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THE MONTGOMERY–HOOLEY THEOREM IN SHORT INTERVALS

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Abstract. In this paper we generalize and extend work by Languasco, Perreli and Zaccagnini on the Montgomery–Hooley theorem in short intervals. We are thus able to obtain asymptotic formulae for the mean-square error in the distribution of general sequences in arithmetic progressions in short intervals. As applications we consider lower and upper bound approximations to the characteristic function of the primes in a short interval and an average over short intervals for the von Mangoldt function.

§1. *Introduction.* The distribution of primes in arithmetic progressions has great interest in number theory, with many applications both within and without the subject field. We write

$$\psi(x; q, a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function. It is then expected that, for all q in some range and every a coprime to q, we have

$$\psi(x; q, a) \sim \frac{x}{\phi(q)},$$

where $\phi(q)$ denotes Euler's function. The difficulty in establishing such a result (except when q is very small relative to x, or unless one assumes the generalized Riemann hypothesis) is well known [**6**, Chs 19–22], but much better results are known "on average". The most celebrated of the averaged results is the Bombieri–Vinogradov theorem [**6**, Ch. 28]. Another averaged result that has attracted substantial attention is the so-called Barban–Davenport–Halberstam theorem [**3**, **4**, **7**] (see also [**6**, Ch. 29]) which may be stated as follows (a slight improvement given by Gallagher [**8**]). Here, and throughout this paper, implied constants in the $\ll$ and O notation depend at most on the exponent of the logarithm in the statements of theorems and lemmas (and these will all ultimately be related back to a given fixed A), and will not be effectively computable on account of appeals (not necessarily always explicit) to the Siegel–Walfisz theorem [**6**, Chs 21 and 22].

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THEOREM 1. *Suppose that for some fixed A we have $x(\log x)^{-A} \leqslant Q \leqslant x$. Then*

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x;\,q,\,a)-\frac{x}{\phi(q)}\right|^2 \ll xQ\log x. \tag{1}$$

This result is actually the best possible, since Montgomery [**18**] has demonstrated that the left-hand side of (1) is asymptotic to $Qx\log x$. Hooley has made many further investigations of this problem; for example see [**14, 15**]. As a result of the work of Montgomery, Hooley, and others (see, for example the contribution of Vaughan [**21, 22**]) it is now known that an asymptotic formula can be obtained for the left-hand side of (1) for a wide variety of counting functions in place of $\psi(x;\,q,\,a)$. More recently, Languasco *et al* [**16**] have obtained analogous results in short intervals. To be precise, they proved the following, where

$$S(x,\,h,\,Q)=\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x+h;\,q,\,a)-\psi(x;\,q,\,a)-\frac{h}{\phi(q)}\right|^2,$$

and

$$\epsilon(h,\,Q)=\exp\left(-\frac{(\log(2h/Q))^{3/5}}{\log\log(3h/Q)^{1/5}}\right).$$

THEOREM 2. *Let $\eta,\,A>$ be given, $x^{7/12+\eta}\leqslant h\leqslant x$ and $Q\leqslant h$. Then*

$$S(x,\,h,\,Q)=hQ\log(xQ/h)+(x+h)Q\log(1+h/x)-\kappa hQ+E(h,\,Q,\,x), \tag{2}$$

where

$$E(h,\,Q,\,x)\ll h^2(\log x)^{-A}+h^{1/2}Q^{3/2}\epsilon(h,\,Q)^c$$

for some positive constant c. Here κ is the positive constant

$$1+\gamma+\log(2\pi)+\sum_p\frac{\log p}{p(p-1)},$$

and γ is Euler's constant.

We see that (2) furnishes us with an asymptotic formula when $h(\log x)^{-A}\ll Q\leqslant h$ with a leading term of size $hQ\log x$. The purpose of this paper is to consider whether one can obtain results of this type for even smaller interval lengths. We shall pursue two questions. First we ask if it is possible to replace $\Lambda(n)$ with sieve upper and lower bound functions for the primes as in [**1, 2**]. Then we consider averaging over the short intervals. In the latter case we will find that the intervals can be almost as small as $x^{1/3}$. It should be possible to combine both of these approaches and so obtain results for still shorter intervals (see remarks after Theorem 4 below), but we shall not follow that line here. We note that Vaughan anticipated such short interval results with his prescient comments in the second paragraph of [**21**, p. 783], and our own work on these

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problems could be considered as a development of the programme suggested there. Indeed, Theorem 5 below is a short interval form of [**21**, Theorem 1.1] from which Theorems 2, 3 and 4 follow as corollaries. We should also mention other recent work of Brüdern and Wooley [**5**] who obtained asymptotic formulae where the moduli q are restricted to the values of a polynomial. It may well be possible to combine our approach with theirs and so obtain short interval results on average with general sequences for sparse moduli.

THEOREM 3. *Let $\theta \in [21/40, 7/12]$, write $h = x^\theta$, and suppose for some $A > 0$ we have $h(\log x)^{-A} \leqslant Q \leqslant h$. There are functions $\rho_\theta^+(n)$ and $\rho_\theta^-(n)$ such that*

$$\rho_\theta^+(n) \geqslant \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_\theta^-(n) \leqslant \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

For these functions there exist functions $f^+(\theta)$ non-increasing, $f^-(\theta)$ non-decreasing, such that

$$\sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{x \leqslant n < x+h} \rho_\theta^\pm(n) - f^\pm(\theta) \frac{h}{\phi(q) \log x} \right|^2 = E^\pm(x, \theta, Q), \tag{3}$$

where

$$E^\pm(x, \theta, Q) = Q \sum_{x \leqslant n < x+h} (\rho_\theta^\pm(n))^2 + (1 - \kappa - \log(h/Q)) Qh \left(\frac{f^\pm(\theta)}{\log x} \right)^2 + O(h^2 (\log x)^{-A} + h^{1/2} Q^{3/2} \epsilon(h, Q)^c (\log x)^{-2}). \tag{4}$$

In particular we have

$$f^\pm(7/12) = 1, \qquad f^+(0.55) < 1.01, \qquad f^-(0.55) > 0.99,$$

and

$$f^-(0.525) > 0.09.$$

Remarks. The above result genuinely gives an asymptotic formula in the range specified since the first term on the right-hand side of (4) has size of order $hQ(\log x)^{-1}$, while the other terms are of smaller magnitude.

THEOREM 4. *Let $\theta \in (1/3, 7/12]$, and suppose that $\mathcal{I}_m = [X_m, X_m + A_m)$ are disjoint intervals contained in $[x, 2x]$ for $1 \leqslant m \leqslant M$ where $x^\theta \leqslant A_m \ll x^\theta$ for all m. Assume that, for some $A > 0$, we have $x^\theta (\log x)^{-A} \leqslant Q \leqslant x^\theta$. Then*

$$\sum_{m \leqslant M} \sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{\substack{n \in \mathcal{I}_m \\ n \equiv a \pmod q}} \Lambda(n) - \frac{A_m}{\phi(q)} \right|^2 = F(x, M), \tag{5}$$

where

$$F(x, M) = \sum_{m \leqslant M} A_m Q(\log(X_m Q/A_m) + 1 - \kappa) + O(Mx^{2\theta} (\log x)^{-A} + Mx^{\theta/2} Q^{3/2} \epsilon(h, Q)^c + x^{4/3+\epsilon}), \tag{6}$$

for some $c > 0$ and every $\epsilon > 0$.

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Remarks. Taking $\theta = 1/2$ in the above, we see that we obtain an asymptotic formula as soon as $M > x^{1/3+\epsilon}$. The reader should note that Theorem 4 is in fact a hybrid of the Montgomery–Hooley theorem with early work of Heath-Brown on sums of differences between consecutive primes [**12**]. So, for example, by putting $Q = x^{\theta}(\log x)^{-3}$, from the terms with $q = 1$ on the left-hand side of (5) we can count the number of intervals $\mathcal{I}_m$ not containing primes for $\theta \geqslant 1/2$. We thus recover Heath-Brown's result

$$\sum_{\substack{p_n \leqslant x \\ p_{n+1}-p_n > x^{1/2}}} (p_{n+1} - p_n) \ll x^{5/6+\epsilon}.$$

This result was improved by Heath-Brown himself [**13**], and then by others using a sieve method (see [**17**]). It should therefore be possible to improve the 4/3 exponent on the right-hand side of (6) slightly, or to introduce upper and lower bound approximations to the characteristic function of the set of primes as in Theorem 3.

Both Theorems 3 and 4 are consequences of the following result, as we shall demonstrate later. It is in a form that may be applicable to other "sifted" sequences, and the reader will see that it also implies Theorem 2. First, for a Dirichlet character χ write

$$\delta_\chi = \begin{cases} 1 & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 5. *Suppose $x \geqslant h \geqslant x^{\delta}$ for some $\delta > 0$ and $h(\log x)^{-A} \leqslant Q \leqslant h$ for some $A > 0$. Let $g(n)$ be a function defined on the positive integers which satisfies the following for some function $u(x)$:*

(i) $|g(n)| \ll \log n$;

(ii) *for any $q \leqslant Q$ we have, for any fixed $D > 0$,*

$$\sum_{\substack{x \leqslant n < x+h \\ (n,q)>1}} g(n)^2 \ll h(\log x)^{-D};$$

(iii) *for fixed B, $C > 0$ and χ a character (modulo r) with $r < (\log x)^C$ we have*

$$\sum_{x \leqslant n < x+y} g(n)\chi(n) = \delta_\chi u(x) y + O(h(\log x)^{-B})$$

for $0 < y \leqslant h$.

Then we have

$$\sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{\substack{x \leqslant n < x+h \\ n \equiv a \pmod q}} g(n) - \frac{u(x)h}{\phi(q)} \right|^2 = G(x, h, Q),$$

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with $G(x, h, Q)$ *given by*

$$Q \sum_{x \leqslant n < x+h} g(n)^2 + (1 - \kappa - \log(h/Q))hQu(x)^2 + O(u(x)^2 h^{1/2} Q^{3/2} \epsilon(h, Q)^c + h^2 (\log x)^{-A}), \tag{7}$$

for some $c > 0$.

Remarks. As mentioned above, this is essentially a short interval form of [**21**, Theorem 1.1]. By restricting our attention in the applications to short intervals we have made some details simpler: we do not require Vaughan's technique of introducing weights. We shall give a short proof of this result in §2. The experienced reader will have already understood that Theorem 5 ought to follow in a similar way to Vaughan's result. We also remark that (ii) automatically follows from (i) and (iii) (taking χ as a principal character) when $g(n)$ is non-negative, but we do not want to put that restriction on $g(n)$. The application to $\rho^-(n)$ requires g to take negative values in any case.

§2. *Proof of Theorem 5.* We adopt the notation of [**16**] *mutatis mutandis*. We write

$$\tau_Q(m) = \sum_{\substack{q|m \\ q \leqslant Q}} 1, \qquad W_Q(\alpha) = \sum_{m \leqslant h} \tau_Q(m) e(-m\alpha),$$

with $e(x) = \exp(2\pi i x)$, and

$$S(\alpha) = \sum_{x \leqslant n < x+h} g(n) e(n\alpha), \qquad T(\alpha) = \sum_{x \leqslant n < x+h} e(n\alpha).$$

Also, put $P = (\log x)^E$, $R = h(\log x)^{-F}$ for some $E, F > 0$ to be determined later. We then have the following two lemmas.

LEMMA 1. *Suppose that* $|\alpha - b/r| < 1/r^2$ *for integers* b, r *with* $(b, r) = 1$, $1 \leqslant r \leqslant Q \leqslant h$. *Then we have*

$$|W_Q(\alpha)| \ll \left(\frac{h}{r} + \min(Q, h^{1/2}) + r \right) \log h. \tag{8}$$

Proof. This is proved *inter alia* in [**9**, pp. 121–122], or can be deduced quickly from [**20**, Lemma 2.2]. □

LEMMA 2. *Let* $G > 0$ *be fixed. Then, for any* $|\eta| < 1/R$, *and every pair of integers* b, r *satisfying* $(b, r) = 1$, $1 < r \leqslant P$, *we have*

$$S(b/r + \eta) = \frac{u(x)\mu(r)}{\phi(r)} T(\eta) + O(h(\log x)^{-G}). \tag{9}$$

Proof. See [**1**, pp. 6–7] after noting that [**1**, Theorem 2(i)] is our Theorem 5(iii). □

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We begin our analysis of $G(x, h, Q)$ in a familiar fashion ([22, p. 786], [16, (2.1)]) by writing

$$G(x, h, Q) = S_1 - S_2 - 2S_3,$$

where

$$S_1 = \sum_{q \leqslant Q} \sum_{\substack{x \leqslant n, m < x+h \\ q \mid (n-m) \\ (nm, q)=1}} g(n)g(m), \qquad S_2 = h^2 u(x)^2 \sum_{q \leqslant Q} \frac{1}{\phi(q)},$$

and

$$S_3 = \sum_{q \leqslant Q} \frac{hu(x)}{\phi(q)} \Bigg(\sum_{\substack{x \leqslant n < x+h \\ (n,q)=1}} g(n) - hu(x) \Bigg).$$

We can quickly deal with S_3. Let $L = (\log x)^{A+4}$. For any q we have

$$S_3'(q) = \sum_{\substack{x \leqslant n < x+h \\ (n,q)>1}} g(n) = \sum_{1 < d \mid q} S_3(d),$$

say, where $(n, q) = d$ in $S_3(d)$. If $d \leqslant L$ then (iii) with χ the principal character (modulo d) and $C - 1 = B = A + 3$ gives $S_3(d) \ll h(\log x)^{-A-3}$. On the other hand, the trivial estimate gives $S_3(d) \leqslant (h \log x)/d$ for $d \geqslant L$. Hence

$$S_3'(q) \ll \tau(q) h (\log x)^{-A-3}.$$

Using (iii) with χ the trivial character then gives

$$\sum_{\substack{x \leqslant n < x+h \\ (n,q)=1}} g(n) - hu(x) \ll h(\log x)^{-A-2} + \tau(q) h (\log x)^{-A-3}.$$

Multiplying by $hu(x)/\phi(q)$ and summing this over $q \leqslant Q$ then gives $S_3 \ll h^2 u(x) (\log x)^{-A-1} \ll h^2 (\log x)^{-A}$ in view of (i). We thus see that S_3 only contributes to the O term in (7).

We next deal with S_1. The terms $m = n$ in this sum contribute

$$\sum_{q \leqslant Q} \sum_{\substack{x \leqslant n < x+h \\ (n,q)=1}} g(n)^2 = Q \sum_{x \leqslant n < x+h} g(n)^2 + O(hQ(\log x)^{-A})$$

from (ii). This provides the first term in (7) with another error contributing to the O term.

The remaining part of S_1 can be written as

$$2 \sum_{q \leqslant Q} \sum_{\substack{x \leqslant m < n < x+h \\ q \mid (n-m) \\ (nm, q)=1}} g(n)g(m) = 2S_4 - S_5,$$

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say, where we have removed the condition $(mn, q) = 1$ in S_4, and so $(mn, q) > 1$ in S_5. We can simply dispose of S_5 since

$$\begin{aligned}|S_5| &\leqslant \sum_{q\leqslant Q}\sum_{\substack{x\leqslant m<x+h\\(m,q)>1}} g(m)^2 \sum_{\substack{x\leqslant n<x+h\\ n\equiv m \pmod q}} 1\\ &\leqslant \sum_{q\leqslant Q}\frac{2h}{q}\sum_{\substack{x\leqslant m<x+h\\(m,q)>1}} g(m)^2\\ &\ll h^2(\log x)^{-A}\end{aligned}$$

using (ii) with $D = A + 1$.

We now use the circle method in what has become a familiar way since [9] to analyze S_4. We have

$$S_4 = \int_0^1 |S(\alpha)|^2 W_Q(\alpha)\, d\alpha.$$

We break up the integration range into major and minor arcs where the major arcs are given by

$$\mathfrak{M}(r, b) = \left(\frac{b}{r} - \frac{1}{rR}, \frac{b}{r} + \frac{1}{rR}\right), \quad \text{with } r \leqslant P,\ (b, r) = 1,$$

and so their union is

$$\bigcup_{r\leqslant P}\bigcup_{\substack{b=1\\(b,r)=1}}^{r} \mathfrak{M}(r, b) = \mathfrak{M},$$

say. We then define the minor arcs by $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. It follows that for every $\alpha \in \mathfrak{m}$ there is a pair of coprime integers b, r satisfying $|\alpha - b/r| < 1/(rR) \leqslant 1/r^2$ and $P < r \leqslant R$. Since

$$\int_0^1 |S(\alpha)|^2\, d\alpha = \sum_{x\leqslant n<x+h} g(n)^2 \ll h(\log x)^2$$

we have

$$S_4 = \sum_{m\leqslant h}\tau_Q(m)\int_{\mathfrak{M}} |S(\alpha)|^2\, e(-m\alpha)\, d\alpha + O\left(h(\log x)^2 \max_{\alpha\in\mathfrak{m}} |W_Q(\alpha)|\right).$$

By Lemma 1 the error term above is suitable provided that $\min(E, F) \geqslant A + 3$.

To calculate the integral over the major arcs we put, for even m,

$$\mathfrak{S}(m) = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid m\\p>2}}\frac{p-1}{p-2},$$

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with $\mathfrak{S}(m) = 0$ if m is odd. By standard techniques (see [16, pp. 237–238]) and using the approximation (9) we have

$$S_4 = u(x)^2 \sum_{m \leqslant h} \tau_Q(m)(h-m)\mathfrak{S}(m) + O(h^2(\log x)^{-A}). \tag{10}$$

To see this, note that

$$|S(\eta + b/r)|^2 = \left(\frac{\mu(r)u(x)}{\phi(r)}\right)^2 |T(\eta)|^2 + O\left(\frac{h|T(\eta)|}{\phi(r)(\log x)^G} + \frac{h^2}{(\log x)^{2G}}\right). \tag{11}$$

The principal contribution for the major arcs is given by $u(x)^2$ times

$$\sum_{r \leqslant P} \sum_{\substack{b=1 \\ (b,r)=1}}^{r} \left(\frac{\mu(r)}{\phi(r)}\right)^2 \int_{-1/rR}^{1/rR} |T(\eta)|^2 e(-m(\eta + b/r))\, d\eta.$$

We complete the integration range to $[-1/2, 1/2]$, which incurs an additional error $\ll PR$ and gives a main term (writing $c_r(-m)$ as customary for the Ramanujan sum)

$$(h-m) \sum_{r \leqslant P} \left(\frac{\mu(r)}{\phi(r)}\right)^2 c_r(-m) = (h-m)\mathfrak{S}(m) + O\left(\frac{h \log^2 x}{P}\right)$$

from [20, p. 35]. Upon multiplying by $u(x)^2\tau_Q(m)$ and summing over m, the errors introduced here will be $\ll h^2(\log x)^{-A}$ so long as

$$PR \ll h(\log x)^{-A-3}, \qquad P \gg (\log x)^{A+5},$$

since $u(x) \ll \log x$. The remaining error terms come from the expressions in the O notation in (11). After integrating over the major arcs and summing over m these lead to terms

$$\ll h^3 PR^{-1}(\log x)^{1-2G} \quad \text{and} \quad \ll h^2 P(\log x)^{2-G}.$$

We can therefore make all the errors $\ll h^2(\log x)^{-A}$ upon choosing

$$E = A + 5, \qquad F = 2A + 8, \qquad G = 2A + 7.$$

To progress from (10) we then note (see [22, p. 549], [16, p. 238]) that, for some $c > 0$,

$$2 \sum_{m \leqslant h} \tau_Q(m)(h-m)\mathfrak{S}(m)$$
$$= h^2 \sum_{q \leqslant Q} \frac{1}{\phi(q)} + (1 - \kappa - \log(h/Q))hQ + O(\epsilon(h, Q)^c).$$

Putting together all the terms then completes the proof of (7) as required.

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§3. *Proof of Theorem 3.* We can use the author's sieve method as developed in [**1, 2**], noting in particular the additional explanation in [**10**, Chs 7 and 10], to construct the functions $\rho_\theta^{\pm}(n)$ satisfying the required properties of $g(n)$ in Theorem 5. We note that both the upper and lower bound functions take the value 1 at primes. In addition, the upper bound function counts certain almost-primes with a positive integral weight, while the lower bound function counts certain almost-primes with a negative integral weight (see the final paragraph of [**10**, § 3.6]). It immediately follows from the construction of these functions that $|g(n)| \leqslant 10$ for $\theta \geqslant 0.525$ and so (i) is satisfied in Theorem 5. As an example, we can take $g(n)$ to be $a_0(n)$ from [**1**] to obtain $\rho_\theta^-(n)$, and to be $a_1(k)$ to produce $\rho_\theta^+(n)$ for $\theta = 0.55$ (and the construction will give better approximations for $\theta > 0.55$ and worse approximations for $\theta < 0.55$). These functions satisfy (i) of [**1**, Theorem 2] which we have already noted corresponds to (iii) of our Theorem 5. Our condition (ii) is easily satisfied since for these ρ we have

$$\rho_\theta^{\pm}(n) \neq 0 \quad \text{only if } p|n \Rightarrow p > x^{1/10}. \tag{12}$$

Hence

$$\sum_{\substack{x \leqslant n < x+h \\ (n,q)>1}} g(n)^2 \ll \frac{h\tau(q)}{x^{1/10}} \ll h(\log x)^{-D}.$$

On the other hand, we can construct $\rho_\theta^-(n)$ for $\theta \geqslant 0.525$ by the method in [**2**]. As explained at the end of [**10**, Ch. 10] we can use the Siegel–Walfisz theorem to obtain a sufficiently good zero-free region for the method to work since $r \leqslant P$ here (see (15) below) and the main result in [**11**] can replace Watt's mean value result when dealing with partial sums of L-functions that arise in our present context. The exponent $1/10$ in (12) is now $1/20$, but this still more than suffices to establish (ii).

§4. *Proof of Theorem 4.* In the case of Theorem 4 hypotheses (i) and (ii) of Theorem 5 are automatically satisfied. In order to verify (iii) for sufficiently many intervals we need to have the following generalized version of the lemma on [**12**, p. 8].

LEMMA 3. *Fix $B, C, \epsilon > 0$. Let χ be a character (modulo r) with $r \ll (\log x)^C$. Then there exists a function $E(y, \zeta, \chi)$ such that*

$$\int_x^{2x} \left| \sum_{y \leqslant n < y+\zeta y} \Lambda(n)\chi(n) - y\zeta\delta_\chi + E(y, \zeta, \chi) \right|^2 dy \ll \zeta x^{7/3+\epsilon}, \tag{13}$$

and

$$E(y, \zeta, \chi) \ll \frac{y\zeta}{(\log x)^B} \tag{14}$$

uniformly for $x \geqslant 2$, $x^{-1} \leqslant \zeta \leqslant x^{-1/10}$, $x \leqslant y \leqslant 2x$.

Remarks. This result is proved in [**12**] for the trivial character, but the proof is easily modified to the more general case since powers of $\log x$ can be absorbed into the x^ϵ term. Also, the zero-density estimates used by Heath-Brown have

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the required analogues for characters to small modulus. Our inequality (14) is stronger than he gives since he used only a simple zero-free region for the Riemann zeta-function. Our result can be obtained using the fact that for any $\lambda > 0$ and for all large x no L-function has a zero $\beta + i\gamma$ with $|\gamma| < x$ and

$$\beta > 1 - \frac{1}{r^{\lambda} + (\log x)^{4/5}}. \tag{15}$$

This is a consequence of [**19**, p. 295, Satz 6.2] combined with Siegel's theorem [**6**, Ch. 21].

To prove Theorem 4, suppose that for some fixed character χ to a modulus $r < (\log x)^C$ we have V of the intervals $\mathcal{I}_m$ containing a subinterval of the form $[Y, Y + Y\zeta]$ where $\zeta = x^{\theta-1}(\log x)^{-B}$ and

$$\left| \sum_{Y \leqslant n < Y + \zeta Y} \Lambda(n)\chi(n) - Y\zeta\delta_{\chi} + E(Y, \zeta, \chi) \right| > \frac{x\zeta}{(\log x)^B}.$$

Then, for

$$|y - Y| < \frac{Y\zeta}{3(\log x)^{B+1}}$$

we have

$$\left| \sum_{y \leqslant n < y + \zeta y} \Lambda(n)\chi(n) - y\zeta\delta_{\chi} + E(y, \zeta, \chi) \right| > \frac{x\zeta}{3(\log x)^B}.$$

Hence we have, applying (13) with ϵ replaced by $\epsilon/2$,

$$V\left(\frac{x\zeta}{(\log x)^{B+1}}\right)\left(\frac{x\zeta}{(\log x)^B}\right)^2 \ll \zeta x^{7/3+\epsilon/2}.$$

We thus have

$$V(x^{1-\theta}\log x)^2(\log x)^{2C+1} \ll x^{4/3+\epsilon}. \tag{16}$$

We can thus assume that, with V satisfying (16), for all but $V(\log x)^{2C}$ of the intervals $\mathcal{I}_m$ for every subinterval $[Y, Y + Y\zeta]$ we have for every character to a modulus $r < (\log x)^C$,

$$\sum_{Y \leqslant n < Y + \zeta Y} \Lambda(n)\chi(n) - Y\zeta\delta_{\chi} = O(x^{\theta}(\log x)^{-2B}). \tag{17}$$

It then follows that, for fixed $B, C > 0$ and χ a character (modulo r) with $r < (\log x)^C$, we have, for $y \leqslant A_m$,

$$\sum_{X_m \leqslant n < X_m + y} g(n)\chi(n) = \delta_{\chi} y + O(x^{\theta}(\log x)^{-B}).$$

We can thus apply Theorem 5 with $u(x) \equiv 1$ for each of these "good" intervals $\mathcal{I}_m$ and apply the trivial bound (the left-hand side of (16)) for the sum over a, q on the remaining intervals. Since $B, C > A$, we then obtain (5) with (6) as desired. Here we have used (17) again with χ the trivial character

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(i.e. Heath-Brown's original result) to give

$$\sum_{n\in\mathcal{I}_m}\Lambda(n)^2 = (\log X_m + O(x^{\theta-1}))\sum_{n\in\mathcal{I}_m}\Lambda(n)$$
$$= A_m \log X_m + O(x^{\theta}(\log x)^{-A})$$

for each of the "good" intervals.

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