Ergod. Th. & Dynam. Sys. (2006), **26**, 1511–1520
doi:10.1017/S0143385706000423
© *2006 Cambridge University Press*
Printed in the United Kingdom

The topology of hyperbolic attractors on compact surfaces

TODD FISHER

University of Maryland, College Park, MD 20742, USA
(e-mail: tfisher@math.umd.edu)

(*Received* 30 *January* 2006 *and accepted in revised form* 30 *May* 2006)

Abstract. Suppose M is a compact surface and $\Lambda \subset M$ is a non-trivial mixing hyperbolic attractor for some $f \in \mathrm{Diff}(M)$. We show that if Λ is a hyperbolic set for some $g \in \mathrm{Diff}(M)$, then Λ is a non-trivial mixing hyperbolic attractor or repeller for g.

1. *Introduction*

Hyperbolic attractors and repellers are central to many problems in dynamical systems and possess many interesting dynamical properties. The structure of hyperbolic attractors on surfaces has been studied extensively by Plykin [**6**], Bonatti and Langevin [**1**], Williams [**9**], Zhirov [**10–11**], and others. Much of the previous work studies the topology of a hyperbolic attractor. In this paper we take a different approach. Namely, we assume some knowledge of the topology of a hyperbolic set Λ contained in a surface, and we conclude that Λ is a hyperbolic attractor or repeller. Specifically, we prove the following result.

THEOREM 1.1. *If M is a compact surface and Λ is a non-trivial mixing hyperbolic attractor for a diffeomorphism f of M, and Λ is hyperbolic for a diffeomorphism g of M, then Λ is either a non-trivial mixing hyperbolic attractor or a non-trivial mixing hyperbolic repeller for g.*

In addition, we are able to show, by counterexample, that the above theorem does not hold if $\dim(M) \geq 3$.

THEOREM 1.2. *If $\dim(M) \geq 3$, then there exists a compact set Λ hyperbolic for $f, g \in \mathrm{Diff}(M)$ such that Λ is a non-trivial mixing attractor for f and neither an attractor nor a repeller for g.*

If the mixing condition in Theorem 1.1 is dropped we have the following weakened result.

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

THEOREM 1.3. *If M is a compact surface and Λ is a non-trivial hyperbolic attractor for a diffeomorphism f of M, and Λ is hyperbolic for a diffeomorphism g of M, then there exist an $n \in \mathbb{N}$ and sets $\Lambda_1, \ldots, \Lambda_N$ where $\Lambda = \bigcup_{i=1}^{N} \Lambda_i$, $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \neq j$, and each Λ_i is a mixing hyperbolic attractor or repeller for g^n.*

We note that in [**3**] and [**2**] topologically one-dimensional hyperbolic sets are constructed on compact surfaces that are neither hyperbolic attractors nor repellers.

There are a number of additional open problems related to Theorems 1.1–1.3.

Problem 1.4. Suppose Λ is a locally maximal hyperbolic set for a diffeomorphism f and hyperbolic for a diffeomorphism g. Does this imply that Λ is locally maximal for g or that Λ is contained in a locally maximal hyperbolic set for g?

Problem 1.5. Suppose M is a compact smooth boundaryless manifold of dimension n and Λ is a mixing hyperbolic attractor for f with $\dim(\mathbb{E}^u) = n - 1$ and hyperbolic for a diffeomorphism g. Does this imply that Λ is a mixing hyperbolic attractor or repeller for g?

Related to the first problem, we note that in [**3**] it is shown that, on any manifold of dimension greater than one, there is an open set of diffeomorphisms containing a hyperbolic set that is not contained in a locally maximal one.

2. *Preliminaries*

We first review some basic facts. A set X is *transitive* for a continuous map f if there exists a point $x \in X$ such that the closure of the orbit of x is X. If X is a compact set of a smooth manifold M and f is a continuous map from M to itself, then $f|_X$ is transitive if and only if for any open sets U and V of X there exist some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. A set X is *mixing* if for any open sets U and V in X there exists an $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

From now on we assume that all the maps are diffeomorphisms. A compact set Λ invariant under the action of f is *hyperbolic* if there exists a splitting of the tangent space $T_\Lambda f = \mathbb{E}^u \oplus \mathbb{E}^s$ and positive constants C and $\lambda < 1$ such that, for any point $x \in \Lambda$ and any $n \in \mathbb{N}$,

$$\|Df_x^n v\| \leq C\lambda^n \|v\| \quad \text{for } v \in E_x^s$$

and

$$\|Df_x^{-n} v\| \leq C\lambda^n \|v\| \quad \text{for } v \in E_x^u.$$

The pioneering article by Smale [**8**] states many of the standard results for hyperbolic sets.

A set $X \subset M$ has an *attracting neighborhood* if there exists a neighborhood V of X such that $X = \bigcap_{n\in\mathbb{N}} f^n(V)$. A set $X \subset M$ has a *repelling neighborhood* if there exists a neighborhood U of X such that $X = \bigcap_{n\in\mathbb{N}} f^{-n}(U)$. A set $\Lambda \subset M$ is called a *hyperbolic attractor* (*hyperbolic repeller*) if Λ is a transitive hyperbolic set for a diffeomorphism f with an attracting neighborhood (a repelling neighborhood). A hyperbolic attractor (repeller) is *non-trivial* if it is not the orbit of a periodic sink (source).

Note, it is always possible to make a smooth change of the metric near the hyperbolic set so that $C = 1$. Such a metric is called an *adapted metric*. For $\epsilon > 0$ sufficiently small

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

and $x \in \Lambda$, the *local stable and unstable manifolds* are respectively

$$W^s_\epsilon(x, f) = \{y \in M \mid \text{ for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}$$

and

$$W^u_\epsilon(x, f) = \{y \in M \mid \text{ for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}.$$

The *stable and unstable manifolds* are respectively

$$W^s(x, f) = \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x), f))$$

and

$$W^u(x, f) = \bigcup_{n \geq 0} f^n(W^u_\epsilon(f^{-n}(x), f)).$$

The stable and unstable manifolds of a hyperbolic set are injectively immersed submanifolds.

A hyperbolic set Λ is said to be *locally maximal* (or *isolated*) if there exists a neighborhood V of Λ in M such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$. A hyperbolic set Λ possesses a *local product structure* provided there exist constants $\delta, \epsilon > 0$ such that if $x, x' \in \Lambda$ and $d(x, x') < \delta$, then $W^s_\epsilon(x, f)$ and $W^u_\epsilon(x, f)$ intersect in exactly one point which is contained in Λ. For a hyperbolic set, locally maximal and possessing a local product structure are equivalent conditions.

We now review properties of Markov partitions. For a hyperbolic set Λ, the continuity of the stable and unstable distributions implies there exist constants $\epsilon \geq \alpha > 0$ such that

$$f^{-1}(W^s_\epsilon(f(p))) \cap f(W^u_\epsilon(f^{-1}(q))) = W^s_\epsilon(p) \cap W^u_\epsilon(q)$$

is a single point, possibly not contained in Λ, whenever $p, q \in \Lambda$ satisfy $d(p, q) \leq \alpha$.

A set $R \subset \Lambda$ is a *rectangle* provided R has a diameter less than α and $p, q \in R$ implies that $W^s_\epsilon(p) \cap W^u_\epsilon(q) \in R$ where ϵ and α are as above. A rectangle is *proper* if $R = \mathrm{cl}(\mathrm{int}(R))$. If R is a rectangle containing a point x, then there exists a natural homeomorphism

$$R \simeq W^s_\epsilon(x, R) \times W^u_\epsilon(x, R),$$

where

$$W^s_\epsilon(x, R) = R \cap W^s_\epsilon(x) \quad \text{and} \quad W^u_\epsilon(x, R) = R \cap W^u_\epsilon(x).$$

Definition 2.1. A *Markov partition* of Λ is a finite collection of proper rectangles $\{R_i\}_{i=1}^M$ covering Λ such that:

(1) if $i \neq j$, then $\mathrm{int}(R_j) \cap \mathrm{int}(R_i) = \emptyset$;

(2) if $z \in \mathrm{int}(R_i)$ and $f(z) \in \mathrm{int}(R_j)$, then $W^u(f(z), R_j) \supset f(W^u(z, R_i))$ and $f(W^s(z, R_i)) \subset W^s(f(z), R_j)$.

Next, we review some facts about subshifts of finite type. Let $A = [a_{ij}]$ be an $n \times n$ matrix with non-negative integer entries. The *graph* of A is the directed graph G_A with vertices $\mathcal{V}(G_A) = \{1, \ldots, n\}$ and a_{ij} distinct edges with initial state i and terminal state j. For the graph G_A with edge set $\mathcal{E}$ and adjacency matrix A, let Σ be the space over $\mathcal{E}$ specified by

$$\Sigma = \{\omega = (\omega_j)_{j \in \mathbb{Z}} \mid t(\omega_j) = i(\omega_{j+1}) \text{ for all } j \in \mathbb{Z}\},$$

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

where $t(\omega_j)$ is the terminal state of edge ω_j and $i(\omega_{j+1})$ is the initial state of edge ω_{j+1}. The map on Σ defined by $\sigma(\omega) = \omega'$, where $\omega'_j = \omega_{j+1}$, is called the *edge shift map*. The *subshift of finite type* (SFT) is the space (Σ, σ). The *stable set* for a point $t \in \Sigma$ is denoted $W^s(t)$ and defined as

$$W^s(t) = \{s \in \Sigma \mid \text{ there exists } k \in \mathbb{Z} \text{ where } s_i = t_i \text{ for all } i \geq k\}.$$

The *unstable set* for a point $t \in \Sigma$ is denoted $W^u(t)$ and defined as

$$W^u(t) = \{s \in \Sigma \mid \text{ there exists } k \in \mathbb{Z} \text{ where } s_i = t_i \text{ for all } i \leq k\}.$$

A matrix A is irreducible if, for each pair $1 \leq i, j \leq n$, there is a $k \in \mathbb{N}$ such that $(A^k)_{ij} > 0$. A standard result states that if A is irreducible, then Σ contains a transitive point (see [**7**, p. 277]).

Every adjacency matrix A can be recoded into a block lower triangular matrix, by relabeling the vertices of the graph representing the matrix, where each block is an irreducible component. An irreducible component is a *sink* (*source*) if there are no outgoing edges from (into) it. There is a dense set of points $S \subset \Sigma_A$ such that for all $s \in S$ there exists a $J \in \mathbb{Z}$, a sink A_i, and a source A_j in A depending on s such that for all $k \geq K$ the transition s_k to s_{k+1} is contained in A_i and s_{-k-1} to s_{-k} is contained in A_j. (See [**5**, p. 117] for details.)

Canonically associated to a Markov partition $\mathcal{M}$ is a SFT (Σ, σ) with alphabet $\mathcal{A}(\Sigma) = \{1, \ldots, n\}$ such that a transition from i to j is allowed if

$$f(\text{int}(R_i)) \cap \text{int}(R_j) \neq \emptyset.$$

Similarly, a SFT (Σ, σ) with factor map $\pi : \Sigma \to \Lambda$ is said to generate a Markov partition $\mathcal{M}$ if the following hold:

(1) the image of a cylinder set $C(i) = \{s \in \Sigma \mid s_0 = i\}$ under π is a proper rectangle for any $i \in \mathcal{A}(\Sigma)$;
(2) a transition from i to j in Σ implies $f(\text{int}(R_i)) \cap \text{int}\, R_j \neq \emptyset$; and
(3) if $x \in \text{int}\, R_i$ and $f(x) \in \text{int}\, R_j$, then

$$f(W^s_\epsilon(x, R_i)) \subset R_j \quad \text{and} \quad f^{-1}(W^u_\epsilon(x, R_j)) \subset R_i.$$

For $i \in \mathcal{A}(\Sigma)$ we denote the proper rectangle R_i as the image of the cylinder set $C(i)$ under the map π.

3. *Proofs*

The following theorem will be useful in the proof of Theorem 1.1.

THEOREM 3.1. [**3**] *If Λ is a hyperbolic set and V is a neighborhood of Λ, then there exists a hyperbolic set $\tilde{\Lambda}$ with a Markov partition such that $\Lambda \subset \tilde{\Lambda} \subset V$.*

We now proceed with the proofs.

Proof of Theorem 1.1. The idea of the proof is the following. The proof is divided into two cases. First, if for all $x \in \Lambda$ either

$$T_x W^u(x, f) = T_x W^u(x, g) \quad \text{or} \quad T_x W^u(x, f) = T_x W^s(x, g),$$

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

then the fact that $W^u(x, f)$ is dense in Λ for all $x \in \Lambda$ will imply that Λ is a mixing hyperbolic attractor or repeller. We do not assume that such a choice is consistent for all $x \in \Lambda$. However, we will show this implies that either

$$T_x W^u(x, f) = T_x W^u(x, g) \quad \text{for all } x \in \Lambda$$

or

$$T_x W^u(x, f) = T_x W^s(x, g) \quad \text{for all } x \in \Lambda.$$

Second, if

$$T_x W^u(x, f) \neq T_x W^u(x, g) \quad \text{and} \quad T_x W^u(x, f) \neq T_x W^s(x, g) \quad \text{for some } x \in \Lambda,$$

then we will show that any hyperbolic set $\tilde{\Lambda} \supset \Lambda$ with a Markov partition will have non-empty interior and contain a mixing hyperbolic attractor and a mixing hyperbolic repeller for some power of g. This will imply that Λ contains a mixing hyperbolic attractor Λ_a and a mixing hyperbolic repeller Λ_r for some power of g. For $x \in \Lambda_a$ we will show that $W^u(x, f) = W^u(x, g)$ so $\Lambda_a = \Lambda$. For $y \in \Lambda_r$ we will show that $W^u(y, f) = W^s(y, g)$ so $\Lambda_r = \Lambda$, a contradiction.

Before proceeding to the proof, we show that the stable and unstable manifolds of Λ for g are each one dimensional. Indeed, suppose $\dim(E^s) = 2$. Then $W^u(x, g) = \{x\}$ for all $x \in \Lambda$. Then for V sufficiently small and $\tilde{\Lambda} \subset V$ a hyperbolic set with a Markov partition for g containing Λ we have a SFT (Σ, σ) and finite-to-one map π from Σ to Λ. This implies that the unstable set for any point $s \in \Sigma$ is finite. Hence, Σ consists of a finite number of points, a contradiction. Similarly we can show that $\dim(E^u) \neq 2$.

We now proceed with the proof. Suppose that for all $x \in \Lambda$ either

$$T_x W^s(x, g) = T_x W^u(x, f) \quad \text{or} \quad T_x W^u(x, g) = T_x W^u(x, f).$$

Fix $x \in \Lambda$ and suppose $T_x W^u(x, f) = T_x W^u(x, g)$. Since $W^u(x, f)$ is homeomorphic to the reals, we know that

$$T_y W^u(y, f) = T_y W^u(y, g) \quad \text{for all } y \in W^u(x, f).$$

The set $W^u(x, f)$ is dense in Λ, see [**7**, p. 424]. Hence,

$$T_y W^u(y, f) = T_y W^u(y, g) \quad \text{for all } y \in \Lambda.$$

A similar argument shows that if $T_x W^u(x, f) = T_x W^s(x, g)$, then

$$T_y W^u(y, f) = T_y W^s(y, g) \quad \text{for all } y \in \Lambda.$$

Therefore, we know that either

$$W^s(x, g) = W^u(x, f) \quad \text{for all } x \in \Lambda$$

or

$$W^u(x, g) = W^u(x, f) \quad \text{for all } x \in \Lambda.$$

If $W^u(x, g) = W^u(x, f)$ for all $x \in \Lambda$, then Λ is locally maximal and has an attracting neighborhood for g. Indeed, if we extend the adapted metric on Λ for g to a neighborhood of Λ and fix $\epsilon > 0$ sufficiently small, then the set

$$V = \bigcup_{x \in \Lambda} W^s_\epsilon(x, g)$$

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

is an open neighborhood of Λ such that $\bigcap_{n\in\mathbb{N}} g^n(V) = \Lambda$. Hence, Λ is locally maximal for g with an attracting neighborhood V.

If Λ is a locally maximal hyperbolic set for g and contains a point $x \in \Lambda$ such that $W^u(x) \subset \Lambda$, it follows from Propositions 5 and 6 of [**2**] that Λ contains a mixing hyperbolic attractor Λ_a for some power n of g. Let $x \in \Lambda_a$. The set $W^u(x, g^n)$ trivially contains $W^u(x, g)$. Since

$$\Lambda \supset \Lambda_a \supset \overline{W^u(x, g^n)} \supset \overline{W^u(x, g)} = \overline{W^u(x, f)} = \Lambda,$$

we know that Λ is a non-trivial mixing hyperbolic attractor for g^n. The set Λ is transitive for g since Λ is mixing for g^n. Hence, Λ is a non-trivial hyperbolic attractor for g. We need to show that Λ is mixing for g.

The set Λ is locally maximal and transitive for g, and hence $\overline{\mathrm{Per}(g|_\Lambda)} = \Lambda$. Fix open sets U and V in Λ and $p \in U \cap \mathrm{Per}(g|_\Lambda)$. Since $\overline{W^u(p, g)} = \Lambda$, we know that there exist a $y \in V \cap W^u(p, g)$ and $N \in \mathbb{N}$ such that $g^{-n}(y) \in U$ for all $n \geq N$. Hence, $g^n(U) \cap V \neq \emptyset$ for all $n \geq N$ and Λ is mixing for g. If $W^s(x, g) = W^u(x, f)$, a similar argument shows that Λ is a non-trivial mixing hyperbolic repeller for g.

Next, suppose for some point $x \in \Lambda$ that

$$T_x W^s(x, g) \neq T_x W^u(x, f) \quad \text{and} \quad T_x W^u(x, g) \neq T_x W^u(x, f).$$

It then follows that there exists a neighborhood U of x in Λ such that for all $y \in U$ we have

$$T_y W^s(y, g) \neq T_y W^u(y, f) \quad \text{and} \quad T_y W^u(y, g) \neq T_y W^u(y, f).$$

Then there exists a constant $\delta_0 > 0$ such that $W^u_{\delta_0}(x, f) \subset U$.

Let V be a neighborhood of Λ and $\tilde{\Lambda} \supset \Lambda$ a hyperbolic set with a Markov partition for g such that $\tilde{\Lambda} \subset V$. Since the rectangles of the Markov partition are proper and there are only a finite number of rectangles we know there exist a point $y \in W^u_{\delta_0}(x, f)$ and $\delta_1 > 0$ such that $W^u_{\delta_1}(y, f) \subset W^u_{\delta_0}(x, f) \cap R_i$ for some rectangle R_i in the Markov partition.

CLAIM 3.2. *The interior of $\tilde{\Lambda}$ is non-empty.*

Proof. Fix $\epsilon \geq \alpha > 0$ such that $\mathrm{diam}(R_i) < \alpha$ for any R_i in the Markov partition and $p, q \in R_i$ implies $W^s_\epsilon(p) \cap W^u_\epsilon(q)$ consists of one point in R_i. Furthermore,

$$f^{-1}(W^s_\epsilon(f(x'))) \cap f(W^u_\epsilon(f^{-1}(x''))) = W^s_\epsilon(x') \cap W^u_\epsilon(x'')$$

is a single point, possibly not contained in $\tilde{\Lambda}$, whenever $x', x'' \in \tilde{\Lambda}$ satisfy $d(x', x'') \leq \alpha$. By the product structure in R_i, we know that for each $y' \in W^u_{\delta_1}(y, f)$ we have the points

$$W^s_\epsilon(y', g) \cap W^u_\epsilon(y, g) \quad \text{and} \quad W^s_\epsilon(y', g) \cap W^u_\epsilon(y, g)$$

that are contained in R_i. Since $W^u_\epsilon(y, f)$ is diffeomorphic to an open interval, this implies that there exist constants $\delta_u, \delta_s > 0$ such that

$$W^u_{\delta_u}(y, g) \subset R_i \quad \text{and} \quad W^s_{\delta_s}(y, g) \subset R_i.$$

Again using the product structure of R_i we have that y is contained in the interior of $\tilde{\Lambda}$. □

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

CLAIM 3.3. *The set $\tilde{\Lambda}$ contains a non-trivial hyperbolic attractor Λ_a and a non-trivial hyperbolic repeller Λ_r.*

Proof. Let (Σ, σ) be the SFT associated with the Markov partition. Then there exist a point $z \in \text{int}(\tilde{\Lambda})$ and $s \in \Sigma$ such that $\pi(s) = z$ and under forward iterates s approaches a sink A_i and under backward iterates s approaches a source A_j. From Lemma 4 in [**2**] it follows that for each $x \in \pi(A_i)$ we have $W^u(x, g) \subset \pi(A_i)$ and for each $x \in \pi(A_j)$ we have $W^s(x, g) \subset \pi(A_j)$. The set $\Lambda_a = \pi(A_i)$ is a transitive set. For ϵ and α as above, we know that if $x, y \in \Lambda_a$, then $W^s_\epsilon(x, g) \cap W^u_\epsilon(y, g)$ consists of one point. Since $W^u(y, g) \subset \Lambda_a \subset \tilde{\Lambda}$ it follows that $W^s_\epsilon(x, g) \cap W^u_\epsilon(y, g) \in \Lambda_a$. Hence, Λ_a has a local product structure and Λ_a is locally maximal. Let V be an isolating neighborhood for Λ_a. Then there exists an $\epsilon'' > 0$ such that

$$V' = \bigcup_{x \in \Lambda_a} W^s_{\epsilon''}(x, g) \subset V$$

is an isolating neighborhood of Λ_a. Fix $y \in \bigcap_{n \in \mathbb{N}} g^n(V')$. If $m \in \mathbb{N}$, then $g^{-m}(y) \in \bigcap_{n \in \mathbb{N}} g^n(V')$ since

$$\bigcap_{n \in \mathbb{N}} g^{m+n}(V') \supset \bigcap_{n \in \mathbb{N}} g^n(V').$$

Hence, $y \in \bigcap_{n \in \mathbb{Z}} g^n(V') = \Lambda_a$ and V' is an attracting set for Λ_a. Therefore, Λ_a is a non-trivial hyperbolic attractor for g. A similar argument shows that $\Lambda_r = \pi(A_j)$ is a non-trivial hyperbolic repeller for g. □

LEMMA 3.4. *If*

$$T_x W^s(x, g) \neq T_x W^u(x, f) \quad \textit{and} \quad T_x W^u(x, g) \neq T_x W^u(x, f)$$

for some $x \in \Lambda$, then the set Λ contains a non-trivial hyperbolic attractor and non-trivial hyperbolic repeller for g.

Proof. From Claim 3.3 we know that arbitrarily close to Λ there exist a non-trivial hyperbolic attractor and a non-trivial hyperbolic repeller.

Suppose Λ does not contain a hyperbolic attractor for g. Then there exists a point $p \in \Lambda$ such that p is a limit point of a sequence x_n where each x_n is contained in a hyperbolic attractor not contained in Λ. For $\epsilon > 0$ sufficiently small the set

$$\bigcup_{x \in W^u_\epsilon(p, f)} W^s_\epsilon(x, g)$$

is a neighborhood of p. Then there exist $x \in W^u_\epsilon(p, f)$ and some $n \in \mathbb{N}$ such that $x_n \in W^s_\epsilon(x, g)$. Hence, x is contained in the basin of attraction for a non-trivial hyperbolic attractor Λ_a for all $n \in \mathbb{N}$. This implies that $g^n(x)$ is contained in the basin of attraction for Λ_a. Let x' be a limit point for a subsequence of $g^n(x)$; then $x' \in \Lambda_a$ and $x' \in \Lambda$ since Λ is closed.

CLAIM 3.5. *If $x \in \Lambda$, then*

$$g^{-1}(W^u_\epsilon(g^{-n}(x), f)) \subset W^u(g^{-n-1}(x), f).$$

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

Proof. Note that for all $n \in \mathbb{N}$ we know that

$$g^{-1}(W^u_\epsilon(g^{-n}(x), f)) \subset \Lambda.$$

Since Λ locally is a Cantor set cross an interval, this implies that

$$g^{-1}(W^u_\epsilon(g^{-n}(x), f)) \subset W^u(g^{-n-1}(x), f). \qquad \square$$

We now return to the proof of the lemma. Suppose that

$$T_{x'}W^u(x', f) = T_{x'}W^u(x', g).$$

Then for n sufficiently large and $\epsilon' > 0$ sufficiently small we know that $W^u_{\epsilon'}(g^n(x), f)$ is transverse to $W^s_{\epsilon'}(x', g)$. It now follows from the continuity of the unstable distribution and Claim 3.5 that $W^u(x', f) = W^u(x', g)$. Hence, $\Lambda_a \subset \Lambda$ since $W^u(\mathcal{O}_g(x'), g)$ is dense in Λ_a, a contradiction.

Suppose that

$$T_{x'}W^u(x', f) \neq T_{x'}W^u(x', g).$$

For any periodic point $q_0 \in \Lambda_a$ we know that $W^u(\mathcal{O}_g(q_0), g)$ is dense in Λ_a. Since the intersection of $W^u(x', f)$ and $W^u(x', g)$ is transverse, this implies there exist a point $q \in \mathrm{Per}(\Lambda_a)$ and $x'' \in W^u(x', f)$ such that $x'' \in W^u(q, g)$. Then $q \in \Lambda$ since $g^{-n}(x'') \in \Lambda$ for all $n \in \mathbb{N}$.

It now follows from the Lambda Lemma and Claim 3.5 that $W^u(q, f) = W^s(q, g)$. Since $W^u(q, f)$ is dense in Λ it now follows that

$$T_yW^u(y, f) = T_yW^s(y, g) \quad \text{for all } y \in \Lambda,$$

a contradiction. Hence, Λ contains a non-trivial hyperbolic attractor. A similar argument shows that Λ contains a non-trivial hyperbolic repeller. $\square$

We now return to the proof of the theorem. Let $\Lambda_a \subset \Lambda$ be a non-trivial hyperbolic attractor for g and $\Lambda_r \subset \Lambda$ be a non-trivial hyperbolic repeller for g. From the spectral decomposition theorem for locally maximal hyperbolic sets [**4**, p. 574] there exist sets $\Lambda_1, \ldots, \Lambda_{m_1}, \Lambda'_1, \ldots, \Lambda'_{m_2}$, and a constant $n \in \mathbb{N}$ satisfying the following:

- $\Lambda_a = \bigcup_{i=1}^{m_1} \Lambda_i$;
- $\Lambda_i \cap \Lambda_j = \emptyset$ if $i \neq j$;
- $g(\Lambda_i) = \Lambda_{\sigma(i)}$ for some permutation σ;
- $g^n(\Lambda_i) = \Lambda_i$;
- Λ_i is mixing for g^n;
- $\Lambda_r = \bigcup_{i=1}^{m_2} \Lambda'_i$;
- $\Lambda'_i \cap \Lambda'_j = \emptyset$ if $i \neq j$;
- $g(\Lambda'_i) = \Lambda'_{\sigma'(i)}$ for some permutation σ';
- $g^n(\Lambda'_i) = \Lambda'_i$; and
- Λ'_j is mixing for g^n.

For each Λ_i there exists a neighborhood U_i of Λ_i such that $U_i \cap \Lambda_j = \emptyset$ if $i \neq j$. Fix V_a as an attracting neighborhood of Λ_a for g. Then $U_i \cap V_a$ is an attracting set for Λ_i under the action of g^n. Hence, each Λ_i is a non-trivial mixing hyperbolic attractor for g^n.

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

Similarly, we can show that each Λ'_j is a non-trivial mixing hyperbolic repeller for g^n. Let $x \in \Lambda_a$, and $y \in \Lambda_r$. Then $W^u(x, g^n) \subset \Lambda_i \subset \Lambda$ for some $1 \leq i \leq m_1$. This implies that $W^u(x, g^n) = W^u(x, f)$ and

$$\Lambda = \overline{W^u(x, f)} = \overline{W^u(x, g^n)} = \Lambda_i.$$

Similarly, $W^s(y, g^n) \subset \Lambda'_j \subset \Lambda$ for some $1 \leq j \leq m_2$, which implies that $W^s(y, g^n) = W^u(y, f)$ and

$$\Lambda_i = \overline{W^u(x, g^n)} = \Lambda = \overline{W^u(y, f)} = \overline{W^s(y, g^n)} = \Lambda'_j,$$

a contradiction. Hence, $W^u(x, f) = W^u(x, g)$ or $W^u(x, f) = W^s(x, g)$ for all $x \in \Lambda$, and Λ is a non-trivial mixing hyperbolic attractor or repeller for g. □

Proof of Theorem 1.2. Let f_0 be a diffeomorphism of the open two-dimensional disk D^2 such that f_0 is the identity on a neighborhood of the boundary and f_0 contains a hyperbolic attractor Λ. In the appendix of [**3**], such a diffeomorphism is constructed. Let $n \geq 3$ and f_1 be a diffeomorphism of the open $(n-2)$-dimensional disk D^{n-2} such that the origin is an attracting fixed point and f_1 is the identity on a neighborhood of the boundary. Let $f = (f_0, f_1)$ and $g = (f_0, f_1^{-1})$.

The set $\Lambda \times \{0\}$ is a hyperbolic attractor for f and hyperbolic for g. However, $\Lambda \times \{0\}$ is neither a hyperbolic attractor nor a hyperbolic repeller on g.

Let M be a compact manifold with $\dim(M) \geq 3$. Then there exist a sufficiently small open set U in M, a local coordinate system $\varphi : U \to D^2 \times D^{n-2}$, and diffeomorphisms F and G of M such that $\varphi^{-1} F \varphi = f$, $\varphi^{-1} G \varphi = g$, and the set $\varphi^{-1}(\Lambda \times \{0\})$ is a hyperbolic for F and G, a hyperbolic attractor for F, and neither a hyperbolic attractor nor a hyperbolic repeller for G. □

Proof of Theorem 1.3. Let Λ be a hyperbolic attractor for f. Then from the spectral decomposition theorem we know that $\Lambda = \Lambda_1, \ldots, \Lambda_N$, where each Λ_i is topologically mixing and the Λ_i are compact disjoint topological sets. Furthermore, there is a power $m \in \mathbb{N}$ of f such that under f^m each of the Λ_i are invariant. Thus each Λ_i is a mixing hyperbolic attractor under f^m.

If there is a power n of g such that under g^n each of the Λ_i are hyperbolic sets, then it follows from Theorem 1.1 that each of the Λ_i are mixing hyperbolic attractors or repellers for g^n.

To complete the proof of Theorem 1.3 it is sufficient to show the following claim.

CLAIM 3.6. *Given* $1 \leq i \leq N$, *the set* $g(\Lambda_i) = \Lambda_j$ *for some* $1 \leq j \leq N$.

Proof. Let $x \in \Lambda_i$. If $g(x) \in \Lambda_j$, then there exists a $\delta > 0$ such that $g(W^u_\delta(x, f)) \subset \Lambda_j$. This follows from the continuity of g since each Λ_i is closed and disjoint from any other set Λ_k where $k \neq i$ and $1 \leq k \leq N$. Suppose $g(W^u(x, f))$ is not a subset of Λ_i. The set $W^u(x, f)$ is a smoothly immersed copy of $\mathbb{R}$, so identifying $W^u(x, f)$ with the reals we can find a first point $y \in W^u(x, f)$ such that $g(y) \notin \Lambda_j$. Let $g(y) \in \Lambda_k$ for $k \neq j$. Then there is a $\delta' > 0$ such that $g(W^u_{\delta'}(y, f)) \subset \Lambda_k$, a contradiction. Hence, $g(W^u(x, f)) \subset \Lambda_j$. Then $g(\Lambda_i) \subset \Lambda_j$ since $W^u(x, f)$ is dense in Λ_i.

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press

We now show $g(\Lambda_i) \supset \Lambda_j$. There exists a sufficiently small neighborhood U of Λ_i such that $U \cap \Lambda_k = \emptyset$ for $k \neq i$. Then $g(U) \cap \Lambda_j = g(\Lambda_i) \cap \Lambda_j$ since Λ is invariant for g. Suppose $x \in g(\Lambda_i) \cap \Lambda_j$. Then there exists an $\epsilon > 0$ such that $W^u_\epsilon(x, f) \subset g(\Lambda_i) \cap \Lambda_j$. As above, suppose there is a first point $y \in W^u(x, f)$ such that y is not in $g(\Lambda_i)$. Then there exists a $\delta' > 0$ such that $g(W^u_{\delta'}(y, f))$ is not contained in Λ_j, contradicting the choice of y. Hence, $W^u(x, f) \subset g(\Lambda_i)$ and $g(\Lambda_i) \supset \Lambda_j$ since $W^u(x, f)$ is dense in Λ_j. Therefore, $g(\Lambda_i) = \Lambda_j$. □

Theorem 1.3 then follows. □

References

[1] C. Bonatti and R. Langevin. Difféomorphismes de Smales des surfaces. *Asterique* **250** (1998).

[2] T. Fisher. Hyperbolic sets with nonempty interior. *Discrete Contin. Dyn. Syst.* **15**(2) (2006), 433–446.

[3] T. Fisher. Hyperbolic sets that are not locally maximal. *Ergod. Th. & Dynam. Sys.* **26** (2006), 1491–1509.

[4] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[5] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

[6] R. V. Plykin. Sources and sinks for a-diffeomorphisms of surfaces. *Math. USSR – Sb* **23** (1974), 233–253.

[7] C. Robinson. *Dynamical Systems Stability, Symbolic Dynamics, and Chaos*. CRC Press, 1999.

[8] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[9] R. Williams. One-dimensional nonwandering sets. *Topology* **6** (1967), 473–487.

[10] A. Yu. Zhirov. Hyperbolic attractors of diffeomorphisms of oriented surfaces. *Russian Acad. Sci. Sb. Math.* **82**(1) (1995), 135–174.

[11] A. Yu. Zhirov. Hyperbolic attractors of diffeomorphisms of oriented surfaces. ii. enumeration and application to pseudo-Anosov diffeomorphisms. *Russian Acad. Sci. Sb. Math.* **83**(1) (1995), 23–65.

[12] A. Yu. Zhirov. Hyperbolic attractors of diffeomorphisms of oriented surfaces. iii. a classification algorithm. *Sb. Math.* **2** (1995), 221–244.

https://doi.org/10.1017/S0143385706000423 Published online by Cambridge University Press