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Zero Sums in Abelian Groups

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Let k be a positive integer and G a finite abelian group of order n, where $n \geqslant k^2 - 4k + 8$. Then every sequence of $2n - \frac{1}{4}k^2 + k - 2$ elements in G assuming k distinct values has an n-subsequence with sum zero. This settles a conjecture of Bialostocki and Lotspeich.

Let G be a finite abelian group of order n. The Erdős–Ginzburg–Ziv theorem [5] states that every sequence of $2n-1$ elements in G contains an n-subsequence with sum zero. This result has received some attention in recent years and several generalizations have been proved or conjectured (see [1–4],[6–9]). Bialostocki and Lotspeich [2] conjectured that every sequence of $2n-k+1$ elements of the cyclic group of order n assuming k distinct values has a subsequence with length n and sum zero; a generalization of this conjecture was proved by Hamidoune, Ordaz and Ortuno [9]. In this paper we prove another conjecture of Bialostocki and Lotspeich [2], that for positive integers n and k, and an abelian group G of order n, every sequence of $2n - \frac{1}{4}k^2 + k - 2$ elements of G assuming k distinct values has an n-subsequence with sum zero, provided that n is sufficiently large compared to k.

Theorem 1. *Let n and k be positive integers, where $n \geqslant k^2-4k+8$, and let G be an abelian group of order n. Then every sequence S of length at least $2n - \frac{1}{4}k^2 + k - 2$ assuming k distinct values has an n-subsequence with sum zero.*

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To prove Theorem 1 we need some preliminaries: the first is from [6] and the second is a slight reformulation of a result from [7].

Lemma 2. *Let k and n be positive integers with $2 \leqslant k \leqslant [n/4]+2$, and let $a_1,\dots,a_{2n-k}$ be a sequence of $2n-k$ elements in $\mathbb{Z}_n$. Suppose that for every n-subset I of $\{1,\dots,2n-k\}$, $\sum_{i\in I} a_i \neq 0$. Then, one can rearrange the sequence to the form*

$$\underbrace{a,\cdots,a}_{u},\underbrace{b,\dots,b}_{v},c_1,\dots,c_{2n-k-u-v},$$

with $u \geqslant v \geqslant n-2k+3$ and $2n-k-u-v \leqslant k-2$ and $a-b$ generating Z_n.

Lemma 3. *Let G be a finite abelian group of order n, let m and h be positive integers, and let g be an element of G, and*

$$S=(a_1,\dots,a_m,\underbrace{0,\dots,0}_{h})$$

be a sequence of elements in G. Suppose that no element occurs more than h times in the sequence $a_1,\dots,a_m$ and suppose that g can be expressed as a sum over a subsequence T of $a_1,\dots,a_m$ with $|T| \geqslant n-h$. Then g can be expressed as a sum over an n-subsequence of S.

Proof of Theorem 1. If G is not cyclic then the theorem follows from a result in [7], since $|S| \geqslant n+D(G)-1$, where $D(G)$ is the Davenport constant on G. So we may assume that $G=\mathbb{Z}_n$, the cyclic group of order n. Assume, to the contrary, that S has no n-subsequence with sum zero. Then by Lemma 2 we may assume that

$$S=(\underbrace{0,\dots,0}_{u},\underbrace{1,\dots,1}_{v},x_1,\dots,x_{2n-l-u-v}),$$

where $l \leqslant \frac{1}{4}k^2-k+2$, $u \geqslant v \geqslant n-2k+3$, $2n-l-u-v \leqslant l-2$ and all $x_i \neq 0,1$.

For every $x \in \mathbb{Z}_n$, we denote by $|x|$ the least nonnegative inverse image under the natural homomorphism from $\mathbb{Z}$ (the additive group of integers) onto $\mathbb{Z}_n$. Without loss of generality, we may assume that

$$S=(\underbrace{0,\dots,0}_{u},\underbrace{1,\dots,1}_{v},y_1,\dots,y_r,z_1,\dots,z_t), \tag{1}$$

with $2 \leqslant |y_i| \leqslant n/2$ for $i=1,\dots,r$ and $1 \leqslant |-z_i| < n/2$ for $i=1,\dots,t$ and $r+t=2n-l-u-v \leqslant l-2$.

Let W be a subsequence of $(y_1,\dots,y_r,z_1,\dots,z_t)$. If $x+\sum W=0$ with $|x| \leqslant v$ and $|x|+|W| \geqslant n-u$, then by Lemma 3 one can find an n-subsequence of S with sum zero, a contradiction. Now we have

$$\text{either } \left|\sum W\right| \leqslant n-v-1 \text{ or } \left|\sum W\right| \geqslant u+1+|W|, \tag{2}$$

from (2); one can easily prove by induction that, for all $r,t \leqslant l \leqslant [n/4]$,

$$\sum_{i=1}^{p}|y_i| = \left|\sum_{i=1}^{p} y_i\right| \leqslant n-v-1, \tag{3}$$

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for $p = 1, \ldots, r$, and

$$\sum_{i=1}^{q} |-z_i| = \left|-\sum_{i=1}^{q} z_i\right| \leqslant n-u-1-q, \tag{4}$$

for $q = 1, \ldots, t$.

Set $e = |\{y_1, \ldots, y_r\}|$ and $f = |\{z_1, \ldots, z_t\}|$. Then by (3) we get

$$2 + 3 + \cdots + (e+1) + 2(r-e) \leqslant n - v - 1,$$

which gives

$$\frac{e^2 - e}{2} + 2r \leqslant n - v - 1 \tag{5}$$

Similarly, one can derive from (4) that

$$\frac{f^2 - f}{2} + t \leqslant n - u - 1 - t, \tag{6}$$

and from (5) and (6) that

$$\frac{e^2 + e}{2} + \frac{f^2 + f}{2} \leqslant n - v - 1 + n - u - 1 - r - t = l - 2.$$

This gives

$$l \geqslant \frac{(e+f)^2}{4} + \frac{e+f}{2} + 2 = \frac{(k-2)^2}{4} + \frac{k-2}{2} + 2 = \frac{k^2}{4} - k/2 + 2 > l,$$

which is a contradiction. This completes the proof. □

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