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# DIMENSIONAL APPROXIMATION OF AN INHOMOGENEOUS ATTRACTOR WITHOUT ANY SEPARATION CONDITION

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**Abstract**

We show that the Hausdorff dimension of the attractor of an inhomogeneous self-similar iterated function system (or self-similar IFS) can be well approximated by the Hausdorff dimension of the attractor of another inhomogeneous self-similar IFS satisfying the strong separation condition. We also determine a formula for the Hausdorff dimension of the algebraic product and sum of the inhomogeneous attractor.

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## 1. Introduction and statements

Let $(X, d)$ be a complete metric space and $N \geq 2$. The system $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ on $X$ is called an iterated function system (IFS) if each $f_i$ is a contraction map on $X$ to $X$, that is,

$$d(f_i(x), f_i(y)) \leq r_i d(x, y) \quad \text{for all } x, y \in X,$$

where $r_i \in (0, 1)$. It is well known that there exists a unique nonempty compact set $K_\phi \subseteq X$ such that

$$K_\phi = \bigcup_{i=1}^{N} f_i(K_\phi).$$

The set $K_\phi$ is called the attractor of the IFS $\mathcal{I}$. The IFS $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ on $X$ is called self-similar if each $f_i$ is a similarity map on $X$, that is, $d(f_i(x), f_i(y)) = r_i d(x, y)$ for all $x, y \in X$, where $r_i \in (0, 1)$. There are several other variants of IFS, such as self-conformal IFS [18], graph-directed IFS (GDIFS) [19], inhomogeneous IFS [4], inhomogeneous GDIFS [7] and many more.

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The fractal dimension is very important in fractal geometry in studying the geometrical behaviour of the fractal sets. There are various types of fractal dimensions, such as the Hausdorff dimension, the box dimension, the packing dimension and the Assouad dimension (see [10, 14]). Our main focus in this article is on the Hausdorff dimension. The Hausdorff dimension is measure theoretic, so it is difficult to compute, but very convenient for analytical purposes.

The determination of the exact value of the fractal dimension of the attractor corresponding to an IFS is a very well-known and challenging problem. The exact value of the Hausdorff dimension of the attractor associated with self-similar IFS, self-conformal IFS and GDIFS consisting of infinitesimal similitude maps was determined under some suitable separation conditions (see, for instance, [3, 15, 17, 19, 20, 25]). In the case of overlapping, the calculation of the exact value of the fractal dimension of the attractor corresponding to an IFS is very difficult (see [1, 8, 14]), but some estimates have been found (see [5, 9, 10, 16, 24]).

Our focus is on the inhomogeneous IFS, defined by Barnsley and Demko [4] as follows. Let $(X, d)$ be a complete metric space and $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ be an IFS on $X$, where each $f_i$ is a contraction on $X$. Let $C \subseteq X$ be a compact set. The system $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ is called an inhomogeneous IFS on $X$. Then, there exists a unique nonempty compact set $K_c \subseteq X$ such that

$$K_c = \bigcup_{i=1}^{N} f_i(K_c) \cup C.$$

The set $K_c$ is called the inhomogeneous attractor and the set $C$ is called the condensation set. This is a general version of IFS with a more complex structure. The system $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ is called an inhomogeneous self-similar IFS on $X$ if each $f_i$ is a similarity map on $X$. The dimensional results for the attractor of an inhomogeneous self-similar IFS under some suitable separation condition can be found in [1, 6, 13, 21].

Firstly, the dimensional approximation results were discussed by Peres and Shmerkin [23]. They showed that the Hausdorff dimension of a self-similar set in the plane satisfying the open set condition (OSC) can be approximated by a well-behaved self-similar set. Orponen [22] showed that the dimensional approximation result of Peres and Shmerkin [23] is also valid without the open set condition. In [11, 12], Farkas extended the result of Orponen [22] to higher dimensions and obtained some dimensional approximation results for the attractor of the graph-directed self-similar IFS.

In this paper, we consider the inhomogeneous self-similar IFS. We are not assuming any separation conditions on the IFS. By adapting the ideas of Peres and Shmerkin [23] and Farkas [11, 12], we prove our main result. To avoid noninteresting and trivial cases, throughout, we assume that $C \neq \emptyset, C \neq K_\phi$ and $K_\phi$ is not a singleton set. Our main result is as follows.

THEOREM 1.1. *Let $(X, d)$ be a complete metric space and $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS on $X$. Let $K_c$ be the inhomogeneous attractor*

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*corresponding to* $\mathcal{J}$ *and* $K_\phi$ *be the attractor corresponding to the homogeneous part* $\{f_1, f_2, \ldots, f_N\}$ *of* $\mathcal{J}$.

(1) *If* $C \cap K_\phi = \emptyset$ *or* $C \subsetneq K_\phi$, *then for each* $\epsilon > 0$, *there exists an inhomogeneous self-similar IFS* $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ *on* $X$ *satisfying SSC, with attractor* $\tilde{K}_c$ *such that*

$$\tilde{K}_c \subseteq K_c \quad \textit{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_c) \geq \dim_H(K_c) - \epsilon,$$

*where the* $\tilde{f}_i$ *are compositions of some of the* $f_j$.

(2) *If* $K_\phi \subsetneq C$, *then for each* $\epsilon > 0$, *there exists an inhomogeneous self-similar IFS* $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, \tilde{C}\}$ *on* $X$ *satisfying SSC, with attractor* $\tilde{K}_{\tilde{c}}$ *such that*

$$\tilde{K}_{\tilde{c}} \cap K_c \neq \emptyset \quad \textit{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_{\tilde{c}}) \geq \dim_H(K_c) - \epsilon,$$

*where* $\tilde{C}$ *is some self-similar copy of* $C$ *and the* $\tilde{f}_i$ *are compositions of some of the* $f_j$.

We use this result to prove further dimensional approximation results. Under some assumptions on the condensation set, we also show that every inhomogeneous self-similar IFS can be well approximated by some self-similar IFS satisfying the SSC. As an application, we determine a formula for the algebraic sum and product of the inhomogeneous attractor.

## 2. Preliminaries

First, we provide the definition of the Hausdorff measure and dimension.

DEFINITION 2.1. Assume that $(X, d)$ is a metric space and let $F \subseteq X$. We say that $\{U_i\}$ is a $\delta$-cover of $F$ if $F \subset \bigcup_{i=1}^{\infty} U_i$ and $0 < |U_i| \leq \delta$ for each $i$, where $|U_i|$ denotes the diameter of the set $U_i$. For each $\delta > 0$ and $s \geq 0$, we define

$$H^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

The quantity $H^s_\delta(F)$ increases as $\delta$ decreases, and thus $\lim_{\delta \to 0+} H^s_\delta(F)$ exists. We write

$$H^s(F) = \lim_{\delta \to 0+} H^s_\delta(F).$$

We call $H^s(F)$ the $s$-dimensional Hausdorff measure of the set $F$. Using this, the Hausdorff dimension of the set $F$ is defined by

$$\dim_H(F) = \inf\{s \geq 0 : H^s(F) = 0\} = \sup\{s \geq 0 : H^s(F) = \infty\}.$$

DEFINITION 2.2. Let $(X, d)$ be a metric space and $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ be an IFS on $X$ such that each $f_i$ is a similarity map on $X$ with similarity ratio $r_i$. Then, there is a unique $s \in \mathbb{R}$ such that

$$\sum_{i=1}^{N} r_i^s = 1.$$

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The quantity $s$ is known as the similarity dimension of the IFS $\mathcal{I}$.

Next, we discuss the concept of code space for an IFS, which is very useful in understanding the structure of an IFS.

**2.1. Code space for an IFS.** Let $(X, d)$ be a complete metric space and $\mathcal{I} := \{f_1, f_2, \ldots, f_N\}$ be an IFS on $X$. Let $K_\phi$ be the attractor of the IFS $\mathcal{I}$. For each $n \in \mathbb{N}$, let $\Xi^n$ denote the collection of all finite sequences of length $n$, that is, $\Xi^n = \{(w_1, w_2, \ldots, w_n) : w_i \in \{1, 2 \ldots, N\}\}$. Let $\Xi^*$ denote the collection of all finite sequences, that is, $\Xi^* = \bigcup_{n=1}^{\infty} \Xi^n$. For each $\sigma \in \Xi^*$, we denote $f_\sigma = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}$, where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$. Let $\Xi$ denote the collection of all infinite sequences, that is, $\Xi = \{(w_1, w_2, w_3, \ldots) : w_i \in \{1, 2, \ldots, N\}\}$. For each $w = (w_1, w_2, w_3, \ldots) \in \Xi$, we define $w|_n = (w_1, w_2, \ldots, w_n)$. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots), w = (w_1, w_2, w_3, \ldots) \in \Xi$. We define a metric $d_\Xi$ on $\Xi$ by

$$d_\Xi(\sigma, \omega) = \begin{cases} 0 & \text{if } \sigma = \omega, \\ 2^{-k} & \text{if } \sigma \neq \omega, \end{cases}$$

where $k$ is the least index for which $\sigma_k \neq \omega_k$ and $\sigma, \omega \in \Xi$. Then, $(\Xi, d_\Xi)$ is a compact metric space. We call $\Xi$ the code space corresponding to the IFS $\mathcal{I}$. For each $\sigma = (\sigma_1, \sigma_2, \ldots) \in \Xi$, the limit

$$\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(a)$$

exists uniquely for any $a \in X$. We define a map $\Pi : \Xi \to K_\phi$ by

$$\Pi(\sigma) = \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(a) = \lim_{k\to\infty} f_{\sigma|_k}(a),$$

where $\sigma = (\sigma_1, \sigma_2, \ldots) \in \Xi$. The map $\Pi$ is continuous and onto. The map $\Pi$ is called the coding map for the IFS $\mathcal{I}$.

Olsen and Snigireva [21] determined the relation between the inhomogeneous attractor, the condensation set and the attractor corresponding to the homogeneous part of inhomogeneous IFS.

THEOREM 2.3 [21]. *Let $(X, d)$ be a complete metric space and $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous IFS on $X$. Let $K_c$ be the inhomogeneous attractor corresponding to $\mathcal{J}$. Assume that $K_\phi$ is the attractor corresponding to the homogeneous part $\{f_1, f_2, \ldots, f_N\}$ of $\mathcal{J}$. Then,*

$$K_c = K_\phi \cup O \quad \text{where} \quad O = C \bigcup \Big( \bigcup_{w\in\Xi^*} f_w(C) \Big).$$

*The set $O$ is known as the orbital set for the inhomogeneous IFS $\mathcal{J}$.*

Next, we describe some separation conditions for the IFS and inhomogeneous IFS.

DEFINITION 2.4. Let $(X, d)$ be a complete metric space. We say that the IFS $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ on $X$ satisfies the open set condition (OSC) if there is a nonempty

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open set $O \subset X$ with $f_i(O) \subset O$ and $f_i(O) \cap f_j(O) = \emptyset$ for $i \neq j \in \{1, 2, \ldots, N\}$. Moreover, if $O \cap K_\phi \neq \emptyset$, where $K_\phi$ is the attractor of the IFS $\mathcal{I}$, then we say that the IFS $\mathcal{I}$ satisfies the strong open set condition (SOSC). If $f_i(K_\phi) \cap f_j(K_\phi) = \emptyset$ for $i \neq j \in \{1, 2, \ldots, N\}$, then we say that the IFS $\mathcal{I}$ satisfies the strong separation condition (SSC).

DEFINITION 2.5. Let $(X, d)$ be a complete metric space. We say that the inhomogeneous IFS $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ on $X$ satisfies the strong separation condition (SSC) if $f_1(K_c), f_2(K_c), \ldots, f_N(K_c)$ and $C$ are pairwise disjoint.

The following result establishes a relationship between the Hausdorff and box dimensions for the attractor of the self-similar IFS.

THEOREM 2.6 [9]. *Let $(X, d)$ be a complete metric space and $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ be a self-similar IFS on $X$. Let $K_\phi$ be the attractor of the IFS $\mathcal{I}$. If $\dim_H(K_\phi) = s$, then $\dim_H(K_\phi) = \dim_B(K_\phi) = s$ and $H^s(K_\phi) < \infty$. Furthermore, if the IFS $\mathcal{I}$ satisfies the SSC, then $H^s(K_\phi) > 0$ and $s$ is the same as the similarity dimension of the IFS $\mathcal{I}$.*

The following lemmas are very useful in proving our main result. The proofs use the same idea as in [11, Lemmas 7.1 and 7.2].

LEMMA 2.7. *Let $(X, d)$ be a complete metric space. Let $f_1 : X \to X$ and $f_2 : X \to X$ be two contractive one-one maps on $X$ with distinct fixed points. Then, the maps $f_1^n \circ f_2$ have distinct fixed points for each $n \in \mathbb{N}$.*

PROOF. We prove the result by contradiction. Assume there exist two distinct $n_1$, $n_2 \in \mathbb{N}$ such that $f_1^{n_1} \circ f_2$ and $f_1^{n_2} \circ f_2$ have the same fixed point $x \in X$. This implies that

$$f_1^{n_1}(f_2(x)) = x \quad \text{and} \quad f_1^{n_2}(f_2(x)) = x.$$

Without loss of generality, let $n_1 < n_2$, say $n_2 = k + n_1$ for some $k \in \mathbb{N}$. Then,

$$f_1^{n_2} \circ f_2(x) = x = f_1^k(f_1^{n_1}(f_2(x))).$$

Thus, we get $f_1^k(x) = x$. By the Banach fixed point theorem, $f_1(x) = x$. Since $f_1^{n_1}(f_2(x)) = x$ and $f_1(x) = x$, we have $f_1^{n_1}(f_2(x)) = f_1^{n_1}(x)$. Since $f_1^{n_1}$ is one-one, we have $f_2(x) = x$. This implies that $f_1$ and $f_2$ have the same fixed point, which is a contradiction. Thus, the maps $f_1^n \circ f_2$ have distinct fixed points for each $n \in \mathbb{N}$. □

LEMMA 2.8. *Let $(X, d)$ be a complete metric space. Let $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ be an IFS on $X$ such that each $f_i$ is a contractive one-one map on $X$. Assume that $f_1$ and $f_2$ have distinct fixed points. Then, there exists an IFS $\mathcal{I}' = \{g_1, g_2, \ldots, g_N\}$ consisting of contractive one-one maps on $X$ with the following properties:*

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(1) $g_1 = f_1$ *and* $g_2 = f_2$;
(2) *for each* $i \in \{3, \ldots, N\}$, *either* $g_i = f_1^n \circ f_i$ *or* $g_i = f_2^n \circ f_i$ *for some* $n \in \mathbb{N}$;
(3) *for any distinct* $i, j \in \{1, 2, \ldots, N\}$, $g_i$ *and* $g_j$ *have distinct fixed points.*

PROOF. The result follows by Lemma 2.7 and an induction argument. □

## 3. Main results

In this section, we prove the main theorem and some other subsequent results. The first proposition proves the dimensional approximation result for the inhomogeneous self-similar IFS under the condition $C \cap K_\phi = \emptyset$.

PROPOSITION 3.1. *Let* $(X, d)$ *be a complete metric space and* $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ *be an inhomogeneous self-similar IFS on* $X$. *Let* $K_c$ *be the inhomogeneous attractor corresponding to* $\mathcal{J}$ *and* $K_\phi$ *be the attractor corresponding to the homogeneous part* $\{f_1, f_2, \ldots, f_N\}$ *of* $\mathcal{J}$. *Assume that* $C \cap K_\phi = \emptyset$. *Then, for each* $\epsilon > 0$, *there exists an inhomogeneous self-similar IFS* $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ *on* $X$ *satisfying SSC with attractor* $\tilde{K}_c$, *such that*

$$\tilde{K}_c \subseteq K_c \quad \text{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_c) \geq \dim_H(K_c) - \epsilon.$$

PROOF. Since the compact set $K_\phi$ has more than 2 elements, there exist two distinct $i, j \in \{1, 2, \ldots, N\}$ such that $f_i$ and $f_j$ have distinct fixed points. By Lemma 2.8, there exist $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_N \in \Xi^*$ such that for distinct $k_1, k_2 \in \{1, 2, \ldots, N\}$, the maps $f_{\mathbf{i}_{k_1}}$ and $f_{\mathbf{i}_{k_2}}$ have different fixed points. Let $x_k$ be the fixed point of $f_{\mathbf{i}_k}$ for $k \in \{1, 2, \ldots, N\}$. Let $d_1 = \min\{d(x_{k_1}, x_{k_2}) : k_1 \neq k_2 \in \{1, 2, \ldots, N\}\}$. Clearly, $d_1 > 0$. Since $C \cap K_\phi = \emptyset$, it follows that $d_2 := d(C, K_\phi) > 0$. Let $r_{\max} := \max\{r_i : i \in \{1, 2, \ldots, N\}\}$ and $r_{\min} := \min\{r_i : i \in \{1, 2, \ldots, N\}\}$, where $r_i$ is the similarity ratio of $f_i$ for $i \in \{1, 2, \ldots, N\}$. We can choose $m_1 \in \mathbb{N}$ such that

$$r_{\max}^{m_1} \operatorname{diam}(K_c) \leq \frac{\min\{d_1, d_2\}}{4}.$$

By the code space theory, since $x_k$ is the fixed point of $f_{\mathbf{i}_k}$, we have $x_k \in K_\phi$ and $x_k \in f_{\mathbf{i}_k}^n(K_\phi)$ for each $n \in \mathbb{N}$. Thus, $x_k \in f_{\mathbf{i}_k}^n(K_c)$ for each $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, N\}$. Since $r_{\max}^{m_1} \operatorname{diam}(K_c) \leq \min\{d_1, d_2\}/4$, we have $f_{\mathbf{i}_k}^{m_1}(K_c) \subset \mathcal{B}(x_k, \min\{d_1, d_2\}/4)$ for each $k \in \{1, 2, \ldots, N\}$. This implies that

$$f_{\mathbf{i}_{k_1}}^{m_1}(K_c) \cap f_{\mathbf{i}_{k_2}}^{m_1}(K_c) = \emptyset \quad \text{and} \quad f_{\mathbf{i}_{k_1}}^{m_1}(K_c) \cap C = \emptyset \quad \text{for all } k_1 \neq k_2 \in \{1, 2, \ldots, N\}.$$

Let $K = \bigcup_{k=1}^N f_{\mathbf{i}_k}^{m_1}(K_\phi)$. If $K = K_\phi$, then we consider the inhomogeneous IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$, where $\tilde{f}_k = f_{\mathbf{i}_k}^{m_1}$ for $k \in \{1, 2, \ldots, N\}$ and $\tilde{N} = N$. In this case, $\tilde{K}_\phi = K_\phi$ and $C \subset \tilde{O} \subset O$. Thus, $\tilde{K}_c \subset K_c$. By using the countable stability of the Hausdorff dimension, we get $\dim_H(\tilde{K}_c) = \dim_H(K_c)$.

Let $K \subsetneq K_\phi$. Then, there exists $x \in K_\phi$ such that $x \notin K$. This implies that $x \neq x_k$ for all $k \in \{1, 2, \ldots, N\}$. Set $d_3 := \min\{d(x, x_k) : k \in \{1, 2, \ldots, N\}\} > 0$. Now, we choose

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$m_2 \in \mathbb{N}$ such that

$$r_{\max}^{m_2}\operatorname{diam}(K_c) \le \frac{\min\{d_1, d_2, d_3\}}{4}.$$

By similar arguments to those above,

$$f_{\mathbf{i}_{k_1}}^{m_2}(K_c) \cap f_{\mathbf{i}_{k_2}}^{m_2}(K_c) = \emptyset \quad \text{and} \quad f_{\mathbf{i}_{k_1}}^{m_2}(K_c) \cap C = \emptyset \quad \text{for all } k_1 \ne k_2 \in \{1, 2, \ldots, N\}.$$

Since $x \in K_\phi$, we get a code $w = (w_1, w_2, \ldots) \in \Xi$ for $x$ such that $x \in f_{w|_n}(K_c)$ for all $n \in \mathbb{N}$. Set $\tilde{w} := w|_{m_2} = (w_1, w_2, \ldots, w_{m_2})$. Since $r_{\max}^{m_2}\operatorname{diam}(K_c) \le \min\{d_1, d_2, d_3\}/4$,

$$f_{\tilde{w}}(K_c) \cap f_{\mathbf{i}_k}^{m_2}(K_c) = \emptyset \quad \text{and} \quad f_{\tilde{w}}(K_c) \cap C = \emptyset \quad \text{for all } k \in \{1, 2, \ldots, N\}.$$

Since $K_\phi \subset K_c$, we have $f_{\tilde{w}}(K_\phi) \subset f_{\tilde{w}}(K_c)$. Let $s = \dim_H(K_\phi)$. Thus, $\dim_H(f_{\tilde{w}}(K_\phi)) = s$ and, by Theorem 2.6, $H^s(K_\phi) < \infty$. Given that $K_\phi$ has at least 2 elements, by Lemma 2.7, $K_\phi$ has infinitely many elements. This implies that $s > 0$. Thus, without loss of generality, we can assume that $\epsilon \in (0, s)$. Since $\dim_H(f_{\tilde{w}}(K_\phi)) = s$, we have $H^{(s-\epsilon/2)}(f_{\tilde{w}}(K_\phi)) = \infty$. Thus, we can find a sufficiently small $\delta > 0$ such that for any $3\delta$-cover $\{U_i\}$ of $f_{\tilde{w}}(K_\phi)$, we have $\sum_{i=1}^{\infty} |U_i|^{(s-\epsilon/2)} > 1$. For each $x \in f_{\tilde{w}}(K_\phi)$, there exist $\sigma = (\sigma_1, \sigma_2, \ldots) \in \Xi$ such that $x \in f_{\tilde{w}}(f_{\sigma|_n}(K_c))$ for all $n \in \mathbb{N}$. Then, there exists $\tilde{n} \in \mathbb{N}$ such that $r_{\min}\delta \le \operatorname{diam}(f_{\tilde{w}}(f_{\sigma|_{\tilde{n}}}(K_c))) < \delta$. This implies that the set

$$\mathcal{T} = \{f_{\tilde{w}}(f_\tau(K_c)) : \tau \in \Xi^* \text{ and } r_{\min}\delta \le \operatorname{diam}(f_{\tilde{w}}(f_\tau(K_c))) < \delta\}$$

forms a $\delta$-cover of $f_{\tilde{w}}(K_\phi)$. One can easily see that the set $\mathcal{T}$ is a collection of finite sets. Let $\tau_1, \tau_2, \ldots, \tau_{\tilde{k}} \in \Xi^*$ be such that $\{f_{\tilde{w}}(f_{\tau_1}(K_c)), f_{\tilde{w}}(f_{\tau_2}(K_c)), \ldots, f_{\tilde{w}}(f_{\tau_{\tilde{k}}}(K_c))\}$ is a collection of maximal pairwise disjoint $\delta$-covers of $f_{\tilde{w}}(K_\phi)$. Let $U_i$ be the $\delta$-neighbourhood of $f_{\tilde{w}}(f_{\tau_i}(K_c))$ for each $i \in \{1, 2, \ldots, \tilde{k}\}$. Due to the maximality of a cover, $\{U_i\}_{i=1}^{\tilde{k}}$ forms a $3\delta$-cover of $f_{\tilde{w}}(K_\phi)$. Thus,

$$\sum_{i=1}^{\tilde{k}} (3\delta)^{(s-\epsilon/2)} \ge \sum_{i=1}^{\tilde{k}} |U_i|^{(s-\epsilon/2)} > 1.$$

This yields $\tilde{k} \ge 3\delta^{-(s-\epsilon/2)}$. Set $\tilde{f}_k := f_{\mathbf{i}_k}^{m_2}$ for $k \in \{1, 2, \ldots, N\}$ and $\tilde{f}_{N+k} := f_{\tilde{w}} \circ f_{\tau_k}$ for $k \in \{1, 2, \ldots, \tilde{k}\}$. We define an inhomogeneous IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{N+\tilde{k}}, C\}$ on $X$. Let $\tilde{K}_c$ be the attractor of the inhomogeneous IFS $\tilde{\mathcal{J}}$. Let $\tilde{K}_\phi$ be the attractor corresponding to the homogeneous part $\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{N+\tilde{k}}\}$ of $\tilde{\mathcal{J}}$ and $\tilde{O}$ be the orbital sets for the inhomogeneous IFS $\tilde{\mathcal{J}}$. It is easy to see that $\tilde{K}_\phi \subset K_\phi$ and $C \subset \tilde{O} \subset O$. This implies that $\tilde{K}_c \subset K_c$. By the definition of $\tilde{f}_i$ for $i \in \{1, 2, \ldots, N + \tilde{k}\}$ and since $\tilde{K}_c \subset K_c$,

$$\tilde{f}_i(\tilde{K}_c) \cap \tilde{f}_j(\tilde{K}_c) = \emptyset \quad \text{and} \quad \tilde{f}_i(\tilde{K}_c) \cap C = \emptyset \quad \text{for all } i \ne j \in \{1, 2, \ldots, N + \tilde{k}\}.$$

Thus, the inhomogeneous IFS $\tilde{\mathcal{J}}$ satisfies the SSC. Now, we will show the dimension aspect of our result. We define an IFS $\tilde{\mathcal{I}} := \{\tilde{f}_{N+1}, \tilde{f}_{N+2}, \ldots, \tilde{f}_{N+\tilde{k}}\}$ on $X$. The similarity ratio for the map $\tilde{f}_{N+i}$ is $r_{\tilde{w}} r_{\tau_i}$ for each $i \in \{1, 2, \ldots, \tilde{k}\}$. Let $K'_\phi$ be the attractor of the IFS $\tilde{\mathcal{I}}$. By observing the code space of the IFS $\tilde{\mathcal{I}}$, we get $K'_\phi \subseteq \tilde{K}_\phi$. By the definition of $\tilde{f}_{N+i}$ for $i \in \{1, 2 \ldots, \tilde{k}\}$ and $K'_\phi \subset \tilde{K}_\phi \subset \tilde{K}_c$, we see that the IFS $\tilde{\mathcal{I}}$ satisfies the SSC. Let

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$s_0$ be the similarity dimension of the IFS $\tilde{\mathcal{I}}$. Thus,

$$\sum_{i=1}^{\tilde{k}} (r_{\tilde{w}} r_{\tau_i})^{s_0} = 1.$$

Since $r_{\min}\delta \le \operatorname{diam}(f_{\tilde{w}}(f_\tau(K_c))) < \delta$ for each $i \in \{1, 2, \ldots, \tilde{k}\}$, we have $r_{\tilde{w}} r_{\tau_i} \ge r_{\min}\delta/\operatorname{diam}(K_c)$. This implies that $s_0 \ge \log(1/\tilde{k})/\log(r_{\min}\delta/\operatorname{diam}(K_c))$. Since $\tilde{k} \ge 3\delta^{-(s-\epsilon/2)}$,

$$s_0 \ge \frac{(s - \frac{\epsilon}{2}) + (s - \frac{\epsilon}{2})\frac{\log 3}{\log \delta}}{1 + \frac{\log r_{\min}}{\log \delta} - \frac{\log(\operatorname{diam}(K_c))}{\log \delta}}.$$

Therefore, for small enough $\delta$, we have $s_0 \ge s - \epsilon$. Since the IFS $\tilde{\mathcal{I}}$ satisfies the SSC, by Theorem 2.6, $\dim_H(K'_\phi) = s_0 \ge s - \epsilon$. Thus, $\dim_H(\tilde{K}_\phi) \ge \dim_H(K_\phi) - \epsilon$ and so

$$\begin{aligned}\dim_H(\tilde{K}_c) &= \max\{\dim_H(\tilde{K}_\phi), \dim_H(\tilde{O})\} \\ &= \max\{\dim_H(\tilde{K}_\phi), \dim_H(C)\} \\ &\ge \max\{\dim_H(K_\phi) - \epsilon, \dim_H(C)\} \ge \dim_H(K_c) - \epsilon.\end{aligned}$$

This completes the proof. □

We apply the above result to demonstrate that every nonempty compact set can be approximated by a well-behaved inhomogeneous self-similar set in the context of the Hausdorff dimension.

COROLLARY 3.2. *Let $(X, d)$ be a complete metric space and $C$ be a nonempty compact subset of $X$. Then, for each $\epsilon > 0$, there exists an inhomogeneous self-similar IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ on $X$ satisfying SSC with attractor $\tilde{K}_c$, such that*

$$C \subsetneq \tilde{K}_c \quad \textit{and} \quad \dim_H(C) = \dim_H(\tilde{K}_c).$$

PROOF. First, we consider a self-similar IFS $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ on $X$ with attractor $K_\phi$ such that

$$K_\phi \cap C = \emptyset \quad \text{and} \quad \dim_H(K_\phi) \le \dim_H(C).$$

We define an inhomogeneous IFS $\mathcal{J} := \{f_1, f_2, \ldots, f_N, C\}$ on $X$ satisfying all the assumptions of Proposition 3.1. By Proposition 3.1, there exists an inhomogeneous self-similar IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ on $X$ satisfying SSC with attractor $\tilde{K}_c$, such that $\tilde{K}_c \subseteq K_c$. Further, $C$ is a proper subset of $\tilde{K}_c$ and $\dim_H(K_c) = \dim_H(C)$. This completes the proof. □

In the next proposition, we prove the dimensional approximation result for the inhomogeneous self-similar IFS under the condition $C \subsetneq K_\phi$.

PROPOSITION 3.3. *Let $(X, d)$ be a complete metric space and $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS on $X$. Let $K_c$ be the inhomogeneous attractor corresponding to $\mathcal{J}$ and $K_\phi$ be the attractor corresponding to the homogeneous*

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*part* $\{f_1, f_2, \ldots, f_N\}$ *of* $\mathcal{J}$. *Assume that* $C \subsetneq K_\phi$. *Then, for each* $\epsilon > 0$, *there exists an inhomogeneous self-similar IFS* $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ *on X satisfying SSC with attractor* $\tilde{K}_c$, *such that*

$$\tilde{K}_c \subseteq K_c \quad \textit{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_c) \geq \dim_H(K_c) - \epsilon.$$

PROOF. Since $C \subsetneq K_\phi$, there exists $x \in K_\phi$ and $x \notin C$. Set $d_1 := d(x, C) > 0$. Let $r_i$ be the similarity ratio of $f_i$ for $i \in \{1, 2, \ldots, N\}$. Set $r_{\max} := \max\{r_i : i \in \{1, 2, \ldots, N\}\}$. We choose $m \in \mathbb{N}$ such that

$$r_{\max}^m \operatorname{diam}(K_c) \leq \frac{d_1}{4}.$$

Since $x \in K_\phi$, we find a code $w = (w_1, w_2, \ldots) \in \Xi$ such that $x \in f_{w|_n}(K_c)$ for all $n \in \mathbb{N}$. Set $\tilde{w} := w|_m = (w_1, w_2, \ldots, w_m)$. Since $r_{\max}^m \operatorname{diam}(K_c) \leq \min\{d_1\}/4$, we have

$$f_{\tilde{w}}(K_c) \cap C = \emptyset.$$

Since $K_\phi \subset K_c$, we get $f_{\tilde{w}}(K_\phi) \subset f_{\tilde{w}}(K_c)$. Let $s = \dim_H(K_\phi)$. Thus, $\dim_H(f_{\tilde{w}}(K_\phi)) = s$ and, by Theorem 2.6, $H^s(K_\phi) < \infty$. Using the same idea as in the proof of the previous proposition, we get an inhomogeneous IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{k}}, C\}$, where $\tilde{f}_i := f_{\tilde{w}} \circ f_{\tau_i}$ for each $i \in \{1, 2, \ldots, \tilde{k}\}$ and $\tilde{f}_i(K_c) \cap \tilde{f}_j(K_c) = \emptyset$ for $i \neq j \in \{1, 2, \ldots, \tilde{k}\}$. Let $\tilde{K}_c$ be the inhomogeneous attractor corresponding to $\tilde{\mathcal{J}}$ and $\tilde{K}_\phi$ be the attractor corresponding to the homogeneous part $\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{k}}\}$ of $\tilde{\mathcal{J}}$. Since $f_{\tilde{w}}(K_c) \cap C = \emptyset$ and $\tilde{K}_c \subseteq K_c$, the inhomogeneous IFS $\tilde{\mathcal{J}}$ satisfies the SSC. The conclusion of the proposition follows from the fact $\dim_H(\tilde{K}_\phi) \geq \dim_H(K_\phi) - \epsilon$. □

By applying this proposition, we prove that the Hausdorff dimension of the self-similar set can be approximated by the Hausdorff dimension of a well-behaved inhomogeneous self-similar set.

PROPOSITION 3.4. *Let* $(X, d)$ *be a complete metric space and* $\mathcal{I} = \{f_1, f_2, \ldots, f_N\}$ *be a self-similar IFS on X. Let* $K_\phi$ *be the attractor of the IFS* $\mathcal{I}$. *Then, for each* $\epsilon > 0$, *there exists an inhomogeneous self-similar IFS* $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, C\}$ *on X satisfying SSC with attractor* $\tilde{K}_c$, *such that*

$$\tilde{K}_c \subsetneq K_\phi \quad \textit{and} \quad \dim_H(K_\phi) \geq \dim_H(\tilde{K}_c) \geq \dim_H(K_\phi) - \epsilon.$$

PROOF. Since the attractor $K_\phi$ has more than 2 elements, we can take a nonempty compact set $C$ such that $C \subsetneq K_\phi$. Now, we define an inhomogeneous IFS $\mathcal{J} := \{f_1, f_2, \ldots, f_N, C\}$. Let $K_c$ be the attractor and $O$ be the orbital set for the inhomogeneous IFS $\mathcal{J}$. Since the attractor $K_\phi$ corresponds to the homogeneous part $\{f_1, f_2, \ldots, f_N\}$ of $\mathcal{J}$ and $O \subset K_\phi$, we have $K_c = K_\phi$. One can easily see that the inhomogeneous IFS $\mathcal{J}$ satisfies all the assumption of Proposition 3.3. By Proposition 3.3, the result follows. □

In the next proposition, we prove the dimensional approximation result for the inhomogeneous self-similar IFS under the remaining condition $K_\phi \subsetneq C$.

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PROPOSITION 3.5. *Let $(X, d)$ be a complete metric space and $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS on $X$. Let $K_c$ be the inhomogeneous attractor corresponding to $\mathcal{J}$ and $K_\phi$ be the attractor corresponding to the homogeneous part $\{f_1, f_2, \ldots, f_N\}$ of $\mathcal{J}$. Assume that $K_\phi \subsetneq C$. Then, for each $\epsilon > 0$, there exists an inhomogeneous self-similar IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, \tilde{C}\}$ on $X$ satisfying SSC with attractor $\tilde{K}_{\tilde{c}}$, such that*

$$\tilde{K}_{\tilde{c}} \cap K_c \neq \emptyset \quad \text{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_{\tilde{c}}) \geq \dim_H(K_c) - \epsilon,$$

*where $\tilde{C}$ is some self-similar copy of $C$.*

PROOF. Let $K_\phi \subsetneq C$. We can choose a similarity map $g$ on $X$ such that $g(C) \cap K_\phi = \emptyset$. Set $\tilde{C} = g(C)$. Clearly, $\dim_H(\tilde{C}) = \dim_H(C)$. Now, we define an inhomogeneous IFS $\mathcal{J}_1 := \{f_1, f_2, \ldots, f_N, \tilde{C}\}$ on $X$. Let $K_{\tilde{c}}$ be the attractor of the inhomogeneous IFS $\mathcal{J}_1$. The inhomogeneous IFS $\mathcal{J}_1$ satisfies all the assumptions of Proposition 3.1. By Proposition 3.1, for each $\epsilon > 0$, there exists an inhomogeneous self-similar IFS $\tilde{\mathcal{J}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}, \tilde{C}\}$ on $X$ satisfying SSC with attractor $\tilde{K}_{\tilde{c}}$, such that

$$\dim_H(K_{\tilde{c}}) \geq \dim_H(\tilde{K}_{\tilde{c}}) \geq \dim_H(K_{\tilde{c}}) - \epsilon.$$

Now, it only remains to show that $\dim_H(K_{\tilde{c}}) = \dim_H(K_c)$. Let $\tilde{O}$ and $O$ be the orbital sets for the inhomogeneous IFS $\mathcal{J}_1$ and $\mathcal{J}$, respectively. Thus,

$$\tilde{O} = \tilde{C} \bigcup \Big( \bigcup_{w \in \Xi^*} f_w(\tilde{C}) \Big).$$

By the countable stability property of the Hausdorff dimension, $\dim_H(\tilde{O}) = \dim_H(\tilde{C}) = \dim_H(C) = \dim_H(O)$. Since $K_{\tilde{c}} = K_\phi \cup \tilde{O}$ and $K_c = K_\phi \cup O$, we have $\dim_H(K_{\tilde{c}}) = \dim_H(K_c)$. This completes the proof. □

PROOF OF THEOREM 1.1. Theorem 1.1(1) follows by combining Propositions 3.1 and 3.3. Theorem 1.1(2) follows from Proposition 3.5. □

For the further discussion, we assume that $(X, d)$ is a complete metric space and $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ is an inhomogeneous IFS on $X$, where each $f_i$ is a similarity map with similarity ratio $r_i \in (0, 1)$. We assume that $C$ is the attractor of some IFS $\mathcal{I}' = \{g_1, g_2, \ldots, g_M\}$ on $X$, where each $g_j$ is a similarity map with similarity ratio $c_j \in (0, 1)$. Let $I = \{1, 2, \ldots, N\}$ and $J = \{1, 2, \ldots, M\}$. Let $K_c$ be the attractor corresponding to inhomogeneous IFS $\mathcal{J}$. Thus,

$$K_c = \bigcup_{i=1}^{N} f_i(K_c) \cup C = \Big( \bigcup_{i=1}^{N} f_i(K_c) \Big) \cup \Big( \bigcup_{j=1}^{M} g_j(C) \Big).$$

In the next result, we show that every inhomogeneous self-similar set defined as above can be approximated by a well-behaved self-similar set in the context of the Hausdorff dimension.

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PROPOSITION 3.6. *Let $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS defined as above. Let $K_c$ be the attractor for the inhomogeneous IFS $\mathcal{J}$. Then, for each $\epsilon > 0$, there exists a self-similar IFS $\tilde{\mathcal{I}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}\}$ satisfying SSC with attractor $\tilde{K}_\phi$, such that*

$$\tilde{K}_\phi \subseteq K_c \quad \textit{and} \quad \dim_H(K_c) \geq \dim_H(\tilde{K}_\phi) \geq \dim_H(K_c) - \epsilon.$$

PROOF. Let $K_\phi$ be the attractor corresponding to the homogeneous part $\{f_1, f_2, \ldots, f_N\}$ of $\mathcal{J}$ and $O$ be the orbital set for the inhomogeneous IFS $\mathcal{J}$. Since $K_c = K_\phi \cup O$, we have $\dim_H(K_c) = \max\{\dim_H(K_\phi), \dim_H(C)\}$. Thus, either $\dim_H(K_c) = \dim_H(K_\phi)$ or $\dim_H(K_c) = \dim_H(C)$.

Assume $\dim_H(K_c) = \dim_H(K_\phi)$. The set $K_\phi$ is the attractor of the IFS $\{f_1, f_2, \ldots, f_N\}$. By [11, Proposition 1.8], for each $\epsilon > 0$, there exists a self-similar IFS $\tilde{\mathcal{I}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}\}$ satisfying SSC with attractor $\tilde{K}_\phi$, such that

$$\tilde{K}_\phi \subseteq K_\phi \quad \text{and} \quad \dim_H(K_\phi) \geq \dim_H(\tilde{K}_\phi) \geq \dim_H(K_\phi) - \epsilon.$$

Since $K_\phi \subset K_c$ and $\dim_H(K_c) = \dim_H(K_\phi)$, the desired result follows.

Alternatively, assume $\dim_H(K_c) = \dim_H(C)$. Since $C$ is the attractor of the IFS $\mathcal{I}'$, we have $C \subset K_c$ and the desired result follows from [11, Proposition 1.8]. □

Let $B_1, B_2 \in \mathbb{R}$. The algebraic product $B_1 \cdot B_2$ of $B_1$ and $B_2$ is defined by $B_1 \cdot B_2 := \{b_1 \cdot b_2 : b_1 \in B_1, b_2 \in B_2\}$. The next result gives the dimension of the algebraic product of the attractor of the inhomogeneous self-similar IFS defined above.

PROPOSITION 3.7. *Let $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS defined as in Proposition 3.6 on $\mathbb{R}$. Let $K_c$ be the attractor for the inhomogeneous IFS $\mathcal{J}$. Let $\dim_H(K_c) = \dim_B(K_c)$. Then,*

$$\dim_H(K_c \cdot K_c) = \min\{2 \dim_H(K_c), 1\}.$$

PROOF. Since $\dim_H(K_c) = \dim_B(K_c)$, by the product formula for the Hausdorff dimension, $\dim_H(K_c \times K_c) = 2 \dim_H(K_c)$. We define a map $\Psi : K_c \times K_c \to K_c \cdot K_c$ by $\Psi(a, b) = a \cdot b$. Since $K_c$ is a nonempty compact set, the map $\Psi$ is a Lipschitz map. This implies that $\dim_H(K_c \cdot K_c) \leq \min\{2 \dim_H(K_c), 1\}$.

For the reverse inequality, Proposition 3.6 shows that for each $\epsilon > 0$, there exists a self-similar IFS $\tilde{\mathcal{I}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{\tilde{N}}\}$ satisfying SSC with attractor $\tilde{K}_\phi$, such that

$$\tilde{K}_\phi \subseteq K_c \quad \text{and} \quad \dim_H(K_c) \leq \dim_H(\tilde{K}_\phi) \geq \dim_H(K_c) - \epsilon.$$

Since $\tilde{K}_\phi \subseteq K_c$, we have $\dim_H(\tilde{K}_\phi \cdot \tilde{K}_\phi) \leq \dim_H(K_c \cdot K_c)$. By [2, Corollary 2.9], we get $\dim_H(\tilde{K}_\phi \cdot \tilde{K}_\phi) = \min\{2 \dim_H(\tilde{K}_\phi), 1\}$. This implies that

$$\dim_H(K_c \cdot K_c) \geq \min\{2 \dim_H(\tilde{K}_\phi), 1\} \geq \min\{2 \dim_H(K_c) - 2\epsilon, 1\}$$

for arbitrary $\epsilon$. Thus, $\dim_H(K_c \cdot K_c) \geq \min\{2 \dim_H(K_c), 1\}$. This completes the proof. □

The final result gives the dimension of the algebraic sum of the attractor of the inhomogeneous self-similar IFS defined above.

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PROPOSITION 3.8. *Let $\mathcal{J} = \{f_1, f_2, \ldots, f_N, C\}$ be an inhomogeneous self-similar IFS defined as in Theorem 3.6 on $\mathbb{R}$. Let $K_c$ be the attractor corresponding to the inhomogeneous IFS $\mathcal{J}$. Let $\dim_H(K_c) = \dim_B(K_c)$. Assume that for some $i, i' \in \{1, 2, \ldots, N\}$ and $j, j' \in \{1, 2, \ldots, M\}$, we have $\log(r_i)/\log(r_{i'}) \notin \mathbb{Q}$ and $\log(c_j)/\log(c_{j'}) \notin \mathbb{Q}$. Then,*

$$\dim_H(K_c + K_c) = \min\{2 \dim_H(K_c), 1\}.$$

PROOF. Since $\dim_H(K_c) = \dim_B(K_c)$, the product formula gives $\dim_H(K_c \times K_c) = 2 \dim_H(K_c)$. The map $\Psi : K_c \times K_c \to K_c + K_c$ defined by $\Psi(a, b) = a + b$ is a Lipschitz map. Thus, $\dim_H(K_c + K_c) \leq \min\{2 \dim_H(K_c), 1\}$.

For the opposite inequality, let $K_\phi$ be the attractor corresponding to the homogeneous part $\{f_1, f_2, \ldots, f_N\}$ of $\mathcal{J}$. Since $\dim_H(K_c) = \max\{\dim_H(K_\phi), \dim_H(C)\}$, we have either $\dim_H(K_c) = \dim_H(K_\phi)$ or $\dim_H(K_c) = \dim_H(C)$. Assume that $\dim_H(K_c) = \dim_H(K_\phi)$. Since $\log(r_i)/\log(r_{i'}) \notin \mathbb{Q}$ for some $i, i' \in \{1, 2, \ldots, N\}$, by [23, Theorem 2], $\dim_H(K_\phi + K_\phi) = \min\{2 \dim_H(K_\phi), 1\}$. Since $K_\phi \subset K_c$ and $\dim_H(K_c) = \dim_H(K_\phi)$, we obtain

$$\dim_H(K_c + K_c) \geq \min\{2 \dim_H(K_c), 1\}.$$

Assume $\dim_H(K_c) = \dim_H(C)$. By similar arguments based on $\log(c_i)/\log(c_{i'}) \notin \mathbb{Q}$ and [23, Theorem 2], we get the same conclusion. This completes the proof. □

## References


[1] S. Baker, J. M. Fraser and A. Máthé, 'Inhomogeneous self-similar sets with overlaps', *Ergodic Theory Dynam. Systems* **39**(1) (2019), 1–18.

[2] B. Bárány, 'On some non-linear projections of self-similar sets in $\mathbb{R}^3$', *Fund. Math.* **237** (2017), 83–100.

[3] B. Bárány, A. Käenmäki and P. Nissinen, 'Covering number on inhomogeneous graph-directed self-similar sets', Preprint, 2023, arXiv:2307.16263.

[4] M. F. Barnsley and S. Demko, 'Iterated function systems and the global construction of fractals', *Proc. Roy. Soc. Lond. Sect. A* **399** (1985), 243–275.

[5] S. Chandra and S. Abbas, 'On fractal dimensions of fractal functions using function spaces', *Bull. Aust. Math. Soc.* **106** (2022), 470–480.

[6] S. Dubey and S. Verma, 'Fractal dimension for a class of inhomogeneous graph-directed attractors', *Discrete and Continuous Dynamical Systems - Series S* (2024), doi:10.3934/dcdss.2024158.

[7] S. Dubey and S. Verma, 'Inhomogeneous graph-directed attractors and fractal measures', *J. Anal.* **32** (2024), 157–170.

[8] K. J. Falconer, 'The Hausdorff dimension of some fractals and attractors of overlapping construction', *J. Stat. Phys.* **47** (1987), 123–132.

[9] K. J. Falconer, *Techniques in Fractal Geometry* (Wiley, Chichester, 1997).

[10] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications* (Wiley, New York, 1999).

[11] Á. Farkas, 'Projections of self-similar sets with no separation condition', *Israel J. Math.* **214**(1) (2016), 67–107.

[12] Á. Farkas, 'Dimension approximation of attractors of graph directed IFSs by self-similar sets', *Math. Proc. Cambridge Philos. Soc.* **167**(1) (2019), 193–207.

[13] J. M. Fraser, 'Inhomogeneous self-similar sets and box dimensions', *Studia Math.* **213** (2012), 133–156.


https://doi.org/10.1017/S0004972725100129 Published online by Cambridge University Press

[14] J. M. Fraser, *Assouad Dimension and Fractal Geometry*, Cambridge Tracts in Pure Mathematics, 222 (Cambridge University Press, Cambridge, 2020).
[15] J. E. Hutchinson, 'Fractals and self-similarity', *Indiana Univ. Math. J.* **30** (1981), 713–747.
[16] S. Jha and S. Verma, 'Dimensional analysis of $\alpha$-fractal functions', *Results Math.* **76**(4) (2021), 1–24.
[17] R. D. Mauldin and M. Urbański, 'Dimensions and measures in infinite iterated function systems', *Proc. Lond. Math. Soc. (3)* **73** (1996), 105–154.
[18] R. D. Mauldin and M. Urbański, 'Conformal iterated function systems with applications to the geometry of continued fractions', *Trans. Amer. Math. Soc.* **351**(12) (1999), 4995–5025.
[19] R. D. Mauldin and S. C. Williams, 'Hausdorff dimension in graph directed construction', *Trans. Amer. Math. Soc.* **309**(2) (1988), 811–829.
[20] R. D. Nussbaum, A. Priyadarshi and S. V. Lunel, 'Positive operators and Hausdorff dimension of invariant sets', *Trans. Amer. Math. Soc.* **364**(2) (2012), 1029–1066.
[21] L. Olsen and N. Snigireva, '$L^q$ spectra and Rényi dimensions of in-homogeneous self-similar measures', *Nonlinearity* **20** (2007), 151–175.
[22] T. Orponen, 'On the distance sets of self-similar sets', *Nonlinearity* **25** (2012), 1919–1929.
[23] Y. Peres and P. Shmerkin, 'Resonance between Cantor sets', *Ergodic Theory Dynam. Systems* **29** (2009), 201–221.
[24] A. Priyadarshi, 'Lower bound on the Hausdorff dimension of a set of complex continued fractions'. *J. Math. Anal. Appl.* **449**(1) (2017), 91–95.
[25] S. Verma, 'Hausdorff dimension and infinitesimal similitudes on complete metric spaces', Preprint, 2021, arXiv:2101.07520.

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