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Remarks on the Ruelle operator and the invariant line fields problem: II

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Abstract. Let R be a rational map. A critical point c is called *summable* if the series $\sum_i (1/(R^i)'(R(c)))$ is absolutely convergent. Under certain topological conditions on the postcritical set we prove that R cannot be structurally stable if it has a summable critical point $c \in J(R)$.

1. *Introduction and main statement*

This work is a continuation and generalization of the works [**Lev, Mak**] and [**Mak1**]. In the summable case (see Definition 1 below) the idea of Levin on the construction of a fixed point for the Ruelle operator allows us to remove certain cumbersome conditions of [**Mak**].

The main aim of this work is to prove Theorem A below. Let c_i denote the critical points of a rational map R and $Pc(R) = \overline{\bigcup_i \bigcup_{n\geq 0} R^n(R(c_i))}$ its postcritical set.

Definition 1. A point $a \in \mathbb{C}$ of a given rational map R is called 'summable' if and only if the series

$$\sum_{i=0}^{\infty} \frac{1}{(R^i)'(R(a))}$$

is absolutely convergent. Here, the derivatives are taken in the spherical metric.

Let $X_a = \overline{\bigcup_{n\geq 0} R^n(a)}$ be the closure of the forward orbit of a point $a \in \overline{\mathbb{C}}$.

Definition 2. Let X be the space of rational maps R that fix $0, 1, \infty$ and have a summable critical point $c \in J(R)$ and satisfy one of the following conditions:

(1) $c \notin X_c$;

(2) the diameters of all components of $\mathbb{C}\backslash X_c$ are uniformly bounded away from zero;

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(3) $m(X_c) = 0$, where m is the Lebesgue measure; or
(4) $c \in \partial D \subset J(R)$, where D is some component of $F(R)$.

Note that the last condition holds for maps with completely invariant domain.

Definition 3. The space of quasiconformal deformations of a rational map R is

$$qc(R) = \{F \in \mathbb{C}P^{2d+1} : \text{ there is a quasiconformal automorphism } h_F \text{ of the Riemann sphere } \overline{\mathbb{C}} \text{ such that } F = h_F \circ R \circ h_F^{-1}\}/PSL_2(\mathbb{C}).$$

The rational map R is called *structurally stable* if there exists a neighborhood $V_R \subset \mathbb{C}P^{2d+1}$ of R such that any map $g \in V_R$ is quasiconformally conjugate to R. Hence R is a structurally stable map if and only if $\dim(qc(R)) = 2\deg(R) - 2$.

We can now formulate the main result of this work.

THEOREM A. *Let $R \in X$. Then R is not structurally stable (i.e. R is an* unstable map*).*

Avila ([**Av**] and see also [**Lev**] and [**Mak**] for polynomials of degree two) did prove Theorem A for rational maps with the following additional conditions on R:

If there exists a summable critical point $c \in J(P)$, such that $\sum_{i=0} 1/(R^i)'(R(c)) \neq 0$, and $\infty \in F(R)$, then R is unstable.

His ideas involve the techniques of polynomial-like maps.

COROLLARY A. *Let R be a rational map with summable critical point $c \in J(R)$. Assume*

$$\sum_{i=0} \frac{1}{(R^i)'(R(c))} \neq 0,$$

then R is unstable.

The next result (Theorem B) is an application of the ideas and arguments of Theorem A.

Definition 4. Let W consist of the rational maps R satisfying the following conditions:
(1) R has no parabolic points; and
(2) every critical point c of R is simple (that is $R''(c) \neq 0$) and its forward orbit is infinite and disjoint with the forward orbit of any other critical point.

Note that any structurally stable map belongs to W.

Definition 5. A rational map R is called *summable* if all critical points belonging to its Julia set are summable.

THEOREM B. *Let $R \in W$ be a summable rational map with completely invariant domain. Then there exists no invariant line field on $J(R)$.*

This result is not new (see the results of Bruin and van Strien [**BS**] and Rivera-Letelier [**RL**]) and is not as strong as [**BS**] and [**RL**]. We are simply giving another approach to the invariant line fields problem.

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In forthcoming papers we shall discuss an application of the proposed approach to the exponential map and to entire functions without asymptotic values.

We now collect some definitions and notation.

1.1. *The Riemann surface $S(R)$.* We recall the definition of $S(R)$ for a map $R \in W$.

By Sullivan's results, see [**S**], only five types of the periodic domains can occur in Fatou set of a rational map R and each type has an associated Riemann surface [**S**]. If $R \in W$, then only three types of periodic domains can occur in $F(R)$, namely: attractive periodic domain, cycle of Siegel's disks and cycle of Hermann's rings.

1.1.1. *The case with fundamental domain.*

(1) Let D be an attractive periodic domain. Let $D(R)$ be the full orbit of D minus closure of the full orbit of the critical points. Then the space of the full orbits of the points on $D(R)$ defines the associated Riemann surface $S(D)$, which is a torus with punctures $a_1, \dots, a_l$, where a_i correspond to the full orbits of the critical points from $D(R)$. Then there is an unbranched holomorphic covering

$$P_D : D(R) \to S(D).$$

1.1.2. *Foliated cases.*

(2) Let D be a cycle of Siegel disks of a period k. Then the associated Riemann surface $S(D)$ is a disk conformally equivalent to D with a marked point $a_1 \in \partial D$ and punctures $a_2, \dots, a_{l+1}$ and equipped with a cyclic group $G(D)$ generated by the rotation on an angle $2\pi i\alpha$, where α is an irrational. Here the points $a_2, \dots, a_{l+1}$ correspond to the first hits of forward orbits of the critical points and $G(D)$ corresponds to the action $R^{kn} : D \to D$. Then there is an unbranched covering P_D from the full orbit of D minus backward orbits of punctures onto $S(D)$ semiconjugating R^k with the generator of $G(D)$.

(3) Let D be a cycle of Hermann's rings of a period k. Then, as above, the associated Riemann surface $S(D)$ is a circular ring conformally equivalent to D with marked points a_1, a_2 belonging to the different components of ∂D and with punctures $a_3, \dots, a_{l+2}$ and equipped with a cyclic group $G(D)$ generated by rotation on an angle $2\pi i\alpha$, where α is an irrational. Here the points $a_3, \dots, a_{l+2}$ again correspond to the first hits of forward orbits of critical points. The group $G(D)$ corresponds to the action $R^{kn} : D \to D$. Then there exists an unbranched covering P_D from the full orbit of D minus backward orbits of punctures on S_D semiconjugating R^k with the generator of $G(D)$.

1.2. *Critical decomposition of the foliated cases.* In both foliated cases the group $G(D)$ is an everywhere dense subgroup in the group of all rotations of $S(D)$ in the topology of uniform convergence on $S(D)$. Hence, for any $z \in S(D)$ the closure of the orbit $G(D)(z)$ yields a circle which is called *a leaf of the invariant foliation*. If the leaf l contains a puncture x, then we call l *a critical leaf* and denote it by l_x. After removing the critical leaves from $S(D)$ we obtain the collection $\bigcup S_i \cup D$ of rings S_i and disk D (in the case of Siegel disks). We call this decomposition *critical decomposition* and denote as $CS(D)$.

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Definition 6. The Riemann surface $S(R)$ for $R \in W$ is the following surface

$$\left(\bigcup_D S(D), \text{where } D \text{ is attracting periodic domain}\right)$$
$$\cup \left(\bigcup_D CS(D), \text{where } D \text{ is a foliated periodic domain}\right).$$

Let us define $\Omega(R) = \left(\overline{\mathbb{C}} \setminus \left(\overline{\bigcup_i R^{-i}(Pc(R))}\right)\right)$ then, by the above, there exists a projection $P : \Omega(R) \mapsto S(R)$ which is unbranched holomorphic covering.

1.3. *The space A(S(R)).* Let $A(S(R))$ be the space of holomorphic integrable quadratic differentials on the disconnected surface $S(R)$ which are invariant under the action of the group $G(D)$ on the foliated subsurfaces.

Assume h is quadratic absolutely integrable holomorphic differential on a foliated surface $S(D)$ with the critical decomposition $CS(D) = \bigcup S_i \cup D$. For this decomposition we have

$$h_{|S_i} = h_i(z) \cdot dz^2 \quad \text{and} \quad \phi_D = h_0 \cdot dz^2$$

where h_0 and h_i are holomorphic absolutely integrable functions on D and S_i, respectively. The calculations show that $h_i(z) = c_i/z^2$ and $h_0 = 0$. From the discussion above we conclude that for a cycle of Hermann's rings with k critical leaves the dimension $\dim(A(CS(D))) = k + 1$. If S corresponds to the cycle of Siegel discs with k critical leaves, then $\dim(A(CS(D))) = k$.

We always assume here that the hyperbolic metric λ on the foliated surface S is the collection of complete hyperbolic metrics λ_i on the components of the critical decomposition of S. For example if $\bigcup_i S_i$ is the critical decomposition of S, then the space $HD(S)$ of harmonic differentials Beltrami differentials on S consists of the elements

$$\lambda^{-2}\overline{\phi} = \sum_i \frac{\overline{c_i}\lambda_i^{-2}}{\overline{z^2}},$$

where $\phi \in A(CS(D))$. Then $\dim(HD(CS(D))) = \dim(A(CS(D)))$ and any harmonic Beltrami differential is invariant with respect to the group $G(D)$.

Let $S \in S(R)$ be component corresponding to the attractive periodic domain. Let $HD(S)$ be the space of harmonic differentials on S: these are differentials which in a local charts have the form $\alpha = \overline{\phi}\, dz^2/\rho^2|dz|^2$, where $\phi\, dz^2 \in A(S(R))$ and $\rho|dz|$ is the Poincaré metric.

Definition 7. Let $S(R) = \left(\bigcup_{i=1}^{l} S_i\right) \cup \left(\bigcup_{i=1}^{k} CS_i\right)$, for the map $R \in W$. Then we call the space $HD(S(R)) = HD(S_1) \times \cdots \times HD(S_l) \times HD(CS_1) \times \cdots \times HD(CS_k)$ *the space of harmonic Beltrami differentials* on the surface $S(R)$.

In fact, the space $HD(S(R))$ is isomorphic to the dual space $A(S(R))^*$ by way of the Petersen inner product

$$\langle \psi, \phi \rangle = \iint_{S(R)} \lambda^{-2}\overline{\psi}\phi.$$

For any measurable set $X \subset \overline{\mathbb{C}}$ let us define $\mathrm{Belt}(X) = L_\infty(X)$.

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PROPOSITION 1. *Assume $R \in W$. Then there exists a continuous linear projection $P^* : A(\Omega(R)) \mapsto A(S(R))$. The dual map $P_* : HD(S(R)) \mapsto \mathrm{Belt}(\Omega(R))$ is an injection which maps into the space of Beltrami differentials on $\Omega(R)$.*

Proof. Let $O(D)$ be the full orbit of a periodic attracting component $D \in \Omega$. If $P : O(D) \mapsto S(D)$ is the projection, then $P^* : A(O(D)) \mapsto S(D)$ is the usual push-forward map which in local charts is the summation over all branches of P^{-1}. The map P_* is the usual pull-back, $P_*(\mu) = \mu \circ P = \mu(P)(\overline{P'}/P')$ locally in charts, and hence P_* is an injection.

Foliated case. Let $X \in CS(D)$ be a component of the critical decomposition and $O(X)$ be its full orbit, then again the projection $P : O(X) \mapsto X$ allows one to construct the push-forward and pull-back maps as above. However, in this situation the push-forward map maps $A(O(X))$ onto the space $A(X)$ of all integrable holomorphic quadratic differentials on X. We need to consider the average with respect to the group $G(D)$.

Now let the surface S be the component X equipped with the group $G(D)$.

CLAIM. *There exists a continuous linear projection $\pi : A(X) \mapsto A(S)$. The dual map π_* is an injection from $HD(S)$ into the space of Beltrami differentials on X.*

First assume that the claim is proved, then we finish our proposition by setting $P^* = \pi \circ$ (push-forward map) and $P_* =$ (pull-back map) $\circ \pi_*$.

In what follows we need the following basic facts about non-expansive operators and rational quadratic differentials.

MEAN ERGODICITY LEMMA. *Let T be a non-expansive (that is, the norm $\|T\| \leq 1$) linear endomorphism of a Banach space B and let $\phi \in B$ be any element.*

(1) *Assume that for the* Cesaro average $A_N(T, \phi) = (1/N) \sum_{i=0}^{N-1} T^i(\phi)$ *there exists a subsequence $\{n_i\}$ such that $A_{n_i}(T, \phi)$ weakly converges to an element $f \in B$. Then f is a fixed point for T and $A_N(\phi)$ converges to f strongly (i.e. in norm). If $f = 0$ then $\phi \in \overline{(I - T)(B)}$ and vice versa, i.e. if $\phi \in \overline{(I - T)(B)}$, then $A_n(T, \phi)$ tends to zero with respect to the norm.*

(2) *A linear continuous operator T on a norm space B is called* mean ergodic *if and only if the Cesaro average $A_N(T, \phi)$ converges with respect to the norm for any element $\phi \in B$. In this case $B = \mathrm{Fix}(T) \times \overline{(I - T)(B)}$ and $A_n(T)$ converges (in the strong topology) to the continuous projection $\pi : B \to \mathrm{Fix}(T)$, where $\mathrm{Fix}(T)$ is the space of fixed elements for T.*

Proof. See the text of Krengel [**Kren**, Theorem 1.1, p. 72 and Theorem 1.3, p. 73]. □

LEMMA. (Bers' density theorem) *Let C be a closed subset of $\overline{\mathbb{C}}$ and C_0 a dense subset of C. If Y is the complement of C in $\overline{\mathbb{C}}$, then let $A(Y) \subset L_1(\overline{\mathbb{C}})$ be the subspace of the functions holomorphic on Y. Let $R(C_0) \subset L_1(\overline{\mathbb{C}})$ be the subspace of rational functions holomorphic outside of C_0. Then $R(C_0)$ is an everywhere dense subspace in $A(Y)$ in the L_1-norm.*

Proof. See the text of Gardiner and Lakic [**GL**]. □

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Let $g(z) = \rho z \in G(D)$ be the generator. Then g defines a non-expansive operator $T : A(X) \mapsto A(X)$ by the formula $T(h)(z) = h(\rho z)\rho^2$ and the space $\mathrm{Fix}(T) = A(S)$.

Our aim is to show that $T : A(X) \mapsto A(X)$ is mean ergodic.

Let $Y \subset A(X)$ be the linear span of quadratic integrable rational differentials on $\overline{\mathbb{C}}$ with poles in $\overline{\mathbb{C}}\backslash\overline{X}$. Then by Bers' density theorem Y is an everywhere dense subset of $A(X)$. If the Cesaro averages $A_n(T, h)$ converge in the strong topology for any element $h \in Y$, then T is mean ergodic and the Cesaro averages form an equicontinuous family of operators.

Hence, by the above it is enough to show the strong convergence of $A_n(T, h)$ for any $h \in Y$. Let $h \in Y$ be an element, then by assumptions h is holomorphic in a ring Z which compactly contains the ring X. Let $h(z) = \sum_{-\infty}^{+\infty} a_i z^i$ be the Laurent series of h, then the decomposition of $A_n(T, h)$ is

$$\sum_{-\infty}^{+\infty} a_i z^i \frac{\sum_{l=0}^{n-1} \rho^{l(i+2)}}{n}.$$

Since $\rho = \exp(2\pi i\alpha) \neq 1$ and the series $\sum_{-\infty}^{+\infty} a_i z^i$ converges absolutely on $\overline{X}$, then the series for $A_n(T, h)$ also converges absolutely on $\overline{X}$. Hence, the calculations by element show that $\lim_{n\to\infty} \left(\sum_{l=0}^{n-1} \rho^{l(i+2)}/n\right) = 0$, for any $i \neq -2$. As result we have

$$\lim_{n\to\infty} A_n(T, h) = \frac{a_{-2}}{z^2}.$$

The uniform convergence in the limit above on $\overline{X}$ implies the strong convergence in $A(X)$.

The dual map $\pi_* : HD(X) \mapsto \mathrm{Belt}(X)$ is simply inclusion so

$$\iint_X \pi(h)\mu = \iint_X h\pi_*(\mu),$$

for any $h \in A(X)$ and $\mu \in HD(X)$. Hence, the claim and Proposition 1 are proved. □

1.4. *The space* $H^1(R)$. Let $M_{\deg(R)}$ be the space of rational maps of degree $\deg(R)$ fixing the points $0, 1, \infty$. Then $M_{\deg(R)}$ may be identified with an open subset in the projective space $\mathbb{C}P^{2\deg(R)-2}$ by the coefficient correspondence. Let $K(R) \subset M_{\deg(R)}$ be the component containing the map R.

Let $T_R(K(R))$ be the complex tangent space to $K(R)$ at the point R. Suppose $R(z) = z(P(z)/Q(z))$ with $\deg(P) = \deg(R) - 1, \deg(Q) \leq \deg(R) - 1$ and $P(1) = Q(1)$. Let $h : D \mapsto K(R)$ be a complex-analytic curve in $K(R)$ with $h(0) = R$, where $D \subset \mathbb{C}$ is the unit disk. Then we can obtain the Taylor-series expansion

$$h(\lambda) = R_\lambda(z) = z\frac{P + \lambda P_1 + \cdots}{Q + \lambda Q_1 + \cdots} = R(z) + \lambda z\frac{P_1 Q - Q_1 P}{Q^2} + \cdots,$$

where $\deg(P_1) \leq \deg(R) - 2, \deg(Q) \leq \deg(R) - 1$. Then we may identify $T_R(K(R))$ with the linear space

$$H^1(R) = \left\{ z\frac{P_1 Q - Q_1 P}{Q^2} : \text{ polynomials } P_1 \text{ and } Q_1 \text{ satisfy } Q_1(1) = P_1(1), \right.$$
$$\left. \deg(Q_1) \leq \deg(R) - 1, \deg(P_1) \leq \deg(R) - 2 \right\},$$

and $\dim(H^1(R)) = 2\deg(R) - 2$.

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Note in the case $\deg(Q) = \deg(R) - 1$, $\deg(P) < \deg(R) - 1$, the polynomials P_1, Q_1 above satisfy $Q_1(1) = P_1(1)$, $\deg(Q_1) \le \deg(R) - 2$, $\deg(P_1) \le \deg(R) - 1$.

1.5. *The Bers map.* Let $\phi \in L_\infty(\mathbb{C})$ and denote the Beltrami operator by $B_R(\phi) = \phi(R)(\overline{R}'/R') : L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C})$. Then the open unit ball B of the space $\mathrm{Fix}(R) \subset L_\infty(\mathbb{C})$ of fixed points of B_R is called *the space of Beltrami differentials for* R; it describes all qc-deformations of R.

Let $\mu \in \mathrm{Fix}(R)$; then for any λ with $|\lambda| < 1/\|\mu\|$, the element $\mu_\lambda = \lambda\mu \in B$. Let f_λ be the family of qc-maps corresponding to the Beltrami differentials μ_λ with $f_\lambda(0, 1, \infty) = (0, 1, \infty)$. Then the map

$$\lambda \to R_\lambda = f_\lambda \circ R \circ f_\lambda^{-1} \in K(R)$$

is a conformal map. Let $R_\lambda(z) = R(z) + \lambda G_\mu(z) + \cdots$. Differentiation with respect to λ at the point $\lambda = 0$ gives the equation

$$F_\mu(R(z)) - R'(z)F_\mu(z) = G_\mu(z),$$

where $F_\mu(z) = (\partial/\partial\lambda) f_\lambda(z)_{|\lambda=0}$ and $G_\mu(z) = (\partial/\partial\lambda)R_\lambda(z)_{|\lambda=0} \in H^1(R)$.

Remark 2. By the theory of qc maps (see, for example, [**Krush**]), for any $\mu \in L_\infty(\mathbb{C})$ and t with $|t| < \epsilon$ and ϵ small, there exists the following formula for the qc-map f_t fixing $0, 1, \infty$:

$$f_t(z) = z - \frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{t\mu}{\xi(\xi-1)(\xi-z)} + C(\epsilon, R)\|\mu\|_\infty^2,$$

where $|z| < R$ and $C(\epsilon, R)$ is a constant that does not depend on μ. In addition,

$$F_\mu(z) = \left.\frac{\partial f_\lambda}{\partial\lambda}\right|_{\lambda=0} = -\frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu}{\xi(\xi-1)(\xi-z)}$$

and $\overline{\partial} F_\mu = \mu$ in the sense of distributions.

Hence, we can define a linear map $\beta : \mathrm{Fix}(R) \to H^1(R)$ by the formula

$$\beta(\mu) = F_\mu(R(z)) - R'(z)F_\mu(z).$$

In analogy with Kleinian groups we call the map β the *Bers' map* (see, for example, [**Kra**]).

We call the space $HD(R) = P_*(HD(S(R)))$ *the space of harmonic differentials*. For any element $\alpha \in HD(R)$ the supporter $\mathrm{supp}(\alpha) \in F(R)$. Then $\dim(HD(R)) = \dim(A(S(R)))$.

Now the push-forward map P^* (which is dual to P_*) maps the space $A(\Omega(R))$ onto $A(S(R))$, where the space $A(\Omega(R))$ is the Banach space of all holomorphic integrable functions on $\Omega(R)$. Hence, by duality we have

$$\iint_{\Omega(R)} P_*(\alpha)\phi = \iint_{S(R)} \alpha P^*(\phi),$$

for any $\alpha \in HD(S(R))$ and $\phi \in A(\Omega(R))$. Thus, the element $\beta = P_*(\alpha)$ presents the trivial functional on the space $A(\Omega(R))$ if and only if $\alpha = 0$.

Let $J_R = \mathrm{Fix}(R)_{|J(R)}$ be the space of invariant Beltrami differentials supported on the Julia set.

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THEOREM 3. *Let $R \in W$ be a rational map. Then:*
(1) *the β restricted on $HD(R) \times J_R$ is an injection;*
(2) *if R is structurally stable, then $\beta : HD(R) \times J_R \to H^1(R)$ is an isomorphism.*

Proof. (1) Let $A(J(R)) \subset L_1(\overline{\mathbb{C}})$ be the subspace of functions which are holomorphic on $F(R)$. Then $A(J(R))$ is a Banach space with L_1-norm. Furthermore, let $A(R) \subset A(J(R))$ be the subspace of rational functions. In other words, $A(R)$ is the linear span of the functions

$$\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)},$$

where $a \in J(R)$. Then by Bers' density theorem $\overline{A(R)} = A(J(R))$.

Now let $0 \neq \mu \in \ker(\beta) \cap (HD(R) \times J_R)$; then we have

$$F_\mu(R(z)) = R'(z)F_\mu(z)$$

and, hence, $F_\mu = 0$ on the set of all non-parabolic periodic points and, hence, is also equal to zero on the Julia set. Now if $F(R) = \emptyset$, then $F_\mu = 0$ identically on $\overline{\mathbb{C}}$; and using the fact that $\mu = \overline{\partial} F_\mu$ (in the sense of distributions) we have $\mu = 0$.

If $F(R) \neq \emptyset$, then on $A(J(R))$ the functional $L_\mu(\phi) = \iint \mu\phi$ satisfies $L_\mu(\gamma_a(z)) = F_\mu(a) = 0$, for any $a \in J(R)$. By Bers' density theorem we have $L_\mu = 0$ on $A(J(R))$, hence $\mu = 0$ almost everywhere on $J(R)$. Then we obtain $\mu \in HD(R)$ represents the zero-functional on the space $A(\Omega(R)) \subset A(J(R))$. By the discussion above, we have $\mu = 0$.

(2) If R is structurally stable, then $\dim(HD(R) \times J_R) = \dim(H^1(R)) = 2\deg(R) - 2$. By (1) the operator β is linear and injective and, hence, an isomorphism. □

1.6. *The Ruelle operator.*

Definition 8. For a rational map R the Ruelle operator R^* is the linear endomorphism of $L_1(\mathbb{C})$ defined

$$R^*(\phi)(z) = \sum \phi(J_i(z))(J_i'(z))^2 = \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{(R'(y))^2},$$

where J_i are branches of R^{-1}.

Then we have the following lemma.

LEMMA 4. *Let $Y \subset \overline{\mathbb{C}}$ be a completely invariant measurable subset with respect to a rational map R. Then:*
(1) *$R^* : L_1(Y) \to L_1 Y$ is a surjective linear endomorphism with $\|R^*\|_{L_1(Y)} \leq 1$;*
(2) *The operator $R_*(\phi) = \phi(R)(R')^2/\deg(R) : L_1(Y) \to L_1(Y)$ is a right inverse to R^*;*
(3) *The Beltrami operator $B_R : L_\infty(Y) \to L_\infty(Y)$ is the dual operator to R^*;*
(4) *If $Y \subset \overline{\mathbb{C}} \setminus \overline{\bigcup_i R^{-i}(Pc(R))}$ is an open subset and $A(Y) \subset L_1(Y)$ is the subset of holomorphic functions, then $R^*(A(Y)) = A(Y)$.*

Proof. All items follow from the definitions. □

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Suppose that all critical points c_i are simple. Then there exists a decomposition $1/R'(z) = \omega + \sum b_i/(z - c_i)$, where $\omega = 1/R'(\infty)$ is the multiplier of ∞, c_i are the critical points, and by the residue theorem $b_i = 1/R''(c_i)$. For $i = 1, \ldots, 2\deg(R) - 2$ let $h_i(z) = 1/R'(z) - b_i/(z - c_i)$.

LEMMA 5. *For any rational map R with simple critical points the following statements are true.*

(1) *Let*

$$\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)} \in L_1(\overline{\mathbb{C}}),$$

where $a \in \mathbb{C}\backslash\{0, 1\}$ is not a critical point. Then

$$R^*(\gamma_a(z)) = \frac{\gamma_{R(a)}(z)}{R'(a)} + \sum_i b_i \gamma_a(c_i)\gamma_{R(c_i)}(z).$$

Let $\tau_a(z) = 1/(z - a)$, where $a \in \mathbb{C}$ is not a critical point. Then

$$R^*(\tau_a(z)) = \frac{\tau_{R(a)}(z)}{R'(a)} + \sum_i b_i \tau_a(c_i)\tau_{R(c_i)}(z).$$

(2) *If $a = c_i$ is a critical point, then*

$$R^*(\gamma_a(z)) = \left(h_i(a) + b_i \frac{2c_i - 1}{c_i(c_i - 1)}\right)\gamma_{R(a)}(z) + \sum_{j \neq i} b_j \gamma_a(c_j)\gamma_{R(c_j)}(z),$$

and

$$R^*(\tau_a(z)) = h_i(a)\tau_{R(a)}(z) + \sum_{j \neq i} b_j \tau_a(c_j)\tau_{R(c_j)}(z),$$

where the coefficient

$$h_i(a) + b_i \frac{2c_i - 1}{c_i(c_i - 1)} = \lim_{a \to c_i}\left(\frac{1}{R'(a)} + b_i \gamma_a(c_i)\right).$$

(3) *If $\mu \in \mathrm{Fix}(R)$, then*

$$F_\mu(a) = -\frac{1}{\pi}\iint \mu\gamma_a(z)$$

and

$$\beta(\mu)(a) = -R'(a)\sum b_i F_\mu(R(c_i))\gamma_a(c_i).$$

Proof. (1) Let ϕ be any C^∞ function with compact support and $\mathrm{supp}(\phi) \in \mathbb{C}\backslash\{0, 1\}$. Then we claim

$$\iint (\phi(R))_{\overline{z}}\gamma_a(z) = \iint \phi_{\overline{z}}\gamma_{R(a)}(z).$$

We prove this claim in the following way. By using the decomposition

$$\gamma_a(z) = \frac{a-1}{z} - \frac{a}{z-1} + \frac{1}{z-a},$$

we obtain

$$\begin{aligned}\iint (\phi(R))_{\overline{z}}\gamma_a(z) &= (a-1)\iint \frac{(\phi(R))_{\overline{z}}}{z} - a\iint \frac{(\phi(R))_{\overline{z}}}{z-1} + \iint \frac{(\phi(R))_{\overline{z}}}{z-a} \\ &= (a-1)\phi(R(0)) - a\phi(R(1)) + \phi(R(a)) = *.\end{aligned}$$

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$\phi(\infty)=\phi(0)=\phi(1)=0$ and $0, 1, \infty$ are fixed, hence we have

$$*=\phi(R(a))=(R(a)-1)\phi(0)-R(a)\phi(1)+\phi(R(a))=\iint \phi_{\overline{z}}\gamma_{R(a)}(z).$$

This proves the claim.

Now let us show (1):

$$\begin{aligned}\iint \phi_{\overline{z}}R^*(\gamma_a(z)) &= \iint \frac{\phi_{\overline{z}}(R)\overline{R'(z)}}{R'(z)}\gamma_a(z)=\iint \frac{(\phi(R))_{\overline{z}}}{R'(z)}\gamma_a(z)\\ &= \omega\iint (\phi(R))_{\overline{z}}\gamma_a(z)+a(a-1)\\ &\quad\times\iint (\phi(R))_{\overline{z}}\sum\frac{b_i}{z(z-1)(z-a)(z-c_i)}\\ &= \omega\iint (\phi(R))_{\overline{z}}\gamma_a(z)+\sum\frac{a(a-1)b_i}{a-c_i}\\ &\quad\times\iint (\phi(R))_{\overline{z}}\left(\frac{1}{z(z-1)(z-a)}-\frac{1}{z(z-1)(z-c_i)}\right)=*.\end{aligned}$$

Then by the claim above we have

$$\begin{aligned}*&=\iint \phi_{\overline{z}}\gamma_{R(a)}(z)\left(\omega+\sum\frac{b_i}{a-c_i}\right)-\sum\frac{b_i a(a-1)}{c_i(c_i-1)(a-c_i)}\iint \phi_{\overline{z}}\gamma_{R(c_i)}(z)\\ &=\iint \phi_{\overline{z}}\left(\frac{\gamma_{R(a)}}{R'(a)}+\sum b_i\gamma_a(c_i)\gamma_{R(c_i)}(z)\right).\end{aligned}$$

Let $H(z)=R^*(\gamma_a(z))-\gamma_{R(a)}/R'(a)-\sum b_i\gamma_a(c_i)\gamma_{R(c_i)}(z)$, then by the above calculations we have $\partial H/\partial\overline{z}=0$ (in the sense of distributions) on $\mathbb{C}\backslash\{0,1\}$. By Weyl's lemma, H is a holomorphic function on $\mathbb{C}\backslash\{0,1\}$, hence $H(z)=C+A/z+B/(z-1)$. Since the function H is integrable over $\overline{\mathbb{C}}$, we have $H=0$.

By the argument above, the calculation of the derivative of $R^*(\tau_a(z))$ (in the sense of distributions) shows that the function $g(z)=R^*(\tau_a(z))-\tau_{R(a)}/R'(a)-\sum_i b_i\tau_a(c_i)\tau_{R(c_i)}(z)$ is holomorphic over $\mathbb{C}$ and $g(z)\to 0$, for $z\to\infty$. Hence, $g(z)=0$.

(2) Note that all integrals in the calculations of (1) depend continuously on the parameter a. Hence, we can consider the limit as $a\to c_i$ in the formulas of (1). Then

$$R^*(\gamma_{c_i}(z))=\left(\lim_{a\to c_i}\left(\frac{1}{R'(a)}+b_i\gamma_a(c_i)\right)\right)\gamma_{R(c_i)}(z)+\sum_{j\neq i}b_j\gamma_{c_i c_j}\gamma_{R(c_j)}(z)$$

and

$$R^*(\tau_{c_i}(z))=\left(\lim_{a\to c_i}\left(\frac{1}{R'(a)}+b_i\tau_a(c_i)\right)\right)\tau_{R(c_i)}(z)+\sum_{j\neq i}b_j\tau_{c_i c_j}\tau_{R(c_j)}(z).$$

We have

$$\frac{1}{R'(a)}+b_i\gamma_a(c_i)=\omega+\sum_{j\neq i}\frac{b_j}{a-c_j}+\frac{b_i}{a-c_i}+\frac{b_i(a-1)}{c_i}-\frac{b_i a}{c_i-1}+\frac{b_i}{c_i-a}.$$

Then we are done.

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(3) Let $\mu \in \mathrm{Fix}(R)$, then by (1) we have

$$\begin{aligned}\pi F_\mu(a) &= -\iint \mu\gamma_a(z) = -\iint \mu R^*(\gamma_a(z)) \\ &= -\frac{1}{R'(a)} \iint \mu\gamma_{R(a)}(z) - \sum_i b_i \gamma_a(c_i) \iint \mu\gamma_{R(c_i)}(z) \\ &= \pi \frac{F_\mu(R(a))}{R'(a)} + \pi \sum_i b_i F_\mu(R(c_i))\gamma_a(c_i)\end{aligned}$$

and, hence,

$$\beta(\mu)(a) = F_\mu(R(a)) - R'(a)F_\mu(a) = -R'(a)\sum_i b_i F_\mu(R(c_i))\gamma_a(c_i). \qquad (*)$$

This finishes the proof. □

Remark 6. Lemma 5 gives a different set of coordinates for the spaces $H^1(R)$ and $HD(R) \times J_R$. Namely the formula $(*)$ above describes the linear injection $\beta^* : HD(R) \times J_R \to \mathbb{C}^{(2\deg(R)-2)}$ by

$$\beta^*(\mu) = (F_\mu(R(c_1)), \ldots, F_\mu(R(c_{2\deg(R)-2}))).$$

When R is a structurally stable map, then β^* is an isomorphism.

1.7. *Formal relation of Ruelle–Poincaré series.* Let again $R \in X$ be a map with simple critical points.

Definition 9. (Ruelle–Poincaré series) We have the following Ruelle–Poincaré series.

(1) *Backward Ruelle–Poincaré series*:

$$RS(x, R, a) = \sum_{n=0}^{\infty} (R^*)^n(\gamma_a)(x),$$

where $a \in \mathbb{C}\backslash\{0, 1\}$ is a parameter.

(2) *Forward Ruelle–Poincaré series*:

$$RP(x, R) = \sum_{n=0}^{\infty} \frac{1}{(R^n)'(R(x))}.$$

(3) *Modified Ruelle–Poincaré series*:

$$A(x, R, a) = \sum_{n=0} \frac{1}{(R^n)'(a)} \gamma_{R^n(a)}(x).$$

Note that the Ruelle–Poincaré series generalizes the Poincaré series introduced by McMullen for rational maps (see [**MM**]).

Proposition 7 below gives a formal relation between Ruelle–Poincaré series.

Definition 10. If $A = \sum_{i=0} a_i$ and $B = \sum_{i=0} b_i$, then the *formal Cauchy product* is $C = A \otimes B = \sum_{i=0} c_i$, where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

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PROPOSITION 7. *Let $R \in X$ be a rational map with simple critical points c_i. Let $d_i = R(c_i)$ be the critical values. Then we have the following formal relation between series:*

$$RS(z, R, a) = A(z, R, a) + \sum_i \frac{1}{R''(c_i)} A(c_i, R, a) \otimes RS(z, R, d_i).$$

This proposition is part of Proposition A of [**Mak**].

Proof. Using Lemma 5, we calculate

$$\begin{aligned}
\gamma_a(z) &= \gamma_a(z) \\
R^*(\gamma_a(z)) &= \frac{1}{R'(a)}\gamma_{R(a)}(z) + \sum_i b_i \gamma_a(c_i)\gamma_{d_i}(z) \\
(R^*)^2(\gamma_a(z)) &= \frac{1}{(R^2)'(a)}\gamma_{R^2(a)}(z) + \sum_i b_i \left(\frac{\gamma_{R(a)}(c_i)}{R'(a)}\gamma_{d_i}(z) + \gamma_a(c_i) R^*(\gamma_{d_i}(z)) \right) \\
&\vdots \\
(R^*)^n(\gamma_a(z)) &= \frac{1}{(R^n)'(a)}\gamma_{R^n(a)}(z) \\
&\quad + \sum_i b_i \left(\frac{\gamma_{R^{n-1}(a)}(c_i)}{(R^{n-1})'(a)}\gamma_{d_i}(z) \right. \\
&\quad \left. + \frac{\gamma_{R^{n-2}(a)}(c_i)}{(R^{n-2})'(a)} R^*(\gamma_{d_i}(z)) + \cdots + \gamma_a(c_i)(R^*)^{n-1}(\gamma_{d_i}(z)) \right).
\end{aligned}$$

Summation of columns gives the desired expression. □

LEMMA 8. *Let R be a rational map and $0, 1, \infty$ be fixed points. Let $a \in \mathbb{C}$ be summable for R. Then:*

(1) *the series $A(z, R, a)$ defines a measurable integrable function on $\mathbb{C}$ that is holomorphic outside of $\overline{\bigcup_n R^n(R(a))}$; and*
(2) *for any critical point c the series $A(c, R, a)$ converges.*

Proof. (1) From the theory of qc-maps (see, for example, [**Kra**] and [**Vek**]) there exists a constant $M < \infty$ which does not depend on a such that we have the estimate $\|\gamma_a(z)\| \leq M|a||\ln|a||$.

Then

$$\iint |A(z, R, a)| \leq M \sum \left| \frac{x^i |R^i(a)||\ln|R^i(a)||}{(R^i)'(a)} \right| < \infty.$$

(2) Let c be a critical point. If $c \notin \overline{\bigcup_n R^n(a)}$, then by (1) above we are done.

Now assume that $c \in \overline{\bigcup_n R^n(a)}$. Let U_ϵ be an ϵ-neighborhood of c. Let the points $R^{n_i}(a) \in U_\epsilon$. Then by the arguments above and the fact that $c \neq R^n(a)$ for all $n \geq 0$, we need to estimate the following expression

$$\sum_i \frac{1}{(R^{n_i})'(a)}\gamma_{R^{n_i}(a)}(c).$$

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By using the equality $R'(z) = (z-c)R''(c) + O(|z-c|^2)$ for $z \in U_\epsilon$, we obtain

$$\frac{1}{|R^{n_i} - c|} \le \frac{|R''(c)| + O(|R^{n_i} - c|)}{|R'(R^{n_i}(a))|} \le C\frac{1}{|R'(R^{n_i}(a))|};$$

hence

$$|\gamma_{R^{n_i}(a)}(c)| \le C\frac{1}{|R'(R^{n_i}(a))|}\frac{|R^{n_i}(a)||R^{n_i}(a)-1|}{|c(c-1)|} \le C_1\frac{1}{|R'(R^{n_i}(a))|}$$

where C and C_1 are constants depending only on ϵ and the point c. As a result we have

$$\left|\sum_i \frac{1}{(R^{n_i})'(a)}\gamma_{R^{n_i}(a)}(c)\right| \le C_1 \sum_i \frac{1}{|(R^{n_i+1})'|} < \infty.$$

This finishes the proof of the lemma. □

2. *Restriction on the image of β^**

Let $R \in W$ and let $c_1 \in J(R)$ be a summable critical point. Let us consider the relation between Ruelle–Poincaré series given by Proposition 7 for the value $a = R(c_1) = d_1$. Let $\mu \in HD(R) \times J_R$ be any element. Let us integrate μ with respect to the expression $(R^*)^n(\gamma_{d_1})$ found in Proposition 7. Then using the invariance of μ, we obtain

$$F_\mu(d_1) = \iint \mu\gamma_{d_1}(z) = \iint \mu(R^*)^n(\gamma_{d_1}(z)) = \frac{1}{(R^n)'(d_1)}\iint \mu\gamma_{R^n(d_1)}(z)$$
$$+ \sum_i b_i \iint \mu\gamma_{d_i}(z)\left(\frac{\gamma_{R^{n-1}(d_1)}(c_i)}{(R^{n-1})'(d_1)} + \frac{\gamma_{R^{n-2}(d_1)}(c_i)}{(R^{n-2})'(d_1)} + \cdots + \gamma_{d_1}(c_i)\right).$$

Taking the limit $n \to \infty$ we obtain

$$F_\mu(d_1) = \sum_j b_j F_\mu(d_j) A(c_j, R, d_1). \tag{1}$$

Let $C_j = A(c_j, R, d_1)$; by Lemma 8 the C_j are well-defined constants that do not depend on μ. Then we may rewrite the relation (1) as follows:

$$F_\mu(d_1) = \sum_i b_i C_i F_\mu(d_i) \text{ or } F_\mu(d_1)(1 - b_1C_1) = \sum_{i\ge 2} b_i C_i F_\mu(d_i), \tag{2}$$

and call relation (2) a *trivial relation* if and only if

$$C_1 = \frac{1}{b_1} \quad \text{and} \quad C_i = 0 \quad \text{for } i > 1.$$

Here we need the following proposition.

PROPOSITION 9. *If relation (2) is non-trivial, then* $\dim(HD(R) \times J_R) \le 2\deg(R) - 3$. *In particular, R is structurally unstable.*

Proof. By Remark 6 the operator β induces an injection.

$$\beta^* : HD(R) \times J_R \to \mathbb{C}^{(2d-2)}.$$

If relation (2) above is non-trivial, then this relation defines a non-trivial lineal equation in $\mathbb{C}^{(2d-2)}$ and the image of β^* is a subset of the set of solutions of this equation. Hence, $\dim(HD(R) \times J_R) = \dim(\Im(\beta^*)) \le 2\deg(R) - 3$ and we are done. □

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PROPOSITION 10. *Let $R \in W$. Then the following hold.*

(1) *If relation (2) is trivial, then*

$$\sum_{i=0} \frac{1}{(R^i)'(d_1)} = \sum_{i=0} \frac{R^i(d_1)}{(R^i)'(d_1)} = 0.$$

(2) *If $\phi(z) = A(z, R, d_1) \neq 0$ identically on $Y = \mathbb{C}\backslash X_{c_1}$, then relation (2) is non-trivial.*

Here we use an idea of Levin [**Lev**] of considering the function $\phi(z) = A(z, R, d_1)$ itself. Levin observes that (in the case $\deg(R) = 2$) that the function $A(z, R, d_1)$ is a fixed point for R^*.

Proof. Assume that (2) is a trivial relation, that is

$$C_1 = A(c_1, R, d_1) = \phi(c_1) = \frac{1}{b_1} \quad \text{and} \quad C_i = A(c_i, R, d_1) = \phi(c_i) = 0 \quad \text{for } i > 1.$$

Then we claim that, assuming the above hypotheses, we have

$$R^*(\phi(z)) = \phi(z).$$

We prove the claim in the following way. By Lemma 5, a calculation shows that

$$R^*(\phi(z)) = \phi(z) - \gamma_{d_1}(z) + \sum_{i}^{2d-2} b_i \gamma_{d_i}(z) A(c_i, R, d_1).$$

However, assuming $\phi(c_i) = C_i = 0$ for $i > 1$ and $\phi(c_1) = 1/b_1$, we have $R^*(\phi(z)) = \phi(z)$. This finishes the claim. □

Now let us check (1). Let

$$A = \left(\sum_{n=0} \frac{R^n(d_1)}{(R^n)'(d_1)}\right) \quad \text{and} \quad B = \left(\sum_{n=0} \frac{1}{(R^n)'(d_1)}\right).$$

Then we have

$$\phi(z) = \frac{1}{z}\left(\sum_{n=0} \frac{R^n(d_1) - 1}{(R^n)'(d_1)}\right) - \frac{1}{z-1}\left(\sum_{n=0} \frac{1}{(R^n)'(d_1)}\right) + \psi(z) = \frac{A-B}{z} - \frac{A}{z-1} + \psi(z),$$

where

$$\psi(z) = \sum_{i} \frac{\tau_{R^i(d_1)}(z)}{(R^i)'(d_1)}.$$

Hence,

$$R^*(\phi(z)) = (A-B)R^*(\tau_0(z)) - BR^*(\tau_1(z)) + \sum_{i} \frac{R^*(\tau_{R(d_1)}(z))}{(R^i)'(d_1)} = *.$$

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By Lemma 5 (calculation of $\tau_a(z)$), we have

$$\begin{aligned}
* &= (A-B)\left(\frac{\tau_0(z)}{R'(0)} + \sum_j b_j \tau_0(c_j)\tau_{R(c_j)}(z)\right) \\
&\quad - B\left(\frac{\tau_1(z)}{R'(1)} + \sum_j b_j \tau_1(c_j)\tau_{R(c_j)}(z)\right) + \psi(z) \\
&\quad - \tau_{d_1}(z) + \sum_j b_j \tau_{R(c_j)}(z)\psi(c_j) \\
&= \sum_j b_j \tau_{R(c_j)}(z)((A-B)\tau_0(c_j) - B\tau_1(c_j) + \psi(c_j)) \\
&\quad + (\psi(z) + (A-B)\tau_0(z) - B\tau_1(z)) \\
&\quad + \left((A-B)\tau_0(z)\left(\frac{1}{R'(0)} - 1\right) - B\tau_1(z)\left(\frac{1}{R'(1)} - 1\right)\right) \\
&\quad - \tau_{d_1}(z) = \phi(z) - \tau_{d_1}(z) + \sum_j b_j \tau_{R(c_j)}(z)\phi(c_j) \\
&\quad + \left((A-B)\tau_0(z)\left(\frac{1}{R'(0)} - 1\right) - B\tau_1(z)\left(\frac{1}{R'(1)} - 1\right)\right) = **.
\end{aligned}$$

Since relation (2) is trivial, we have

$$** = \phi(z) + (A-B)\tau_0(z)\left(\frac{1}{R'(0)} - 1\right) - B\tau_1(z)\left(\frac{1}{R'(1)} - 1\right) \overset{\text{(by the claim above)}}{=} \phi(z).$$

The fixed points 0 and 1 are non-parabolic and hence $A = B = 0$. This takes care of (1).

(2) We need the following lemma.

LEMMA 11. *Let R be a rational map and $R^*(f) = f$, for an element $f \in L_1(\mathbb{C})$. Then the measure $\sigma(A) = \iint_A |f(z)|$ is a non-negative invariant absolutely continuous probability measure.*

Proof. We use an argument here which in reality arises from the Hamilton–Krushkal extremality condition and is mentioned in statements such as Lemma 10 of Douady and Hubbard [**DH**] and Epstein [**Eps**]. Invariance of the measure σ means that

$$|f| = |R^*||f| = \sum_i^{\deg(R)} |f(J_i)||J_i'|^2$$

where the operator $|R^*|$ is called *the modulus of the Ruelle operator*.

Let us show invariance. For any fixed $1 \le j \le \deg(R)$ we have

$$\begin{aligned}
\|f(z)\| = \|R^*(f)\| &= \iint \left|\sum_i f(J_i)(J_i')^2\right| \\
&\le \iint |f(J_j)(J_j')^2| + \iint \left|\sum_{i\neq j} f(J_i)(J_i')^2\right| \le \|f\|.
\end{aligned}$$

In the above expression we have that the left-hand side is equal to the far right-hand side. Hence, all inequalities are in fact equalities. Let us define $\alpha_j(z) = f(J_j(z))(J_j(z)')^2$ and

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$\beta_j(z) = f(z) - \alpha_j(z) = \sum_{i\neq j} f(J_i(z))(J_i(z)')^2$. Then by the above we have

$$\iint |\alpha_j + \beta_j| = \iint |\alpha_j| + \iint |\beta_j|.$$

We deduce that $|\alpha_j + \beta_j| = |\alpha_j| + |\beta_j|$ almost everywhere. Indeed, suppose not; let $A = \{z\colon |\alpha_j(z) + \beta_j(z)| < |\alpha_j(z)| + |\beta_j(z)|\}$ with $m(A) > 0$, where m is the Lebesgue measure. Then $\|f\| = \iint_A |f| + \iint_{\overline{\mathbb{C}}\setminus A} |f(z)| < \|f\|$, a contradiction. Lemma 11 is then finished by induction. □

COROLLARY 12. *If $\phi(z) \neq 0$ identically on Y, then $J(R) = \overline{\mathbb{C}}$ and $\overline{\phi}/|\phi|$ defines an invariant Beltrami differential.*

Proof. Recall that a measurable positive Lebesgue measure set $A \subset \overline{\mathbb{C}}$ is *wandering* for a map R if and only if $m(R^{-n}(A) \cap R^{-k}(A)) = 0$ for $k \neq n$. If λ is a non-negative invariant probability measure, then $\lambda(A) = 0$ for any wandering set A. Then we have:

(1) if $F(R) \neq \emptyset$, then by Lemma 11 and the above argument, $\sigma(F(R)) = 0$ and hence $\phi = 0$ identically on Y;

(2) if $J(R) = \overline{\mathbb{C}}$, then either $\phi = 0$ identically on Y or $\phi \neq 0$ identically on every component of Y.

In the last case let us show that $\mu = \overline{\phi}/|\phi| \in \mathrm{Fix}(R)$. Indeed in the notation of Lemma 11 let us show that

$$\frac{\beta_j}{\alpha_j} = k_j \geq 0 \quad \text{is a non-negative function.}$$

Actually, we have $|1 + \beta_j/\alpha_j| = 1 + |\beta_j/\alpha_j|$; then if $\beta_j/\alpha_j = \gamma_1^j + i\gamma_2^j$, we obtain

$$(1 + (\gamma_1^j))^2 + (\gamma_2^j)^2 = (1 + \sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2})^2 = 1 + (\gamma_1^j)^2 + (\gamma_2^j)^2 + 2\sqrt{(\gamma_1^j)^2 + (\gamma_2^j)^2}.$$

Hence, $\gamma_2^j = 0$ and $\alpha_j/\beta_j = \gamma_1^j$ is a real-valued function but α_j/β_j is a meromorphic function. So $\gamma_1^j = k_j$ is constant on every connected component of Y and condition $|1 + k_j| = 1 + |k_j|$ shows $k_j \geq 0$.

Then for any $1 \leq j \leq \deg(R)$ we have

$$\frac{R^*(\phi)}{\alpha_j} = \frac{\alpha_j + \beta_j}{\alpha_j} = 1 + k_j \geq 1$$

and

$$\phi = R^*(\phi) = (1 + k_j)\phi(J_j)(J_j')^2.$$

That means for any j

$$\mu(z) = \frac{(1 + k_j)\overline{\phi(J_j)(J_j')^2}}{|(1 + k_j)\phi(J_j)(J_j')^2|} = \mu(J_j)\frac{\overline{J_j'}}{J_j'}$$

and we obtain

$$\mu(R)\frac{\overline{R'}}{R'} = \mu.$$

This completes the proof of the corollary. □

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To finish Proposition 10 we use a result of McMullen [**MM1**, Lemma 3.16] which states that if $\overline{\phi}/|\phi| \in \mathrm{Fix}$ for $\phi(z) \neq 0$ an integrable holomorphic function on a domain $U \subset J(R)$, then R is *double covered by an integral torus endomorphism* and in particular is a postcritically finite map which contradicts with assumptions on the maps from W. Proposition 10 is finished.

COROLLARY A. *Let R be a rational map with summable critical point $c \in J(R)$. If the sum*

$$\sum_{i=0} \frac{1}{(R^i)'(R(c))} \neq 0,$$

then R is unstable.

Proof. By hypotheses and Proposition 10 we have that relation (2) cannot be trivial. An application of Proposition 9 finishes the proof of this corollary. □

3. *Proof of Theorem A*

Let $R \in X$ be a rational map. Our aim is to show that the hypothesis of Theorem A implies that $\dim(qc(R)) < 2\deg(R) - 2$ (which is incompatible with structural stability of R).

THEOREM 13. *Let $R \in X$ be a rational map. Then* $\dim(HD(R) \times J_R) < 2d - 2$.

Proof. Let $c_1 \in J(R)$ be a summable critical point. Assume $\dim(HD(R) \times J_R) = 2d - 2$; then R is a structurally stable map. then by Proposition 9 and 10, $\phi(z) = A(z, R, d_1) = 0$ identically on $Y = \mathbb{C}\backslash X_{c_1}$.

To finish the proof of Theorem 13 we need the following.

PROPOSITION 14. *Let $a_i \in \mathbb{C}, a_i \neq a_j$, for $i \neq j$ be a sequence of the points such that $Z = \overline{\bigcup_i a_i} \subset \mathbb{C}$ is a compact set with empty interior. Let $b_i \neq 0$ be complex numbers such that the series $\sum b_i$ is absolutely convergent. Then the function $l(z) = \sum_i b_i/(z - a_i)$ is not identically zero on $Y = \mathbb{C}\backslash Z$ in the following cases:*

(1) *the set Z has zero Lebesgue measure;*
(2) *if the diameters of the components of $\mathbb{C}\backslash Z$ are uniformly bounded from below away from zero; or*
(3) *if O_j denote the components of Y, then $\bigcup_i a_i \in \bigcup_j \partial O_j$.*

Proof. Assume that $l(z) = 0$ on Y. Let us calculate the derivative $\overline{\partial} l$ in the sense of distributions; then $\omega = \overline{\partial} l = \sum_i b_i \delta_{a_i}$ and by standard arguments

$$l(z) = -\int \frac{d\omega(\xi)}{\xi - z}.$$

Since $a_i \neq a_j$ for $i \neq j$, the measure $\omega = 0$ if and only if all coefficients $b_i = 0$.

Let us check (1). Otherwise, in this case we have that the function l is locally integrable and $l = 0$ almost everywhere. Thus, $\omega = \overline{\partial} l = 0$ in the sense of distributions and $\omega = 0$ as a functional on the space of all continuous functions on Z. However, this contradicts the arguments above.

(2) Assume that $l = 0$ identically outside of Z. Let $R(Z) \subset C(Z)$ denote the algebra of all uniform limits of rational functions with poles outside of Z in the sup-topology.

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Here $C(Z)$ denotes as usual the space of all continuous functions on Z with the sup-norm. Then the measure ω induces a linear functional on $R(Z)$. Items (2) and (3) are based on the generalized Mergelyan theorem (see [**Gam**, Theorem 10.4]) which states that '*If diameters of all components of* $\mathbb{C}\setminus Z$ *are bounded uniformly from below away from 0, then every continuous function holomorphic on the interior of* Z *belongs to* $R(Z)$'.

Let us show that ω annihilates the space $R(Z)$. Indeed, let $r(z) \in R(Z)$ be a rational function and γ a curve enclosing Z close enough to Z such that $r(z)$ does not have poles in the interior of γ. Then since $l = 0$ outside of Z we need only apply Fubini's theorem:

$$\int r(z)\,d\omega(z) = \int d\omega(z)\frac{1}{2\pi i}\int_\gamma \frac{r(\xi)\,d\xi}{\xi - z} = \frac{1}{2\pi i}\int_\gamma r(\xi)\,d\xi \int \frac{d\omega(z)}{\xi - z}$$
$$= \frac{1}{2\pi i}\int_\gamma r(\xi) l(\xi)\,d\xi = 0.$$

By the generalized Mergelyan theorem, we have $R(Z) = C(Z)$ and $\omega = 0$, a contradiction.

Now let us check (3). We claim that $l = 0$ almost everywhere on $\bigcup_i \partial O_i$.

We prove the claim as follows. Let $E \subset \bigcup_i \partial O_i$ be any measurable subset with positive Lebesgue measure. Then the function

$$F_E(z) = \iint_E \frac{dm(\xi)}{\xi - z}$$

is continuous on $\mathbb{C}\setminus\bigcup_i O_i$ and is holomorphic on the interior of $\mathbb{C}\setminus\bigcup_i O_i$. Again, by the generalized Mergelyan theorem, $F_E(z)$ can be approximated on $\mathbb{C}\setminus\bigcup_i O_i$ by functions from $R\big(\mathbb{C}\setminus\bigcup_i O_i\big)$; and hence by the above arguments and by our hypotheses we have $\int F_E(z)\,d\omega(z) = 0$. However, another application of Fubini's theorem gives

$$0 = \int F_E(z)\,d\omega(z) = \int d\omega(z)\iint_E \frac{dm(\xi)}{\xi - z}$$
$$= \iint_E dm(\xi)\int d\omega(z)\frac{1}{\xi - z} = \iint_E l(\xi)\,dm(\xi).$$

Hence, for any measurable $E \subset \bigcup_i \partial O_i$, we have $\iint_E l(z) = 0$. This proves the claim.

Now for any component $O \in Y$ and any measurable $E \subset \partial O$ we have $\iint_E l(z) = 0$. By assumption $l = 0$ almost everywhere on $\mathbb{C}$, a contradiction. This finishes the proof of the proposition. □

PROPOSITION 15. *Under the hypotheses of Theorem 13, the function* $A(z, R, d_1) = \phi(z) \neq 0$ *identically on* Y *in the following cases:*

(1) *if* $c_1 \notin X_{c_1}$;
(2) *if the diameters of components of* Y *are uniformly bounded from below away from 0;*
(3) *if* $m(X_{c_1}) = 0$, *where* m *is the Lebesgue measure on* $\mathbb{C}$;
(4) *if* $X_{c_1} \subset \bigcup_i \partial D_i$, *where* D_i *are the components of Fatou set.*

Proof. (1) If R is structurally stable then relation (2) is trivial. In particular, $\phi(c_1) = 1/b_1 \neq 0$ and we are done with case (1).

Now assume that the set X_{c_1} is bounded. Then by Proposition 11 we have that $\phi(z) = \psi(z)$. Hence, by Proposition 14, we are finished with the last three cases.

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Now let X_{c_1} be unbounded. Let $y \in \mathbb{C}$ be a point such that the point $1 - y \in Y$. Then the map

$$g(z) = \frac{yz}{z + y - 1}$$

maps X_{c_1} into $\mathbb{C}$. Consider the function

$$f(z) = \sum \frac{1}{(R^i)'(d_1)(z - g(R^i(d_1)))};$$

by Proposition 14 $f(z) \neq 0$ identically on $g(Y)$.

Now we claim that under the hypotheses of Theorem 9,

$$f(g(z)) = \frac{(z + y - 1)^2}{y(y - 1)}\phi(z) = \frac{\phi(z)}{g'(z)}.$$

We prove the claim as follows. We have for any n

$$\begin{aligned}\frac{1}{g(z) - g(R^n(d_1))} &= \frac{(z + y - 1)(R^n(d_1) + y - 1)}{y(y - 1)(z - R^n(d_1))} \\ &= \frac{1}{y(y - 1)}\left(\frac{(z + y - 1)^2}{z - R^n(d_1)} + 1 - y - z\right),\end{aligned}$$

then

$$\begin{aligned}f(g(z)) &= \sum \frac{1}{(R^i)'(d_1)(g(z) - g(R^i(d_1)))} \\ &= \frac{1}{y(y - 1)}\left((1 - y - z)\sum \frac{1}{(R^i)'(d_1)}\right. \\ &\quad \left. + (z + y - 1)^2 \sum \frac{1}{(R^i)'(d_1)(z - g(R^i(d_1)))}\right) = *\end{aligned}$$

and by Proposition 10 we have

$$* = \frac{(z + y - 1)^2}{y(y - 1)}\phi(z).$$

This proves the claim.

If $\phi = 0$ on Y, then $f = 0$ on $g(Y)$, which contradicts Proposition 14. Proposition 15 is now proved. □

To finish Theorem 13 we use Propositions 9, 10 and 15.

4. *Proof of Theorem B*

Let us formulate again the main observation of this paper.

MAIN OBSERVATION. *Each summable critical point $c_1 \in J(R)$ gives a restriction in the image of the operator β. Namely for all $\mu \in HD(R) \times J_R$, we have*

$$F_\mu(R(c_1))(1 - b_1C_1) = \sum_{i>1} b_iC_iF_\mu(R(c_i)),$$

and hence

$$\beta(\mu)(a) = -R'(a)\sum_{i\geq 2} b_iF_\mu(R(c_i))\left(\gamma_a(c_i) + \frac{C_ib_1}{1 - b_1C_1}\gamma_a(c_1)\right).$$

Thus $\dim(\beta(HD(R) \times J_R)) \leq 2\deg(R) - 3$.

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By the above arguments, we obtain that the image $\beta(HD(R) \times J_R)$ belongs to the solution set of the following system of equations

$$F_\mu(R(c_i))(1 - b_i A(c_i, R, R(c_i))) = \sum_{j \neq i} b_j F_\mu(R(c_j))A(c_j, R, R(c_i)),$$

for all $c_i \in J(R)$, $i = 1, \ldots, k$. Hence if this system is linearly independent, then

$$\dim(\beta(HD(R) \times J_R)) = \dim(qc(R)) - \#\{\text{summable critical points on } J(R)\}$$

and by using injectivity of β we have $J_R = \emptyset$. This complete the proof of the theorem.

Now assume that the system above is linearly dependent. This means that there exist numbers $B_i \neq 0$ such that the function

$$\alpha(z) = \sum B_i A(z, R, R(c_i))$$

is a fixed point for the Ruelle operator R^*. By the arguments of Proposition 10, Lemma 11 and Corollary 12 we obtain that $\alpha(z) = 0$ almost everywhere on $\mathbb{C}$. This means that the measure

$$\frac{\partial \alpha}{\partial \overline{z}} = \sum_i B_i \sum_n \frac{\delta_{R^n(R(c_i))}}{(R^n)'(R(c_i))} = 0.$$

Hence, $B_i = 0$ for all i. This contradiction finishes Theorem B.

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