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Rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$

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Abstract. We investigate groups of rotations of $\mathbb{R}^3$ and $\mathbb{R}^4$ that act transitively on the unit sphere.

1 Introduction

The following theorem was proved by Montgomery and Zippin in 1940 [3].

Theorem 1.1 *If G is a group of rotations of $\mathbb{R}^3$ that acts transitively on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, then G is the group of all rotations of $\mathbb{R}^3$.*

The proof is based on the construction of a curve of subgroups, each of which is homeomorphic to the circle $\mathbb{S}^1$, in the topological group G. In this short note, we:

(1) give an elementary proof of Theorem 1.1;
(2) prove the analogous result for the group generated by reflections across planes through the origin;
(3) show that the statement corresponding to Theorem 1.1 in $\mathbb{R}^4$ is false.

In terms of matrices, these refer to the orthogonal groups $\mathrm{SO}(3)$, $\mathrm{O}(3)$, and $\mathrm{SO}(4)$, respectively.

2 An elementary proof of Theorem 1.1

First, we recall that each rotation of $\mathbb{R}^3$ is the composition of two rotations of order two. Clearly, we need only prove this for a rotation R whose axis A is the line joining the points $\mathbf{e}_3$ and $-\mathbf{e}_3$, where $\mathbf{e}_3 = (0,0,1)$. Now $R = R_1R_2$, for some reflections R_1 and R_2, where each R_j is the reflection across a vertical plane that contains A. Then $R = (R_1R_0)(R_0R_2)$, where R_0 is the reflection across the equatorial plane of $\mathbb{S}^2$, and each factor is a rotation of order two.

Now suppose that G is a group of rotations of $\mathbb{R}^3$ about the origin 0 that acts transitively on the unit sphere $\mathbb{S}^2$. Then there is some g in G such that $g(\mathbf{e}_3) = -\mathbf{e}_3$. As rotations are linear maps, they map antipodal (i.e., diametrically opposite) points to antipodal points, so we see that g interchanges $\mathbf{e}_3$ and $-\mathbf{e}_3$. Now this can only happen if g is a rotation of order two, say with fixed points $\mathbf{v}$ and $-\mathbf{v}$. Now take any rotation R of order two, say with fixed points $\mathbf{w}$ and $-\mathbf{w}$, and note that R is uniquely determined by $\mathbf{w}$. As G is transitive, there is some h in G with $h(\mathbf{v}) = \mathbf{w}$. As hgh^{-1} is a rotation of

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order two which fixes $\mathbf{w}$ and $-\mathbf{w}$, we see that $R = hgh^{-1}$. This shows that G contains every rotation of $\mathbb{R}^3$ of order two; hence G is the group of all rotations of $\mathbb{R}^3$.

3 The orthogonal group $\mathrm{O}(3)$

The orthogonal group $\mathrm{O}(3)$ is the group generated by reflections across all planes through the origin in $\mathbb{R}^3$, and we have the following result.

Theorem 3.1 *If G is a subgroup of $\mathrm{O}(3)$ that acts transitively on $\mathbb{S}^2$, then G is $\mathrm{O}(3)$ or $\mathrm{SO}(3)$.*

Proof Let $\mathbf{e}_1 = (1, 0, 0)$. Exactly as in the proof of Theorem 1.1 there are maps g and h in $\mathrm{O}(3)$ such that g interchanges $\mathbf{e}_3$ and $-\mathbf{e}_3$, and h interchanges $\mathbf{e}_1$ and $-\mathbf{e}_1$. Now each of g and h is either a rotation of order two or a reflection. If g and h are reflections, then gh is a rotation of order two so that, in all cases, G contains a rotation of order two. By transitivity, G contains all rotations of order two; hence all rotations of $\mathbb{R}^3$. If G contains a reflection, then $G = \mathrm{O}(3)$; otherwise, $G = \mathrm{SO}(3)$. ■

4 Rotations of $\mathbb{R}^4$

It is well known that the orthogonal maps of $\mathbb{R}^4$ onto itself can be described in terms of quaternions; for more details, see [1, 2, 4]. In particular, the elements of $\mathrm{SO}(4)$ are the maps of the form $x \mapsto axb$, where x is a quaternion (identified as a point in $\mathbb{R}^4$), and a and b are unit quaternions. Now it is shown in [1, p. 142] that the set of such maps of the form $x \mapsto ax$ forms a subgroup of $\mathrm{SO}(4)$, as do the maps of the form $x \mapsto xb$, and, moreover that the intersection of these two subgroups contains only the two maps $x \mapsto x$ and $x \mapsto -x$. This shows that the set of such maps $x \mapsto ax$ forms a *proper* subgroup of $\mathrm{SO}(4)$, and this acts transitively on $\mathbb{S}^3$ since the orbit of 1 is the set of unit quaternions (i.e., $\mathbb{S}^3$).

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