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A QUANTITATIVE DYNAMICAL BOREL–CANTELLI LEMMA FOR TWISTED RECURRENCE ON $[0,1]^d$

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Abstract

Let $T : [0,1]^d \to [0,1]^d$ be a piecewise expanding linear map such that each domain of linearity is a coordinate-parallel hyperrectangle that T sends to $[0,1]^d$. Let $\{\mathbf{r}_n\}$ be a sequence of vectors with $\mathbf{r}_n = (r_{n,1}, \dots, r_{n,d}) \in (\mathbb{R}_{\geq 0})^d$. Let m_d denote d-dimensional Lebesgue measure. Let $\{f_n\}$ be a sequence of piecewise Lipschitz coordinate-wise vector functions on $[0,1]^d$ with a uniform Lipschitz constant. For any given $\epsilon > 0$, we obtain a unified counting function for the shrinking target and recurrence problems:

$$\sum_{n=1}^{N} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) = \Psi(N) + O(\Psi^{1/2}(N)(\log \Psi(N))^{3/2+\varepsilon})$$

for m_d-almost every $\mathbf{x} \in [0,1]^d$ and for all $N \in \mathbb{N}$, where $\Psi(N) := 2^d \sum_{n=1}^{N} r_{n,1} \cdots r_{n,d}$ and $R(f_n(\mathbf{x}), \mathbf{r}_n)$ is a coordinate-parallel hyperrectangle centred at $f_n(\mathbf{x})$. Our result gives a unified statement to show that the quantitative dynamical Borel–Cantelli lemma can govern both the shrinking target and recurrence problems, thereby eliminating the need for separate treatments and different proofs. This result can also apply to the situation where both problems are present simultaneously along different directions in $[0,1]^d$.

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1. Introduction

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with a compatible metric ρ, that is, (X, ρ) is a metric space, $\mathcal{B}$ is a Borel σ-algebra of X and μ is a T-invariant probability measure. Suppose that (X, ρ) is complete and separable. Poincaré's recurrence theorem, one of the most fundamental results in dynamical systems and ergodic theory (see [3, Theorem 2.11]), states that μ-almost every point returns close to itself under a measure-preserving map T, that is,

$$\liminf_{n\to\infty} \rho(T^n x, x) = 0.$$

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Boshernitzan proved the first general quantitative recurrence result in [2, Theorem 1.2]. Specifically, assume that the α-dimensional Hausdorff measure of X is σ-finite for some $\alpha > 0$. Then,

$$\liminf_{n\to\infty} n^{1/\alpha}\rho(T^n x, x) < \infty \quad \text{for } \mu\text{-a.e. } x \in X.$$

Here, μ-a.e. denotes 'almost every' with respect to μ. Put differently, for μ-a.e. $x \in X$, there is a constant $c(x) > 0$ such that $\rho(T^n x, x) < c(x)n^{-1/\alpha}$ for infinitely many $n \in \mathbb{N}$. This result motivates the study of the size of the set

$$\mathcal{R}(\psi) := \{x \in X : \rho(T^n x, x) < \psi(n) \text{ for i.m. } n \in \mathbb{N}\}.$$

Here, $\psi : \mathbb{N} \to \mathbb{R}^+$ is a positive function and i.m. denotes 'infinitely many'. The zero-full law for the μ-measure of the recurrence set $\mathcal{R}(\psi)$, often referred to as the quantitative recurrence property or dynamical Borel–Cantelli lemma for recurrence theory, has been extensively studied in various dynamical systems (see, for example, [4–6]). In other words, Boshernitzan's result states $\sum_{n=1}^{\infty} \mathbb{1}_{B(x,c(x)n^{-1/\alpha})}(T^n x) = \infty$ for μ-a.e. $x \in X$, where $\mathbb{1}_{B(x,c(x)n^{-1/\alpha})}$ denotes the characteristic function of the open ball $B(x, c(x)n^{-1/\alpha})$ with centre $x \in X$ and radius $c(x)n^{-1/\alpha}$. This result motivates us to study a significantly stronger quantitative statement for the counting function

$$\sum_{n=1}^{N} \mathbb{1}_{B(x,\psi(n))}(T^n x) := \#\{1 \le n \le N : T^n x \in B(x, \psi(n))\},$$

where $N \in \mathbb{N}$, often referred to as the quantitative dynamical Borel–Cantelli lemma (for related results, see [4, 8, 9]). Of particular relevance to our work is the recent study by Levesley *et al.* [8], which gives the recurrence counting function for piecewise expanding linear maps on $[0, 1]^d$, and clarifies the distinct proofs for the recurrence counting function and the shrinking target problem introduced as follows.

The shrinking target problem involves determining the rate at which the orbit of a μ-typical point accumulates near a given point $y \in X$. More specifically, for $\psi : \mathbb{N} \to \mathbb{R}^+$, define

$$\mathcal{R}^y(\psi) := \{x \in X : \rho(T^n x, y) < \psi(n) \text{ for i.m. } n \in \mathbb{N}\}.$$

In [9], Li *et al.* prove that the results for the counting function hold for a large class of measure-preserving dynamical systems and the 'target' sets can be any measurable subset centred at a given point of X.

These two problems are among the most commonly studied, but they are typically treated separately using different methods, as in [4, 8]. A more general definition, known as the twisted recurrence or modified shrinking target problem, unifies the two problems. Specifically, for $\psi : \mathbb{N} \to \mathbb{R}^+$ and a sequence $\{f_n\}$ of Borel measurable functions on X, define

$$\mathcal{R}^f(\psi) := \{x \in X : \rho(T^n x, f_n(x)) < \psi(n) \text{ for i.m. } n \in \mathbb{N}\}.$$

It follows that if $f_1 = f_2 = \cdots = f_n = \cdots = \mathrm{Id}_X$, then $\mathcal{R}^f(\psi) = \mathcal{R}(\psi)$. Moreover, if $f_1 = f_2 = \cdots = f_n = \cdots \equiv y$, then $\mathcal{R}^f(\psi) = \mathcal{R}^y(\psi)$. The zero-full law for the μ-measure

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of the twisted recurrence set has been the subject of significant study in various dynamical systems (see [7, 10, 11, 14]). For Hausdorff dimension results, see [11–16]. However, the quantitative dynamical Borel–Cantelli lemma has not yet been established for twisted recurrences.

Following the same setup and similar methods as in [8], our goal in this paper is to establish, via a unified proof, the counting function for both the shrinking target and recurrence problems. Throughout the paper, we take X to be the unit hypercube $[0,1]^d$ endowed with a metric ρ. Let m_d denote the d-dimensional Lebesgue measure. Let $\mathcal{P}_m$ denote the collection of cylinder sets J_m of order m that partition $[0,1]^d$. Let $D_x T$ denote the derivative of T at $x \in X$.

The following definition introduces the class of measure-preserving systems $([0,1]^d, T, m_d)$, as defined in [8].

DEFINITION 1.1 (Piecewise expanding linear map [8]). We say that $T : [0,1]^d \to [0,1]^d$ is a piecewise expanding linear map if each domain of linearity J_1 is a coordinate-parallel hyperrectangle that T sends to $[0,1]^d$ such that $D_{\mathbf{x}} T|_{J_1}$ is a diagonal matrix and the following conditions hold.

(i) For general $m \in \mathbb{N}$, $J_m = \prod_{i=1}^d J_m^{(i)} \in \mathcal{P}_m$ is the coordinate-parallel hyperrectangle of linearity for T^m such that $D_{\mathbf{x}} T$ is the same for any $\mathbf{x} \in J_m$, so we simply write $T'|_{J_m}$. For $1 \le i \le d$, let $K_{J_m}^{(i)}$ be the (i,i)th component of the diagonal matrix $(T^m)'|_{J_m}$ and $K_{J_m} := \prod_{i=1}^d K_{J_m}^{(i)}$ so that $T^m|_{J_m}$ is a similarity with the dilatation factor $|K_{J_m}|$.

(ii) T is expanding in all directions, that is, there exists a constant $\lambda > 1$ such that for $1 \le i \le d$, $m \in \mathbb{N}$ and $J_m \in \mathcal{P}_m$,

$$|K_{J_m}^{(i)}| \ge \lambda^m. \tag{1.1}$$

(iii) T is a measure-preserving transformation with respect to m_d, that is, m_d is a T-invariant probability measure. For any measurable set $F \subseteq J_m$, due to linearity,

$$m_d(T^m F) = |K_{J_m}| m_d(F).$$

(iv) [8, Proposition 4.1]. T is hyperrectangular exponentially mixing with respect to m_d, that is, there is a constant γ with $0 < \gamma < 1$ such that for any $n \in \mathbb{N}$, any coordinate-parallel hyperrectangle $R \subset [0,1]^d$ and any measurable set $F \subset [0,1]^d$,

$$m_d(R \cap T^{-n} F) = m_d(R) m_d(F) + O(\gamma^n) m_d(F).$$

REMARK 1.2. Given the set up as above, for $1 \le i \le d$ and any $\mathbf{x}, \mathbf{y} \in J_m$ with $x_i \ne y_i$,

$$|K_{J_m}^{(i)}| = \frac{1}{m_1(J_m^{(i)})} = \frac{\rho((T^m \mathbf{x})_i, (T^m \mathbf{y})_i)}{\rho(x_i, y_i)}$$

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so that $|K_{J_m}| = 1/m_d(J_m)$. As a result, $\sum_{J_m \in \mathcal{P}_m} |K_{J_m}|^{-1} = 1$. From [8, Proposition 4.1], we can take $\gamma := 1/\lambda$ and rewrite (1.1) as

$$|K^{(i)}_{J_m}| \geq \gamma^{-m} \quad (1 \leq i \leq d).$$

Well-known dynamical systems that satisfy the conditions of Definition 1.1 include the tent map, the Lüroth map and expanding, diagonal integer matrix transformations of the d-dimensional torus $\mathbb{T}^d$.

Next, we specify the class of functions f that can be treated by our technique.

DEFINITION 1.3 (Piecewise Lipschitz coordinate-wise vector function). Let $p > 0$. We say that $f : [0,1]^d \to [0,1]^d$ is a Lipschitz coordinate-wise vector function, if for $1 \leq i \leq d$ and any $\mathbf{x}, \mathbf{y} \in [0,1]^d$,

$$f(\mathbf{x}) = (f_1(x_1), f_2(x_2), \ldots, f_d(x_d)) \quad \text{and} \quad \rho(f_i(x_i), f_i(y_i)) \leq p\rho(x_i, y_i).$$

Furthermore, f is piecewise Lipschitz if there exist finitely many measurable subsets U_j in $[0,1]^d$ such that $m_d([0,1]^d \setminus \bigcup_j U_j) = 0$ and $f|_{U_j} : U_j \to [0,1]^d$ is a Lipschitz function for each j.

For example, when $d = 1$, the function $f(x) = 2x \bmod 1$ is piecewise Lipschitz but not Lipschitz on $[0,1)$.

Motivated by the weighted theory of Diophantine approximation, for a sequence of piecewise Lipschitz coordinate-wise vector functions $\{f_n : [0,1]^d \to [0,1]^d\}$ with a uniform Lipschitz constant p and a sequence $\{\mathbf{r}_n\}$ of vectors with $\mathbf{r}_n = (r_{n,1}, \ldots, r_{n,d}) \in (\mathbb{R}_{\geq 0})^d$, define

$$\begin{aligned}\mathcal{R}^f(\{\mathbf{r}_n\}) &:= \{\mathbf{x} \in [0,1]^d : T^n\mathbf{x} \in R(f_n(\mathbf{x}), \mathbf{r}_n) \text{ for i.m. } n \in \mathbb{N}\} \\ &= \left\{\mathbf{x} \in [0,1]^d : T^n\mathbf{x} \in \prod_{i=1}^{d} B(f_n^{(i)}(x_i), r_{n,i}) \text{ for i.m. } n \in \mathbb{N}\right\} \\ &= \{\mathbf{x} \in [0,1]^d : \rho((T^n\mathbf{x})_i, f_n^{(i)}(x_i)) \leq r_{n,i} \text{ for } 1 \leq i \leq d \text{ and i.m. } n \in \mathbb{N}\},\end{aligned}$$

and similarly for $N \in \mathbb{N}$, define

$$\sum_{n=1}^{N} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) := \#\{1 \leq n \leq N : T^n\mathbf{x} \in R(f_n(\mathbf{x}), \mathbf{r}_n)\}.$$

We now proceed to our main theorem.

THEOREM 1.4. *Let $T : [0,1]^d \to [0,1]^d$ be a piecewise expanding linear map. Let $\{f_n\}$ be a sequence of piecewise Lipschitz coordinate-wise vector functions on $[0,1]^d$ with a uniform Lipschitz constant p. For any sequence of vectors $\{\mathbf{r}_n\}$ in $(\mathbb{R}_{\geq 0})^d$ and any given $\varepsilon > 0$,*

$$\sum_{n=1}^{N} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) = \Psi(N) + O(\Psi^{1/2}(N)(\log \Psi(N))^{3/2+\varepsilon})$$

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for m_d-a.e. $\mathbf{x} \in [0,1]^d$, where $\Psi(N) := 2^d \sum_{n=1}^{N} r_{n,1} \cdots r_{n,d}$. In particular,

$$m_d(\mathcal{R}^f(\{\mathbf{r}_n\})) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} = \infty. \end{cases}$$

2. Proof of Theorem 1.4

For a sequence of piecewise Lipschitz coordinate-wise vector functions $\{f_n\}$ on $[0,1]^d$ with a uniform Lipschitz constant p and a sequence $\{\mathbf{r}_n\}$ of vectors with $\mathbf{r}_n = (r_{n,1}, \ldots, r_{n,d}) \in (\mathbb{R}_{\geq 0})^d$, define

$$A_n := \{\mathbf{x} \in [0,1]^d : T^n\mathbf{x} \in R(f_n(\mathbf{x}), \mathbf{r}_n)\}.$$

Unlike the shrinking target case, the sets A_n cannot be expressed in the form $T^{-n}R_n$ for some hyperrectangles R_n. To address this, we follow the approach from [10, Lemma 3.2], which considers A_n locally by studying its intersection with

$$A_n \cap f_n^{-1}R(f_n(\mathbf{x}), p\mathbf{r}),$$

where $\mathbf{r} = (r_1, \ldots, r_d) \in (\mathbb{R}_{\geq 0})^d$. This intersection can be approximated by the preimages of hyperrectangles under T.

LEMMA 2.1 [10, Lemma 3.2]. *Let $R(f_n(\mathbf{x}), p\mathbf{r})$ be a coordinate-parallel hyperrectangle centred at $f_n(\mathbf{x}) \in [0,1]^d$, where $p\mathbf{r} = (pr_1, \ldots, pr_d) \in (\mathbb{R}_{\geq 0})^d$. For any $n \in \mathbb{N}$ and any subset S of $f_n^{-1}R(f_n(\mathbf{x}), p\mathbf{r})$, we have*

$$S \cap A_n \subset S \cap T^{-n}R(f_n(\mathbf{x}), \mathbf{r}_n + p\mathbf{r}).$$

Furthermore, if $pr_i \leq r_{n,i}$ for $1 \leq i \leq d$, then

$$S \cap T^{-n}R(f_n(\mathbf{x}), \mathbf{r}_n - p\mathbf{r}) \subset S \cap A_n.$$

The following result [1, Lemma 1.5] serves as a key tool in metric Diophantine approximation for proving counting statements.

LEMMA 2.2 [1, Lemma 1.5]. *Let $(X, \mathcal{B}, \mu)$ be a probability space, let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of nonnegative μ-measurable functions defined on X and let $(c_n)_{n\in\mathbb{N}}, (\phi_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that $0 \leq c_n \leq \phi_n$ $(n = 1, 2, \ldots)$. Suppose that for arbitrary $a, b \in \mathbb{N}$ with $a < b$,*

$$\int_X \Big(\sum_{n=a}^{b} (f_n(x) - c_n) \Big)^2 d\mu(x) \leq C \sum_{n=a}^{b} \phi_n \tag{2.1}$$

for an absolute constant $C > 0$. Then, for any given $\varepsilon > 0$,

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} c_n + O\Big(\Phi(N)^{1/2} \log^{3/2+\varepsilon} \Phi(N) + \max_{1\leq k\leq N} c_k\Big)$$

for μ-a.e. $x \in X$, where $\Phi(N) := \sum_{n=1}^{N} \phi_n$.

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To establish our result, we apply Lemma 2.2 with

$$X := [0,1]^d, \quad f_n(x) := \mathbb{1}_{A_n}(\mathbf{x}) \quad \text{and} \quad c_n := \phi_n := m_d(A_n),$$

so that $\Phi(N) := \sum_{n=1}^{N} m_d(A_n)$. For any $a, b \in \mathbb{N}$ with $a < b$,

$$\begin{aligned}\Big(\sum_{n=a}^{b}(f_n(x)-c_n)\Big)^2 &= \Big(\sum_{n=a}^{b}(\mathbb{1}_{A_n}(\mathbf{x})-m_d(A_n))\Big)^2\\ &= \Big(\sum_{n=a}^{b}\mathbb{1}_{A_n}(\mathbf{x})\Big)^2 + \Big(\sum_{n=a}^{b}m_d(A_n)\Big)^2 - 2\sum_{n=a}^{b}\mathbb{1}_{A_n}(\mathbf{x})\cdot\sum_{n=a}^{b}m_d(A_n)\\ &= \sum_{n=a}^{b}\mathbb{1}_{A_n}(\mathbf{x}) + 2\sum_{a\le m<n\le b}\sum\mathbb{1}_{A_m}(\mathbf{x})\mathbb{1}_{A_n}(\mathbf{x}) + \Big(\sum_{n=a}^{b}m_d(A_n)\Big)^2 - 2\sum_{n=a}^{b}m_d(A_n)\cdot\sum_{n=a}^{b}\mathbb{1}_{A_n}(\mathbf{x}),\end{aligned}$$

so the left-hand side of (2.1) is equal to

$$\sum_{n=a}^{b}m_d(A_n) + 2\sum_{a\le m<n\le b}\sum m_d(A_m\cap A_n) - \Big(\sum_{n=a}^{b}m_d(A_n)\Big)^2. \tag{2.2}$$

2.1. Calculating the measure of A_n. In this section, we aim to obtain the relationship between $\Phi(N)$ and $\Psi(N)$, thereby connecting the above lemma with our result.

PROPOSITION 2.3. *For arbitrary $N \in \mathbb{N}$,*

$$\Phi(N) = \sum_{n=1}^{N} m_d(A_n) = 2^d\sum_{n=1}^{N} r_{n,1}\cdots r_{n,d} + O(1) = \Psi(N) + O(1).$$

PROOF. Consider a sequence of real numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \in (0, 1/p)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\varepsilon_n = 0$, where p is the Lipschitz constant in Definition 1.3. For sufficiently large $n \in \mathbb{N}$ and $1 \le i \le d$, we have $2\varepsilon_n r_{n,i} < r_{n,i} < 1$. Accordingly, for $1 \le i \le d$, we partition the unit interval $[0,1]$ into $\lceil 1/(2\varepsilon_n r_{n,i})\rceil$ pairwise disjoint subintervals of equal length

$$l_{n,i} := \frac{1}{\lceil 1/(2\varepsilon_n r_{n,i})\rceil}. \tag{2.3}$$

Taking the product, we obtain a decomposition of the unit hypercube $[0,1]^d$ into

$$\mathcal{I} := \prod_{i=1}^{d}\left\lceil\frac{1}{2\varepsilon_n r_{n,i}}\right\rceil$$

pairwise disjoint coordinate-parallel hyperrectangles

$$\{R(\mathbf{x}_j, \mathbf{t}_n) : \mathbf{x}_j \in [0,1]^d\}_{j\in\mathcal{I}},$$

where $\mathbf{t}_n = (t_{n,1}, \ldots, t_{n,d}) := \frac{1}{2}\mathbf{l}_n = (\frac{1}{2}l_{n,1}, \ldots, \frac{1}{2}l_{n,d})$, and the sidelength $l_{n,i}$ of each coordinate-parallel hyperrectangle in the ith direction is defined in (2.3) and satisfies $l_{n,i} \le 2\varepsilon_n r_{n,i}$ for $1 \le i \le d$.

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For any $\mathbf{y} \in R(\mathbf{x}, \mathbf{t}_n)$, we have $f_n(\mathbf{y}) = (f_n^{(1)}(y_1), f_n^{(2)}(y_2), \dots, f_n^{(d)}(y_d))$ and $\rho(f_n^{(i)}(y_i), f_n^{(i)}(x_i)) \le p\rho(y_i, x_i) \le pt_{n,i}$ for $1 \le i \le d$ so that $\mathbf{y} \in f_n^{-1}R(f_n(\mathbf{x}), \mathbf{t}_n)$. Consequently, for each $j \in \mathcal{I}$, we have $R(\mathbf{x}_j, \mathbf{t}_n) \subset f_n^{-1}R(f_n(\mathbf{x}_j), p\mathbf{t}_n)$.

Since $\varepsilon_n \in (0, 1/p)$ for all $n \in \mathbb{N}$, we have $p\varepsilon_n r_{n,i} < r_{n,i}$ for $1 \le i \le d$. By Lemma 2.1 and since $t_{n,i} = \frac{1}{2}l_{n,i} \le \varepsilon_n r_{n,i}$ for $1 \le i \le d$,

$$\begin{aligned} R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), (1 - p\varepsilon_n)\mathbf{r}_n) &\subset R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), \mathbf{r}_n - p\mathbf{t}_n) \\ &\subset R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n \subset R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), \mathbf{r}_n + p\mathbf{t}_n) \\ &\subset R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), (1 + p\varepsilon_n)\mathbf{r}_n). \end{aligned}$$

By the hyperrectangular exponentially mixing property, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} m_d(R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n) &\ge m_d(R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), (1 - p\varepsilon_n)\mathbf{r}_n)) \\ &\ge (m_d(R(\mathbf{x}_j, \mathbf{t}_n)) - C_1\gamma^n)m_d(R(f_n(\mathbf{x}_j), (1 - p\varepsilon_n)\mathbf{r}_n)) \\ &= m_d(R(\mathbf{x}_j, \mathbf{t}_n))2^d(1 - p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} - C_1\gamma^n 2^d(1 - p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} \\ &\ge m_d(R(\mathbf{x}_j, \mathbf{t}_n))2^d(1 - p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} - C_1\gamma^n 2^d r_{n,1} \cdots r_{n,d}. \end{aligned}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, for sufficiently large $n \in \mathbb{N}$, we have $(1 - p\varepsilon_n)^d \ge 1 - dp\varepsilon_n$. Together with $\#\mathcal{I} \ge 1/(2^d\varepsilon_n^d r_{n,1} \cdots r_{n,d})$, we consequently obtain

$$\begin{aligned} m_d(A_n) &= \sum_{j\in\mathcal{I}} m_d(R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n) \\ &\ge 2^d(1 - p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} - \#\mathcal{I}C_1\gamma^n 2^d r_{n,1} \cdots r_{n,d} \\ &\ge 2^d r_{n,1} \cdots r_{n,d} - 2^d dp\varepsilon_n r_{n,1} \cdots r_{n,d} - C_1\gamma^n/\varepsilon_n^d. \end{aligned}$$

Similarly, for the other side,

$$\begin{aligned} m_d(R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n) &\le m_d(R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), (1 + p\varepsilon_n)\mathbf{r}_n)) \\ &\le (m_d(R(\mathbf{x}_j, \mathbf{t}_n)) + C_1\gamma^n)m_d(R(f_n(\mathbf{x}_j), (1 + p\varepsilon_n)\mathbf{r}_n)) \\ &= m_d(R(\mathbf{x}_j, \mathbf{t}_n))2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} + C_1\gamma^n 2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} \\ &\le m_d(R(\mathbf{x}_j, \mathbf{t}_n))2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} + C_1\gamma^n 2^{2d} r_{n,1} \cdots r_{n,d}. \end{aligned}$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, for sufficiently large $n \in \mathbb{N}$, we have $(1 + p\varepsilon_n)^d \le 1 + 2^d p^d \varepsilon_n$. Together with the upper bound

$$\#\mathcal{I} = \prod_{i=1}^{d} \left\lceil \frac{1}{2\varepsilon_n r_{n,i}} \right\rceil \le \prod_{i=1}^{d} \left(\frac{1}{2\varepsilon_n r_{n,i}} + 1 \right) \le \frac{1}{\varepsilon_n^d r_{n,1} \cdots r_{n,d}},$$

this yields

$$\begin{aligned} m_d(A_n) &\le 2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} + \#\mathcal{I}C_1\gamma^n 2^{2d} r_{n,1} \cdots r_{n,d} \\ &\le 2^d r_{n,1} \cdots r_{n,d} + 2^{2d} p^d \varepsilon_n r_{n,1} \cdots r_{n,d} + 2^{2d} C_1\gamma^n/\varepsilon_n^d. \end{aligned}$$

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Since $r_{n,i} < 1$ for $1 \le i \le d$,

$$|m_d(A_n) - 2^d r_{n,1} \cdots r_{n,d}| \le 2^{2d} p^d \varepsilon_n r_{n,1} \cdots r_{n,d} + 2^{2d} C_1 \gamma^n / \varepsilon_n^d \le 2^{2d} p^d \varepsilon_n + 2^{2d} C_1 \gamma^n / \varepsilon_n^d.$$

The proof is completed by setting $\varepsilon_n = 1/((p+1)n^2)$ and noting that both series $\sum_{n\in\mathbb{N}} \varepsilon_n$ and $\sum_{n\in\mathbb{N}} \gamma^n/\varepsilon_n^d$ converge. □

2.2. Estimating the measure of $A_m \cap A_n$. To see if condition (2.1) of Lemma 2.2 holds, we must estimate (2.2), that is, $2\sum\sum_{a\le m<n\le b} m_d(A_m \cap A_n)$. Accordingly, we consider a coordinate-parallel hyperrectangle of linearity J_m for a sufficiently large $m \in \mathbb{N}$.

LEMMA 2.4. *For a sufficiently large $m \in \mathbb{N}$ and each coordinate-parallel hyperrectangle of linearity J_m for T^m, the coordinate-parallel hyperrectangle $I_{J_m} := J_m \cap A_m$ has sidelengths $\ell_m = (\ell_{m,1}, \ldots, \ell_{m,d}) \in (\mathbb{R}_{\ge 0})^d$, where the i-dimensional sidelength $\ell_{m,i}$ satisfies*

$$\ell_{m,i} \le \frac{2r_{m,i}}{|K_{J_m}^{(i)}| - p} \quad \text{for } 1 \le i \le d,$$

where p is the Lipschitz constant in Definition 1.3.

PROOF. Choose any $\mathbf{x}, \mathbf{y} \in J_m \cap A_m$. From Remark 1.2,

$$\rho((T^m\mathbf{x})_i, (T^m\mathbf{y})_i) = |K_{J_m}^{(i)}|\rho(x_i, y_i).$$

However, by Definition 1.3,

$$\begin{aligned}\rho((T^m\mathbf{x})_i, (T^m\mathbf{y})_i) &\le \rho((T^m\mathbf{x})_i, f_m^{(i)}(x_i)) + \rho(f_m^{(i)}(x_i), f_m^{(i)}(y_i)) + \rho((T^m\mathbf{y})_i, f_m^{(i)}(y_i)) \\ &\le 2r_{m,i} + p\rho(x_i, y_i).\end{aligned}$$

Thus, $|K_{J_m}^{(i)}|\rho(x_i, y_i) \le 2r_{m,i} + p\rho(x_i, y_i)$. For sufficiently large $m \in \mathbb{N}$, we have $|K_{J_m}^{(i)}| \ge p$, so that $\rho(x_i, y_i) \le 2r_{m,i}/(|K_{J_m}^{(i)}| - p)$. Consequently, $I_{J_m} := J_m \cap A_m$ is a coordinate-parallel hyperrectangle and

$$\prod_{i=1}^d I_{J_m}^{(i)} := \prod_{i=1}^d J_m^{(i)} \cap B\Big(\frac{x_i + y_i}{2}, \frac{r_{m,i}}{|K_{J_m}^{(i)}| - p}\Big).$$ □

PROPOSITION 2.5. *For arbitrary $a, b \in \mathbb{N}$ with $a < b$,*

$$2\sum_{a\le m<n\le b}\sum m_d(A_m \cap A_n) \le \Big(\sum_{n=a}^b m_d(A_n)\Big)^2 + O\Big(\sum_{n=a}^b m_d(A_n)\Big) + O(1).$$

PROOF. Fix $m < n$ and a coordinate-parallel hyperrectangle of linearity $J_m = \prod_i J_m^{(i)}$ for T^m. We estimate the measure of the intersection $I_{J_m} \cap A_n := (J_m \cap A_m) \cap A_n$. Throughout the proof, $\{\varepsilon_n\}_{n\in\mathbb{N}}$ denotes a sequence with each $\varepsilon_n \in (0, 1/p)$ and $\lim_{n\to\infty} \varepsilon_n = 0$, in accordance with the construction used in the proof of

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Proposition 2.3. From Lemma 2.4, $I_{J_m} = \prod_{i=1}^{d} I_{J_m}^{(i)}$, where each $I_{J_m}^{(i)}$ is an interval of length $\ell_{m,i} \le 2r_{m,i}/(|K_{J_m}^{(i)}| - p)$. For $1 \le i \le d$, partition the interval $I_{J_m}^{(i)}$ into

$$\mathcal{N}_{J_m}^{(i)} := \left\lceil \frac{r_{m,i}}{\varepsilon_n r_{n,i}(|K_{J_m}^{(i)}| - p)} \right\rceil$$

pairwise disjoint subintervals of equal length

$$l_{n,i} := \frac{\ell_{m,i}}{\mathcal{N}_{J_m}^{(i)}} \le 2\varepsilon_n r_{n,i}.$$

Taking the product, we obtain a decomposition of the coordinate-parallel hyperrectangle I_{J_m} into

$$\mathcal{I} := \prod_{i=1}^{d} \mathcal{N}_{J_m}^{(i)} = \prod_{i=1}^{d} \left\lceil \frac{r_{m,i}}{\varepsilon_n r_{n,i}(|K_{J_m}^{(i)}| - p)} \right\rceil$$

pairwise disjoint coordinate-parallel hyperrectangles $\{R(\mathbf{x}_j, \mathbf{t}_n) : \mathbf{x}_j \in I_{J_m}\}_{j \in \mathcal{I}}$, where $\mathbf{t}_n = (t_{n,1}, \dots, t_{n,d}) = \frac{1}{2}\mathbf{l}_n = (\frac{1}{2}l_{n,1}, \dots, \frac{1}{2}l_{n,d})$ and the sidelength of each coordinate-parallel hyperrectangle in the ith direction is $l_{n,i}$, which by construction satisfies $l_{n,i} \le 2\varepsilon_n r_{n,i}$ for $1 \le i \le d$. Following the same argument as in Proposition 2.3, $R(\mathbf{x}_j, \mathbf{t}_n) \subset f_n^{-1}R(f_n(\mathbf{x}_j), p\mathbf{t}_n)$ for each $j \in \mathcal{I}$. Applying Lemma 2.1,

$$\begin{aligned} R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n &\subset R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), \mathbf{r}_n + p\mathbf{t}_n) \\ &\subset R(\mathbf{x}_j, \mathbf{t}_n) \cap T^{-n}R(f_n(\mathbf{x}_j), (1 + p\varepsilon_n)\mathbf{r}_n), \end{aligned}$$

so that

$$T^m(R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n) \subset T^m(R(\mathbf{x}_j, \mathbf{t}_n)) \cap T^{-(n-m)}R(f_n(\mathbf{x}_j), (1 + p\varepsilon_n)\mathbf{r}_n).$$

By the hyperrectangular exponentially mixing property, there exists a constant $C_1 > 0$ such that

$$m_d(T^m(R(\mathbf{x}_j, \mathbf{t}_n) \cap A_n)) \le (m_d(T^m(R(\mathbf{x}_j, \mathbf{t}_n))) + C_1\gamma^{n-m})2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d}.$$

Summing over $j \in \mathcal{I} = \prod_{i=1}^{d} \lceil r_{m,i}/(|K_{J_m}^{(i)}| - p)\varepsilon_n r_{n,i} \rceil$,

$$\begin{aligned} &m_d(T^m(I_{J_m} \cap A_n)) \\ &\le \left(m_d(T^m(I_{J_m})) + \prod_{i=1}^{d} \left\lceil \frac{r_{m,i}}{\varepsilon_n r_{n,i}(|K_{J_m}^{(i)}| - p)} \right\rceil C_1\gamma^{n-m} \right) \cdot 2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d}. \end{aligned}$$

According to Definition 1.1(iii), for any measurable set $F \subseteq J_m$, due to linearity, we have $m_d(T^m F) = |K_{J_m}| m_d(F)$, giving

$$\begin{aligned} &m_d(I_{J_m} \cap A_n) \\ &\le \left(m_d(I_{J_m}) + \prod_{i=1}^{d} \left\lceil \frac{r_{m,i}}{\varepsilon_n r_{n,i}(|K_{J_m}^{(i)}| - p)} \right\rceil |K_{J_m}|^{-1} C_1\gamma^{n-m} \right) \cdot 2^d(1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d}. \end{aligned}$$

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Next, we estimate $m_d(A_m \cap A_n)$ by summing over all cylinder sets $J_m \in \mathcal{P}_m$. Recall that $|K_{J_m}| = 1/m_d(J_m)$, $\sum_{J_m \in \mathcal{P}_m} |K_{J_m}|^{-1} = 1$ and $|K^{(i)}_{J_m}| \geq \gamma^{-m}$ for $1 \leq i \leq d$ from Remark 1.2. It follows that there exists a constant $C_2 > 1$ such that $(|K^{(i)}_{J_m}| - p)^{-1} \leq (\gamma^{-m} - p)^{-1} = \gamma^m/(1 - p\gamma^m) \leq C_2\gamma^m$ whenever $m \in \mathbb{N}$ is large enough so that $1 - p\gamma^m \geq 1/C_2$. Since $r_{n,i} < 1$ for $1 \leq i \leq d$, we obtain

$$
\begin{aligned}
m_d(A_m \cap A_n) &\leq \Big(m_d(A_m) + \prod_{i=1}^{d} \Big\lceil \frac{C_2\gamma^m r_{m,i}}{\varepsilon_n r_{n,i}} \Big\rceil C_1\gamma^{n-m}\Big) 2^d (1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} \\
&\leq \Big(m_d(A_m) + \prod_{i=1}^{d} \Big(1 + \frac{C_2\gamma^m r_{m,i}}{\varepsilon_n r_{n,i}}\Big) C_1\gamma^{n-m}\Big) 2^d (1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} \\
&\leq \Big(m_d(A_m) + \Big(1 + 2^d C_2^d \gamma^m \prod_{i=1}^{d} \frac{1}{\varepsilon_n r_{n,i}}\Big) C_1\gamma^{n-m}\Big) 2^d (1 + p\varepsilon_n)^d r_{n,1} \cdots r_{n,d} \\
&\leq m_d(A_m) 2^d (1 + 2^d p^d \varepsilon_n) r_{n,1} \cdots r_{n,d} + \Big(1 + (2C_2)^d \gamma^m \prod_{i=1}^{d} \frac{1}{\varepsilon_n r_{n,i}}\Big) C_1\gamma^{n-m} 2^{2d} r_{n,1} \cdots r_{n,d} \\
&\leq m_d(A_m) 2^d r_{n,1} \cdots r_{n,d} + C_3\varepsilon_n m_d(A_m) r_{n,1} \cdots r_{n,d} + C_4\gamma^{n-m} r_{n,1} \cdots r_{n,d} + C_5\gamma^n/\varepsilon_n^d,
\end{aligned}
$$

where $C_3, C_4, C_5 > 0$ are absolute constants. By setting $\varepsilon_n = 1/(p+1)n^2$ and using Proposition 2.3, for arbitrary $a, b \in \mathbb{N}$ with $a < b$,

$$
\begin{aligned}
&2 \sum_{a \leq m < n \leq b} m_d(A_m \cap A_n) \\
&\quad \leq 2 \sum_{a \leq m < n \leq b} m_d(A_m) 2^d r_{n,1} \cdots r_{n,d} + \frac{2C_3}{p+1} \sum_{a \leq m < b} m_d(A_m) \sum_{n=1}^{\infty} n^{-2} \\
&\qquad + 2C_4 \sum_{a < n \leq b} r_{n,1} \cdots r_{n,d} \sum_{m=1}^{\infty} \gamma^m + 2C_5 (p+1)^d \sum_{n=1}^{\infty} n^{2d} \gamma^n \\
&\quad \leq 2 \sum_{a \leq m < n \leq b} m_d(A_m) 2^d r_{n,1} \cdots r_{n,d} + C_3' \sum_{a \leq m < b} m_d(A_m) + C_4' 2^d \sum_{a < n \leq b} r_{n,1} \cdots r_{n,d} + C_5' \\
&\quad \leq 2 \sum_{a \leq m < n \leq b} m_d(A_m) m_d(A_n) + C_6' \sum_{a \leq m < b} m_d(A_m) + C_4' \sum_{a < n \leq b} m_d(A_n) + C_7' \\
&\quad \leq \Big(\sum_{n=a}^{b} m_d(A_n) \Big)^2 + O\Big(\sum_{n=a}^{b} m_d(A_n) \Big) + O(1),
\end{aligned}
$$

where $C_3', C_4', C_5', C_6', C_7' > 0$ are absolute constants. □

Proof of Theorem 1.4. It follows from Proposition 2.5 that there exists a constant $C > 0$ such that (2.2) is less than $C \sum_{n=a}^{b} m_d(A_n)$. Therefore, (2.1) holds and for any given $\varepsilon > 0$,

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$$\sum_{n=1}^{N} \mathbb{1}_{A_n}(\mathbf{x}) = \Phi(N) + O(\Phi(N)^{1/2} \log^{3/2+\varepsilon} \Phi(N) + \max_{1\le k\le N} m_d(A_k))$$

for m_d-a.e. $\mathbf{x} \in [0,1]^d$. By Proposition 2.3, for any given $\varepsilon > 0$,

$$\sum_{n=1}^{N} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) = \Psi(N) + O(\Psi^{1/2}(N)(\log \Psi(N))^{3/2+\varepsilon})$$

for m_d-a.e. $\mathbf{x} \in [0,1]^d$, where $\Psi(N) = 2^d \sum_{n=1}^{N} r_{n,1} \cdots r_{n,d}$.

For any such $\mathbf{x}$, if $\sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} = \infty$, we have $\sum_{n=1}^{\infty} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) = \infty$ so that $\mathbf{x} \in \mathcal{R}^f(\{\mathbf{r}_n\})$. If $\sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} < \infty$, then $\sum_{n=1}^{\infty} \mathbb{1}_{R(f_n(\mathbf{x}),\mathbf{r}_n)}(T^n\mathbf{x}) < \infty$ so that $m_d(\mathcal{R}^f(\{\mathbf{r}_n\})) = 0$. Hence,

$$m_d(\mathcal{R}^f(\{\mathbf{r}_n\})) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} r_{n,1} \cdots r_{n,d} = \infty. \end{cases}$$

This completes the proof of Theorem 1.4. □

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