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Stability, corners, and other two-dimensional shapes

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Abstract. We introduce a relaxation of stability, called almost sure stability, which is insensitive to perturbations by subsets of Loeb measure 0 in a non-standard finite group. We show that almost sure stability satisfies a stationarity principle in the sense of geometric stability theory for measure-independent elements. We apply this principle to deduce the existence of squares in dense almost surely stable subsets of Cartesian products of non-standard finite groups, possibly non-abelian. Our results imply qualitative asymptotic versions for Cartesian products of finite groups. In the final section, we establish the existence of 3×2-grids (and thus of L-shapes) in dense almost surely stable two-dimensional subsets of finite abelian groups of odd order.

1 Introduction

In the last few years, local stability, as developed by Hrushovski and Pillay [21], has found several applications [11, 12, 30, 42] to questions on asymptotic behavior in additive combinatorics. Recall that a subset A of a group G is *k-stable* if its Cayley graph

$$\mathrm{Cay}(G, A) = \{(g, h) \in G \times G : \ g^{-1} \cdot h \in A\}$$

induces no half-graph of height k, that is, there is no sequence $(a_1, b_1, \dots, a_k, b_k)$ in G^{2k} such that the pair $a_i^{-1} \cdot b_j$ belongs to A if and only if $i \leq j$.

Although the notion of stability has played and continues to play a crucial role in the development of model theory, from a combinatorial perspective, it is exceptionally sensitive: the slightest perturbation of the set (adding a single instance of a half-graph) has the potential to destroy it. Motivated by this, we introduce a relaxation of stability which takes such perturbations into account. Given finite sets X and Y, a relation $S \subseteq X \times Y$ is *almost surely k-stable* if there are few (in the sense of the counting measure associated with $X \times Y$) half-graphs of height k induced by S (see Definition 4.3).

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Similar attempts to quantify stability appeared in earlier work of Terry and the third author [43, Section 5] and Coregliano and Malliaris [13, Definition 3.5].[1]

Our purpose for this work is twofold: on the one hand, we aim to develop the notion of almost sure stability from a model-theoretic point of view.[2] On the other hand, we hope to further cement the relevance of stability to classical questions in arithmetic combinatorics (in both the abelian and the non-commutative case). The article thus focuses on the existence of certain two-dimensional shapes, such as corners and squares, in dense subsets of groups.

Given an abelian group G, a (nontrivial) *corner* in a subset $S \subseteq G \times G$ is given by a pair (x, y) in G^2 along with $d \neq 0_G$ in G with the property that all three pairs (x, y), $(x + d, y)$, and $(x, y + d)$ lie in S. It is well known [41] that the existence of nontrivial corners for dense subsets in $G \times G$ implies Roth's theorem [36] on 3-term arithmetic progressions in dense subsets A of G. Indeed, a nontrivial corner in $\mathrm{Cay}(G, A) \subseteq G \times G$ determined by (x, y) and d gives rise, by a simple projection, to the nontrivial 3-term arithmetic progression $x - y - d, x - y, x - y + d$ in A.

The existence of corners in subsets S of $G \times G$ of positive density follows straight from the multidimensional Szemerédi theorem. The latter was first proved qualitatively in [16], and quantitative versions (albeit with poor bounds) follow from the hypergraph regularity lemmas of Gowers [18] and Nagle, Rödl, Schacht, and Skokan [32, 35]. However, finding corners is not quite as difficult as resolving the full multidimensional Szemerédi theorem, and indeed some remarkable progress has been made toward a strongly quantitative resolution [1, 19, 27, 39, 40].[3]

In the case of a non-abelian group G, we distinguish between so-called *naive corners*, that is, configurations of the form $(x, y), (x \cdot g, y), (x, y \cdot g)$ with $g \neq 1_G$, and *BMZ corners* (for Bergelson, McCutcheon, and Zhang [4]), which are of the form $(x, y), (x \cdot g, y), (x \cdot g, y \cdot g)$ with $g \neq 1_G$. Note that the latter configuration can also be written as $(x, y), (g \cdot x, y), (x, y \cdot g)$, by first writing $z = x \cdot g$ and then replacing the set $S \subseteq G \times G$ with its image after applying the inverse operation in the first coordinate. Similarly, naive corners can be expressed as $(x, y), (g \cdot x, y), (g \cdot x, y \cdot g)$.

In abelian groups, naive and BMZ corners are easily seen to be equivalent. Some abelian techniques can be adapted to the non-abelian context for BMZ corners, but do not seem to apply to naive corners [4], and an extensive literature now exists [2, 5, 41].

Other two-dimensional shapes are of interest in abelian groups, for instance, *squares*, which are configurations given by $(x, y) \in G^2$ together with $d \neq 0_G$ in G such that all of (x, y), $(x + d, y)$, $(x, y + d)$, $(x + d, y + d)$ lie in S. We observe that the non-abelian formulation of a square, where we require

$$(x, y), (x \cdot g, y), (x, y \cdot g), (x \cdot g, y \cdot g)$$

to lie in S, actually contains both a naive and a BMZ corner (and could have been expressed in a number of equivalent ways).

[1] More recently, and subsequent to the present work, this notion also appeared as μ-stability in work of Chernikov and Towsner [10, Remark 2.18].

[2] Almost sure stability is taken further in the recent preprints [9, 17].

[3] See also the spectacular very recent preprint [23].

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In the presence of almost sure stability, that is, when few half-graphs of height k are induced by the subset S of $G \times G$, we are able to obtain qualitative asymptotic bounds on the density of S that guarantees the existence of a square in S. In fact, our methods work in both abelian and non-abelian finite groups (see Corollary 5.5).

Moreover, we prove the following strengthening in the setting of finite groups of bounded exponent (see Corollary 5.6).

Theorem A *Given integers $k, r \geq 1$ and real numbers $\delta, \varepsilon > 0$, there exists an integer $N = N(k, r, \delta, \varepsilon) \geq 1$ and real numbers $\theta = \theta(k, r, \delta, \varepsilon) > 0$ and $\eta = (k, r, \delta, \varepsilon) > 0$ with the following property.*

Let G be a finite group of exponent at most r, and consider a relation $S \subseteq G \times G$ of size $|S| \geq \delta |G|^2$ such that the collection $\mathcal{H}_k(S)$ of all half-graphs of height k induced by S on G has size $|\mathcal{H}_k(S)| \leq \theta |G|^{2k}$. For an element g in G, set

$$\Lambda_{\square}(S)_g = \{(a, b) \in G^2 : (a, b), (a \cdot g, b), (a, b \cdot g), (a \cdot g, b \cdot g) \in S\}.$$

Then, there exists a subgroup H of G of index at most N such that

$$\left|\left\{g \in H : \ |\Lambda_{\square}(S)_g| < \eta |S|\right\}\right| < \varepsilon |H|.$$

Going beyond corners, in a recent breakthrough, Peluse [34] obtained the first reasonable bound for the existence of so-called *L-shapes* (where each of the pairs $(x, y), (x + d, y), (x, y + d), (x, y + 2d)$ belongs to S) in dense subsets of $\mathbb{F}_p^n \times \mathbb{F}_p^n$.

In the presence of almost sure stability, we are able to establish the existence of even larger two-dimensional patterns, including 3×2-grids (see Corollary 6.4).

Theorem B *Given an integer $k \geq 1$ and a real number $\delta > 0$, there is an integer $N(k, \delta) \geq 1$ and real numbers $\theta = \theta(k, \delta) > 0$ and $\varepsilon = \varepsilon(k, \delta) > 0$ with the following property.*

Let G be a finite abelian group of odd order with $|G| \geq N$, and consider a relation $S \subseteq G \times G$ of size $|S| \geq \delta |G|^2$ such that the collection $\mathcal{H}_k(S)$ of all half-graphs of height k induced by S on G has size $|\mathcal{H}_k(S)| \leq \theta |G|^{2k}$.

Then, the set

$$\Lambda_{3\times 2}(S) = \left\{(a, b, g) \in G^3 : \begin{array}{l} (a,\ b),\ (a+\ g,\ b),\ (a+2g,\ b),\ (a,\ b+g), \\ (a+g,\ b+\ g)\ \ and\ \ (a+\ 2g,\ b+\ g)\ \ all\ lie\ in\ \ S \end{array}\right\}$$

has size $|\Lambda_{3\times 2}(S)| \geq \varepsilon |G|^3$. In particular, the relation S contains an L-shape.

The techniques used to prove Theorems A and B are of a model-theoretic flavor, and rely heavily on a stationarity result for almost surely stable relations (see Theorem 4.7). In forthcoming work, we establish some of the results presented here in the language of additive combinatorics. While the techniques used will be different, the structure of the proofs will follow a similar pattern. The present article should therefore be viewed as an effort to highlight the close interactions and foster stronger synergies between model theory and additive combinatorics.

In order to render the presentation of this first work more accessible to an audience who may not be versed in the language of model theory, we have adapted the results using some of the (perhaps more familiar) machinery of ultraproducts, with the aim of keeping the presentation as self-contained as possible. This choice should not present

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an obstacle for the model-theoretic reader, who will easily translate the terminology and techniques to the more general setting.

Outline of the article. Sections 2 and 3 introduce the background needed for the remainder of the article. In Section 4, we define almost sure stability and prove a stationarity principle for almost stable relations. The main theorems concerning corners and squares are derived in Section 5, while Section 6 considers more complex two-dimensional configurations.

2 Non-standard finite groups and Loeb measures

Recall that a non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is a non-empty collection of infinite sets closed under finite intersections and with the property that either a subset of $\mathbb{N}$ or its complement belongs to $\mathfrak{U}$. Such ultrafilters exist and each one induces a finitely additive probability measure on all subsets of $\mathbb{N}$, taking values 0 and 1 only, such that no finite subset has measure 1.

Definition 2.1 A *non-standard finite group* (sometimes referred to in the literature as a *hyperfinite group*) has as underlying set $\prod_{n\to\mathfrak{U}} G_n$ for some collection $(G_n)_{n\in\mathbb{N}}$ of finite groups of strictly increasing size (that is, $|G_n| < |G_{n+1}|$ for all n in $\mathbb{N}$) and some non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, where the set $\prod_{n\to\mathfrak{U}} G_n$ consists of the infinite Cartesian product $\prod_{n\in\mathbb{N}} G_n$ modulo the equivalence relation

$$(g_n)_{n\in\mathbb{N}} \sim_{\mathfrak{U}} (h_n)_{n\in\mathbb{N}} \iff \{n \in \mathbb{N} : \ g_n = h_n\} \in \mathfrak{U}.$$

That is, we identify two sequences if they are equal $\mathfrak{U}$-almost everywhere (with respect to the measure induced by $\mathfrak{U}$ on all subsets of $\mathbb{N}$).

If we denote the class of $(g_n)_{n\in\mathbb{N}}$ modulo $\sim_{\mathfrak{U}}$ by $[(g_n)_{n\in\mathbb{N}}]_{\mathfrak{U}}$, it follows immediately from the definition of the non-principal ultrafilter $\mathfrak{U}$ that the quotient $\prod_{n\to\mathfrak{U}} G_n$ is a group, where

$$[(g_n)_{n\in\mathbb{N}}]_{\mathfrak{U}} \cdot [(h_n)_{n\in\mathbb{N}}]_{\mathfrak{U}} = [(g_n \cdot h_n)_{n\in\mathbb{N}}]_{\mathfrak{U}}$$

is well-defined and has neutral element $1_{\prod_{\mathfrak{U}} G_n} = [(1_{G_n})_{n\in\mathbb{N}}]_{\mathfrak{U}}$. Furthermore, the inverse $[(g_n)_{n\in\mathbb{N}}]_{\mathfrak{U}}^{-1} = [(g_n^{-1})_{n\in\mathbb{N}}]_{\mathfrak{U}}$.

Note that a finite Cartesian product of non-standard finite groups is again a non-standard finite group.

Definition 2.2 A subset X of a non-standard finite group $\prod_{n\to\mathfrak{U}} G_n$ is *internal* if there exists a collection $(X_n)_{n\in\mathbb{N}}$ such that each X_n is a subset of G_n and

$$X = \left\{ [(g_n)_{n\in\mathbb{N}}]_{\mathfrak{U}} \in \prod_{n\to\mathfrak{U}} G_n : \ g_n \text{ belongs to } X_n \text{ for } \mathfrak{U}\text{-almost all } n \text{ in } \mathbb{N} \right\}.$$

Given an internal subset X of $G = \prod_{n\to\mathfrak{U}} G_n$ induced from the collection $(X_n)_{n\in\mathbb{N}}$ as above, we will denote the subset X_n of G_n by $X(G_n)$. Likewise, we will sometimes denote the internal set X by $\prod_{n\to\mathfrak{U}} X_n$.

Remark 2.1

(1) Every finite subset of a non-standard finite group is internal.
(2) The collection of internal sets is closed under Boolean combinations and projections. Given a (group) word $w = w(u, v_1, \ldots, v_n)$ and a tuple $\bar{a} = (a_1, \ldots, a_n)$ in

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a non-standard finite group G, the set

$$X = \{g \in G : \ w(g, \bar{a}) = 1_G\}$$

is internal.

(3) Given internal subsets $X_1, \dots, X_n$ of a non-standard finite group G, the Cartesian product $X_1 \times \cdots \times X_n$ is an internal subset of the nth Cartesian product $G^n = G \times \cdots \times G$.

(4) Given an internal set X of G, the subset

$$\{(x, y) \in G \times G : \ x \cdot y \in X\}$$

is an internal subset of $G \times G$. In particular, every translate (left or right) of an internal set is internal.

(5) In this particular set-up, Łoś's Theorem becomes tautologically immediate: given an internal set X in a non-standard finite group $G = \prod_{n \to \mathfrak{U}} G_n$, we have that

$$X \neq \varnothing \iff X(G_n) \neq \varnothing \text{ for } \mathfrak{U}\text{-almost all } n \text{ in } \mathbb{N}.$$

Every non-standard finite group arising as a limit of finite groups has cardinality continuum and so it has continuum many internal sets. However, whenever we restrict our attention to only countably many internal sets non-standard finite groups have the following remarkable property, which in model-theoretic terms is called $\aleph_1$-saturation.

Fact 2.2 ($\aleph_1$-saturation) *Let G be a non-standard finite group. Every internal countable cover of an internal set admits a finite sub-covering: given an internal subset X of G and a countable family $(Y_n)_{n \in \mathbb{N}}$ of internal subsets of G such that $X = \bigcup_{n \in \mathbb{N}} Y_n$, there is some natural number k such that*

$$X = Y_{n_1} \cup \cdots \cup Y_{n_k}.$$

Equivalently, a countable intersection of internal sets is non-empty whenever every finite sub-intersection is.

In particular, every infinite internal subset of a non-standard finite group must be uncountable.

Every non-standard finite group is equipped with a finitely additive probability measure on the Boolean algebra of internal sets, induced by the normalized counting measure on every finite group.

Definition 2.3 The *Loeb* or *non-standard counting measure* of the non-standard finite group $G = \prod_{n \to \mathfrak{U}} G_n$ is defined for every internal subset X of G as

$$\mu_G(X) = \lim_{n \to \mathfrak{U}} \frac{|X(G_n)|}{|G_n|}.$$

In an abuse of notation, we call the value $\mu_G(X)$ the density of X.

Note that for every real number r in the interval $[0,1]$, the measure $\mu_G(X) \geq r$ whenever $|X(G_n)| \geq r|G_n|$ for $\mathfrak{U}$-almost all n in $\mathbb{N}$.

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Remark 2.3 Given an internal subset Z of the non-standard finite group G^{k+m} and a real number r in $[0,1]$, there is an internal subset Y_r of G^m such that

$$\left\{y \in G^m : \mu_{G^k}(Z_y) > r\right\} \subseteq Y_r \subseteq \left\{y \in G^m : \mu_{G^k}(Z_y) \geq r\right\},$$

where $Z_y = \{x \in G^k : (x,y) \in Z\}$ denotes the fiber of Z over y. Namely, set

$$Y_r = \prod_{n \to \mathfrak{U}} \left\{y \in G_n^m : |Z_y(G_n^k)| \geq r|G_n^k|\right\}.$$

We include the following easy result to demonstrate how information about the measure of non-standard finite objects allows us to provide asymptotic bounds on the cardinalities of the corresponding sequence of finite objects.

Lemma 2.4 *Consider a family $(G_n, X_n)_{n\in\mathbb{N}}$, where X_n is a subset of the finite group G_n. The following are equivalent for every real number r in $[0,1]$:*

(1) *For every $\varepsilon > 0$, there is some $n_0 = n_0(r,\varepsilon)$ in $\mathbb{N}$ such that $|X_n| \leq (r+\varepsilon)|G_n|$ for all $n > n_0$.*
(2) *For every non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, we have that $\mu_G(X) \leq r$ for the internal set $X = \prod_{n\to\mathfrak{U}} X_n$ in the non-standard finite group $\prod_{n\to\mathfrak{U}} G_n$.*

Proof $(a) \Rightarrow (b)$: No finite set lies in a non-principal ultrafilter $\mathfrak{U}$. Hence, by the way, the non-standard counting measure on the non-standard finite group $G = \prod_{n\to\mathfrak{U}} G_n$ has been defined, it follows from (a) that $\mu_G(X) \leq r + \varepsilon$ for every $\varepsilon > 0$, so $\mu_G(X) \leq r$, as desired.

$(b) \Rightarrow (a)$: If (a) is false, negating quantifiers, there is some value $\varepsilon > 0$ such that for every natural number $n_0 \geq 1$, we find some $n > n_0$ with $|X_n| > (r+\varepsilon)|G_n|$. Hence, the subset

$$Q_\varepsilon = \{n \in \mathbb{N} : |X_n| > (r+\varepsilon)|G_n|\}$$

is infinite. Choose a non-principal ultrafilter $\mathfrak{U}$ containing each Q_ε (which exists). By definition of the non-standard counting measure on the non-standard finite group $G = \prod_{n\to\mathfrak{U}} G_n$, we deduce

$$\mu_G(X) = \lim_{n\to\mathfrak{U}} \frac{|X(G_n)|}{|G_n|} \geq \lim_{n\to\mathfrak{U}} r + \varepsilon = r + \varepsilon.$$

Thus, $\mu_G(X) > r$, which yields the desired contradiction. ∎

Remark 2.5 Taking set-theoretic complements, it follows from Lemma 2.4 that the following two conditions are equivalent:

(1) For every $\varepsilon > 0$, there is some $n_0 = n_0(r,\varepsilon)$ in $\mathbb{N}$ such that $|X_n| \geq (r-\varepsilon)|G_n|$ for all $n > n_0$.
(2) For every non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, we have that $\mu_G(X) \geq r$ for the internal set $X = \prod_{n\to\mathfrak{U}} X_n$ in the non-standard finite group $\prod_{n\to\mathfrak{U}} G_n$.

Remark 2.6 (cf. [6, Section 2]) Carathéodory's Extension Criterion is satisfied by the Loeb measure by $\aleph_1$-saturation (Fact 2.2). Therefore, the Loeb measure μ_G extends to a unique σ-additive measure on the σ-algebra generated by the internal subsets of G. Abusing notation, we will not distinguish between μ_G and its unique extension.

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Furthermore, the family of measures $\{\mu_{G^k}\}_{k\geq 1}$ satisfies the Fubini–Tonelli Theorem [6, Theorem 19], meaning that for any internal subset Z of G^{n+m}, the following holds:

- The function $y \mapsto \mu_{G^n}(Z_y)$, resp., $x \mapsto \mu_{G^m}(Z_x)$, is μ_{G^m}-measurable, resp., μ_{G^n}-measurable.
- We have the equality

$$\mu_{G^{n+m}}(Z) = \int_{G^n} \mu_{G^m}(Z_x)\,\mathrm{d}\mu_{G^n} = \int_{G^m} \mu_{G^n}(Z_y)\,\mathrm{d}\mu_{G^m}.$$

3 Tame families and dense subsets

In view of $\aleph_1$-saturation (Fact 2.2), we introduce the following notion, which appears in [3] as $\bigwedge$-internal and in [14] as Π-definable. For readers familiar with model theory, this corresponds to type-definable sets over a countable parameter set in a countable language.

Definition 3.1 A subset X of G is *ω-internal* if it is a countable intersection of internal subsets of G.

Remark 3.1

(1) Internal sets are ω-internal.
(2) Every countable decreasing chain $(Y_n)_{n\in\mathbb{N}}$ of infinite internal subsets of a non-standard finite group G yields a non-empty ω-internal set. By $\aleph_1$-saturation (Fact 2.2), this ω-internal set is uncountable.
(3) The projection of an ω-internal subset of G^{n+m} onto the first n coordinates is ω-internal in G^n. Indeed, if X is a countable intersection of a decreasing family $(X_k)_{k\in\mathbb{N}}$ of internal subsets of G^{n+1}, Fact 2.2 gives immediately that $\pi(X)$ equals $\bigcap_{k\in\mathbb{N}} \pi(X_k)$.

Example 3.2 Let G be the non-standard finite group $\prod_{n\to\mathfrak{U}} \mathbb{Z}/2^n\mathbb{Z}$, for some non-principal ultrafilter $\mathfrak{U}$. For every natural number $k \geq 1$, the function $f_k : x \mapsto 2^k \cdot x$ from G to G is an internal function, that is, its graph is an internal set. Each internal subgroup $\mathrm{Im}(f_k)$ has finite index in G and they form a strictly decreasing chain, which yields an ω-internal uncountable subgroup of G which is 2-divisible.

In view of Fact 2.2, we will need to restrict our attention to suitable countable families of internal sets. Model-theoretically, this is done by fixing a suitable countable first-order language and considering the corresponding definable sets, which are always internal (see, for example, [3, 20, 33, 14]). For the sake of the presentation, we will avoid introducing the terminology of first-order formulas and provide an approach tailored to our purposes.

Definition 3.2 Let $G = \prod_{n\to\mathfrak{U}} G_n$ be a non-standard finite group. A family $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, where each $\mathcal{F}_n$ consists of internal subsets of G^n, is *tame* if it satisfies the following conditions:

(1) Every internal set of the form:

$$X_w = \{(g_1, \ldots, g_n) \in G^n : w(g_1, \ldots, g_n) = 1_G\},$$

where $w = w(u_1, \ldots, u_n)$ is a (group) word, belongs to $\mathcal{F}_n$.

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(2) $\mathcal{F}$ is closed under finite Cartesian products, Boolean combinations, projections, and permutations of the coordinates.
(3) Given an internal subset Z of G^n in $\mathcal{F}_n$, the internal set

$$\{(x, y) \in G^n \times G^n : x \cdot y \in Z\}$$

belongs to $\mathcal{F}_{2n}$.
(4) Given an internal subset Z of G^{k+m} in $\mathcal{F}_{k+m}$ and a rational number q in $[0,1]$, there is an internal set Y_q in $\mathcal{F}_m$ such that

$$\left\{y \in G^m : \mu_{G^k}(Z_y) > q\right\} \subseteq Y_q \subseteq \left\{y \in G^m : \mu_{G^k}(Z_y) \geq q\right\},$$

as in Remark 2.3. In particular, for every internal subset Z of G^{k+m} in $\mathcal{F}_{k+m}$, the subset

$$\left\{y \in G^m : \mu_{G^k}(Z_y) = 0\right\} = \bigcap_{n \geq 1} \left(G^m \setminus Y_{\frac{1}{n}}\right)$$

is ω-internal and given by a countable intersection of internal subsets in $\mathcal{F}_m$.

Remark 3.3 The reader familiar with the notion of first-order formulas in a given language containing the language of groups will notice that a tame family $\mathcal{F}$ induces a first-order language $\mathcal{L}$ containing the language of groups after adding, for every internal subset X of G^n in $\mathcal{F}$, a distinguished predicate R_X such that its interpretation in G equals X.

Analogously, every language $\mathcal{L}_0$ containing the language of groups can be enlarged to a language $\mathcal{L} \supseteq \mathcal{L}_0$ (as in [20, Section 2.6]) with $|\mathcal{L}| \leq \max(|\mathcal{L}_0|, \aleph_0)$ such that $\mathcal{L}$ induces a tame family of internal sets, by setting

$$X_\psi = \{(a_1, \ldots, a_n) \in G^n : \psi(a_1, \ldots, a_n) \text{ holds in the } \mathcal{L}\text{-structure } G\}$$

for every $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$.

With this translation in mind, we see that certain internal sets automatically belong to a tame family $\mathcal{F}$, without having to write them explicitly as a Boolean combination of suitable projections. For example, if the subset S of $G \times G$ is in $\mathcal{F}$ (and thus *definable*), then so is the set of BMZ-corners

$$\{(a, b) \in G^2 : \text{ for some } g \text{ in } G \setminus \{1_G\}, \text{ all of } (a, b), (g \cdot a, b), (a, g \cdot b) \text{ lie in } S\}.$$

To obtain an example of a countable tame family of internal sets, it suffices to close the family of internal sets given by group words under countably many instances of properties (II)–(IV). More generally, the following holds.

Fact 3.4 *Given any countable family $\mathcal{F}_0$ of internal sets, there is a countable tame family $\mathcal{F}$ of internal sets extending $\mathcal{F}_0$.*

Remark 3.5 By Fact 3.4, whenever we want to apply Lemma 2.4 to a distinguished internal set X in a non-standard finite group, we may always assume that X belongs to a countable tame family.

If the tame family $\mathcal{F}$ arises as the family of definable sets with respect to a fixed language $\mathcal{L}$ as explained in Remark 3.3, then the following notion of richness

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corresponds to the classical model-theoretic notion of an elementary substructure in that particular language.

Definition 3.3 A subset M of a non-standard finite group G is *rich with respect to the tame family* $\mathcal{F}$ if every non-empty fiber X_a of G^n, where a is tuple in M and X is an internal subset of $G^{n+|a|}$ in $\mathcal{F}$, contains an n-tuple whose coordinates all lie in M, i.e., $X_a(M) = X_a \cap M^n \neq \emptyset$.

A straightforward chain argument, mimicking the proof of Downward Löwenheim–Skolem for first-order languages, yields the following result.

Remark 3.6 (Downward Löwenheim–Skolem) Given a countable tame family $\mathcal{F}$ and a countable subset A of a non-standard finite group G, there is a countable subset M of G containing A which is rich with respect to $\mathcal{F}$.

Notice that every rich subset is, in particular, an infinite subgroup of G. However, a countable rich subset M no longer satisfies the $\aleph_1$-saturation condition for covers of internal subsets. Indeed, the M-points of the internal set given by $x = x$ can be covered by countably many singletons (running through every point in M), yet it does not admit a finite cover.

Henceforth, we fix a countable tame family $\mathcal{F}$ of internal sets. All internal and ω-internal sets we shall consider belong to this particular family.

Given a subset A of a non-standard finite group G, we will denote by $\mathcal{F}(A)$ the collection of all fibers Y_a of G^n, with n running over all possible natural numbers, where a is a tuple in A and Y is an internal subset of $G^{n+|a|}$ in $\mathcal{F}$.

Definition 3.4 With the previous notation, the internal subsets in $\mathcal{F}(A)$ are said to be *defined over A*.

An ω-internal set $X = \bigcap_{n\in\mathbb{N}} X_n$ is *defined over A* if each one of the internal sets X_n is defined over the parameter set A.

Remark 3.7 Note that for every internal set Z of G^{k+m} and every rational number q in $[0,1]$, there exists an internal set Y_q as in Condition (IV) of Definition 3.2, which is defined over the same parameters as Z.

Definition 3.5 Consider a non-standard finite group G and a countable subset A of G. Given a tuple b of elements of G, its *type over A* is the ω-internal set

$$\mathrm{tp}(b/A) = \bigcap_{\substack{b\in X \\ X\in\mathcal{F}(A)}} X.$$

Since the set $\mathcal{F}(A)$ is countable, by Remark 3.1(b), the type $\mathrm{tp}(b/A)$ is an uncountable ω-internal set whenever b does not lie in a finite fiber defined over A of some internal set in $\mathcal{F}$ (that is, in model-theoretic terms, whenever the type is not algebraic). Note that $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$ if and only if b' belongs to $\mathrm{tp}(b/A)$.

Remark 3.8 Consider a countable subset A of parameters in a non-standard finite group G. Given two tuples (b,c) and b' such that b' belongs to $\mathrm{tp}(b/A)$, there is some tuple c' of length $k = |c|$ such that (b',c') belongs to $\mathrm{tp}((b,c)/A)$ (denoted henceforth, as is standard, by $\mathrm{tp}(b,c/A)$ for brevity). While for model-theorists, this

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is an easy consequence of $\aleph_1$-saturation, we will include a quick proof for the sake of completeness. Indeed, by $\aleph_1$-saturation, it suffices to show that every finite intersection

$$(Y_1 \cap \cdots \cap Y_n)_{b'} = (Y_1)_{b'} \cap \cdots \cap (Y_n)_{b'} \subseteq G^k$$

is non-empty, where each of the Y_j varies over all internal sets in $\mathrm{tp}(b, c/A)$. Now, the intersection $\bigcap_{j=1}^n Y_j$ belongs to $\mathrm{tp}(b, c/A)$ and thus contains the tuple (b, c). Hence, by Definition 3.2, the projection onto the first $|b|$ coordinates of $\bigcap_{j=1}^n Y_j$ is an internal set defined over A which contains b. Since b' belongs to $\mathrm{tp}(b/A)$, there exists some d such that (b', d) belongs to $\bigcap_{j=1}^n Y_j$, as desired (note that the element d need not be c').

A classical Ramsey argument and the $\aleph_1$-saturation argument in Remark 3.8 allow us to produce, out of a given infinite sequence and a countable set of parameters, a new sequence with a remarkable property known as *indiscernibility*.

Definition 3.6 We say that a sequence $(a_i)_{i\in\mathbb{N}}$ is *indiscernible over* the countable subset A (or A-*indiscernible*) if for every natural number n and every increasing enumeration $i_0 < \cdots < i_{n-1}$, we have that $\mathrm{tp}(a_0, \ldots, a_{n-1}/A)$ equals $\mathrm{tp}(a_{i_0}, \ldots, a_{i_{n-1}}/A)$, that is,

$$(a_0, \ldots, a_{n-1}) \in X \iff (a_{i_0}, \ldots, a_{i_{n-1}}) \in X$$

for every internal subset X in $\mathcal{F}(A)$ of the appropriate arity.

We also define a notion of *density* for a non-standard finite group.

Definition 3.7 Let G be a non-standard finite group.

- An ω-internal subset is *dense* if it is not contained in any internal set of density 0 (see Definition 2.3).
- Given a countable subset A of G and a tuple b of elements of G, we say that b is *dense over* A if the ω-internal subset $\mathrm{tp}(b/A)$ is dense.

Remark 3.9

(1) An internal subset X is dense if and only if it intersects every internal subset of density 1. Notice that the internal set X is dense (seen as an ω-internal subset) if and only if it has positive density. However, an ω-internal set can be dense and yet have density 0 (with respect to the extension of the Loeb measure to the σ-algebra generated by all internal sets, see [6, Remark 16]).

(2) If an ω-internal set $X = \bigcap_{n\in\mathbb{N}} X_n$ is given by a decreasing chain, then the set X is dense if and only if each internal set X_n is. Indeed, if X were not dense, this would be witnessed by an internal set Y of density 0. Thus, the decreasing intersection

$$\bigcap_{n\in\mathbb{N}} (X_n \setminus Y) = \varnothing,$$

which, by $\aleph_1$-saturation, would yield that some $X_n \setminus Y$ must be empty.

(3) If b is dense over A, then so is every element in $\mathrm{tp}(b/A)$.

(4) If b is dense over $A \cup \{g\}$, then so are $b \cdot g$ and $g \cdot b$.

Lemma 3.10 *Let X be an ω-internal dense subset of a non-standard finite group G defined over a countable subset A. If B is a countable subset of*

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parameters extending A, then there exists some element in X which is dense over B.

In particular, if c is dense over A, then there is some c' in $\mathrm{tp}(c/A)$ which is dense over B.

Proof Write $X = \bigcap_{n\in\mathbb{N}} X_n$. By $\aleph_1$-saturation, it suffices to show that

$$X_1 \cap \cdots \cap X_m \cap (G \backslash Y_1) \cap \cdots \cap (G \backslash Y_m) \neq \varnothing$$

for every $m \geq 1$, where $Y_1, \ldots, Y_m$ are fibers of internal sets in $\mathcal{F}$ defined over B and of density 0. If the above intersection were empty, then $X_1 \cap \cdots \cap X_m$ would have density 0, and X would not be dense. ■

Lemma 3.11 *Consider a countable set of parameters A and a finite tuple (b, c) in a non-standard finite group G such that b is dense over $A \cup \{c\}$. If (b', c') belongs to $\mathrm{tp}(b, c/A)$, then b' is dense over $A \cup \{c'\}$.*

Proof Assume for a contradiction that b' is not dense over $A \cup \{c'\}$. By Remark 3.9, there exists a fiber X of an internal set defined over $A \cup \{c'\}$ of density 0 and containing b'. Write $X = Z_{c'}$ for some internal set $Z \subset G^{k+m}$ defined over A, where $k = |b| = |b'|$ and $m = |c| = |c'|$. By Remark 3.7 (after taking set-theoretic complements), there is for every natural number $n \geq 1$ an internal set $Y_{\frac{1}{n}}$ defined over A such that

$$\left\{ y \in G^m : \mu_{G^k}(Z_y) < \frac{1}{n} \right\} \subseteq Y_{\frac{1}{n}} \subseteq \left\{ y \in G^m : \mu_{G^k}(Z_y) \leq \frac{1}{n} \right\}.$$

Now, the internal set $\widetilde{Z}_n = Z \cap \left(G^k \times Y_{\frac{1}{n}} \right)$ is defined over A for every natural number $n \geq 1$ and contains (b', c'), so (b, c) also belongs to the intersection of all the $\widetilde{Z}_n$. We conclude that b is not dense over $A \cup \{c\}$, witnessed by the fiber $\widetilde{Z}_c$, as desired. ■

Model-theoretically, it is convenient to capture the global behavior of an internal (or definable) set in terms of a suitable (dense) element in the set. An example of this translation is the following result, which follows from the Fubini–Tonelli Theorem and Caratheodory's Extension Theorem for the Loeb measure.

Fact 3.12 [28, Lemma 1.10 as well as Remarks 1.13 and 1.14 and Lemma 1.15] *Let G be a non-standard finite group and X be an ω-internal subset of G^n defined over a countable subset A. The following are equivalent:*

(1) *There exists some $(b_1, \ldots, b_n)$ in X in good position over A, that is, each b_i is dense over $A \cup \{b_j : j < i\}$.*
(2) *The set X is dense.*

Furthermore, every tuple as in (a) is dense over A with respect to the Loeb measure on G^n.

Note that if a tuple is in good position, then so is every subtuple.

Definition 3.8 Two internal subsets X and Y of G^n are comparable if the set $X \triangle Y$ has Loeb measure 0.

Remark 3.13 By Lemma 3.10, two internal sets X and Y, both defined over A, are comparable if and only if they contain the same dense elements over A.

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By the Fubini–Tonelli Theorem, two internal subsets X and Y of G^{m+n} are comparable if and only if for μ_{G^n}-almost all elements b in G^n, the fibers X_b and Y_b are comparable.

4 Stability and almost sure stability

Let $k \geq 1$ be a natural number. A *half-graph of height k induced by* the relation $S \subseteq X \times Y$ consists of a sequence $(a_1, b_1, \ldots, a_k, b_k)$ with a_i in X and b_i in Y for each $i = 1, \ldots, k$, such that the pair (a_i, b_j) belongs to S if and only if $i \leq j$. We denote by $\mathcal{H}_k(S)$ the collection of all half-graphs of height k induced by S. Note that for each sequence $(a_1, b_1, \ldots, a_k, b_k)$ in $\mathcal{H}_k(S)$, the elements a_i must be pairwise distinct and similarly for the elements b_i.

Definition 4.1 The relation $S \subseteq X \times Y$ is *k-stable* if it induces no half-graph of height k, or equivalently, if $\mathcal{H}_k(S) = \varnothing$. A subset A of a group G is *k-stable* if the relation given by the Cayley graph of A in G

$$\mathrm{Cay}(G, A) = \{(g, h) \in G \times G : \ g^{-1} \cdot h \in A\}$$

is k-stable, or equivalently, if the relation

$$\Gamma(G, A) = \{(g, h) \in G \times G : \ h^{-1} \cdot g \in A\}$$

is k-stable.

Example 4.1 A non-empty subset A of a group G is 2-stable if and only if it is a coset of a subgroup of G. Indeed, cosets are clearly 2-stable. To verify the other direction, it suffices to show that $a \cdot b^{-1} \cdot c$ belongs to A for every a, b, and c in A. Sidon subsets of abelian groups, such as $2^{\mathbb{N}}$ in $\mathbb{Z}$, are easily seen to be 3-stable [37, Lemma 1.3].

Remark 4.2 Consider a non-standard finite group $G = \prod_{n \to \mathfrak{U}} G_n$ and an internal relation S on $G \times G$ defined over a countable subset A. By Łoś' Theorem (see Remark 2.1(e)), the relation S is k-stable if and only if $S(G_n)$ is k-stable in G_n for $\mathfrak{U}$-almost all n in $\mathbb{N}$. Furthermore, a straightforward $\aleph_1$-saturation argument yields that the internal relation S is k-stable for some $k \geq 2$ if and only if there is no A-indiscernible sequence $(a_i, b_i)_{i \in \mathbb{N}}$ such that the pair (a_i, b_j) belongs to S if and only if $i \leq j$.

The previous remark motivates the following definition.

Definition 4.2 Consider a non-standard finite group $G = \prod_{n \to \mathfrak{U}} G_n$. An ω-internal relation S on $G \times G$ defined over the countable subset A is *stable* if there is no infinite A-indiscernible sequence $(a_i, b_i)_{i \in \mathbb{N}}$ such that the pair (a_i, b_j) belongs to S if and only if $i \leq j$.

Using the Krein–Milman theorem on the locally compact Hausdorff topological real vector space of all σ-additive probability measures, Hrushovski [20, Proposition 2.25] proved the following result. Roughly speaking, it asserts that the relation $R(a, b)$ defined by requiring the measure of the intersection of two associated internal sets X_a and Y_b to exceed a certain threshold is stable. For the purpose of this work, we will state an adapted version of [20, Lemma 2.10], extracting it from the formulation in [29, Fact 2.2 and Corollary 2.3].

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Fact 4.3 *Let G be a non-standard finite group and let α be a real number in $[0,1]$.*

(1) *Given two internal subsets X of G^{n+r} and Y of G^{n+s} in $\mathcal{F}$, the relation $R^{\alpha}_{X,Y}$ of $G^r \times G^s$ defined by*

$$R^{\alpha}_{X,Y}(a,b) \;\Leftrightarrow\; \mu_{G^n}(X_a \cap Y_b) \leq \alpha$$

is stable. Notice that $R^{\alpha}_{X,Y}$ is ω-internal by Definition 3.2(IV), and definable over the same parameters needed to define both X and Y, by Remark 3.7.

(2) *If the two subsets X of G^{n+r} and Y of G^{n+s} are ω-internal and defined over A, the relation $R_{X,Y}$ of $G^r \times G^s$ defined by*

$$R_{X,Y}(a,b) \;\Leftrightarrow\; \text{the } \omega - \text{internal subset } X_a \cap Y_b \text{ of } G^n \text{ is not dense}$$

is equational, that is, there is no A-indiscernible sequence $(a_i, b_i)_{i\in\mathbb{N}}$ such that $X_{a_0} \cap Y_{b_0}$ is dense yet $X_{a_0} \cap Y_{b_1}$ is not.

(3) *Equationality implies the following (see [29, Remark 2.1 and Corollary 2.3]). For every ω-internal subset X of G^{n+r} defined over a countable rich subset M, if the fiber X_a is dense for some a which is itself dense over M, then so is the intersection $\bigcap_{i=0}^{m} X_{b_i}$ for every m in $\mathbb{N}$ and every tuple $(b_0, \ldots, b_m)$ of elements in $\mathrm{tp}(a/M)$ in good position over M, as in Fact 3.12(a), that is, the element b_i is dense over $M \cup \{b_j : j < i\}$ for $i \leq m$.*

While the notion of stability has proved highly profitable in model theory, from a more combinatorial perspective, it is open to the criticism that whether or not a relation is stable is very sensitive to minor perturbations. In particular, modifications by an internal set of measure 0 can destroy stability, as the following example illustrates.

Example 4.4 For some non-principal ultrafilter $\mathfrak{U}$, consider the non-standard finite group $G = \prod_{n\to\mathfrak{U}} \mathbb{Z}/n^2\mathbb{Z}$ as well as the internal set $X = \prod_{n\to\mathfrak{U}} \{\overline{0}, \ldots, \overline{n-1}\}$ and the internal relation S on $X \times X$ arising from the standard linear order of $\mathbb{Z}$ restricted to the set of representatives $\{0, \ldots, n-1\}$. By Łoś's theorem (see Remark 2.1(e)), the collection $\mathcal{H}_k(S)$ of induced half-graphs of height k is non-empty for every natural number $k \geq 1$, so the internal relation S is never stable. However, for every index $k \geq 1$, the internal set $\mathcal{H}_k(S)$ has $\mu_{G^{2k}}$-measure 0, so it is comparable to the empty relation, which in turn is k-stable.

Motivated by the above example, we introduce a weakening of stability, which is combinatorially robust with respect to perturbations by sets of measure 0.

Definition 4.3 Consider a non-standard finite group G as well as two internal subsets X of G^n and Y of G^m. Given a natural number $k \geq 1$, an internal relation $S \subseteq X \times Y$ is *almost surely k-stable* if $\mu_{G^{(n+m)k}}(\mathcal{H}_k(S)) = 0$, that is, the collection of all half-graphs of height k induced by S is negligible in $G^{(n+m)k}$ with respect to the non-standard counting measure.

Remark 4.5

(1) It is immediate that almost sure stability is preserved by finite Boolean combinations as well as by permutation of the variables: if $S \subseteq X \times Y$ is almost surely k-stable, then so is the inverse relation

$$S^{\mathrm{opp}} = \{(y,x) \in Y \times X : \; (x,y) \in S\}.$$

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(2) Fact 3.12 yields that the internal relation S defined over the countable subset A is not almost surely k-stable if and only if it induces a half-graph of height k witnessed by a dense tuple $(a_1, b_1, \dots, a_k, b_k)$ in good position over A.

As a consequence, whenever a relation S is comparable to an internal k-stable or even an almost surely k-stable relation S', Remark 3.13 yields that S is almost surely k-stable.

We will see in Proposition 4.9 below that every almost surely stable relation $S \subseteq X \times Y$ can be approximated for every $\varepsilon > 0$ by a finite union of internal boxes of the form $X' \times Y'$. Hence, the almost surely k-stable relation is ε-comparable to a stable relation S', although our methods do not allow us to compute the degree of stability of S'.

In fact, the (bipartite version of the) induced removal lemma [15] implies that every almost surely k-stable must be comparable to a k-stable relation, but we do not know how to give a model-theoretic proof of this result. So we ask the following:

Question *Is there a model-theoretic account that every almost surely k-stable internal relation is comparable (or ε-comparable) to an internal k-stable relation?*

A straightforward application of Lemma 2.4 (with $r = 0$) yields an analog of almost sure stability for families of finite groups in terms of their asymptotic behavior.

Remark 4.6 Consider a family $(G_\ell, X_\ell, Y_\ell, S_\ell)_{\ell \in \mathbb{N}}$, where for each $\ell \in \mathbb{N}$, G_ℓ is a finite group, $X_\ell \subseteq G_\ell^n$ and $Y_\ell \subseteq G_\ell^m$ for some $n, m \in \mathbb{N}$, and $S_\ell \subseteq X_\ell \times Y_\ell$. The following are equivalent for every natural number $k \geq 1$:

(1) For every $\theta > 0$, there is some $\ell_0 = \ell_0(k, \theta)$ in $\mathbb{N}$ such that $|\mathcal{H}_k(S_\ell)| \leq \theta |G_\ell|^{k(n+m)}$ for all $\ell \geq \ell_0$, where $\mathcal{H}_k(S_\ell)$ is the collection of all half-graphs of height k induced by S_ℓ on the finite group G_ℓ.

(2) For every non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, the internal relation $S = \prod_{n \to \mathfrak{U}} S_\ell$ is almost surely k-stable.

If (a) holds, we say that $(G_\ell, X_\ell, Y_\ell, S_\ell)_{\ell \in \mathbb{N}}$ is an *almost surely k-stable family*.

A key feature of stable relations is that they are *stationary* [20, Lemma 2.3], in the sense that their truth value is constant along the set of pairs of realizations which are *non-forking independent*. Non-forking independence is a fundamental notion in model theory, originally due to Shelah, defined in combinatorial terms for any structure [38, Chapter III.1, Definition 1.4]. We will not need to introduce non-forking independence in this article. Instead, inspired by classical results in model theory, we will show that the truth value of an almost surely stable relation remains constant along the set of pairs of realizations for which one coordinate is dense over the other.

Before stating the corresponding result, we introduce some notation.

Definition 4.4 Fix two types $\mathrm{tp}(a/M)$ and $\mathrm{tp}(b/M)$ over a countable rich subset M. We denote by $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ the set of all pairs (a', b') with a' in $\mathrm{tp}(a/M)$ and b' in $\mathrm{tp}(b/M)$ such that (a', b') or (b', a') is in good position over M (see Fact 3.12).

Note that the set $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ is empty exactly if one of the types $\mathrm{tp}(a/M)$ or $\mathrm{tp}(b/M)$ is not dense, by Lemma 3.10. Moreover,

$$(a', b') \in \mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M)) \iff (b', a') \in \mathrm{GP}(\mathrm{tp}(b/M), \mathrm{tp}(a/M)).$$

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Theorem 4.7 *Consider a non-standard finite group G as well as two internal subsets X of G^n and Y of G^m and an internal relation S on $X \times Y$, all defined over a countable rich subset M.*

If S is almost surely k-stable for some $k \geq 1$, then for every two (dense) types $\mathrm{tp}(a/M)$ and $\mathrm{tp}(b/M)$, the set $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ is either contained in S or disjoint from S.

Proof If a does not lie in X, neither does any element of $\mathrm{tp}(a/M)$, and likewise if b does not lie in Y. In this case, $X \times Y$ (and thus S) is clearly disjoint from $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$. So we may assume that $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ is a subset of $X \times Y$.

Suppose now that S is not disjoint from $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$. Up to relabeling, we may assume by Remark 4.5(a) that (a, b) lies in S, with (a, b) in good position, that is, the element b is dense over $M \cup \{a\}$ (keeping in mind that a is already dense over M).

Claim *All pairs (c, d) in $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ with (d, c) in good position over M are contained in S.*

Proof of Claim Suppose to the contrary that there is a pair (c, d) with c dense over $M \cup \{d\}$ not contained in S. By Remark 3.8 (using the fact that d belongs to the type $\mathrm{tp}(b/M)$), there is a pair (a', b) in $\mathrm{tp}(c, d/M)$. So the original tuple (a, b) belongs to S but (a', b) does not lie in S (for S is definable over M). In particular, we have two pairs (a, b) and (a', b) satisfying the following:

- the type $\mathrm{tp}(a/M) = \mathrm{tp}(a'/M)$ is dense;
- the element b is dense over $M \cup \{a\}$ and a' is dense over $M \cup \{b\}$ (by Lemma 3.11);
- the pair (a, b) belongs to S but (a', b) does not.

We will now construct inductively, for every n in $\mathbb{N}$, a dense sequence $(a_0, b_0, \ldots, a_n, b_n)_{0 \leq i \leq n}$ in good position over M witnessing that S induces a half-graph of height n such that a_i belongs to $\mathrm{tp}(a/M)$ and b_i to $\mathrm{tp}(b/M)$ for each $i \leq n$. In particular, the case $n = k$ contradicts the almost sure stability of S by Remark 4.5(b).

For $n = 0$, set $a_0 = a$ and $b_0 = b$, so (a_0, b_0) is in good position over M and lies in S, as desired. Suppose now that the sequence $(a_0, b_0, \ldots, a_r, b_r)$ has already been constructed for some $r < n$. Consider the ω-internal set

$$Z = \{(x, y) \in X \times Y : \ x \in \mathrm{tp}(a/M) \text{ and } (x, y) \notin S\},$$

which is defined over M. Clearly, (a', b) belongs to Z, so the fiber $Z_b \subseteq X$ is dense. By Fact 4.3(c), so is the intersection $Z_{b_0} \cap \cdots \cap Z_{b_r}$, since the subtuple $(b_0, \ldots, b_r)$ is also in good position over M and all coordinates lie in $\mathrm{tp}(b/M)$. Hence, there is an element a_{r+1} in the above intersection which is dense over $M \cup \{a_i, b_i\}_{0 \leq i \leq r}$. In particular, the element a_{r+1} lies in $\mathrm{tp}(a/M)$ but (a_{r+1}, b_i) does not lie in S for $i \leq r$.

Similarly, the ω-internal set

$$W = \{(x, y) \in X \times Y : \ y \in \mathrm{tp}(b/M) \text{ and } (x, y) \in S\}$$

is also defined over M and the fiber $W_a \subseteq Y$ is dense, since (a, b) lies in W. Thus, there exists an element b_{r+1} in $\mathrm{tp}(b/M)$ contained in $W_{a_0} \cap \cdots \cap W_{a_{r+1}}$ dense over

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$M \cup \{a_i, b_i\}_{0 \leq i \leq r} \cup \{a_{r+1}\}$, which yields that the tuple

$$(a_0, b_0, \ldots, a_{r+1}, b_{r+1})$$

is in good position over M by Fact 3.12 and satisfies the desired properties. ■Claim

By the previous claim, choose any pair (a', b') in $S \cap \mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ such that a' is dense over $M \cup \{b'\}$. The pair (b', a') is in good position over M and it belongs to the inverse relation S^{opp}, which is again almost surely k-stable, by Remark 4.5(a). We conclude by the Claim (applied to S^{opp} inverting the roles of X and Y) that all pairs (b'', a'') in $\mathrm{GP}(\mathrm{tp}(b/M), \mathrm{tp}(a/M))$ with b'' dense over $M \cup \{a''\}$ must be contained in S^{opp}. Hence, all pairs (a'', b'') in $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ in good position over M belong to S, so the set $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$ is fully contained in S, as desired. ■

Corollary 4.8 *Consider a non-standard finite group G as well as two internal subsets X of G^n and Y of G^m and an internal relation S on $X \times Y$, all defined over a countable rich subset M. If S is almost surely k-stable for some $k \geq 1$, then the fibers S_a and $S_{a'}$ are comparable whenever a' belongs to the dense type* $\mathrm{tp}(a/M)$.

Proof If the set-theoretic difference $S_a \backslash S_{a'}$ were dense for some choice a and a', both in the same dense type over M, then it would contain some element b which is dense over $M \cup \{a, a'\}$. In particular, both (a, b) and (a', b) would lie in $\mathrm{GP}(\mathrm{tp}(a/M), \mathrm{tp}(b/M))$, yet (a, b) belongs to S but (a', b) does not, which contradicts Theorem 4.7. ■

Proposition 4.9 *Consider a non-standard finite group G as well as two internal subsets X of G^n and Y of G^m and an internal relation S on $X \times Y$, all defined over a countable rich subset M. If S is almost surely k-stable for some $k \geq 1$, then for every $\varepsilon > 0$, there is a finite (possibly empty) union of boxes*

$$U = \bigcup_{i=1}^{\ell} (X_i \times Y_i),$$

with all $X_i \times Y_i \subseteq X \times Y$, defined over M, such that $\mu(U \triangle S) < \varepsilon$.

Note that every box is 2-stable and thus the finite union U is r-stable for some r, which can be explicitly computed in terms of the number ℓ of boxes occurring in U. However, at the time of writing, we are unable to give an explicit bound on ℓ or show that U and S are actually comparable.

Proof If S is not dense in G^{n+m}, then it suffices to take U to be the empty union of boxes. Thus, we may assume that the internal subset S of G^{n+m} is dense.

Claim 1 *Consider an arbitrary subset Z of S in the σ-algebra of internal sets defined over M. (Note that Z need not be ω-internal, but it is measurable with respect to the extension of the Loeb measure.) If Z has positive measure, then there exists a dense element a in G^n such that the fiber Z_a is dense and satisfies the following property ($\star$): for every internal subset W of Y defined over M, if $\mu_{G^m}(W \backslash S_a) = 0$, then*

$$\mu_{G^n}\big(\{x \in G^n : \mu_{G^m}(W \backslash S_x) = 0\}\big) > 0.$$

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Proof of Claim 1 By Fubini–Tonelli (Remark 2.6), the measurable set

$$Z_1 = \{x \in G^n : \mu_{G^m}(Z_x) > 0\}$$

has positive measure (since $\mu_{G^{n+m}}(Z)$ is not zero). For every internal subset W of Y defined over M, set

$$D_W = \{x \in G^n : \mu_{G^m}(W \backslash S_x) = 0\}.$$

Notice that there are only countably many such sets D_W. Consider the subfamily $\mathcal{D}$ of all sets D_W of measure 0. Similarly, let $\mathcal{E}$ be the countable family of all internal subsets X' of G^n defined over M of measure 0. By σ-additivity of the (extension of the) Loeb measure, the set

$$Z_1 \backslash \Big(\bigcup_{D_W \in \mathcal{D}} D_W \cup \bigcup_{X' \in \mathcal{E}} X' \Big)$$

has positive measure, so it is non-empty. Now choose some element a in the above set-theoretic difference. By the choice of $\mathcal{E}$, the type $\mathrm{tp}(a/M)$ is dense and so is the fiber Z_a (since a lies in Z_1). Furthermore, the element a satisfies $(\star)$ by our choice of the family $\mathcal{D}$. ■Claim 1

As the internal set S is dense, it has positive measure. By the previous claim, there is an element a which is dense over M satisfying $(\star)$ with respect to $Z = S$ such that the fiber S_a is dense. Thus, the fiber S_a contains an element b which is dense over $M \cup \{a\}$. By Fact 3.12, the dense pair (a, b) is in good position over M and clearly lies in S.

Claim 2 *For every pair (a, b) in S in good position over M such that a satisfies $(\star)$, there are an internal subset Y' of Y and an ω-internal subset X' of X, both defined over M, such that (a, b) lies in $X' \times Y'$ and $\mu((X' \times Y') \backslash S) = 0$.*

Proof of Claim 2 By Theorem 4.7, there is no element in $\mathrm{tp}(b/M)$ dense over $M \cup \{a\}$ outside of S_a, so the ω-internal subset $\mathrm{tp}(b/M) \backslash S_a$ of Y is not dense. By $\aleph_1$-saturation, there is some internal subset Y' of Y defined over M containing b with $\mu_{G^m}(Y' \backslash S_a) = 0$. We deduce by $(\star)$ that $\mu_{G^n}(X') > 0$, where

$$X' = \{x \in X : \mu_{G^m}(Y' \backslash S_x) = 0\}$$

is an ω-internal subset (by Definition 3.2(IV)) containing a (this set is $D_{Y'}$ with $Z = S$ in the notation of Claim 1).

A straightforward computation using Fubini–Tonelli yields

$$\mu_{G^{n+m}}((X' \times Y') \backslash S) = \int_{X'} \mu_{G^m}(Y' \backslash S_x) \, \mathrm{d}\mu_{G^n} = 0,$$

as desired. ■Claim 2

Consider now the countable collection $\mathcal{B}_S$ of all subsets $X' \times Y'$ as in Claim 2, that is, the set Y' is internal while X' is ω-internal, both are defined over M and

$$X' = \{x \in X : \mu_{G^m}(Y' \backslash S_x) = 0\},$$

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so $\mu_{G^{n+m}}((X' \times Y')\backslash S) = 0$. By σ-additivity, we have that

$$\left(\bigcup_{X' \times Y' \in \mathcal{B}_S} (X' \times Y') \right) \backslash S$$

has measure 0 (with respect to the extension of the Loeb measure). Observe further that the set

$$Z = S \backslash \bigcup_{X' \times Y' \in \mathcal{B}_S} (X' \times Y')$$

belongs to the σ-algebra of internal sets defined over M. If Z has positive measure, we deduce as in the discussion right after Claim 1 that Z contains a pair (a, b) in good position over M with a satisfying $(\star)$. It follows from Claim 2 that (a, b) lies in some box $X' \times Y'$ in $\mathcal{B}_S$, contradicting the choice of Z. Thus, S and $\bigcup_{X' \times Y' \in \mathcal{B}_S} (X' \times Y')$ are comparable with respect to the extension of the Loeb measure. We need only show that we can replace the latter union, modulo an ε-error, by an actual finite union of internal boxes defined over M.

Given $\varepsilon > 0$, by continuity of the Loeb measure, there is some integer ℓ and subsets $X'_1 \times Y_1, \ldots, X'_\ell \times Y_\ell$ in $\mathcal{B}_S$ such that

$$\mu_{G^{n+m}} \left(S \backslash \bigcup_{i=1}^{\ell} (X'_i \times Y_i) \right) < \frac{\varepsilon}{2}.$$

Note that $\ell > 0$, since $\mathcal{B}_S$ is non-empty, for S is assumed to be dense. For each $1 \leq i \leq \ell$, the ω-internal subset $(X'_i \times Y_i)\backslash S$ has measure 0, so we can find internal subsets X_i, with each X_i containing X'_i, such that

$$\mu_{G^{n+m}}\big((X_i \times Y_i)\backslash S\big) < \frac{\varepsilon}{2\ell}.$$

Note that X_i is again internal and defined over M. Now set $U = \bigcup_{i=1}^{\ell} (X_i \times Y_i)$, so U contains $\bigcup_{i=1}^{\ell} (X'_i \times Y_i)$, and it follows that

$$\begin{aligned} \mu_{G^{n+m}}(S \triangle U) &= \mu_{G^{n+m}}(S \backslash U) + \mu_{G^{n+m}}(U \backslash S) \\ &\leq \mu_{G^{n+m}} \left(S \backslash \bigcup_{i=1}^{\ell} (X'_i \times Y_i) \right) + \sum_{i=1}^{\ell} \mu_{G^{n+m}} ((X_i \times Y_i) \backslash S) \\ &< \frac{\varepsilon}{2} + \ell \cdot \frac{\varepsilon}{2\ell} = \varepsilon. \end{aligned}$$

This completes the proof of Proposition 4.9. ∎

5 Corners and squares

Observe that the results in Section 4 did not make use of the underlying group structure. In this section, on the other hand, the group structure will play a fundamental role in guaranteeing the existence of certain two-dimensional patterns in a dense almost surely stable relation.

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Given a countable rich subset M, the group $G(M) = G \cap M$ of M-rational points of G naturally acts on the collection of types over M by

$$\mathrm{tp}(a/M) \star g = \mathrm{tp}(a \cdot g/M).$$

This is a right action, but there is also a natural left action, defined analogously. With respect to the previous right action, we can thus consider the stabilizer in $G(M)$ of a type. This subgroup need not be definable in general. However, in the presence of stability, the stabilizer of a type becomes ω-internal (or type-definable), though we will not need this for the purpose of this article.

Since the seminal work of Kim and Pillay [25] on simple theories, many notions and techniques from geometric stability have been adapted successfully to other contexts. Pushing beyond simplicity, in groundbreaking work [20], Hrushovski established the existence of an ω-internal subgroup of a non-standard finite group that plays the role of the *stabilizer* of *every* dense type over the countable rich substructure M. This subgroup is known in model theory as the *connected component* of G over M and denoted by G^{00}_M.

The next fact summarizes the content of Hrushovski's *stabilizer theorem*, tailored to our particular context. For ease of reference, the presentation largely follows the formulation of [31, Theorem 2.12].

Fact 5.1 ([20, Theorem 3.5] & [31, Theorem 2.12]) *Let M be a countable rich subset of a non-standard finite group G. Then, there exists an ω-internal normal subgroup G^{00}_M defined over M with the following properties:*

(1) *The ω-internal subset G^{00}_M equals a countable intersection of internal generic symmetric neighborhoods Z_n of the identity, each defined over M, that is,*
   - *each $Z_n = Z_n^{-1}$ is symmetric and contains the identity element 1_G;*
   - *finitely many translates of each Z_n cover the group G;*
   - *$Z_{n+1} \cdot Z_{n+1} \subseteq Z_n$ for all n in $\mathbb{N}$.*

(2) *Whenever (a, b) is in good position over M with b in $\mathrm{tp}(a/M)$, the dense element $b \cdot a^{-1}$ over M belongs to G^{00}_M. In particular, the ω-internal set G^{00}_M is dense.*

(3) *Whenever a is dense over M and g in G^{00}_M is dense over M, the ω-internal set*

$$\mathrm{tp}(a/M) \cap \mathrm{tp}(a/M) \cdot g$$

is again dense, that is, there exists some a' in $\mathrm{tp}(a/M)$ which is dense over $M \cup \{g\}$ such that $a' \cdot g$ belongs to $\mathrm{tp}(a/M)$.

Similarly, the ω-internal set $\mathrm{tp}(a/M) \cap g \cdot \mathrm{tp}(a/M)$ is also dense, so there exists some a'' in $\mathrm{tp}(a/M)$ which is dense over $M \cup \{g\}$ such that $g \cdot a''$ belongs to $\mathrm{tp}(a/M)$.

(4) *If N is a countable rich subset of G containing M, then G^{00}_N is a subgroup of G^{00}_M.*

Furthermore, it follows that if G has finite exponent, then G^{00}_M is a countable intersection of internal subgroups, each defined over M and of finite index: Indeed, we can equip the quotient group G/G^{00}_M with the so-called logic topology, according to which a subset in the quotient is closed if its preimage is ω-internal and defined over M. The quotient group G/G^{00}_M is a compact Hausdorff topological group of finite exponent, and thus profinite by [22, Theorem 4.5]. Since M and $\mathcal{F}$ are countable, the logic topology is second countable, so there is a fundamental system of open normal subgroups H_n/G^{00}_M with $G^{00}_M = \bigcap_n H_n$.

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Now, each quotient H_n/G_M^{00} is closed of finite index, by topological compactness, and thus the preimage H_n is internal and defined over M, as desired.

Since the ω-internal normal subgroup G_M^{00} equals a countable intersection of internal generic symmetric neighborhoods Z_n, each defined over M, the elements a' of any given type $\mathrm{tp}(a/M)$ all lie in the same coset $a \cdot G_M^{00}$. Indeed, we need only show that $a^{-1} \cdot a'$ belongs to Z_n for every n in $\mathbb{N}$. Now, the subset M is rich, so finitely many translates of each Z_n by elements of $G(M)$ cover G. In particular, the element a belongs to $g \cdot Z_{n+1}$ for some g in $G(M)$ and therefore so does a'. Hence, we deduce that

$$a^{-1} \cdot a' = a^{-1} \cdot (g \cdot g^{-1}) \cdot a' = (g^{-1} \cdot a)^{-1} \cdot (g^{-1} \cdot a')$$

belongs to $(Z_{n+1})^{-1} \cdot Z_{n+1} = Z_{n+1} \cdot Z_{n+1} \subset Z_n$, as desired.

We now have all the ingredients in order to state our main result.

Theorem 5.2 *Consider a non-standard finite group G along with an internal relation S on $G \times G$ defined over a countable rich subset M. If S is dense and almost surely k-stable for some $k \geq 1$, then for every g in G_M^{00} which is dense over M, the internal set*

$$\Lambda_\square(S)_g = \{(x, y) \in G \times G : \ (x, y), (x \cdot g, y), (x, y \cdot g) \text{ and } (x \cdot g, y \cdot g) \text{ all lie in } S\}$$

has positive density.

Proof Fix some element g in G_M^{00} which is dense over M. In order to show that the internal set $\Lambda_\square(S)_g$ has positive density, we need only show that $\Lambda_\square(S)_g$ contains a pair (a, b) dense over $M \cup \{g\}$. We do so with the help of the following auxiliary claims.

Claim 3 *There is a tuple (c, d) in good position over M with c and d in $\mathrm{tp}(g/M)$ such that $g = c^{-1} \cdot d$.*

Proof of Claim 3 By Fact 5.1 applied to the dense element g^{-1} of G_M^{00} and the dense type $\mathrm{tp}(g/M)$, we deduce that there exists some element a' in $\mathrm{tp}(g/M)$ dense over $M \cup \{g\}$ such that $g'' = g^{-1} \cdot a'$ belongs to $\mathrm{tp}(g/M)$. Hence, for the element g'' in $\mathrm{tp}(g/M)$, there are two elements $c' = g$ and $d' = a'$ in $\mathrm{tp}(g/M)$ with (c', d') in good position over M such that $g'' = (c')^{-1} \cdot d'$. By Remark 3.8 and Lemma 3.11, we deduce that the same is true for every element in $\mathrm{tp}(g''/M) = \mathrm{tp}(g/M)$. We thus obtain the desired tuple (c, d) as in the statement with $g = c^{-1} \cdot d$. ■Claim 3

Claim 4 *The internal set $S \cap S \cdot (1_G, g^{-1})$ is dense.*

Proof of Claim 4 Recall that S is dense and thus so is $S \cdot (1_G, g^{-1})$. Hence, by Fact 4.3(c), so is the intersection $S \cdot (1_G, c^{-1}) \cap S \cdot (1_G, d^{-1})$, since the tuple (c, d) as in Claim 3 is in good position over M and both c and d lie in the dense type $\mathrm{tp}(g/M)$. Multiplying on the right by $(1_G, c)$, we deduce that $S \cap S \cdot (1_G, g^{-1})$ is dense, as desired. ■Claim 4

By Claim 4 and Fact 3.12, there is a pair (a_0, b) in good position over $M \cup \{g\}$ contained in $S \cap S \cdot (1_G, g^{-1})$. That is, both pairs (a_0, b) and $(a_0, b \cdot g)$ belong to S.

Note that we have not yet used that S is almost surely stable. The idea is to replace a_0 by a suitable realization a such that the product $a \cdot g$ lies again in $\mathrm{tp}(a/M)$.

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Indeed, Fact 5.1(c) yields that there is some a in $\mathrm{tp}(a_0/M)$ dense over $M \cup \{g\}$ such that $a \cdot g$ lies again in $\mathrm{tp}(a_0/M)$. We may assume that a is dense over $M \cup \{b, g\}$, by Lemma 3.10. Clearly, each of the pairs (b, a), $(b, a \cdot g)$, $(b \cdot g, a)$, and $(b \cdot g, a \cdot g)$ is in good position over M.

Theorem 4.7 implies that

$$\mathrm{GP}(\mathrm{tp}(a_0/M), \mathrm{tp}(b/M)) \subseteq S$$

since the pair (a_0, b) lies in S and is in good position over M. Thus, both pairs (a, b) and $(a \cdot g, b)$ lie in S.

Now, the pair $(a_0, b \cdot g)$ also lies in S and is in good position over M. Again by Theorem 4.7, the pairs $(a, b \cdot g)$ and $(a \cdot g, b \cdot g)$ of $\mathrm{GP}(\mathrm{tp}(a'/M), \mathrm{tp}(b \cdot g/M))$ must lie in S. In particular, the pair (a, b) is dense over $M \cup \{g\}$ and belongs to the internal set $\Lambda_{\square}(S)_g$, as desired. ∎

A straightforward application of Fubini–Tonnelli (Remark 2.6) yields the following corollary.

Corollary 5.3 *Consider a non-standard finite group G along with an internal relation S on $G \times G$ defined over a countable rich subset M. If S is dense and almost surely k-stable for some $k \geq 1$, then the internal set*

$$\Lambda_{\square}(S) = \left\{(x, y, g) \in G^3 : \ (x, y), (x \cdot g, y), (x, y \cdot g), \text{ and } (x \cdot g, y \cdot g) \text{ all lie in } S\right\}$$

has positive density.

In particular, S contains a dense collection of both BMZ and naive corners, as remarked in the introduction. Moreover, if G is abelian, then there is a dense collection of triples (x, y, d) in G^3 such that (x, y), $(x + d, y)$, $(x, y + d)$, $(x + d, y + d)$ form a square in S.

In the special case of a binary relation given by the Cayley graph of a subset A of G, the proof of Theorem 5.2 yields a stronger result, since every dense element g in G_M^{00} is a popular side length for many corners.

Proposition 5.4 *Consider a non-standard finite group G and an internal almost surely k-stable dense subset A of G defined over a countable rich subset M. For every g in G_M^{00} which is dense over M, the sets A and $g \cdot A$ are comparable, that is,*

$$\mu_G(A \triangle (g \cdot A)) = 0.$$

In particular, whenever the element a in A is dense over $M \cup \{g\}$, $g^m \cdot a$ belongs to A for all m in $\mathbb{Z}$. Hence, the ω-internal set $\bigcap_{m \in \mathbb{Z}} g^m \cdot A$ is dense with measure $\mu_G(A) > 0$.

Proof As in Claim 3 of Theorem 5.2, write $g = c^{-1} \cdot d$, where (c, d) are in good position over M and both lie in the same type $\mathrm{tp}(g/M)$. Suppose toward a contradiction that $\mu_G(A \triangle (g \cdot A)) > 0$, or equivalently, that

$$\mu_G\big((c \cdot A) \triangle (d \cdot A)\big) > 0.$$

Without loss of generality, we may assume that $(c \cdot A) \backslash (d \cdot A)$ is dense, so it contains some element b which is dense over $M \cup \{c, d\}$. This means that $c^{-1} \cdot b$ belongs to A but $d^{-1} \cdot b$ does not, or equivalently, the pair (c, b) belongs to the almost

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surely k-stable relation $\mathrm{Cay}(G, A)$, but (d, b) does not. Since both pairs belong to $\mathrm{GP}(\mathrm{tp}(g/M), \mathrm{tp}(b/M))$, this contradicts Theorem 4.7.

To prove the final part, observe that since A and $g \cdot A$ are comparable, so are A and $g^{-1} \cdot A$. By continuity, we need only show that each intersection

$$\bigcap_{-m \leq i \leq m} g^i \cdot A$$

has constant measure $\mu_G(A)$ for all m in $\mathbb{Z}$. Otherwise, the internal set

$$A \setminus \bigcap_{-m \leq i \leq m} g^i \cdot A$$

is dense, so by Lemma 3.10, we may choose an element a in A which is dense over $M \cup \{g\}$. Remark 3.13 implies that a must belong to both $g \cdot A$ and $g^{-1} \cdot A$, and inductively, we conclude that a belongs to $g^i \cdot A$ for all $-m \leq i \leq m$, which gives the desired contradiction. ∎

We conclude this section by deducing a finitary (albeit ineffective) version of Corollary 5.3, using Lemma 2.4 and the accompanying Remark 2.5, and strengthening it in two special cases.

Corollary 5.5 *Given an integer $k \geq 1$ and a real number $\delta > 0$, there is an integer $\ell_0(k, \delta) \geq 1$ and real numbers $\theta = \theta(k, \delta) > 0$ and $\varepsilon = \varepsilon(k, \delta) > 0$ with the following property.*

Let G be a finite group of order $|G| \geq \ell_0$ and let $S \subseteq G \times G$ be a relation of size $|S| \geq \delta |G|^2$ such that the collection $\mathcal{H}_k(S)$ of all half-graphs of height k induced by S on G has size $|\mathcal{H}_k(S)| \leq \theta |G|^{2k}$. Then, $\Lambda_\square(S)$ has size at least $\varepsilon |G|^3$.

In particular, the relation S contains a (nontrivial) square provided that $|G| \geq 1/\varepsilon$.

Proof The proof proceeds by contradiction. Suppose there are $k \geq 1$ and $\delta > 0$ such that for every $\ell \geq 1$ (setting $\theta = \varepsilon = 1/\ell$), we can find a finite group G_ℓ of order at least ℓ and a relation $S_\ell \subseteq G_\ell \times G_\ell$ of density δ such that $\mathcal{H}_k(S_\ell) \leq \frac{1}{\ell} |G_\ell|^{2k}$, yet $|\Lambda_\square(S_\ell)| < |G_\ell^3|/\ell$.

In particular, the family $(G_\ell, S_\ell)_{\ell \in \mathbb{N}}$ is almost surely k-stable, as in Remark 4.6. Take a non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and consider the non-standard finite group $G = \prod_{\ell \to \mathfrak{U}} G_\ell$ as well as the internal set $S = \prod_{\ell \to \mathfrak{U}} S_\ell$. By Remark 2.5, the set S has Loeb measure $\mu_{G^2}(S) \geq \delta$ and is almost surely k-stable by Remark 4.6. Choose any countable rich subset M. Corollary 5.3 yields that $\mu_{G^3}(\Lambda_\square(S))$ is at least η for some $\eta > 0$.

Remark 2.4 applied to $r = \eta$ yields the desired contradiction, since $\Lambda_\square(S_\ell)$ has size at most $\eta |G_\ell|^3/2$ for sufficiently large ℓ. ∎

In the particular case that the group in question has bounded exponent, we can strengthen Corollary 5.5 using Fact 5.1 to obtain a subgroup of bounded index almost all of whose elements witness the existence of a square.

Corollary 5.6 *Given integers $k, r \geq 1$ and real numbers $\delta, \varepsilon > 0$, there is an integer $\ell = \ell(k, r, \delta, \varepsilon) \geq 1$ and real numbers $\theta = \theta(k, r, \delta, \varepsilon) > 0$ and $\eta = \eta(k, r, \delta, \varepsilon) > 0$ with the following property.*

Let G be a finite group of exponent bounded by r, and consider a relation $S \subseteq G \times G$ of size $|S| \geq \delta |G|^2$ such that the collection $\mathcal{H}_k(S)$ of all half-graphs of height k induced

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by S on G has size $|\mathcal{H}_k(S)| \leq \theta|G|^{2k}$. Then, there exists a subgroup H of G of index at most ℓ such that

$$\left|\left\{g \in H : \ |\Lambda_\square(S)_g| < \eta|S|\right\}\right| < \varepsilon|H|,$$

where $\Lambda_\square(S)_g \subseteq S$ is as in Theorem 5.2.

Proof As in the proof of Corollary 5.5, negating quantifiers, we deduce that for a fixed choice of k, r, δ, and ε, for every ℓ in $\mathbb{N}$ (setting $\theta = \eta = 1/\ell$), there is a finite group G_ℓ of exponent at most r and a relation $S_\ell \subseteq G_\ell \times G_\ell$ of density δ such that

$$|\mathcal{H}_k(S_\ell)| \leq \frac{1}{\ell}|G_\ell|^{2k},$$

yet for every subgroup $H \leqslant G_\ell$ of index at most ℓ, the subset of elements g in H with

$$|\Lambda_\square(S)_g| < \frac{|S|}{\ell}$$

has size at least $\varepsilon|H|$. In particular, the group G_ℓ has size at least ℓ, by taking H the trivial subgroup of G_ℓ.

Take a non-principal ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and consider the non-standard finite group $G = \prod_{\ell\to\mathfrak{U}} G_\ell$ as well as the internal set $S = \prod_{\ell\to\mathfrak{U}} S_\ell$. By Remark 2.5, the set S has Loeb measure $\mu_{G^2}(S) \geq \delta$ and is almost surely k-stable by Remark 4.6. Choose any countable rich subset M. Notice that the exponent of the non-standard finite group G is bounded by r by Łoś's Theorem (Theorem 2.1). Fact 5.1 yields that the subgroup G^{00}_M is a countable intersection of internal subgroups H_n, each of finite index and defined over M. We may assume that $H_{n+1} \subseteq H_n$ for every n in $\mathbb{N}$.

Definition 3.2 tells us that the set

$$Z = \{g \in G^{00}_M : \ \mu_{G^2}(\Lambda_\square(S)_g) = 0\} = \bigcap_{\substack{n\in\mathbb{N}\\ 0\neq m\in\mathbb{N}}} \left\{g \in H_n : \ \mu_{G^2}(\Lambda_\square(S)_g) < \frac{1}{m}\right\}$$

is ω-internal.

Fact 3.12 and Theorem 5.2 imply that Z is not dense, so by $\aleph_1$-saturation and Remark 3.9, we deduce that for some $0 \neq m$ in $\mathbb{N}$ and some finite-index internal subgroup $H = H_n$ defined over M, the set $\{g \in H : \ \mu_{G^2}(\Lambda_\square(S))_g) < \frac{1}{m}\}$ is contained in some internal set of density 0.

Choose some ℓ_0 in $\mathbb{N}$ with

$$0 < \frac{\mu_{G^2}(S)}{\ell_0} < \frac{1}{2m}$$

such that for $\mathfrak{U}$-almost all $\ell \geq \ell_0$, the subgroup $H(G_\ell)$ has index $[G : H] \leq \ell_0$. Now, Definition 3.2 yields an internal set W with

$$\left\{g \in H : \ \mu_{G^2}(\Lambda_\square(S))_g) < \frac{1}{2m}\right\} \subseteq W \subseteq \left\{g \in H : \ \mu_{G^2}(\Lambda_\square(S))_g) \leq \frac{1}{2m}\right\},$$

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whence $\mu_G(W) = 0$ and $\mu_G(W) < \varepsilon \cdot \mu_G(H) = \varepsilon/[G : H]$. Therefore, for $\mathfrak{U}$-almost all $\ell \geq \ell_0$, we have that

$$\left|\left\{g \in H(G_\ell) : \ |\Lambda_\square(S_\ell)_g| < \frac{|S_\ell|}{\ell}\right\}\right| \leq |W(G_\ell)| < \frac{\varepsilon}{[G : H]}|G_\ell| = \varepsilon|H(G_\ell)|.$$

This contradicts our choice of G_ℓ, as desired. ∎

Mimicking the proof above, together with Proposition 5.4, we deduce the following finitary statement concerning the existence of arbitrary long arithmetic progressions in almost surely stable dense sets.

Corollary 5.7 *Given integers $k, m, r \geq 1$ and real numbers $\delta, \varepsilon, \eta > 0$, there is an integer $n = n(k, m, r, \delta, \varepsilon, \eta) \geq 1$ and a real number $\theta = \theta(k, m, r, \delta, \varepsilon, \eta) > 0$ with the following property.*

Let G be a finite group of exponent bounded by r, and let $A \subseteq G$ be subset of G of size $|A| \geq \delta|G|$ such that the collection $\mathcal{H}_k(\mathrm{Cay}(G, A))$ of all half-graphs of height k induced by its Cayley graph on G has size $|\mathcal{H}_k(\mathrm{Cay}(G, A))| \leq \theta|G|^{2k}$. Then, there exists a subgroup H of G of index at most n such that

$$|\{h \in H : \ |m\text{-AP}(A)_h| < (1 - \eta)|A|\}| < \varepsilon|H|,$$

where m-$\mathrm{AP}(A)_h = \{a \in A : \ \{a, h \cdot a, \ldots, h^{m-1} \cdot a\} \subseteq A\}$.

6 Grids and *L*-shapes

In the preceding section, we saw that almost sure stability was sufficient to imply the existence of squares in dense subsets of Cartesian products of arbitrary finite groups. In order to extend the previous results to other two-dimensional shapes, such as L-shapes or 3×2-grids, we will need to impose some (mild) conditions on the nature of the groups in question.

As mentioned in the introduction, in recent work [34], Peluse obtained the first reasonable upper bound on the density of two-dimensional subsets of $\mathbb{F}_p^n$ without L-shaped configurations. We will show how to obtain a qualitative version of her result valid in finite abelian groups of odd order under the assumption of almost sure stability.

Given a subset X of G, every solution (x, y, z) in X^3 to the equation

$$x \cdot y = z^2$$

with $x \neq y$ determines a *generalized arithmetic progression of length* 3 in X. Indeed, if $x \cdot y = z^2$ with $x \neq y$, then $g = x^{-1} \cdot z = y \cdot z^{-1}$ is nontrivial and satisfies that all three elements x, $x \cdot g$ and $g \cdot x \cdot g$ belong to X. If the group G is abelian, this is an arithmetic progression in the classical sense, for $g \cdot x \cdot g = g^2 \cdot x$. In what follows, we denote by $\mathcal{E}(X)$ the set

$$\mathcal{E}(X) = \{(x, y) \in X^2 : \ x \cdot y = z^2 \text{ for some } z \text{ in } X\}.$$

Definition 6.1 The non-standard finite group G is *suitable* for the equation $x \cdot y = z^2$ if the following conditions hold:

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(i) *Squaring preserves dense elements* (cf. [28, Definition 3.12]), that is, for every element a and every countable set of parameters B, we have that a is dense over B if and only if the element a^2 is dense over B.
(ii) For every dense internal subset X of G, the internal set $\mathcal{E}(X)$ has positive density in G^2.

While (i) implies (ii) in any non-standard finite group G (see [28, Theorem 3.14]), for the sake of a self-contained presentation, we have decided to impose (ii) as an additional condition, since the examples we are most interested in already verify this condition by classical results in additive combinatorics, as the next remark shows.

Remark 6.1 Every non-standard finite group G obtained as a non-principal ultraproduct of a family $(G_\ell)_{\ell\in\mathbb{N}}$ of finite abelian groups of odd order is suitable for the equation $x \cdot y = z^2$. Indeed, in such a non-standard finite abelian group G as above, there are no involutions, so a is the only element in the fiber of a^2 with respect to the group homomorphism $x \mapsto x^2$. Thus, density of elements is preserved by Fubini–Tonelli (Remark 2.6).

There is a plethora of explicit lower bounds for $\mathcal{E}(X)$ for finite abelian groups of odd order. For instance, Bloom and Sisask showed in [7, Theorem 2.1] that whenever a finite subset X_ℓ of a finite abelian group G_ℓ of odd order has density $|X_\ell|/|G_\ell| \geq \sigma$, then $|\mathcal{E}(X_\ell)| \geq f(\sigma)|G_\ell|^2$, with

$$f(\sigma) = \sigma^2 \exp(-C\sigma^{-1}\log(\sigma^{-1})^C)$$

for some absolute constant $C > 0$.[4] It follows easily from Łoś's Theorem that for every internal subset X of density $\mu_G(X) \geq \sigma$ in the non-standard finite abelian group G, $\mu_{G^2}(\mathcal{E}(X)) \geq f(\sigma) > 0$, as desired.

Theorem 6.2 *Consider a non-standard finite group G suitable for the equation $x \cdot y = z^2$ along with an internal relation S on $G \times G$ defined over a countable rich subset M of G. If S is dense and almost surely k-stable for some $k \geq 1$, then there is a tuple (a, g, b) of G^3 in good position over M such that (a, b, g) belongs to the internal set*

$$\Lambda_{3\times 2}(S) = \left\{(x, y, h) \in G^3 : \begin{array}{c}(x, y), (x \cdot h, y), (h \cdot x \cdot h, y), (x, y \cdot h),\\ (x \cdot h, y \cdot h) \text{ and } (h \cdot x \cdot h, y \cdot h) \text{ all lie in } S\end{array}\right\}.$$

In particular, the set $\Lambda_{3\times 2}(S)$ has positive density in G^3, by Fact 3.12 and Definition 3.2(II).

Proof Set $\delta = \mu_{G^2}(S)/2 > 0$. By Definition 3.2, there is an internal set Y_δ defined over M such that

$$\{y \in G : \mu_G(S_y) > \delta\} \subseteq Y_\delta \subseteq \{y \in G : \mu_G(S_y) \geq \delta\},$$

where $S_y = \{x \in G : (x, y) \in S\}$ denotes the fiber of S over y. By Fubini–Tonelli (Remark 2.6), the internal set Y_δ is dense. Hence, choose some c in G dense over M with $\mu_G(S_c) \geq \delta$. By Remark 3.6, there is a countable rich subset N containing $M \cup \{c\}$. Notice that the internal dense subset $S_c \subseteq G$ is defined over N.

[4] For the best bound available at the time of writing, see [8], which is based on the breakthrough work of Kelley and Meka [24].

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Claim *There exists a tuple* (a_1, a_2, a_3) *in* S_c^3 *such that*

(i) $a_1 \cdot a_2 = a_3^2$;
(ii) *the difference* $g = a_1^{-1} \cdot a_3 = a_2 \cdot a_3^{-1}$ *is dense over* N *and belongs to* G_N^{00};
(iii) *the pair* (a_1, g) *is in good position over* N.

Proof of Claim It suffices to find a triple satisfying (i) and (ii) in the statement of the claim with (a_1, a_3) in good position over N, or equivalently, with (a_1, a_2) in good position over N, since squaring preserves dense elements.

Write G_N^{00} as a countable intersection of internal generic symmetric neighborhoods Z_j, with j in $\mathbb{N}$, defined over N as in Fact 5.1(a). In particular, for each Z_j, there are finitely many elements $t_1(j), \ldots, t_{n_j}(j)$ in G such that

$$G = Z_j \cdot t_1(j) \cup \cdots \cup Z_j \cdot t_{n_j}(j).$$

Note that we may find such elements $t_r(j)$ in N, since N is a rich subset of G. Moreover, the product set $Z_{j+1} \cdot Z_{j+1}$ is a subset of Z_j. Assume toward a contradiction that there is no triple as required, and consider the ω-internal set

$$\mathcal{Z} = \bigcap_{\substack{j \in \mathbb{N} \\ \mathcal{X} \in \mathcal{F}}} \left\{ (x, y, z) \in S_c^3 : \; x \cdot y = z^2 \text{ with} x^{-1} \cdot z = y \cdot z^{-1} in Z_j \text{ and } (x, y) \notin \mathcal{X} \right\},$$

where $\mathcal{F}$ is the countable collection of all internal subsets $\mathcal{X}$ of G^2 defined over N of density 0. Notice that the projection π of $\mathcal{Z}$ onto the first two coordinates is again ω-internal by Remark 3.1 and empty, for otherwise, it would contain a pair (a_1, a_2) in good position over N, by Fact 3.12, but such a pair would yield a triple as in the statement. Therefore, the ω-internal set $\mathcal{Z}$ must be empty as well, since $\pi(\mathcal{Z}) = \varnothing$. By Fact 2.2, there are finitely many sets $\mathcal{X}_1, \ldots, \mathcal{X}_r$ in $\mathcal{F}$ and some j in $\mathbb{N}$ such that the internal set

$$\widetilde{\mathcal{Z}} = \left\{ (x, y, z) \in S_c^3 : \; x \cdot y = z^2 \text{ with } x^{-1} \cdot z = y \cdot z^{-1} \text{ in } Z_j \right\}$$

is covered by $\bigcup_{i=1}^r \pi^{-1}(\mathcal{X}_i)$. Now, the internal set S_c is dense, so there is some $t_r(j+1)$ in N such that $S_c \cap (Z_{j+1} \cdot t_r(j+1))$ is dense as well. It follows from the suitability of G that the corresponding set $\mathcal{E}(S_c \cap (Z_{j+1} \cdot t_r(j+1))$ must have positive density. Hence, there is a dense pair (u_1, u_2) over N with each u_i in $S_c \cap (Z_{j+1} \cdot t_r(j+1))$ such that

$$u_1 \cdot u_2 = u_3^2 \text{ for some } u_3 \text{ in } S_c \cap (Z_{j+1} \cdot t_r(j+1)).$$

Now, the pair (u_1, u_2) avoids all $\mathcal{X}_i$ and the common difference

$$\begin{aligned} u_1^{-1} \cdot u_3 = u_2 \cdot u_3^{-1} &= u_2 \cdot (t_r(j+1)^{-1} \cdot t_r(j+1)) \cdot u_3^{-1} \\ &= (u_2 \cdot t_r(j+1)^{-1}) \cdot (u_3 \cdot t_r(j+1)^{-1})^{-1} \end{aligned}$$

belongs to $Z_{j+1} \cdot Z_{j+1} \subseteq Z_j$. Thus, the triple (u_1, u_2, u_3) belongs to $\widetilde{\mathcal{Z}}$ and thus to some $\pi^{-1}(\mathcal{X}_j)$. This implies that the dense pair (u_1, u_2) over N belongs to the internal set $\mathcal{X}_j$ defined over N of density 0, which yields the desired contradiction. ■Claim

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Choose now a tuple (a_1, a_2, a_3) in S_c^3 as in the Claim with $a_1 \cdot a_2 = a_3^2$, where $g = a_1^{-1} \cdot a_3 = a_2 \cdot a_3^{-1}$ lies in G_N^{00} and (a_1, g) in good position over N. By Fact 5.1(d), we have that g, viewed as an element of G_M^{00}, is dense over M. Fact 5.1(c) yields that for some b in $\mathrm{tp}(c/M)$ dense over $M \cup \{a_1, a_2, a_3\}$, the product $b \cdot g$ also belongs to $\mathrm{tp}(c/M)$ (and is dense over $M \cup \{a_1, a_2, a_3\}$). Hence, the triple (a_1, g, b) is in good position over M. We need only verify that (a_1, b, g) belongs to $\Lambda_{3\times 2}(S)$.

By construction, the points

$$(a_1, c), (a_1 \cdot g, c) \text{ and } (g \cdot a_1 \cdot g, c)$$

all lie in the almost surely k-stable internal relation S. As each pair (with the order of coordinates reversed) is in good position over M, Theorem 4.7 implies that for each $i = 1, 2, 3$,

$$\mathrm{GP}(\mathrm{tp}(a_i/M), \mathrm{tp}(c/M)) \subseteq S,$$

so the pairs

$$(a_1, b), (a_1 \cdot g, b) \text{ and } (g \cdot a_1 \cdot g, b)$$

all lie in the almost surely k-stable internal relation S. Analogously, using now that the element $b \cdot g$ of $\mathrm{tp}(b/M)$ is also dense over $M \cup \{a_1, a_1 \cdot g, g \cdot a_1 \cdot g\}$, we conclude again by Theorem 4.7 that each of the pairs

$$(a_1, b \cdot g), (a_1 \cdot g, b \cdot g) \text{ and } (g \cdot a_1 \cdot g, b \cdot g)$$

lies in the almost surely k-stable internal relation S, as desired. ∎

Remark 6.1 immediately yields the following result.

Corollary 6.3 *If the non-standard finite group G is obtained as an ultraproduct of finite abelian groups of odd order, then for every almost surely k-stable dense internal relation S of $G \times G$, the collection of 3×2 grids*

$$\Lambda_{3\times 2}(S) = \left\{ (a, b, g) \in G^3 : \begin{array}{c} (a, b),\ (a+g, b),\ (a+2g, b),\ (a, b+g), \\ (a+g, b+g) \text{ and } (a+2g, b+g) \text{ all lie in } S \end{array} \right\}$$

has positive density in G^3.

In particular, every dense almost surely k-stable subset Z of G (see Definition 4.1) contains a 4-term arithmetic progression $\{z, z+d, z+2d, z+3d\}$ given by a dense pair (z, d) (with $d \neq 0$).

Just like Theorem 5.2 gives rise to the finitary Corollary 5.6, Theorem 6.2 yields a finitary version for sufficiently large abelian groups of odd order.

Corollary 6.4 *Given an integer $k \geq 1$ and a real number $\delta > 0$, there is an integer $\ell_0(k, \delta) \geq 1$ and real numbers $\theta = \theta(k, \delta) > 0$ and $\varepsilon = \varepsilon(k, \delta) > 0$ with the following property.*

Let G be a finite abelian group of odd order with $|G| \geq \ell_0$, and consider a relation $S \subseteq G \times G$ of size $|S| \geq \delta |G|^2$ such that the collection $\mathcal{H}_k(S)$ of all half-graphs of height k induced by S on G has size $|\mathcal{H}_k(S)| \leq \theta |G|^{2k}$. Then, the set

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$$\Lambda_{3\times 2}(S) = \left\{(a,b,g) \in G^3 : \begin{array}{l}(a,\,b),\,(a+g,\,b),\,(a+2g,\,b),\,(a,\,b+g),\\ (a+g,\,b+g)\ \text{and}\ (a+2g,\,b+g)\ \text{all lie in}\ S\end{array}\right\}$$

has size $|\Lambda_{3\times 2}(S)| \geq \varepsilon |G|^3$. *In particular, the relation S contains an L-shape.*

Remark 6.5 The astute reader will have noticed that we used very little about the particular equation $x \cdot y = z^2$ in Theorem 6.2. Indeed, if the non-standard finite group G is built as an ultraproduct of finite groups $(G_\ell)_{\ell\in\mathbb{N}}$ such that for every finite subset X_ℓ of G_ℓ of density σ, the collection of tuples $(a_1, \ldots, a_{m+1})$ in X_ℓ^{m+1} satisfying a certain pattern has size at least $f(\sigma)|G_\ell^m|$ for some function $f : \mathbb{R} \to \mathbb{R}$ which is uniform in the family $(G_\ell)_{\ell\in\mathbb{N}}$, then we could reproduce the proof of Theorem 6.2 verbatim to obtain grids using differences $g = a_i - a_j$ for any suitable choice of coordinates.

This applies in particular to the pattern given by the equation $n_1x_1 + \cdots + n_mx_m = kz$ with $k = \sum_{j=1}^m n_j$ in $\mathbb{Z}/p\mathbb{Z}$, asymptotically as p is large (see, for instance, [26, Theorem 3]).

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