac{1}{\sqrt{a_{33}}}p^{\alpha i}\dot{\kappa}_{\alpha i}+\sum_{n=2}^{\infty}\pi^{\alpha_1\dots\alpha_n i}\dot{d}_{\alpha_1\dots\alpha_n i}$$
$$+\frac{1}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}p^{\alpha_1\dots\alpha_n i}\dot{\lambda}_{\alpha_1\dots\alpha_n i}-\frac{1}{\sqrt{a_{33}}}\frac{\partial h}{\partial\theta}=0, \tag{7·14}$$

where

$$\bar{\pi}^{(\lambda\mu)}=\tfrac{1}{2}(\pi^{\lambda\mu}+\pi^{\mu\lambda})-\frac{1}{2\sqrt{a_{33}}}(p^{\gamma\lambda}\kappa_\gamma^{\cdot\mu}+p^{\gamma\mu}\kappa_\gamma^{\cdot\lambda})$$
$$-\frac{1}{2}\sum_{n=2}^{\infty}(\pi^{\alpha_1\dots\alpha_n\lambda}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \mu}+\pi^{\alpha_1\dots\alpha_n\mu}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda})$$
$$-\frac{1}{2\sqrt{a_{33}}}\sum_{n=2}^{\infty}(p^{\alpha_1\dots\alpha_n\lambda}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \mu}+p^{\alpha_1\dots\alpha_n\mu}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}), \tag{7·15}$$

$$\bar{\pi}^\lambda=\frac{2}{\sqrt{a_{33}}}(n^\lambda-p^{\gamma 3}\kappa_\gamma^{\cdot\lambda})-2\sum_{n=2}^{\infty}\pi^{\alpha_1\dots\alpha_n 3}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}-\frac{2}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}p^{\alpha_1\dots\alpha_n 3}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ \lambda}, \tag{7·16}$$

$$\bar{\pi}=\frac{1}{\sqrt{a_{33}}}(n^3-p^{\gamma 3}\kappa_\gamma^{\cdot 3})-\sum_{n=2}^{\infty}\pi^{\alpha_1\dots\alpha_n 3}d_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ 3}-\frac{1}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}p^{\alpha_1\dots\alpha_n 3}\lambda_{\alpha_1\dots\alpha_n\cdot}^{\ \ \ \ \ 3}, \tag{7·17}$$

and the entropy inequality is

$$-\rho(\dot{A}+S\dot{T})-\frac{1}{\sqrt{a_{33}}}\frac{h}{T}\frac{\partial T}{\partial\theta}+\bar{\pi}^{(\lambda\mu)}\eta_{\lambda\mu}+\bar{\pi}^\lambda\eta_{\lambda 3}$$
$$+\bar{\pi}\eta_{33}+\frac{1}{\sqrt{a_{33}}}p^{\alpha i}\dot{\kappa}_{\alpha i}+\sum_{n=2}^{\infty}\pi^{\alpha_1\dots\alpha_n i}\dot{d}_{\alpha_1\dots\alpha_n i}+\frac{1}{\sqrt{a_{33}}}\sum_{n=2}^{\infty}p^{\alpha_1\dots\alpha_n i}\dot{\lambda}_{\alpha_1\dots\alpha_n i}\geqslant 0. \tag{7·18}$$

The above results are valid whatever the material properties of the rod. For an elastic rod we may follow the procedure of Green and Laws (1) to produce constitutive equations

$$S=-\frac{\partial A}{\partial T}, \tag{7·19}$$

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$$\left.\begin{aligned} \bar{\pi} &= 2\rho\frac{\partial A}{\partial \gamma_{33}}, \quad \bar{\pi}^{\lambda} = 2\rho\frac{\partial A}{\partial \gamma_{\lambda 3}}, \\ \bar{\pi}^{(\lambda\mu)} &= 2\rho\frac{\partial A}{\partial \gamma_{\lambda\mu}}, \end{aligned}\right\} \tag{7·20}$$

$$\frac{1}{\sqrt{a_{33}}}p^{\alpha i} = \rho\frac{\partial A}{\partial \kappa_{\alpha i}}, \tag{7·21}$$

$$\pi^{\alpha_1 \dots \alpha_n i} = \rho\frac{\partial A}{\partial d_{\alpha_1 \dots \alpha_n i}} \quad (n \geqslant 2), \tag{7·22}$$

$$\frac{1}{\sqrt{a_{33}}}p^{\alpha_1 \dots \alpha_n i} = \rho\frac{\partial A}{\partial \lambda_{\alpha_1 \dots \alpha_n i}} \quad (n \geqslant 2), \tag{7·23}$$

where† $$\left.\begin{aligned} A &= A(T, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1 \dots \alpha_n i}, \lambda_{\alpha_1 \dots \alpha_n i}) \quad (n \geqslant 2), \\ \gamma_{ij} &= a_{ij} - A_{ij}. \end{aligned}\right\} \tag{7·24}$$

and A also depends on values of the stated variables in the initial state, and on A_{ij}‡. In evaluating (7·20) we regard A as a function of $\gamma_{\lambda 3}$, γ_{33}, $\frac{1}{2}(\gamma_{\lambda\mu}+\gamma_{\mu\lambda})$. The value of A in (7·24) is found from (6·11) and (6·28) by using the expansion (6·9) and the value of A^* expressed as a function of three-dimensional strains.

With the help of (7·20)–(7·23) the energy equation (7·14) reduces to

$$\rho r - \rho T\dot{S} - \frac{1}{\sqrt{a_{33}}}\frac{\partial h}{\partial \theta} = 0. \tag{7·25}$$

8. *Approximations for rods.* Given the basic postulates of Green and Laws (1) the resulting theory of a rod may be regarded as an exact one in which no attempt is made to relate the theory to three dimensions. Here we start with the three-dimensional theory of section 7 and by systematic approximations recover a theory of the same form as that given by Green and Laws (1). In the theory of sections 6, 7 an approximation has already been made for the temperature but the rest of the theory is exact. It consists, however, of an infinite system of equations in an infinite number of unknowns.

As a first step we assume that the base vectors in (6·9) may be written as

$$\left.\begin{aligned} \mathbf{g}_{\beta} &= \mathbf{a}_{\beta} + \sum_{n=2}^{\infty} n\theta^{\alpha_2} \dots \theta^{\alpha_n}\mathbf{d}_{\beta\alpha_2 \dots \alpha_n}, \\ \mathbf{g}_3 &= \mathbf{a}_3 + \theta^{\alpha}\partial\mathbf{d}_{\alpha}/\partial\theta, \end{aligned}\right\} \tag{8·1}$$

where $\mathbf{g}_{\beta}$ is exact and $\mathbf{g}_3$ is represented approximately by only the first three terms in the expansion (8·1). For an elastic body the Helmholtz function A^* is a function of temperature and the strains (5·2). From (6·11), (5·2), (8·1) and the conservation of mass it follows that A, given by (6·28), reduces to

$$A = A(T, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1 \dots \alpha_n i}) \quad (n \geqslant 2), \tag{8·2}$$

dependence on A_{ij} and initial values of the other arguments in (8·2) being understood. Hence, from (7·23),

$$p^{\alpha_1 \dots \alpha_n i} = 0 \quad (n \geqslant 2). \tag{8·3}$$

† To avoid ambiguity A is expressed as a completely symmetric function of $d_{\alpha_1 \dots \alpha_n i}$ and $\lambda_{\alpha_1 \dots \alpha_n i}$ as far as the indices $\alpha_1 \dots \alpha_n$ are concerned.

‡ A_{ij} is the value of a_{ij} in the reference state.

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The definitions (7·13) now reduce to

$$\pi^{\alpha i} = \frac{1}{\sqrt{a_{33}}} \frac{\delta p^{\alpha i}}{\delta \theta} + \rho q^{\alpha i}, \tag{8·4}$$

and

$$\pi^{\alpha_1 \dots \alpha_n i} = \rho q^{\alpha_1 \dots \alpha_n i} \quad (n \geqslant 2). \tag{8·5}$$

Inspection of (7·7), (7·8), (6·14) and (6·19) shows that $q^{\alpha_1 \dots \alpha_n i}$ involve the moments of all orders through a section of the rod, of body forces and applied surface forces, and inertia terms. In addition to (8·1) we now assume that†

$$q^{\alpha_1 \dots \alpha_n i} = 0 \quad (n \geqslant 2) \tag{8·6}$$

so that, from (8·5),

$$\pi^{\alpha_1 \dots \alpha_n i} = 0 \quad (n \geqslant 2). \tag{8·7}$$

Equations (7·22) are an infinite system of equations which we assume can be inverted to express $d_{\alpha_1 \dots \alpha_n i}\,(n \geqslant 2)$ as a function of T, γ_{ij}, $\kappa_{\alpha i}$ and $\pi^{\alpha_1 \dots \alpha_n i}\,(n \geqslant 2)$, together with certain reference state variables. Omitting reference state variables, it follows that

$$A(T, \gamma_{ij}, \kappa_{\alpha i}, d_{\alpha_1 \dots \alpha_n i}) = A'(T, \gamma_{ij}, \kappa_{\alpha i}, \pi^{\alpha_1 \dots \alpha_n i}) \quad (n \geqslant 2), \tag{8·8}$$

and

$$\left.\begin{aligned} \rho \frac{\partial A'}{\partial T} &= \rho \frac{\partial A}{\partial T} + \sum_{n=2}^{\infty} \pi^{\alpha_1 \dots \alpha_n i} \frac{\partial d_{\alpha_1 \dots \alpha_n i}}{\partial T}, \\ \rho \frac{\partial A'}{\partial \gamma_{ij}} &= \rho \frac{\partial A}{\partial \gamma_{ij}} + \sum_{n=2}^{\infty} \pi^{\alpha_1 \dots \alpha_n k} \frac{\partial d_{\alpha_1 \dots \alpha_n k}}{\partial \gamma_{ij}}, \\ \rho \frac{\partial A'}{\partial \kappa_{\alpha i}} &= \rho \frac{\partial A}{\partial \kappa_{\alpha i}} + \sum_{n=2}^{\infty} \pi^{\alpha_1 \dots \alpha_n k} \frac{\partial d_{\alpha_1 \dots \alpha_n k}}{\partial \kappa_{\alpha i}}. \end{aligned}\right\} \tag{8·9}$$

If the coefficients of $\pi^{\alpha_1 \dots \alpha_n i}$ in (8·9) are finite when $\pi^{\alpha_1 \dots \alpha_n i} = 0\ (n \geqslant 2)$, it follows, using (8·7), (8·3), (7·15)–(7·17), and (7·19)–(7·23), that

$$S = -\frac{\partial A'}{\partial T}, \tag{8·10}$$

$$\frac{1}{\sqrt{a_{33}}}(n^3 - p^{\gamma 3} \kappa_{\gamma}^{\cdot 3}) = 2\rho \frac{\partial A'}{\partial \gamma_{33}}, \tag{8·11}$$

$$\frac{1}{\sqrt{a_{33}}}(n^{\alpha} - p^{\gamma 3} \kappa_{\gamma}^{\cdot \alpha}) = \rho \frac{\partial A'}{\partial \gamma_{\alpha 3}}, \tag{8·12}$$

$$\tfrac{1}{2}(\pi^{\lambda\mu} + \pi^{\mu\lambda}) - \frac{1}{2\sqrt{a_{33}}}(p^{\gamma\lambda} \kappa_{\gamma}^{\cdot \mu} + p^{\gamma\mu} \kappa_{\gamma}^{\cdot \lambda}) = 2\rho \frac{\partial A'}{\partial \gamma_{\lambda\mu}}, \tag{8·13}$$

$$\frac{1}{\sqrt{a_{33}}} p^{\alpha i} = \rho \frac{\partial A'}{\partial \kappa_{\alpha i}}, \tag{8·14}$$

$$A' = A'(T, \gamma_{ij}, \kappa_{\alpha i}), \tag{8·15}$$

† If body forces are zero and the surface of the rod is free from applied forces then the moments of body and surface forces are zero.

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dependence of A' on reference state variables being understood. Moreover, equations (7·11) and (7·12) reduce to

$$\left.\begin{aligned} \pi^{\mu\lambda} - \pi^{\lambda\mu} + \frac{1}{\sqrt{a_{33}}}(p^{\gamma\lambda}\kappa_{\gamma}^{\cdot\mu} - p^{\gamma\mu}\kappa_{\gamma}^{\cdot\lambda}) = 0, \\ \pi^{\lambda 3} - \frac{1}{\sqrt{a_{33}}}(n^{\lambda} + p^{\gamma\lambda}\kappa_{\gamma}^{\cdot 3} - p^{\gamma 3}\kappa_{\gamma}^{\cdot\lambda}) = 0. \end{aligned}\right\} \tag{8·16}$$

Finally, in view of (8·6) we replace $\bar{\mathbf{f}}$ and $\mathbf{q}^{\alpha}$ in (7·7) by

$$\left.\begin{aligned} \bar{\mathbf{f}} &= \mathbf{f} - \dot{\mathbf{v}}, \\ \mathbf{q}^{\alpha} &= \mathbf{l}^{\alpha} - k^{\alpha:\beta}\dot{\mathbf{w}}_{\beta}. \end{aligned}\right\} \tag{8·17}$$

Equations (7·10), (7·25), (8·4), (8·10) to (8·17) are equivalent to those obtained by Green and Laws (1). All the basic three-dimensional equations have been considered systematically, the only ones not used being

$$0 = \pi^{\alpha_1 \dots \alpha_n i} = \rho \frac{\partial A}{\partial d_{\alpha_1 \dots \alpha_n i}} \quad (n \geqslant 2), \tag{8·18}$$

but these merely serve to determine $d_{\alpha_1 \dots \alpha_n i}\,(n \geqslant 2)$.

If desired, further approximations of the complete set of three-dimensional equations may be obtained systematically by retaining more terms in $(8·1)_2$ and adopting a procedure similar to that described above. However, the theory of Green and Laws (1) reflects satisfactorily the basic properties of deformations of rods and is sufficiently general for most purposes. We observe that in practice it will be impossible to compute A' in (8·15) from the three-dimensional Helmholtz function A^*; and A' must be regarded as an arbitrary function of its arguments. Some of the elastic coefficients of A', at least in the linear theory, can, however, be identified by studying known exact solutions of the three-dimensional equations.

Acknowledgement. The work of one of us (P.M.N.) was supported by the U.S. Office of Naval Research under Contract Nonr-222 (69) with the University of California, Berkeley.

REFERENCES

(1) Green, A. E. and Laws, N. *Proc. R. Soc. Ser. A* **283** (1966), 145.
(2) Green, A. E., Naghdi, P. M. and Wainwright, W. L. *Arch. Rational Mech. Anal.* **20** (1965), 287.
(3) Laws, N. *Quart. J. Mech. Appl. Math.* **20** (1967), 167.
(4) Green, A. E. and Naghdi, P. M. *Proc. Cambridge Philos. Soc.* **63** (1967), 537 and **63** (1967), 922.
(5) Green, A. E., Laws, N. and Naghdi, P. M. *Arch. Rational Mech. Anal.* **25** (1967), 285.
(6) Cohen, H. *Internat. J. Engng Sci.* **4** (1966), 511.
(7) Cohen, H. and De Silva, C. N. *J. Mathematical Phys.* **7** (1966), 960.
(8) Antman, S. S. and Warner, W. H. *Arch. Rational Mech. Anal.* **23** (1966), 135.
(9) Green, A. E. and Naghdi, P. M. *Quart. J. Mech. Appl. Math.* (In the press.)

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