Ergod. Th. & Dynam. Sys. (2006), **26**, 1235–1256
doi:10.1017/S0143385706000290
© *2006 Cambridge University Press*
Printed in the United Kingdom

On the ergodic theory of synchronized systems

KLAUS THOMSEN

Institut for Matematiske Fag, Ny Munkegade, 8000 Aarhus C, Denmark
(e-mail: matkt@imf.au.dk)

(*Received* 29 *September* 2004 *and accepted in revised form* 14 *March* 2006)

Abstract. This is a study of the invariant measures of a synchronized subshift (in particular, a sofic shift). Among other results we give necessary and sufficient conditions for the existence of a maximal measure of full support, and a method to determine if the shift is intrinsically ergodic.

1. *Introduction*

This is a study of the set of shift-invariant Borel probability measures $M(X)$ on a synchronized system X, i.e. a transitive shift space which has synchronized words. The emphasis is put on the existence and entropy of measures of full support, and on the ergodic measures of maximal entropy. Thanks to the work of Doris and Ulf-Rainer Fiebig [**FF**], synchronized systems admit a canonical presentation via labeled graphs in much the same way as sofic shifts, and this makes these shift spaces amenable to the approach developed by the author in [**Th1**]. It turns out that the structure which was described in most detail for sofic shift spaces in [**Th1**] is also fundamental for the understanding of the ergodic properties. For general synchronized systems, however, a part of the structure is hidden in a canonical subshift which is empty for sofic shift spaces. We call this *the Markov boundary* of the synchronized system X and denote it by $\partial_M X$. The Markov boundary can be defined for arbitrary shift spaces, but among the transitive shift spaces it is only the non-sofic synchronized shift spaces for which it is not trivial, i.e. neither empty nor the whole shift.

The main results can be listed as follows. Let X be a synchronized system and G be the Fischer graph of X. Then the following results hold.

(a) We have $h(X) = \max\{h(\partial_M X), h(G)\} = \max\{h(\partial X), h(G)\}$, where $h(G)$ is the Gurevič entropy of G and ∂X is the derived shift space of X.

(b) The Gurevič entropy $h(G) = \sup\{h_\mu(X) : \mu \in M(X) \text{ has full support}\}$. This supremum is attained (i.e. is a maximum) if and only if G is positive recurrent.

(c) The synchronized system X has a maximal measure of full support if and only if G is positive recurrent and $h(X) = h(G)$.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

(d) Assume that $h(\partial_M X) < h(G)$. Then G is strongly positive recurrent, and there is an $\epsilon > 0$ such that every ergodic measure $\mu \in M(X)$ with $h_\mu(X) \geq h(X) - \epsilon$ is the push-forward of an ergodic measure ν on Σ_G with $h_\nu(\Sigma_G) = h_\mu(X)$. In particular, X is intrinsically ergodic.

(e) Assume that $h(\partial_M X) = h(G)$, and that G is positive recurrent. Then the maximal ergodic measures on X are the maximal ergodic measures on the Markov boundary $\partial_M X$ and the push-forward of the maximal measure on Σ_G. In particular, X is *not* intrinsically ergodic.

(f) Assume that $h(\partial_M X) = h(G)$, and that G is not positive recurrent. Then the maximal ergodic measures on X are the maximal ergodic measures on the Markov boundary $\partial_M X$. In particular, X is intrinsically ergodic if and only if $\partial_M X$ is.

(g) Assume that $h(\partial_M X) > h(G)$. Then every ergodic measure $\mu \in M(X)$ of entropy $h_\mu(X) > h(G)$ is supported on the Markov boundary $\partial_M X$. In particular, X is intrinsically ergodic if and only if $\partial_M X$ is.

A major part of the proof is a study of the irreducible components of X, in the sense of [**Th1**], that are not subsets of the Markov boundary. These components are called *the visible components*. It turns out that for any ergodic measure μ in $M(X)$ which is not supported by the Markov boundary of X, there is a visible component X_c of X such that $\mu(\tilde{X}_c) = 1$, where $\tilde{X}_c$ is a Borel subset of the forward and backward recurrent points in the closure $\overline{X_c}$ of X_c. Together with the Markov boundary, these sets give rise to a partition of X into disjoint shift-invariant Borel sets modulo a set which is a null set for every element of $M(X)$, and we obtain therefore a corresponding decomposition of every measure in $M(X)$, cf. Theorem 5.6. The results (a)–(g) are obtained by combining this result on the measures in $M(X)$ with a study of the visible components and their Fischer graphs.

2. *The Markov boundary of a shift space*

Let X be a shift space on the finite alphabet $\mathcal{A}$ and $\mathbb{W}(X)$ the language of X, i.e. $\mathbb{W}(X)$ consists of the finite words in the alphabet $\mathcal{A}$ which can occur in an element of X. For every $w \in \mathbb{W}(X)$, let $P(w)$ be the *predecessor set* of w, i.e. $P(w) = \{u \in \mathbb{W}(X) : uw \in \mathbb{W}(X)\}$, and let $F(w)$ be the *follower set* of w, i.e. $F(w) = \{u \in \mathbb{W}(X) : wu \in \mathbb{W}(X)\}$. Set

$$\mathbb{W}_\infty(X) = \{w \in \mathbb{W}(X) : \#\{F(xw) : x \in P(w)\} = \infty\}.$$

LEMMA 2.1. *$\mathbb{W}_\infty(X)$ is the language of a shift space $\partial_M X \subseteq X$.*

Proof. By Proposition 1.3.4 of [**LM**] we must show that $\mathbb{W}_\infty(X)$ is a factorial and prolongable language. Let $w \in \mathbb{W}_\infty(X)$. For each letter $a \in \mathcal{A}$ let $P_a(w)$ denote the set of elements of $P(w)$ which have a as a suffix. Then $\bigcup_{a\in\mathcal{A}} P_a(w) = P(w)$ and since $\#\{F(xw) : x \in P(w)\} = \infty$ there must be a letter $a \in \mathcal{A}$ such that $\#\{F(xw) : x \in P_a(w)\} = \infty$. It follows that $aw \in \mathbb{W}_\infty(X)$, showing that elements of $\mathbb{W}_\infty(X)$ are prolongable to the left. To see that w can also be prolonged to the right, note that $F(xw) = \{aF(xwa) : a \in \mathcal{A},\ xwa \in \mathbb{W}(X)\}$ for all $x \in P(w)$. Since $\#\{F(xw) : x \in P(w)\} = \infty$ this equality shows that, at least for one $a \in \mathcal{A}$, the set $\{F(xwa) : x \in P(wa)\}$ must be infinite, showing that $wa \in \mathbb{W}_\infty(X)$. It follows that $\mathbb{W}_\infty(X)$ is prolongable.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

To see that $\mathbb{W}_\infty(X)$ is factorial, let $u \subseteq w \in \mathbb{W}_\infty(X)$. There are, possibly empty, words $a, b \in \mathbb{W}(X)$ such that $w = aub$. When $x \in P(w)$, it follows that $xa \in P(u)$ and $\{z \in F(xau) : b \text{ is a prefix of } z\} = bF(xw)$. This shows that $\#\{F(yu) : y \in P(u)\} \geq \#\{F(xau) : x \in P(w)\} = \infty$ since $\#\{F(xw) : x \in P(w)\} = \infty$. It follows that $u \in \mathbb{W}_\infty(X)$. □

The shift space $\partial_M X$ of Lemma 2.1 will be called *the Markov boundary* of X. Although it will not be used in the following, it seems appropriate to point out the following alternative description of the Markov boundary:

$$W_\infty(X) = \{w \in \mathbb{W}(X) : \#\{P(wx) : x \in F(w)\} = \infty\}.$$

The identity stresses the naturalness of the Markov boundary by showing that it respects 'time reversal'. Furthermore, it is useful for the comparison of this paper with the approach of Buzzi in [**Bu**].

The next lemma shows in which way a factor map must respect the Markov boundary. For the proof we point out that the number of letters in a word $w \in \mathbb{W}(X)$ is its *length* and that it will be denoted by $|w|$. The set of words in X of length n will be denoted by $\mathbb{W}_n(X)$.

LEMMA 2.2. *Let $\psi : Y \to X$ be a factor map between shift spaces, i.e. X and Y are shift spaces and ψ is a continuous shift-commuting surjection. Then $\partial_M X \subseteq \psi(\partial_M Y)$.*

Proof. Since ψ is continuous and shift-commuting there is an $n \in \mathbb{N}$, a map $\tilde{\psi} : \mathbb{W}_n(Y) \to \mathbb{W}_1(X)$ and an $m \in \mathbb{Z}$ such that, in standard notation, $(\psi \circ \sigma^m(y))_i = \tilde{\psi}(y_{[i,i+n[})$, for all $i \in \mathbb{Z}$ and all $y \in Y$, cf. [**LM**]. The map $\tilde{\psi}$ extends to maps $\tilde{\psi} : \mathbb{W}_k(Y) \to \mathbb{W}_{k-n+1}(X)$, $k \geq n$, in the natural way. Furthermore, for every $z \in \mathbb{W}_k(Y), k \geq n$, we can define a map $\psi_z : F(z) \to F(\tilde{\psi}(z))$ such that $\tilde{\psi}(z)\psi_z(a) = \tilde{\psi}(za)$, $a \in F(z)$. Let $w \in \mathbb{W}(X)$ and consider $x \in P(w)$. For each $u \in \tilde{\psi}^{-1}(w) \subseteq \mathbb{W}_{|w|+n-1}(Y)$, set $A(x,u) = \{v \in P(u) : vu \in \tilde{\psi}^{-1}(xw)\}$. It is easy to see that because ψ is surjective we have the equality

$$F(xw) = \bigcup_{u \in \tilde{\psi}^{-1}(w)} \left(\bigcup_{v \in A(x,u)} \psi_{vu}(F(vu)) \right). \tag{2.1}$$

Now observe that the map ψ_{vu} is the restriction of ψ_u to $F(vu)$. Because of this we see from (2.1) that $\{F(vu) : v \in P(u)\}$ must be infinite for at least one $u \in \tilde{\psi}^{-1}(w)$ when $w \in \mathbb{W}_\infty(X)$. Hence $\tilde{\psi}^{-1}(w) \cap \mathbb{W}_\infty(Y) \neq \emptyset$ when $w \in \mathbb{W}_\infty(X)$. This implies that $\partial_M X \subseteq \psi(\partial_M Y)$. □

It follows from Lemma 2.2 that conjugate shift spaces have conjugate Markov boundaries.

The following observation shows that the Markov boundary is only relevant beyond the sofic case.

THEOREM 2.3. *Let X be a shift space. Then X is sofic if and only if $\partial_M X = \emptyset$.*

Proof. This follows straightforwardly from the well-known fact, cf. e.g. Theorem 3.2.10 of [**LM**], that X is sofic if and only if $\{F(w) : w \in \mathbb{W}(X)\}$ is finite. □

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

3. *The Markov boundary of a synchronized system*

Recall that a shift space X is *transitive* when X contains an element x whose forward orbit $\{\sigma^j(x) : j \in \mathbb{N}\}$ is dense in X. This happens if and only if any pair of words $w, u \in \mathbb{W}(X)$ can be connected in the sense that there is a word $v \in \mathbb{W}(X)$ such that $wvu \in \mathbb{W}(X)$. A *synchronized system* is a transitive shift space X which contains a *synchronizing word*, i.e. a word $v \in \mathbb{W}(X)$ such that $uv, vw \in \mathbb{W}(X) \Rightarrow uvw \in \mathbb{W}(X)$. The first objective in this section is to prove the following theorem.

THEOREM 3.1. *A transitive shift space X is synchronized if and only if $\partial_M X \neq X$.*

By definition $F(xv) = F(v)$ for all $x \in P(v)$ when v is a synchronized word, so it is clear that $\partial_M X \neq X$ when X is a synchronized system. For the proof of the other implication we need to introduce some terminology and notation which will also be fundamental in the rest of the paper.

In the following *a graph*, $G = (V, E)$, will always be an oriented graph with at most countably many vertices, V, and at most countably many edges, E. An edge will be referred to as an *arrow*, e, going from the *start vertex* (or *source*), $s(e) \in V$, to the *terminal vertex* (or *target*), $t(e) \in V$. We will almost exclusively work with graphs which have uniformly bounded out-degree, i.e. graphs with the property that there is a uniform bound to the number of arrows starting at a given vertex. By a *finite path* γ in G we mean a finite sequence, $\gamma = (e_1, e_2, \ldots, e_n)$, of arrows $e_i \in E$ such that $s(e_{i+1}) = t(e_i), i = 1, 2, \ldots, n-1$. Here n is the *length* of γ, and we denote the set of paths of length n in G by $\mathbb{W}_n(\Sigma_G)$. The start vertex $s(\gamma)$ and the terminal vertex $t(\gamma)$ of the path γ are the vertices $s(e_1)$ and $t(e_n)$, respectively. Left-infinite, right-infinite and bi-infinite paths in G are defined similarly. The graph G is *connected* when every pair of vertices in G can be connected by a finite path, i.e. for all $v, w \in V$ there is a $\gamma \in \bigcup_n \mathbb{W}_n(\Sigma_G)$ such that $s(\gamma) = v$ and $t(\gamma) = w$.

The set Σ_G of bi-infinite paths in G is a subset of the topological product space $E^{\mathbb{Z}}$. When we equip E with the metric d_0 determined by the condition that $d_0(e, f) = 1$ when $e \neq f$, we obtain a metric d on $E^{\mathbb{Z}}$, and hence also on Σ_G, such that

$$d((e_i)_{i\in\mathbb{Z}}, (f_i)_{i\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} 2^{-|i|} d_0(e_i, f_i).$$

In the general case Σ_G is neither compact nor locally compact in the topology given by this metric, but it is always a Polish space, i.e. a separable and complete metric space. Note that the shift σ, given by $\sigma((e_i)_{i\in\mathbb{Z}})_j = e_{j+1}$, is a homeomorphism of Σ_G, and we can consider (Σ_G, σ) as a dynamical system.

A *labeled graph*, $G = (V, E, L)$, is a graph $G = (V, E)$, together with a labeling map $L : E \to \mathcal{A}$. It defines a continuous map $L : \Sigma_G \to \mathcal{A}^{\mathbb{Z}}$ in the natural way: $L((e_i)_{i\in\mathbb{Z}}) = (L(e_i))_{i\in\mathbb{Z}}$. Similarly, the labeling takes finite paths in G to words in the alphabet $\mathcal{A}$, as well as left- or right-infinite paths in G to left- or right-infinite words, in a self-explanatory way. The closure $\overline{L(\Sigma_G)}$ of $L(\Sigma_G)$ in $\mathcal{A}^{\mathbb{Z}}$ is then a shift space. We say that the labeled graph $G = (V, E, L)$ *presents* $\overline{L(\Sigma_G)}$. Note that the words $\mathbb{W}(\overline{L(\Sigma_G)})$ of $\overline{L(\Sigma_G)}$ are the labels of finite paths in G which sit as a piece of a bi-infinite path in G. If G has no sources or sinks, in particular, if G is connected, then $\mathbb{W}(\overline{L(\Sigma_G)}) = \bigcup_{n\in\mathbb{N}}\{L(\gamma) : \gamma \in \mathbb{W}_n(\Sigma_G)\}$.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

The labeled graph $G = (V, E, L)$ is *right-resolving* when $e, f \in E$, $s(e) = s(f)$, $L(e) = L(f) \Rightarrow e = f$, and *follower separated* when different vertices have different follower sets, where the *follower set* $F(v)$ of the vertex $v \in V$ is the set $F(v) = \bigcup_{n\in\mathbb{N}}\{L(\gamma) : \gamma \in \mathbb{W}_n(\Sigma_G), s(\gamma) = v\}$. A word $w \in \mathbb{W}(\overline{L(\Sigma_G)})$ is said to be *magic* for G when all finite paths in G with label w terminate at the same vertex, i.e. when $\#\{t(\gamma) : \gamma \in \mathbb{W}_{|w|}(\Sigma_G),\ L(\gamma) = w\} = 1$. We will say that w is *essentially magic* when the terminal vertices of paths in G with label w all have the same follower set.

LEMMA 3.2. *Let X be a transitive shift space such that $\partial_M X \neq X$. There is a countable labeled graph $H = (V, E, L)$ which is connected, right-resolving, follower separated, presents X and has a magic word.*

Proof. Let G_X be the follower set graph of X, as described for example in [**LM**, pp. 73–74]. Then G_X is a labeled graph, the vertices of G_X are $\{F(w) : w \in \mathbb{W}(X)\}$ and there is an arrow with label $a \in \mathcal{A}$ from $F(w)$ to $F(v)$, viz. $F(w) \xrightarrow{a} F(v)$, if and only if $a \in F(w)$ and $F(v) = F(wa)$. Note that the labeling is right-resolving. Since $\partial_M X \neq X$ there is a word $w \in \mathbb{W}(X)\backslash\mathbb{W}_\infty(X)$. Let $u_1, u_2, u_3, \ldots$ be a list of the words of X, each word repeated infinitely many times. It follows from the transitivity of X that we can find words $z_i, y_i \in \mathbb{W}(X)$ such that

$$wz_1u_1y_1wz_2u_2y_2w \ldots wz_nu_ny_nw \in \mathbb{W}(X)$$

for all $n \in \mathbb{N}$. There is then a right-infinite path γ in G_X which starts at $F(w)$ and has the sequence $z_1u_1y_1wz_2u_2y_2w \ldots$ as its label. Since $w \notin \mathbb{W}_\infty(X)$, the set

$$A_0 = \{F(wz_1u_1y_1wz_2u_2y_2w \ldots wz_nu_ny_nw) : n \in \mathbb{N}\}$$

is a finite set of vertices in G_X. Let A be the subset of A_0 consisting of the elements which are visited infinitely many times by γ. Then A is finite and not empty. Let H' be the connected component of G_X containing one (and hence all) element(s) of A. Then H' is a countable connected graph and the labeling of G_X restricts to a right-resolving labeling L' of H' such that $X = \overline{L'(\Sigma_{H'})}$ because all words of X occur infinitely many times in $\{u_n\}$. Since $\#\{F(xw) : x \in P(w)\} < \infty$, there is a finite set A_1 of vertices in H' with the property that any finite path in H' with label w must terminate at one of the vertices in this finite set, A_1 (A_1 contains A, but may be bigger). Let H be the merged graph of H', as defined [**LM**, p. 78]. Then H is a countable connected graph with a right-resolving and follower separated labeling L such that $\overline{L(\Sigma_H)} = \overline{L'(\Sigma_{H'})} = X$, cf. Lemma 3.3.8 of [**LM**]. It follows from the definition of the merged graph that, when A_2 denotes the image of the vertices in A_1 under the graph map $H' \to H$ which identifies vertices in H' with the same follower set, any finite path in H with label w must terminate at one of the vertices in this finite set, A_2. It follows then from the proof of Proposition 3.3.16 of [**LM**] that we can prolong w to the right by a word $x \in \mathbb{W}(X)$ to obtain the word $wx \in \mathbb{W}(X)$ which is magic for H. □

Theorem 3.1 follows from Lemma 3.2 because a magic word for a connected labeled graph is obviously synchronizing for the shift space it presents.

By combining Theorems 2.3 and 3.1 we conclude that among the transitive shift spaces the Markov boundary is only an interesting object for synchronized systems that are not sofic.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

Let $G = (E, V, L)$ be a labeled graph. When $F \subseteq V$ is a finite set of vertices and $n \in \mathbb{N}$, let $\mathbb{W}_n(\Sigma^L_{G\setminus F})$ be the labels of paths in G of length n which avoid F, i.e.

$$\mathbb{W}_n(\Sigma^L_{G\setminus F}) = \{L(\gamma) : \gamma = (e_1, e_2, \ldots, e_n) \in \mathbb{W}_n(\Sigma_G),\ s(e_i), t(e_i) \notin F, i = 1, \ldots, n\}.$$

Set

$$\mathbb{W}_n(\Sigma^L_G(\infty)) = \bigcap_F \mathbb{W}_n(\Sigma^L_{G\setminus F}),$$

where we take the intersection over all finite set of vertices F in G. Then $\mathbb{W}(\Sigma^L_G(\infty)) = \bigcup_n \mathbb{W}_n(\Sigma^L_G(\infty))$ is a factorial language, i.e. a subword of a word in $\mathbb{W}(\Sigma^L_G(\infty))$ is itself in $\mathbb{W}(\Sigma^L_G(\infty))$. This property implies that

$$\#\mathbb{W}_{n+k}(\Sigma^L_G(\infty)) \leq \#\mathbb{W}_n(\Sigma^L_G(\infty))\#\mathbb{W}_k(\Sigma^L_G(\infty))$$

for all n, k, and it follows that the limit

$$h_\infty(\Sigma^L_G) = \lim_{n\to\infty} \frac{1}{n} \log(\#\mathbb{W}_n(\Sigma^L_G(\infty)))$$

exists and equals $\inf_n (1/n) \log(\#\mathbb{W}_n(\Sigma^L_G(\infty)))$. In general $\mathbb{W}(\Sigma^L_G(\infty))$ is not a prolongable language, not even when G is connected and the labeling is right-resolving, so in general $\mathbb{W}(\Sigma^L_G(\infty))$ is not the language of a shift space. But we have the following lemma.

LEMMA 3.3. *Let $G = (V, E, L)$ be a labeled graph which is connected, right-resolving and follower separated. Then $\mathbb{W}(\Sigma^L_G(\infty))$ is the language of a shift space $\Sigma^L_G(\infty)$ such that $\partial_M \overline{L(\Sigma_G)} \subseteq \Sigma^L_G(\infty) \subseteq \overline{L(\Sigma_G)}$.*

Proof. We may assume that G has infinitely many vertices. By Proposition 1.3.4 of [**LM**] we must show that $\Sigma^L_G(\infty)$ is factorial and prolongable. As we have already used, the language $\mathbb{W}(\Sigma^L_G(\infty))$ is clearly factorial. We will show that it is prolongable to the left because the labeling is right-resolving and to the right because it is also follower separated. Consider a word $w \in \mathbb{W}(\Sigma^L_G(\infty))$, and a finite set B of vertices in G. It suffices to find letters $a, b \in \mathcal{A}$ and paths γ' and γ'' in G which do not visit B and have labels aw and wb, respectively. To this end, note first that because G is right-resolving there is a finite set B_1 of vertices such that $B \subseteq B_1$ and no vertex outside B_1 is the terminal vertex of an arrow starting in B. Since $w \in \mathbb{W}(\Sigma^L_G(\infty))$, there is a sequence $\gamma_i, i = 1, 2, \ldots$, of paths in G with different terminal vertices such that $L(\gamma_i) = w$, and γ_i avoids B_1 for all i. Since G is connected, so that all vertices have at least one incoming arrow, and since $\mathcal{A}$ is finite, at least one letter a must be the label of an arrow going in to the starting vertex of one of the γ_i. Then aw is the label of a path γ' in G which avoids B. To prove the existence of γ'', it suffices to show that at least one of the γ_i can be prolonged by an arrow which does not terminate in B. Assume not. Then the follower set of any of the terminal vertices of the γ_i is a union

$$\bigcup_{a\in A, v\in B_0} aF(v)$$

for some set of letters $A \subseteq \mathcal{A}$ and some set of vertices $B_0 \subseteq B$. This gives only a finite number of possible follower sets, contradicting that the terminal vertices of the γ_i are different and G is follower separated. With this contradiction we have shown that

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

$\mathbb{W}(\Sigma_G^L(\infty))$ is the language of a shift space $\Sigma_G^L(\infty)$. The inclusion $\Sigma_G^L(\infty) \subseteq \overline{L(\Sigma_G)}$ follows because G is connected.

To show that the Markov boundary of $\overline{L(\Sigma_G)}$ is contained in $\Sigma_G^L(\infty)$, consider a word $w \in \mathbb{W}(\overline{L(\Sigma_G)}) \backslash \mathbb{W}(\Sigma_G^L(\infty))$. There is then a finite set K of vertices in G such that every path in G with label w terminates in K. For any predecessor $x \in P(w)$ the follower set $F(xw)$ is then the intersection $\bigcap_{v \in K_x} F(v)$, for some subset $K_x \subseteq K$. This gives only a finite number of possibilities, so we conclude that $\#\{F(xw) : x \in P(w)\} < \infty$, i.e. $w \in \mathbb{W}(\overline{L(\Sigma_G)}) \backslash \mathbb{W}_\infty(\overline{L(\Sigma_G)})$. It follows that $\partial_M \overline{L(\Sigma_G)} \subseteq \Sigma_G^L(\infty)$. □

Let X be a synchronized system. A *Fischer cover* of X is a countable connected labeled graph $H = (V, E, L)$ such that:

(1) $\overline{L(\Sigma_H)} = X$, i.e. H presents X;
(2) L is right-resolving;
(3) H follower separated; and
(4) a word $w \in \mathbb{W}(X)$ which is synchronizing for X is magic for H.

It was shown in [**FF**] that every synchronized system X has a Fischer cover, and that a Fischer cover is unique up to an isomorphism of labeled graphs. We may therefore talk about *the* Fischer cover of X. The graph G itself (without the labeling) will be called *the Fischer graph* of X.

The uniqueness of the Fischer cover can be established exactly as in the sofic case, cf. [**FF**] or Proposition 3.3.17 of [**LM**]. To construct it one can proceed as in the proof of Lemma 3.2 with w a synchronizing word, i.e. the Fischer cover is the merged graph of the irreducible component of the follower set graph which contains the follower set of one (and hence all) synchronizing words. By realizing the Fischer cover in this way it is apparent that, for every word w of X which is not in $\mathbb{W}_\infty(X)$, there is a finite set of vertices in the Fischer cover where any path with label w must terminate. By combining this observation with Lemma 3.3 we obtain the following result.

THEOREM 3.4. *Let X be a synchronized system and $G = (V, E, L)$ the Fischer cover of X. Then $\partial_M X = \Sigma_G^L(\infty)$.*

This identification of $\partial_M X$ will be used tacitly in the following.

4. *The Markov boundary and the derived shift spaces*

Recall that the derived shift spaces of a shift space X was introduced in [**Th1**] as follows. The set of periodic points in X is denoted by $\operatorname{Per} X$, and $R(X) = \overline{\operatorname{Per} X}$ is the closure of $\operatorname{Per} X$. The *derived shift space* of X is

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\}.$$

The derived shift space ∂X is itself a shift space, a subshift of X, and the construction can be iterated:

$$\partial^j X = \partial(\partial^{j-1} X),$$

$j \geq 2$. Thus we have a (possibly trivial) sequence

$$\partial^0 X = X \supseteq \partial X \supseteq \partial^2 X \supseteq \partial^3 X \supseteq \cdots$$

of subshifts inside X, the derived shift spaces of X.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

Let X be a synchronized system. We will relate the Markov boundary of X to the derived shift spaces of X. For this purpose we need a version of Proposition 6.1 of [**Th1**] for synchronized systems.

Let $G = (V, E, L)$ be a right-resolving labeled graph. For each $k = 1, 2, 3, \ldots$, we define a labeled graph $G_k^{\#} = (V_k, E_k, L)$ in the following way. The vertices V_k are the subsets F of vertices in G of cardinality k, and there is an arrow labeled $a \in \mathcal{A}$ from $F \in V_k$ to $F' \in V_k$ when $F' = \{t(e) : e \in E,\ L(e) = a,\ s(e) \in F\}$. The disjoint union $\bigcup_{j\geq 1} G_j^{\#}$ is then a right-resolving labeled graph.

A graph H is *non-wandering* when it is the union of its connected components, and a subgraph $H_0 \subseteq H$ of H is a *non-wandering subgraph* when it is a union of connected components of H.

LEMMA 4.1. *Let X be a synchronized system with Fischer cover $G = (V, E, L)$. For each $k \in \mathbb{N}$ there is a non-wandering subgraph H_{k+1} of $\bigcup_{j\geq k+1} G_j^{\#}$ such that*

$$R(\partial^k X) \cup R(\partial_M X) = R(\partial_M X) \cup \overline{L(\Sigma_{H_{k+1}})}. \tag{4.1}$$

Furthermore, the graph H_{k+1} can be chosen to be component reduced in the sense that there is no connected component A of H_{k+1} for which $L(\Sigma_A) \subseteq R(\partial_M X) \cup \overline{L(\Sigma_{H_{k+1}\setminus A})}$. With such a choice, a word $w \in \mathbb{W}(R(\partial^k X))\setminus\mathbb{W}(R(\partial_M X))$ is synchronizing for $R(\partial^k X)$ if and only if it is essentially magic for H_{k+1}.

Proof. The proof is by induction on k. To start the induction, note that $G_1^{\#} = G$ is a non-wandering subgraph of $\bigcup_k G_k^{\#}$, and that $R(\partial^0 X) \cup R(\partial_M X) = X = R(\partial_M X) \cup \overline{L(\Sigma_G)}$. This yields the $k = 0$ case of the lemma. To handle the induction step, let H_k be a non-wandering subgraph of $\bigcup_{j\geq k} G_k^{\#}$ such that $R(\partial^{k-1} X) \cup R(\partial_M X) = R(\partial_M X) \cup \overline{L(\Sigma_{H_k})}$. By removing redundant components from H_k we may assume that H_k is component reduced. For each vertex N in H_k, we let $F(N)$ denote the follower set $F(N) \in 2^{\mathbb{W}(X)}$ defined by H_k. Let H'_{k+1} be the subgraph of $\bigcup_{j\geq 1} G_j^{\#}$ obtained by removing all vertices except those M for which there are subsets N_1, N_2 in M, both of which are vertices in H_k and $F(N_1) \neq F(N_2)$. Since $N_1 \neq N_2$, and both sets contain at least k elements, $H'_{k+1} \subseteq \bigcup_{j\geq k+1} G_j^{\#}$. We let H_{k+1} be the disjoint union of the connected components of H'_{k+1}. The same argument as in the proof of Proposition 6.1 in [**Th1**] shows that H_{k+1} is a non-wandering subgraph of $\bigcup_{j\geq k+1} G_j^{\#}$.

Let $w \in \mathbb{W}(\partial^k X)$, and assume that $w \notin \mathbb{W}(\partial_M X)$. Then $w \in \mathbb{W}(\overline{L(\Sigma_{H_k})})$. Furthermore, there must be two paths in H_k labeled w which terminate in vertices of H_k with different follower sets. Otherwise w would be synchronizing for $R(\partial^{k-1} X)$, and this is not the case since $w \in \mathbb{W}(\partial^k X)$. By taking the union of the vertices from $\bigcup_{j\geq k} G_j^{\#}$ occurring in the paths, both labeled w and terminating in vertices with different follower sets, we obtain a path in H'_{k+1} labeled w. Thus every $w \in \mathbb{W}(\partial^k X)\setminus\mathbb{W}_\infty(X)$ is the label of a path in H'_{k+1}. Consider then a periodic point $x \in \mathrm{Per}(\partial^k X)$ which is not in $\partial_M X$. Write $x = p^\infty$, where $p \in \mathbb{W}_{\mathrm{period}(x)}(\partial^k X)$. For some $N \in \mathbb{N}$, $p^N \notin \mathbb{W}_\infty(X)$, so the number of vertices in H'_{k+1} which are the terminal vertex of a path in H'_{k+1} with label p^N must be finite. As we have just shown there is a path in H'_{k+1} with label p^{Ni} for all $i \in \mathbb{N}$, it follows that there is a loop in H'_{k+1} with label p^{Nj} for some $j \in \mathbb{N}$,

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

and we conclude that $p^\infty \in L(\Sigma_{H_{k+1}})$. Thus $\operatorname{Per}\partial^k X \subseteq \operatorname{Per}\partial_M X \cup L(\Sigma_{H_{k+1}})$ and $R(\partial^k X) \subseteq R(\partial_M X) \cup \overline{L(\Sigma_{H_{k+1}})}$.

To prove the desired equality, (4.1), it suffices now to show that $\overline{L(\Sigma_{H_{k+1}})} \subseteq R(\partial^k X) \cup R(\partial_M X)$, and since H_{k+1} is non-wandering it suffices to consider a loop γ_0 in H_{k+1} and show that $L(\gamma_0)^\infty \in \partial_M X \cup \partial^k X$. This will be done by contradiction, so assume that $L(\gamma_0)^\infty \notin \partial_M X \cup \partial^k X$. There is then an n so large that $w = L(\gamma_0)^n \notin \mathbb{W}_\infty(X)$ and w is synchronizing for $R(\partial^{k-1}X)$. By definition of H_{k+1} there are paths w_1, w_2 in H_k, both labeled w, which terminate in vertices, v_1 and v_2, respectively, with $F(v_1) \neq F(v_2)$. Assume that $F(v_2)$ contains a word b which is not in $F(v_1)$. (If no such word exists, we interchange the roles of w_1 and w_2.) Since H_k is non-wandering the vertices of w_1 are contained in the same connected component A of H_k. Since H_k is component reduced, $L(\Sigma_A)$ is not contained in $\partial_M X \cup \overline{L(\Sigma_{H_k \setminus A})}$. There is therefore a path x_0 in A such that $L(x_0) \notin \mathbb{W}(\overline{L(\Sigma_{H_k \setminus A})}) \cup \mathbb{W}_\infty(X)$. In particular, since $L(x_0) \notin \mathbb{W}_\infty(X)$, the number of terminal vertices of paths with label $L(x_0)$ in A is finite. We can then use the procedure from Proposition 3.3.16 of [**LM**] to prolong $L(x_0)$ to a word $L(x_0)z$ such that $L(x_0)z = L(x)$, where x is a path in A with the property that any other path in A with label $L(x)$ terminates at a vertex with the same follower set as the terminal vertex of x. Since A is connected, there is a path a_0 in A such that the concatenation xa_0w_1 is a path in A. Set $a = L(xa_0)$ and note that $aw = L(xa_0w_1) \in \mathbb{W}(R(\partial^{k-1}X))$. Since $b \in F(w_2)$, then $wb \in \mathbb{W}(R(\partial^{k-1}X))$. We claim that $awb \notin \mathbb{W}(R(\partial^{k-1}X))$. Indeed, if d is a path in H_k with $L(d) = awb$, d must be a path in A since $L(x_0) \subseteq a$. Furthermore, the $|x|$th arrow in d must terminate in a vertex with the same follower set as the terminal vertex of x. Since the labeling is right-resolving it follows that the terminal vertex of the jth vertex in d must have the same follower set as the terminal vertex of the jth arrow in xa_0w_1 when $|x| \leq j \leq |xa_0w_1|$. In particular, when $j = |xa_0w_1|$, this implies that b must be in the follower set of the terminal vertex of w_1. But this is not the case by the choice of b, and hence d cannot exist. Consequently, w cannot be synchronizing for $R(\partial^{k-1}X)$. This contradiction finishes the proof of (4.1).

For the proof of the last statement, note that the preceding argument shows that a word from $\mathbb{W}(R(\partial^{k-1}X)) \setminus \mathbb{W}(R(\partial_M X))$ can only be synchronizing for $R(\partial^{k-1}X)$ when it is essentially magic for H_k, provided H_k is component reduced. Conversely, it follows from the $k-1$ case of (4.1) that an essentially magic word for H_k, which is not a word of $R(\partial_M X)$, must be synchronizing for $R(\partial^{k-1}X)$. □

In the following we will always assume that the graph H_{k+1} of Lemma 4.1 is chosen to be component reduced. It will then be referred to as *the component graph at level k*. Note that H_1 is the Fischer graph of X, and that we have the inclusion

$$\overline{L(\Sigma_A)} \subseteq R(\partial^k X) \tag{4.2}$$

for every connected component A of H_{k+1}.

PROPOSITION 4.2. *In a synchronized system X we have the inclusion $\bigcap_{k=0}^\infty \partial^k X \subseteq \partial_M X$.*

Proof. Since $\partial^{k+1}X \subseteq R(\partial^k X)$, we see from Lemma 4.1 that

$$\bigcap_{k=0}^{\infty} \partial^k X \subseteq R(\partial_M X) \cup \bigcap_{k=1}^{\infty} \overline{L(\Sigma_{H_k})},$$

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

where H_k is the component graph at level $k-1$. Since $\bigcap_{k=1}^{\infty} \overline{L(\Sigma_{H_k})} \subseteq \partial_M X$ by Theorem 3.4, we obtain the stated inclusion. □

The inclusion of Proposition 4.2 is generally a strict inclusion, but can also be an equality. By using the description of the derived shift space of β-shifts given in [**Th2**], β-shifts can be used to give examples of both possibilities.

5. *Invariant measures of a synchronized system*

Let X be a shift space. We denote by $\mathrm{Rec}_{\pm}(X)$ the elements of X that are both forward and backward recurrent under σ, i.e. $\mathrm{Rec}_{\pm}(X)$ consists of the elements $x \in X$ with the property that for all $n \in \mathbb{N}$ the set $\{i \in \mathbb{Z} : x_{[i,i+2n[} = x_{[-n,n[}\}$ is neither bounded below nor bounded above. Observe that $\mathrm{Rec}_{\pm}(X)$ is a shift-invariant Borel subset of X, cf. e.g. the remark [**W**, p. 157]. For any subset $A \subseteq X$, we set $\mathrm{Rec}_{\pm}(A) = A \cap \mathrm{Rec}_{\pm}(X)$.

Let X be a synchronized system and $G = (V, E, L)$ the Fischer cover of X.

LEMMA 5.1. *For each* $j = 0, 1, 2, 3, \ldots,$

$$\mathrm{Rec}_{\pm}(\partial^{j+1}X \backslash \partial_M X) \subseteq R(\partial^{j+1}X).$$

Proof. Let $z \in \mathrm{Rec}_{\pm}(\partial^{j+1}X \backslash \partial_M X)$. Then $z \in R(\partial^j X)$, and it follows from Lemma 4.1 that $z \in \overline{L(\Sigma_{H_{j+1}})}$, where H_{j+1} is the component graph at level j. Since $z \in \partial^{j+1}X$, it follows from Lemma 4.1 that there is no N such that $z_{[-N,N[}$ is essentially magic for H_{j+1}, and since $z \notin \partial_M X$ there is a $K \in \mathbb{N}$ such that $z_{[-K,K[} \notin \mathbb{W}_\infty(X)$. Since z is both forward and backward recurrent, there are sequences $K < j_1 < j_2 < j_3 < \cdots$ and $-K > i_1 > i_2 > i_3 > \cdots$ in $\mathbb{Z}$ such that $z_{[i_k-2K,i_k[} = z_{[-K,K[} = z_{[j_k-2K,j_k[}$ for all k. For each $k \in \mathbb{N}$ there are, under the present assumption, two paths, $(e_i^k)_{i=i_k-2K}^{j_k-1}$ and $(f_i^k)_{i=i_k-2K}^{j_k-1}$, which both have label $z_{[i_k-2K,j_k[}$, but terminate at vertices of H_{j+1} with different follower sets. Since $z_{[-K,K[} \notin \mathbb{W}_\infty(X)$, there are only a finite number of vertices in H_{j+1} which are the terminal vertex of a path in H_{j+1} with label $z_{[-K,K[}$. By passing to subsequences we can therefore arrange that $s(e_{i_k}^m) = s(e_{i_k}^k)$, $s(f_{i_k}^m) = s(f_{i_k}^k)$ and $t(e_{j_k-1}^m) = t(e_{j_k-1}^k)$, $t(f_{j_k-1}^m) = t(f_{j_k-1}^k)$ when $m \geq k$. We can then piece the paths together to get elements $e, f \in \Sigma_{H_{j+1}}$ such that $e_i = e_i^k$ and $f_i = f_i^k$ for all $i = i_k, i_k+1, \ldots, j_k-1$, and all k. By construction the follower sets of $t(e_i)$ and $t(f_i)$ are different for all i, so we see that $z \in L(\Sigma_{H'_{j+2}})$, where H'_{j+2} is the graph used in the proof of Lemma 4.1. In fact $z = L(x)$, where x_i is the edge in H'_{j+2} starting at $s(e_i) \cup s(f_i)$ and terminating at $s(e_{i+1}) \cup s(f_{i+1})$. Note that by construction there are connected components $A \subseteq H'_{j+2}$ and $B \subseteq H'_{j+2}$, and a $T \in \mathbb{N}$ so large that $x_i \in A, i \leq -T$, and $x_i \in B, i \geq T$. Since z is both forward and backward recurrent, this implies that $z \in \overline{L(\Sigma_A)} \cap \overline{L(\Sigma_B)} \subseteq \overline{L(\Sigma_{H_{j+2}})}$. It follows then from Lemma 4.1 that $z \in R(\partial^{j+1}X)$. □

LEMMA 5.2. *For every* $j = 0, 1, 2, \ldots,$ *and every shift-invariant Borel probability measure* μ *on* X,

$$\mu(\partial^j X \backslash \partial_M X) = \mu(R(\partial^j X) \backslash \partial_M X). \tag{5.1}$$

Proof. The result (5.1) is trivial for $j = 0$ since $R(X) = X$. When $j \geq 1$, (5.1) follows from Lemma 5.1 and the Poincaré recurrence theorem. □

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

We recall now the definition of the irreducible components of a shift space X, as introduced in [**Th1**]. For each j, let $\mathbb{S}_j$ be the synchronizing words of $R(\partial^j X)$, and introduce an equivalence relation $\sim$ on $\mathbb{S}_j$ such that $u \sim v$ when $uxv, vyu \in \mathbb{W}(R(\partial^j X))$ for some $x, y \in \mathbb{W}(R(\partial^j X))$. Let $\mathcal{C}_j = \mathbb{S}_j/\sim$. The disjoint union $\bigsqcup_{j=0}^{\infty} \mathcal{C}_j$ will be denoted by $\mathcal{C}$. For each $c \in \mathcal{C}_j$, let X_c be the set of elements x of $R(\partial^j X)$ which 'contains elements from c with bounded gaps', i.e. satisfy that $\sup_{i\in\mathbb{Z}}(\inf\{j - i > 0 : x_{[i,j[} \in c\}) < \infty$. Then X_c is a (usually non-closed) subshift of $R(\partial^j X)$ which is the union of an increasing sequence of irreducible subshifts of finite type, $\overline{X_c}$ is a synchronized system, $\operatorname{Per} \partial^j X = \bigcup_{c\in\mathcal{C}_j} \operatorname{Per} X_c$, and $\overline{X_c} \cap X_{c'} = \emptyset$ when $c \neq c'$, $c, c' \in \mathcal{C}$ (cf. [**Th1**]). The X_c, $c \in \mathcal{C}_j$, are called *the irreducible components of X at level j*. An *irreducible component of X* is then an irreducible component of X at some level.

When $w \in \mathbb{W}(X)$ we set

$$C_w = \bigcup_{j\in\mathbb{Z}} \sigma^j(\{x \in X : x_{[0,|w|[} = w\}).$$

Thus C_w is the open subset of X consisting of the elements which contain the word w. When X_c is an irreducible component of X, set

$$\tilde{X}_c = \bigcup_{v\in c} \operatorname{Rec}_{\pm}(\overline{X_c} \cap C_v).$$

Note that

$$\tilde{X}_c \subseteq R(\partial^j X)\backslash\partial^{j+1} X, \tag{5.2}$$

when X_c is an irreducible component at level j.

LEMMA 5.3. *Let X_c and $X_{c'}$ be different irreducible components in X. Then $\tilde{X}_c \cap \tilde{X}_{c'} = \emptyset$.*

Proof. Assume that $z \in \tilde{X}_c \cap \tilde{X}_{c'}$. It follows from (5.2) that X_c and $X_{c'}$ must be components at the same level. Since z contains words from c as well as c', and is both forward and backward recurrent, it follows from the definition of $\sim$ that $c = c'$. □

LEMMA 5.4. $\operatorname{Rec}_{\pm}(R(\partial^j X)\backslash\partial^{j+1} X) = \bigcup_{c\in\mathcal{C}_j} \tilde{X}_c$ *for all* $j = 0, 1, 2, \ldots$.

Proof. Each $x \in R(\partial^j X)\backslash\partial^{j+1} X$ must contain a word $w \in \mathbb{S}_j$. If in addition x is also both forward and backward recurrent, it follows that $x \in \tilde{X}_c$, where $c \in \mathcal{C}_j$ is the equivalence class containing w. This proves that $\operatorname{Rec}_{\pm}(R(\partial^j X)\backslash\partial^{j+1} X) \subseteq \bigcup_{c\in\mathcal{C}_j} \tilde{X}_c$, and the reversed inclusion follows from (5.2). □

PROPOSITION 5.5. *The equality*

$$X = \partial_M X \cup \bigcup_{c\in\mathcal{C}} (\tilde{X}_c \backslash \partial_M X)$$

holds modulo a Borel set which is a null set for every shift-invariant Borel probability measure on X.

Proof. Note first that it follows from Proposition 4.2 that

$$X = \partial_M X \cup \bigcup_{j=0}^{\infty} \partial^j X \backslash (\partial^{j+1} X \cup \partial_M X).$$

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

And then Lemma 5.2 shows that

$$X = \partial_M X \cup \bigcup_{j=0}^{\infty} R(\partial^j X)\backslash(\partial^{j+1} X \cup \partial_M X)$$

modulo a Borel set which is a null set for every shift-invariant Borel probability measure on X. By Poincaré recurrence

$$\bigcup_{j=0}^{\infty} R(\partial^j X)\backslash(\partial^{j+1} X \cup \partial_M X) = \bigcup_{j=0}^{\infty} \mathrm{Rec}_{\pm}(R(\partial^j X)\backslash(\partial^{j+1} X \cup \partial_M X))$$

modulo a Borel set which is a null set for every shift-invariant Borel probability measure on X, cf. the Remark [**W**, p. 157]. The desired conclusion follows then from Lemma 5.4. □

Observe that the decomposition of X given in Proposition 5.5 is a partition (modulo a universal null set) by Lemma 5.3.

Note that $\tilde{X}_c\backslash\partial_M X \neq \emptyset$ if and only if $\overline{X_c}\backslash\partial_M X \neq \emptyset$ if and only if $\mathrm{Per}\, X_c \not\subseteq \partial_M X$. It follows that, when $\tilde{X}_c\backslash\partial_M X \neq \emptyset$, there is an element of $M(X)$ whose support is contained in X_c. An irreducible component X_c of X will be called *visible* when $\overline{X_c} \not\subseteq \partial_M X$. Then $\mathcal{C}_M = \{c \in \mathcal{C} : \overline{X_c}\backslash\partial_M X \neq \emptyset\}$ is the index set for the visible components of X.

THEOREM 5.6. *Let μ be a shift-invariant Borel probability measure on X. There are then numbers $\lambda_M, \lambda_c \in [0,1]$, and measures $\mu_M, \mu_c \in M(X)$, $c \in \mathcal{C}_M$, such that $\lambda_M + \sum_{c\in\mathcal{C}_M} \lambda_c = 1$, $\mu_M(\partial_M X) = 1$, $\mu_c(\tilde{X}_c) = 1$, $c \in \mathcal{C}_M$, and*

$$\mu = \lambda_M \mu_M + \sum_{c\in\mathcal{C}_M} \lambda_c \mu_c.$$

Furthermore, $h_\mu(X) = \lambda_M h_{\mu_M}(X) + \sum_{c\in\mathcal{C}_M} \lambda_c h_{\mu_c}(X)$.

Proof. The existence of the numbers and the measures follows from the partition of X given by Proposition 5.5. To prove the equality involving entropy, observe that the map $\nu \mapsto h_\nu(X)$ is affine, cf. e.g. Theorem 8.1 of [**W**]. This gives the stated equality when $\{c \in \mathcal{C}_M : \lambda_c > 0\}$ is a finite set. Assume therefore that this is not the case. Consider a finite subset F of $\mathcal{C}_M$, and set $\kappa = 1 - \lambda_M - \sum_{c\in F} \lambda_c$ and $\nu = (1/\kappa)\sum_{c\in\mathcal{C}_M\backslash F} \lambda_c \mu_c$. Then $\nu \in M(X)$ and

$$h_\mu(X) = \lambda_M h_{\mu_M}(X) + \sum_{c\in F} \lambda_c h_{\mu_c}(X) + \kappa h_\nu(X).$$

Since $h_\nu(X) \leq h(X) < \infty$ and κ can be made arbitrarily small by choosing F large, we get the stated equality by approximation. □

Given a synchronized system Y, we will call a measure $\mu \in M(Y)$ *synchronized* when $\mu(Y\backslash\partial Y) = 1$. It follows from Lemma 3.10 of [**Th1**] and Theorem 5.6 that every shift-invariant Borel probability of a synchronized system X is a (possibly infinite) convex combination of a measure supported on the Markov boundary and measures that are synchronized measures for the closure of some visible component of X. The next section is devoted to a study of the latter class of measures.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

6. *The synchronized measures of the visible components*

Let Y be a synchronized system and $H = (V, E, L)$ the Fischer cover of Y. Every Borel probability measure ν on Σ_H gives rise to the measure $\mu = \nu \circ L^{-1}$ on Y. We say that μ is *the push-forward* of ν. Since $L : \Sigma_H \to Y$ is not surjective and Σ_H not compact, the operation does not give us all elements of $M(Y)$, but it does give us all synchronized measures, in an entropy-preserving way, as we shall now show. Set

$$\Sigma_H^{\mathbb{S}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{n\in\mathbb{N}} \sigma^j(\{x \in \Sigma_H : L(x_{[0,n[}) \text{ is synchronizing for } Y\}),$$

which is a dense open subset of Σ_H. In particular, $\mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}})$ is a Borel set in Σ_H.

LEMMA 6.1. *The map $L : \Sigma_H \to Y$ restricts to a homeomorphism*

$$L : \mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}}) \to \mathrm{Rec}_{\pm}(Y\backslash\partial Y).$$

Proof. To see that L is injective on $\mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}})$, consider two elements $x, y \in \mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}})$ such that $L(x) = L(y)$. There is then an $n \in \mathbb{N}$ such that $L(x)_{[n,n[}$ is synchronizing for Y. Since x is recurrent under the inverse shift, there is a sequence $m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ such that $L(x)_{[-m_i,-m_i+2n[} = L(x)_{[-n,n[}$ for all i. Since $L(x) = L(y)$, and the labeling is right-resolving, this implies that $x_k = y_k, k \geq -m_i + 2n$. This is true for all i, so we conclude that $x = y$.

It is clear that $L(x)$ is both forward and backward recurrent since x is, and, as $L(x)$ contains a synchronizing word for Y by definition of $\Sigma_H^{\mathbb{S}}$, we conclude that $L(\Sigma_H^{\mathbb{S}}) \subseteq \mathrm{Rec}_{\pm}(Y\backslash\partial Y)$. To obtain the reversed inclusion, consider $z \in \mathrm{Rec}_{\pm}(Y\backslash\partial Y)$. Since $z \notin \partial Y$, there is a synchronizing word $w \in \mathbb{W}(Y)$ such that $z_{[i,i+|w|[} = w$ for some $i \in \mathbb{Z}$. In fact, since z is both forward and backward recurrent there is a sequence $m_i \in \mathbb{Z}, i \in \mathbb{Z}$, such that $\lim_{i\to\infty} m_i = \infty$, $\lim_{i\to-\infty} m_i = -\infty$ and $z_{[m_i,m_i+|w|[} = w$ for all i. For each $i \geq 0$, consider a path x_i in G with label $L(x_i) = z_{[m_{-i},m_i[}$. Because w is magic for the Fischer cover, and the labeling is right-resolving, the x_i must fit together to form an element $x \in \Sigma_H$ such that $L(x) = z$. Since w is magic for H and the labeling of H is right-resolving, it follows straightforwardly that x is forward and backward recurrent because z is.

The map L is continuous so it remains now only to show that $L^{-1} : \mathrm{Rec}_{\pm}(Y\backslash\partial Y) \to \mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}})$ is continuous. To this end let $V \subseteq \Sigma_H$ be open. We must show that $L(V \cap \mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}}))$ is open in $\mathrm{Rec}_{\pm}(Y\backslash\partial Y)$, so let $x \in V \cap \mathrm{Rec}_{\pm}(\Sigma_H^{\mathbb{S}})$ and set $y = L(x)$. There is then an $n \in \mathbb{N}$ so big that $\{z \in \Sigma_H : z_{[-n,n[} = x_{[-n,n[}\} \subseteq V$ and $L(x)_{[-n,n[}$ is synchronizing for Y. Since x is backward recurrent there is a $k < -3n$ such that $L(x_{[k,k+2n[}) = L(x_{[-n,n[})$. Then

$$\begin{aligned} y \in \{z \in \mathrm{Rec}_{\pm}(Y\backslash\partial Y) : z_{[k,|k|[} = y_{[k,|k|[}\} &\subseteq L(\{u \in \Sigma_H^{\mathbb{S}} : u_{[k+2n,|k|[} = x_{[k+2n,|k|[}\}) \\ &\subseteq L(V \cap \Sigma_H^{\mathbb{S}}), \end{aligned}$$

proving that $L(V \cap \Sigma_H^{\mathbb{S}})$ is open in $\mathrm{Rec}_{\pm}(Y\backslash\partial Y)$, as desired. □

A Borel probability measure μ on Σ_G will be called *synchronized* when $\mu(\Sigma_H^{\mathbb{S}}) = 1$.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

PROPOSITION 6.2. *Let Y be a synchronized system and $H = (V, E, L)$ the Fischer cover of Y. The labeling L induces an affine bijection $\mu \to \mu \circ L^{-1}$ between the synchronized measures on Σ_H and the synchronized measures on Y such that $h_{\mu \circ L^{-1}}(X) = h_\mu(\Sigma_H)$.*

Proof. This follows straightforwardly from Lemma 6.1 by use of the Poincaré recurrence theorem. □

Before we go on to a study of the visible components of a synchronized system, we point out a series of consequences of Proposition 6.2 which are independent of the results of the previous section, in particular concerning the elements of $M(Y)$ of full support. For this we need definitions, terminology and results about infinite graphs coming from the work of several mathematicians, in particular, Vere-Jones and Gurevič. For a recent introduction we refer the reader to [**Ru**]. Let $G = (V, E)$ be a graph. For each pair of vertices $v, u \in V$, let $r_n^G(v, u)$ denote the number of paths of length n which start at v and end at u. Then

$$h(G) = \sup_{u \in V} \left(\limsup_{n \to \infty} \frac{1}{n} \log r_n^G(u, u) \right)$$

is *the Gurevič entropy* of G. When G is connected,

$$h(G) = \limsup_{n \to \infty} \frac{1}{n} \log r_n^G(u, v)$$

for any pair of vertices u, v. In this case $h(G)$ is the supremum of all the measure-theoretic entropies, $h_\mu(\Sigma_G)$, when μ ranges over all the shift-invariant Borel probability measures on Σ_G. A connected graph G is *positive recurrent* when

$$\limsup_{n \to \infty} r_n^G(u, v) e^{-nh(G)} > 0$$

for some and hence all pairs $u, v \in V$, and it is a result of Gurevič [**Gu2**] that there is a shift-invariant Borel probability measure μ on Σ_G with $h_\mu(\Sigma_G) = h(G)$ if and only if G is positive recurrent. When such a maximal measure exists on Σ_G, it is unique and of full support.

For each pair of vertices $v, u \in V$, let $l_n^G(v, u)$ denote the number of paths of length n which start at v, end at u and do not visit u in between. A connected graph is *strongly positive recurrent* when

$$\limsup_{n \to \infty} \frac{1}{n} \log l_n^G(u, u) < \limsup_{n \to \infty} \frac{1}{n} \log r_n^G(u, u)$$

for some (or, equivalently, every) vertex $u \in V$. A strongly positive recurrent graph is positive recurrent and has other nice properties, some of which will be used later on.

LEMMA 6.3. *Let Y be a synchronized system with Fischer graph G. Then $h_\mu(Y) \leq h(G)$ for every synchronized measure μ in $M(Y)$.*

Proof. This follows from Proposition 6.2 and Gurevič's variational principle, cf. [**Gu1**]. □

LEMMA 6.4. *Let Y be a synchronized system with Fischer graph G. There is a synchronized measure $\mu \in M(Y)$ such that $h_\mu(Y) = h(G)$ if and only if G is positive recurrent. When it exists, it is unique and of full support.*

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

Proof. This follows from Proposition 6.2 and Gurevič's work in [**Gu2**]. □

LEMMA 6.5. *Let G be a connected graph with finite Gurevič entropy. There is then a strongly positive recurrent graph $\tilde{G}$ such that $G \subseteq \tilde{G}$, and a graph homomorphism $g : \tilde{G} \to G$ whose restriction to G is the identity on G.*

Proof. Pick a vertex v_0 in G and a loop $\gamma = (e_1, e_2, \ldots, e_N)$ in G with $s(\gamma) = s(e_1) = t(\gamma) = t(e_N) = v_0$. We add to G the vertices $x_1, x_2, \ldots, x_{N-1}$, and the arrows $f_1, f_2, \ldots, f_N$ such that $s(f_1) = t(f_N) = v_0$ while $t(f_i) = x_i, i = 1, 2, \ldots, N-1$, and $s(f_i) = x_{i-1}, i = 2, 3, \ldots, N$. The resulting graph G_1 is connected and we can define a graph homomorphism $g_1 : G_1 \to G$ such that g_1 is the identity on G while $g(x_i) = t(e_i)$, and $g(f_i) = e_i, i = 1, 2, \ldots, N-1$. Since $l_k^G = l_k^{G^1}$ when $k > N$, we see that

$$\limsup \frac{1}{n} \log l_n^G = \limsup \frac{1}{n} \log l_n^{G_1}.$$

Note that in G_1 there are two disjoint loops of length N based at v_0, and hence $h(G_1) \geq (1/N) \log 2$. If $h(G) < (1/N) \log 2$, we take $\tilde{G} = G_1$ and $g = g_1$. If $h(G) \geq (1/N) \log 2$ we repeat the construction with G replaced by G_1 to obtain a connected graph G_2 such that $G_1 \subseteq G_2$, a graph homomorphism $g_2 : G_2 \to G_1$ which is the identity on G_1 and such that the exponential growth of the number of first return loops to v_0 in G_2 is the same as in G_1 and G, and $h(G_2) \geq (1/N) \log 3$. By repeating the construction in this way, we obtain the desired graph $\tilde{G}$ after no more than $n-1$ steps, where n is a natural number such that $(1/N) \log n > h(G)$. □

LEMMA 6.6. *Let Y be a synchronized system. There is then an invariant Borel probability measure on Y of full support.*

Proof. Let $G = (V, E, L)$ be the Fischer cover of Y. Since G is right-resolving, $h(G) \leq \log \#\mathcal{A}$, and hence $h(G) < \infty$. Then Lemma 6.5 gives us a strongly positive recurrent graph $\tilde{G}$ and a shift-commuting continuous surjection $\pi : \Sigma_{\tilde{G}} \to \Sigma_G$. Being strongly positive recurrent, there is a maximal measure on $\Sigma_{\tilde{G}}$, and this measure has full support. Set $\nu(B) = \mu((L \circ \pi)^{-1}(B))$, and note that ν is then an element of $M(Y)$ of full support. □

THEOREM 6.7. *Let Y be a synchronized system with Fischer graph G. Then*

$$h(G) = \sup\{h_\mu(Y) : \mu \in M(Y) \text{ has full support}\}.$$

This supremum is attained (i.e. is a maximum) if and only if G is positive recurrent.

Proof. Thanks to Lemmas 6.3 and 6.4 it remains only to consider an $\epsilon > 0$ and show that there is a measure $\mu \in M(Y)$ of full support such that $h_\mu(Y) \geq h(G) - \epsilon$. To this end, note that it follows from Gurevič's work (cf. [**Gu1**]) that there is a sequence $G_n, n \in \mathbb{N}$, of finite subgraphs in G such that $\lim_{n\to\infty} h(G_n) = h(G)$. For each n the Parry measure gives us a Borel probability measure μ_n on Σ_G such that the support of μ_n is the compact shift space $\Sigma_{G_n} \subseteq \Sigma_G$, and such that $h_{\mu_n}(\Sigma_G) = h(G_n)$. Each μ_n can be pushed forward to a measure ν_n on Y, supported on $L(\Sigma_{G_n})$. Since L is right-resolving,

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

the map $L : \Sigma_{G_n} \to L(\Sigma_{G_n})$ is finite-to-one, and it follows that $h_{\nu_n}(Y) = h_{\mu_n}(\Sigma_G) = h(G_n)$. By Lemma 6.6 there is an element $\nu \in M(Y)$ of full support. Set

$$\nu_n' = \left(1 - \frac{1}{n}\right)\nu_n + \frac{1}{n}\nu$$

and note that ν_n' has full support. Since

$$h_{\nu_n'}(Y) = \left(1 - \frac{1}{n}\right)h_{\nu_n}(Y) + \frac{1}{n}h_\nu(Y)$$

converges to $h(G)$ as n tends to infinity, the proof is complete. □

In the following a shift-invariant Borel probability measure $\mu \in M(Y)$ is said to be *maximal* when $h_\mu(Y) = h(Y)$.

COROLLARY 6.8. *Let Y be a synchronized system with Fischer graph G. Then Y has a maximal measure of full support if and only if G is positive recurrent and $h(G) = h(Y)$.*

Proof. This follows from Theorem 6.7 because the estimate $h(G) \leq h(Y)$ always holds by Proposition 6.1 of [**Pe**]. □

When it exists, a maximal measure of full support is unique and must be the push-forward of the maximal measure on Σ_G. However, there may be other measures, not of full support, that are also maximal.

We turn now to a more detailed study of the visible components of a synchronized system X.

LEMMA 6.9. *Let X_c and $X_{c'}$ be irreducible components of X at the same level. It follows that $\tilde{X}_c \cap \overline{X_{c'}} = \emptyset$ unless $X_c = X_{c'}$.*

Proof. If $z \in \tilde{X}_c \cap \overline{X_{c'}}$, there is a word $w \in c$ which occurs in z. And since $z \in \overline{X_{c'}}$ this implies that there is a word in $\mathbb{W}(\overline{X_{c'}})$ of the form $uxwyu'$, where $u, u' \in c'$ and $x, y \in \mathbb{W}(\overline{X_{c'}})$. But this implies that $u \sim w$ so that $c = c'$. □

LEMMA 6.10. *Let X_c be a visible component of X at level j. It follows that there is a unique irreducible component A of the component graph H_{j+1} at level j such that $\overline{X_c} = \overline{L(\Sigma_A)}$.*

Proof. Since X_c is a visible component, there is a word $w \in \mathbb{W}(\overline{X_c})$ which does not occur in $\partial_M X$. By transitivity of $\overline{X_c}$ we can construct an element $x \in \overline{X_c}$ such that all words of $\mathbb{W}(\overline{X_c})$ occur in x and such that $\{j \in \mathbb{Z} : x_{[j,j+|w|[} = w\}$ is neither bounded below nor above. Then $x \in R(\partial^j X)\backslash\partial_M X$. Since there are only finitely many vertices in H_{j+1} which are the terminal vertex of a path with label w, it follows that there is an element $z \in \Sigma_{H_{j+1}}$ with $L(z) = x$. (The argument for this is part of the proof of Lemma 5.1.) Since H_{j+1} is non-wandering, there is a component $A \subseteq H_{j+1}$ such that $z \in \Sigma_A$. Since every word of $\mathbb{W}(\overline{X_c})$ is the label of a piece in z, we conclude that $\overline{X_c} \subseteq \overline{L(\Sigma_A)}$. In particular, we conclude that $\overline{L(\Sigma_A)} \not\subseteq \partial^{j+1}X$. Thus $\operatorname{Per} L(\Sigma_A) \not\subseteq \partial^{j+1}X$, and it follows then from Lemma 4.2 of [**Th1**] that there is an irreducible component $X_{c'} \subseteq \partial^j X\backslash\partial^{j+1}X$ such that $\operatorname{Per} L(\Sigma_A) \cap X_{c'} \neq \emptyset$. There is therefore a path γ in A such that $L(\gamma) \in c'$. Let $v \in \mathbb{W}(\overline{L(\Sigma_A)})$. By using that A is connected we can then construct a word

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

$u \in \mathbb{W}(\overline{L(\Sigma_A)})$ such that $u^\infty \in L(\Sigma_A)$ and $v \subseteq u$ and $L(\gamma) \subseteq u$. Since $\overline{L(\Sigma_A)} \subseteq R(\partial^j X)$ by (4.2), this implies that $v \in \mathbb{W}(\overline{X_{c'}})$. Since $v \in \mathbb{W}(\overline{L(\Sigma_A)})$ was arbitrary, this implies that $\overline{L(\Sigma_A)} \subseteq \overline{X_{c'}}$. It follows then from Lemma 3.4 of [**Th1**] (or Lemma 6.9 above) that $c = c'$. This proves this existence part. The uniqueness follows because H_{j+1} is component reduced: there is a word $\overline{L(\Sigma_A)}$ which does not occur in $\partial_M X \cup L(\Sigma_{H_{j+1}\setminus A})$. Since this word occurs in X_c, no other component in H_{j+1} will do. □

LEMMA 6.11. *Let A be a connected component of the component graph at level k. Then $\overline{L(\Sigma_A)} \not\subseteq \partial^{k+1} X$.*

Proof. Since H_{k+1} is component reduced, $\overline{L(\Sigma_A)} \not\subseteq \partial_M X$, so there is a word in $\mathbb{W}(\overline{L(\Sigma_A)})\setminus\mathbb{W}_\infty(X)$ with the property that only finitely many vertices in H_{k+1} are the terminal vertex of a path in H_{k+1} with that word as label. We can then use the argument from Proposition 3.3.16 of [**LM**] to prolong the word to obtain a word in $\mathbb{W}(\overline{L(\Sigma_A)})\setminus\mathbb{W}_\infty(X)$ which is essentially magic for H_{k+1}. Such a word is synchronizing for $R(\partial^j X)$ by Lemma 4.1, and hence $\overline{L(\Sigma_A)} \not\subseteq \partial^{k+1} X$. □

PROPOSITION 6.12. *There is a bijection, $c \mapsto A_c$, from the visible irreducible components in X at level j to the irreducible components of the component graph at level j such that $\overline{X_c} = \overline{L(\Sigma_{A_c})}$, $c \in \mathcal{C}_j \cap \mathcal{C}_M$. The Fischer cover of $\overline{X_c}$ is the merged graph of A_c.*

Proof. The map is well defined by Lemma 6.10 and injective by Lemma 6.9. To see that it is surjective, let A be an irreducible component of H_{j+1}. Thanks to Lemma 6.11 we can argue as in the proof of Lemma 6.10 to find an irreducible component X_c at level j such that $\overline{L(\Sigma_A)} \subseteq \overline{X_c}$. Since H_{j+1} is component reduced, $L(\Sigma_A)$, and hence also X_c, must contain a word w which does not occur in $\partial_M X \cup L(\Sigma_{H_{j+1}\setminus A})$. It follows first that X_c is a visible component, and then from Lemma 6.10 that $\overline{X_c} = \overline{L(\Sigma_B)}$ for some irreducible component B of H_{j+1}. Since $w \in \mathbb{W}(\overline{L(\Sigma_B)})$, it follows that $A = B$.

To see that the merged graph of A_c is the Fischer cover of $\overline{X_c}$ it suffices by Theorem 2.16 of [**FF**] to show that the merged graph of A_c presents $\overline{X_c}$, is right-resolving and follower separated and has a magic word. The first three properties follow from Lemma 3.3.8 of [**LM**], and the last property follows from Lemma 4.1. □

With Proposition 6.12 at hand we have a reasonable tool for finding the visible components of a synchronized system. As we shall see, it allows us to obtain information on where the entropy of a synchronized system is located, and on the set of maximal measures.

LEMMA 6.13. *Let $G = (V, E, L)$ be a connected and right-resolving labeled graph. Assume that $\mathbb{W}(\Sigma_G^L(\infty)) \neq \mathbb{W}(\overline{L(\Sigma_G)})$, and let $\widehat{G}$ be the merged graph of G. It follows that $h(\widehat{G}) \leq h(G)$. Furthermore, when $h(\widehat{G}) = h(G)$ and $\widehat{G}$ is positive recurrent, then so is G.*

Proof. Let w be a word in $\overline{L(\Sigma_G)}$ which is not in $\Sigma_G^L(\infty)$. Since G is connected we may assume that w is the label of a loop in G, and hence also the label of a loop in $\widehat{G}$. Let v be a vertex in $\widehat{G}$ which is the terminal vertex of such a loop. Since $w \notin \mathbb{W}(\Sigma_G^L(\infty))$, the set

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

of vertices in G which are the terminal vertex of a path with label w is a finite set F. By construction of the merged graph, any loop in $\widehat{G}$ which starts and ends at v can be lifted to a path in G which starts and ends in F. It follows that

$$r_n^{\widehat{G}}(v, v) \leq \sum_{x,y\in F} r_{n+|w|}^G(x, y).$$

Hence

$$h(\widehat{G}) = \limsup_n \frac{1}{n} \log r_n^{\widehat{G}}(v, v) \leq \limsup_n \frac{1}{n} \log\Big(\sum_{x,y\in F} r_{n+|w|}^G(x, y) \Big) = h(G).$$

The last statement follows from the estimate

$$\limsup_{n\to\infty} r_n^{\widehat{G}}(v, v)e^{-nh(G)} \leq e^{-|w|h(G)} \max_{x,y\in F}\Big(\limsup_{n\to\infty} r_n^G(x, y)e^{-nh(G)}\Big). \qquad \square$$

LEMMA 6.14. *Let G be a right-resolving labeled graph. It follows that*

$$h\Big(\bigcup_{j\geq 1} G_j^{\#}\Big) \leq h(G). \tag{6.1}$$

Furthermore, if there is a connected subgraph A of $\bigcup_{j\geq 1} G_j^{\#}$ which is positive recurrent with $h(A) = h(G)$, then G must contain a positive recurrent connected component A' with $h(A') = h(G)$.

Proof. A vertex v in $\bigcup_{j\geq 1} G_j^{\#}$ consists of a finite set F of vertices in G, and

$$r_n^{\bigcup_{j\geq 1} G_j^{\#}}(v, v) \leq \sum_{(x,y)\in D} r_n^G(x, y),$$

where D is the subset of pairs $(x, y) \in F \times F$ with x and y in the same connected component of G. This yields (6.1), and the last statement follows from the estimate

$$\limsup_{n\to\infty} r_n^{\bigcup_{j\geq 1} G_j^{\#}}(v, v)e^{-nh(G)} \leq \max_{x,y\in D}\Big(\limsup_{n\to\infty} r_n^G(x, y)e^{-nh(G)}\Big). \qquad \square$$

COROLLARY 6.15. *Let X be a synchronized system with Fischer graph G. Let H be the Fischer graph of a visible component in X. Then $h(H) \leq h(G)$.*

Proof. This follows from Proposition 6.12 and Lemmas 6.13 and 6.14. $\square$

It is essential that the component is visible; it is easy to construct examples where the inequality of Corollary 6.15 is violated by the Fischer graph of a component sitting inside $\partial_M X$.

THEOREM 6.16. *Let Y be a synchronized system with Fischer graph G. Then*

$$h(Y) = \max\{h(G), h(\partial_M Y)\} = \max\{h(G), h(\partial Y)\}.$$

Proof. Since $\partial_M X \subseteq \partial Y$, it suffices to prove the first equality. It is clear that $h(Y) \geq h(\partial_M Y)$, and the estimate $h(Y) \geq h(G)$ follows from Proposition 6.1 of [**Pe**]. So it remains to show that $h(Y) \leq \max\{h(G), h(\partial_M Y)\}$. Since Y is expansive and of finite entropy,

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

there is an ergodic element μ of $M(Y)$ with $h_\mu(Y) = h(Y)$. If μ is supported on the Markov boundary we have that $h(Y) = h(\partial_M Y)$. Consider therefore the case when $\mu(\partial_M X) = 0$. It follows then from Theorem 5.6 that there is a visible component Y_c of Y such that $\mu(\tilde{Y}_c) = 1$. Then the restriction of μ to $\overline{Y_c}$ is a synchronized element of $M(\overline{Y_c})$, and by Proposition 6.2 it is therefore the push-forward of a Borel probability measure with the same entropy on Σ_{G_c}, where G_c is the Fischer graph of the synchronized system $\overline{Y_c}$. Hence $h(Y) = h_\mu(Y) \leq h(G_c)$. By Corollary 6.15, $h(G_c) \leq h(G)$. □

It follows from Theorem 6.16 that $h(\partial_M X) = h(\partial X)$ when $h(\partial X) > h(G)$. This is very unlike cases where $h(\partial_M X) < h(G)$, and it is generally not because $\partial_M X$ and ∂X are identical when $h(\partial X) > h(G)$. In fact, $\partial X \backslash \partial_M X$ can have non-zero entropy in this case, although it cannot exceed $h(G)$.

7. *Maximal ergodic measures*

PROPOSITION 7.1. *Let $G = (E, V, L)$ be a connected right-resolving labeled graph. Assume that $h_\infty(\Sigma_G^L) < h(G)$. It follows that G is strongly positive recurrent.*

Proof. Let $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ be a sequence of finite connected subgraphs, $G_n = (V_n, E_n)$, of G such that $G = \bigcup_n G_n$. Then $h(G) = \lim_{n\to\infty} h(G_n)$; cf. [**Gu1**]. For each n there is a finite subset $F_n \subseteq V$ such that $\bigcap_F \mathbb{W}_n(\Sigma_{G\backslash F}^L) = \mathbb{W}_n(\Sigma_{G\backslash F_n}^L)$. It follows that

$$\frac{1}{n}\log\left(\#\bigcap_F \mathbb{W}_n(\Sigma_{G\backslash F}^L)\right) \geq \inf_k \frac{1}{k}\log(\#\mathbb{W}_k(\Sigma_{G\backslash F_n}^L)). \tag{7.1}$$

Since $\#\mathbb{W}_{k+l}(\Sigma_{G\backslash F_n}^L) \leq \#\mathbb{W}_k(\Sigma_{G\backslash F_n}^L)\#\mathbb{W}_l(\Sigma_{G\backslash F_n}^L)$ for all $k, l \in \mathbb{N}$, it follows from (7.1) that

$$\frac{1}{n}\log\left(\#\bigcap_F \mathbb{W}_n(\Sigma_{G\backslash F}^L)\right) \geq \lim_{k\to\infty} \frac{1}{k}\log(\#\mathbb{W}_k(\Sigma_{G\backslash F_n}^L))$$

for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{1}{n}\log\left(\#\bigcap_F \mathbb{W}_n(\Sigma_{G\backslash F}^L)\right) < h(G)$$

by assumption, we conclude that, if n_0 is large enough, then

$$\lim_{k\to\infty} \frac{1}{k}\log(\#\mathbb{W}_k(\Sigma_{G\backslash F_{n_0}}^L)) \leq \frac{n_0 - 1}{n_0 + 1} h(G). \tag{7.2}$$

Since $\lim_{n\to\infty} h(G_n) = h(G)$ and F_{n_0} is finite, there is an m so large that

$$h(G_m) \geq \frac{n_0}{n_0 + 1} h(G) \tag{7.3}$$

and

$$F_{n_0} \subseteq V_m. \tag{7.4}$$

Let u, v be vertices in G_m and let t_k be the number of paths $\gamma = (e_1, e_2, \ldots, e_k)$ in G of length k such that $s(e_1) = u$, $t(e_k) = v$, and $s(e_i) \notin V_m, i = 2, 3, \ldots, k$. Since L is a right-resolving labeling,

$$t_k \leq (\#\mathcal{A})(\#\mathbb{W}_{k-2}(\Sigma_{G\backslash V_m}^L)). \tag{7.5}$$

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

It follows from (7.4) that $V\setminus V_m \subseteq V\setminus F_{n_0}$, so we conclude from (7.5) and (7.2) that

$$\limsup_k \frac{1}{k}\log t_k \leq \lim_k \frac{1}{k}\log(\#\mathbb{W}_k(\Sigma^L_{G\setminus V_m})) \leq \frac{n_0-1}{n_0+1}h(G).$$

Comparing with (7.3) we deduce that $\limsup_k (1/k)\log t_k \leq h(G_m)$. Thanks to a result of Gurevič and Zargaryan [**GZ**], formulated as Theorem 3.2 in [**Ru**], this implies that G is strongly positive recurrent. □

LEMMA 7.2. *Let X be a synchronized system with Fischer cover $G = (V, E, L)$. Assume that G is strongly positive recurrent. It follows that there is an $\epsilon > 0$ such that $h(G_k^{\#}) \leq h(G) - \epsilon$ for all $k \geq 2$.*

Proof. Let p be a finite path in G of length $|p| \geq 2$ whose label is magic for G. Set $m = |p| + 1$. The mth higher edge graph $G^{[m]}$ of G, as defined for example in Definition 2.3.4 of [**LM**], is labeled in the natural way: when $(e_1, e_2, \ldots, e_{m-1})$ and $(f_1, f_2, \ldots, f_{m-1})$ are paths of length $m-1$ in G, defining vertices of $G^{[m]}$, such that $(e_2, e_3, \ldots, e_{m-1}) = (f_1, f_2, \ldots, f_{m-2})$, we label the corresponding arrow in $G^{[m]}$ by $L(f_{m-1})$. It is then easy to see that the union of the connected components in $(G_k^{\#})^{[m]}$ is the same as the union of the connected components in $(G^{[m]})_k^{\#}$. Thus $h(G_k^{\#}) = h((G_k^{\#})^{[m]}) = h((G^{[m]})_k^{\#})$. Let G' be the subgraph of $G^{[m]}$ obtained by removing the vertex of $G^{[m]}$ given by p. Since the label of p is magic for G, it follows that $h((G^{[m]})_k^{\#}) = h((G')_k^{\#})$ when $k \geq 2$. By Lemma 6.14, $h((G')_k^{\#}) \leq h(G')$. Since $h(G^{[m]}) = h(G)$, it suffices now to show that

$$h(G') < h(G^{[m]}). \tag{7.6}$$

To prove this, note first that $G^{[m]}$ is strongly positive recurrent since G is. This follows from Proposition 2.12 of [**GS**]. If G' was connected, we could now deduce (7.6) from Theorem 3.15 of [**GS**] or from Proposition 2.6(ii) of [**Ru**]. But we need the conclusion in the general case. Let u be a vertex in G'. Since $G^{[m]}$ is connected there are paths γ_1 and γ_2 in $G^{[m]}$ such that $s(\gamma_1) = u,\ t(\gamma_1) = p,\ s(\gamma_2) = p$ and $t(\gamma_2) = u$. If we choose γ_1 and γ_2 to be the shortest possible, we see that $r_n^{G'}(u,u) \leq l^{G^{[m]}}_{n+|\gamma_1|+|\gamma_2|}(p,p)$ for all n. It follows that

$$h(G') \leq \limsup_{n\to\infty} \frac{1}{n}\log l_n^{G^{[m]}}(p,p) < h(G^{[m]})$$

since $G^{[m]}$ is strongly positive recurrent, proving (7.6). □

LEMMA 7.3. *Let X be a synchronized system with Fischer graph G. If X_c is a visible component in X such that the Fischer graph of $\overline{X_c}$ has entropy $h(G)$ and is positive recurrent, then X_c is the top component of X.*

Proof. Assume that X_c is *not* the top component of X. It follows from Proposition 6.12 and Lemma 6.13 that there is a $k \geq 2$ and a positive recurrent connected component A of $G_k^{\#}$ such that $h(A) = h(G)$. By Lemma 6.14, G is also positive recurrent in this case, and, by Proposition 2.12 of [**GS**], so is $G^{[n]}$ for all $n \in \mathbb{N}$. Working with a synchronizing word for X in the same way as in the proof of Lemma 7.2 we obtain an $m \in \mathbb{N}$ and a proper subgraph G' of $G^{[m]}$ such that $A^{[m]} \subseteq (G')_k^{\#}$. Then

$$h(G) = h(A^{[m]}) \leq h((G')_k^{\#}) \leq h(G') \leq h(G^{[m]}) = h(G),$$

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

where we have used Lemma 6.14 for the second estimate. Thus $h((G')_k^{\#}) = h(A^{[m]})$ and it follows from the definition of positive recurrence that $A^{[m]}$ is contained in a positive recurrent connected component of $(G')_k^{\#}$. But then Lemma 6.14 shows that G' contains a positive recurrent connected component A', also with entropy $h(G)$. This is impossible because A' is a proper connected subgraph of the connected positive recurrent graph $G^{[m]}$ with the same entropy as $G^{[m]}$, and A' must therefore be transient by a result of Salama (cf. [**Ru**]). □

THEOREM 7.4. *Let X be a synchronized system with Fischer graph G.*

(a) *Assume that $h(\partial_M X) < h(G)$. Then G is strongly positive recurrent, and there is an $\epsilon > 0$ such that every ergodic measure $\mu \in M(X)$ with $h_\mu(X) \geq h(X) - \epsilon$ is the push-forward of a synchronized measure ν on Σ_G with $h_\nu(\Sigma_G) = h_\mu(X)$. In particular, X is intrinsically ergodic.*

(b1) *Assume that $h(\partial_M X) = h(G)$, and that G is positive recurrent. Then the maximal ergodic measures on X are the maximal ergodic measures on the Markov boundary $\partial_M X$ and the push-forward of the maximal measure on Σ_G. In particular, X is* not *intrinsically ergodic.*

(b2) *Assume that $h(\partial_M X) = h(G)$, and that G is not positive recurrent. Then the maximal ergodic measures on X are the maximal ergodic measures on the Markov boundary $\partial_M X$. In particular, X is intrinsically ergodic if and only if $\partial_M X$ is.*

(c) *Assume that $h(\partial_M X) > h(G)$. Then every ergodic measure $\mu \in M(X)$ of entropy $h_\mu(X) > h(G)$ is supported on the Markov boundary $\partial_M X$. In particular, X is intrinsically ergodic if and only if $\partial_M X$ is.*

Proof. (a) The Fischer graph G is positive recurrent by Proposition 7.1. By Lemma 7.2 we can choose $\epsilon > 0$ so small that $h(G) - \epsilon > h(\partial_M X)$ and $h(G_k^{\#}) \leq h(G) - \epsilon$ when $k \geq 2$. It follows then from Theorem 5.6 that an ergodic $\mu \in M(X)$ with $h_\mu(X) \geq h(G) - \epsilon$ must be a synchronized measure on a visible component. Since $h(G_k^{\#}) \leq h(G) - \epsilon$ when $k \geq 2$ it follows from Proposition 6.2 and Corollary 6.15 that this component must be the top component, i.e. μ must be a synchronized measure on X. By Proposition 6.2 it is then the push-forward of a synchronized measure ν of the same entropy on Σ_G. The intrinsic ergodicity follows since $h(G) = h(Y)$ by Theorem 6.16.

(b1) It follows from Theorem 5.6 that a maximal ergodic measure μ which is not supported on $\partial_M X$ is a synchronized measure on a visible component. It follows then from Proposition 6.2 and Corollary 6.15 that μ is the push-forward of a maximal measure on the Fischer graph of this component. It follows then from Gurevič's work that this Fischer graph is positive recurrent, and then Lemma 7.3 tells us that the component must be the top component. By using Proposition 6.2 again we conclude that the measure is the push-forward of the maximal measure on G.

(b2) The same reasoning as in (b1) shows that in this case there are no maximal ergodic measures which are not supported on the Markov boundary.

(c) This follows from Theorem 5.6, Proposition 6.2 and Corollary 6.15. □

It seems appropriate to point out that all four cases of Theorem 7.4 occur in abundance. Any irreducible sofic shift space of positive entropy is an example of case (a), and this

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press

only scratches the surface. Examples of case (c) were first given by Petersen in [**Pe**]. More examples of this case and of cases (b1) and (b2) are easily constructed by labeling infinite graphs. Observe that any connected, right-resolving and follower separated labeled graph with a magic word presents a synchronized system whose Markov boundary can be identified by using Theorem 3.4. It is then easy to construct a wealth of examples. It may demand some creativity to show that case (b1) occurs both with G strongly positive recurrent and with G *not* strongly positive recurrent. Still, we leave this to the interested reader.

8. *Concluding remark*

In this study we have treated the Markov boundary as a 'black hole'. And in the general case the Markov boundary *is* a part of the shift space which is inaccessible to the approach taken. But in many cases the Markov boundary is again a synchronized system, or at least essentially composed of synchronized systems, and then the ergodic structure of the shift space can be further analyzed by applying the above methods and results to the Markov boundary. In general, however, any further insight calls for other methods.

Acknowledgements. A part of this work was carried out at the Max Planck Institute in Bonn, during the activity on Algebraic and Topological Dynamics in 2004. I take this opportunity to thank the MPI, and in particular S. Kolyada, for the invitation to participate in this activity.

References

[Bu] J. Buzzi. Subshifts of quasi-finite type. *Invent. Math.* **159** (2005), 369–406.

[FF] D. Fiebig and U. Fiebig. Covers for coded systems. *Symbolic Dynamics and Its Applications (Contemporary Mathematics, 135)*. Ed. P. Walters. American Mathematical Society, Providence, RI, 1992, pp. 139–180.

[Gu1] B. M. Gurevič. Topological entropy of enumerable Markov chains (in Russian). *Dokl. Akad. Nauk SSSR* **187** (1969), 715–718 (English transl. *Sov. Math. Dokl.* **10**(4) (1969), 911–915).

[Gu2] B. M. Gurevič. Shift entropy and Markov measures in the path space of a denumerable graph (in Russian). *Dokl. Akad. Nauk SSSR* **192** (1970), 963–965 (English transl. *Sov. Math. Dokl.* **11**(3) (1970), 744–747).

[GS] B. M. Gurevič and S. V. Savchenko. Thermodynamic formalism for countable symbolic Markov chains (in Russian). *Uspekhi Mat. Nauk* **53**(2) (1998), 3–106 (English transl. *Russian Math. Surveys* **53**(2) (1998), 245–344).

[GZ] B. M. Gurevič and A. S. Zargaryan. Existence conditions of a maximal measure for a countable symbolic Markov chain (in Russian). *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **43**(5) (1988), 14–18 (English transl. *Moscow Univ. Math. Bull.* (1988), 18–23).

[LM] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[Pe] K. Petersen. Chains, entropy, coding. *Ergod. Th. & Dynam. Sys.* **6** (1986), 415–448.

[Ru] S. Ruette. On the Vere-Jones classification and existence of maximal measures for countable topological Markov chains. *Pac. J. Math.* **209** (2003), 365–380.

[Th1] K. Thomsen. On the structure of a sofic shift space. *Trans. Amer. Math. Soc.* **356** (2004), 3557–3619.

[Th2] K. Thomsen. On the structure of beta shifts. *Algebraic and Topological Dynamics (Contemporary Mathematics, 385)*. American Mathematical Society, Providence, RI, 2005, pp. 321–332.

[W] P. Walters. *An Introduction to Ergodic Theory*. Springer, Berlin, 1982.

https://doi.org/10.1017/S0143385706000290 Published online by Cambridge University Press