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The h-spacing distribution between Farey points

BY VOLKER AUGUSTIN
Richter Weg 12*d*, 92421 *Schwandorf, Germany.*
e-mail: snowbird@bigfoot.de

FLORIN P. BOCA†
*School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF*2 4*YH.*
e-mail: BocaFP@cardiff.ac.uk
On leave from: Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, *Bucharest* 70700, *Romania.*

CRISTIAN COBELI
Institute of Mathematics of the Romanian Academy, P.O. Box 1-764,
Bucharest 70700, *Romania.*
e-mail: ccobeli@stoilow.imar.ro

AND ALEXANDRU ZAHARESCU
Institute of Mathematics of the Romanian Academy, P.O. Box 1-764,
Bucharest 70700, *Romania.*
Institute for Advanced Study, School of Mathematics, Olden Lane,
Princeton, NJ 08540, *U.S.A.*
e-mail: zaharesc@math.uiuc.edu

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1. *Introduction and statement of results*

Let $I = [\alpha, \beta]$ be a subinterval of $[0, 1]$. For each positive integer Q, we denote by $\mathscr{F}_I(Q)$ the set of Farey fractions of order Q from I, that is

$$\mathscr{F}_I(Q) = \{\gamma = a/q \in I\,;\, 1 \leqslant a \leqslant q \leqslant Q, \quad \gcd(a, q) = 1\}, \tag{1·1}$$

and order increasingly its elements $\gamma_j = a_j/q_j$ as $\alpha \leqslant \gamma_1 < \gamma_2 < \cdots < \gamma_{N_I(Q)} \leqslant \beta$. The number of elements of $\mathscr{F}_I(Q)$ is

$$N_I(Q) = |I|N(Q) + O(Q \log Q) = 3|I|Q^2/\pi^2 + O(Q \log Q). \tag{1·2}$$

We simply let $\mathscr{F}(Q) = \mathscr{F}_{[0,1]}(Q)$, $N(Q) = N_{[0,1]}(Q)$.

Farey sequences have been studied for a long time, mainly because of their role in problems related to diophantine approximation. There is also a connection with the Riemann zeta function which has motivated their study. Farey sequences seem to

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be distributed as uniformly as possible along $[0,1]$; a way to prove it is to show that

$$\sum_{\gamma_n \in \mathscr{F}(Q)} \left| \gamma_n - \frac{n}{N} \right| = O_\varepsilon(Q^{\frac{1}{2}+\varepsilon}) \tag{1·3}$$

for all $\varepsilon > 0$, as $Q \to \infty$. Yet this is a very strong statement, as Franel and Landau [**3**, **4**] have shown that (1·3) is equivalent to the Riemann Hypothesis.

Our object here is to investigate the distribution of spacings between Farey points in subintervals of $[0,1]$. Various results related to this problem have been obtained by [**2**, **3**, **5**–**8**, **10**–**13**].

Speaking about the distribution of spacings between elements of a given sequence, there are two standard ways to proceed. One is to study the hth level consecutive spacing measure for any $h \geqslant 1$. The other is to investigate, for $m \geqslant 2$, the m-level correlation measure. Although these approaches are related, it might be easier in specific problems, for technical reasons, to use one rather than the other. Here, we choose the first one with the aim to determine the h-spacing distribution between the points of $\mathscr{F}_I(Q)$.

To make things precise, let $\mathscr{T}$ denote the Farey triangle

$$\{(x,y)\,;\, 0 < x \leqslant 1,\, 0 < y \leqslant 1,\, x + y > 1\}$$

and consider, for each $(x,y) \in \mathbf{R}^2$, the sequence $(L_i(x,y))_{i \geqslant 0}$ defined by $L_0(x,y) = x$, $L_1(x,y) = y$ and then recursively, for $i \geqslant 2$,

$$L_i(x,y) = \left[\frac{1 + L_{i-2}(x,y)}{L_{i-1}(x,y)} \right] L_{i-1}(x,y) - L_{i-2}(x,y). \tag{1·4}$$

Consider as well the map $\Phi_h \colon \mathscr{T} \to (0,\infty)^h$ given by

$$\Phi_h(x,y) = \frac{3}{\pi^2} \left(\frac{1}{L_0(x,y)L_1(x,y)}, \frac{1}{L_1(x,y)L_2(x,y)}, \dots, \frac{1}{L_{h-1}(x,y)L_h(x,y)} \right). \tag{1·5}$$

For each box $\mathscr{B} = \prod_{i=1}^h (\alpha_i, \beta_i) \subset (0,\infty)^h$, we define the following subset of $\mathscr{T}$:

$$\widetilde{\Omega}_{\mathscr{B}} = \bigcap_{i=1}^h \left\{ (x,y) \in \mathscr{T};\, \frac{3}{\pi^2 \beta_i} < L_{i-1}(x,y)L_i(x,y) < \frac{3}{\pi^2 \alpha_i} \right\} = \Phi_h^{-1}(\mathscr{B}). \tag{1·6}$$

Given N real numbers $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_N$, whose mean spacing is about 1, we consider the hth level consecutive spacing (probability) measure ν_h on $[0,\infty)^h$ defined, for $f \in C_c([0,\infty)^h)$, by

$$\int_{[0,\infty)^h} f \, d\nu_h = \frac{1}{N-h} \sum_{j=1}^{N-h} f(x_{j+1} - x_j, x_{j+2} - x_{j+1}, \dots, x_{j+h} - x_{j+h-1}).$$

We shall normalize the sequence $\mathscr{F}_I(Q)$ and consider $x_n = N_I(Q)\gamma_n/|I|$ to get, for each Q, the sequence $(x_n)_{n=1}^{N_I(Q)}$ with mean spacing unity. Accordingly, we obtain a sequence $(\mu_Q^{h,I})_{Q \geqslant 1}$ of probability measures on $[0,\infty)^h$. The object of this paper is to study the convergence of this sequence, proving

Theorem 1·1. *The sequence $(\mu_Q^{h,I})_{Q \geqslant 1}$ converges weakly to a probability measure μ_h, which is independent of I. Moreover, the support $\mathscr{D}_h$ of μ_h coincides with the closure of the*

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range of the function Φ_h and the repartition of μ_h is given, for every box $\mathscr{B} \subset (0,\infty)^h$, by

$$\mu_h(\mathscr{B}) = 2\,\mathrm{Area}\,(\widetilde{\Omega}_{\mathscr{B}}).$$

It is easily seen that $\mathscr{D}_1 = [3/\pi^2, \infty)$, whereas $\mathscr{D}_h$ is strictly smaller than $[3/\pi^2,\infty)^h$ for $h \geqslant 2$.

For randomly distributed numbers, the m-level correlation functions are all constants equal to 1, whereas the hth level consecutive spacing limiting measures μ_h are given by $d\mu_h(\lambda_1,\dots,\lambda_h) = e^{-(\lambda_1+\cdots+\lambda_h)}d\lambda_1\cdots d\lambda_h$. Since $\mathscr{D}_1 = [3/\pi^2,\infty)$, we find out that the Farey numbers from any subinterval of $[0,1]$ do not behave as random numbers. Actually, the proportion of differences between consecutive numbers of $\mathscr{F}(Q)$ which are larger than the average is

$$G_1(1) = 6(1-\log(3/\pi^2))/\pi^2 - 1 = 0{\cdot}3318\ldots \tag{1·7}$$

as $Q \to \infty$, which is smaller than the value $1/e = 0{\cdot}36787\ldots$ expected in the random case.

The case $h = 1$ and $I = [0,1]$ has been previously considered in [**5**]. With the notation from (3·1) and formula (3·2), Theorem 1·1 shows that the repartition function of μ_1 is given by

$$\begin{aligned} G_1(t) &= \int_t^\infty d\mu_1(x) = 1 - 2\,\mathrm{Area}\,(\{(x,y)\in\mathscr{T};\, xy > 3/(\pi^2 t)\}) \\ &= \begin{cases} 1, & \text{for } 0 \leqslant t \leqslant \dfrac{3}{\pi^2}, \\ -1 + \dfrac{6}{\pi^2 t} - \dfrac{6}{\pi^2 t}\log\dfrac{3}{\pi^2 t}, & \text{for } \dfrac{3}{\pi^2} \leqslant t \leqslant \dfrac{12}{\pi^2}, \\ -1 + \dfrac{6}{\pi^2 t} + \sqrt{1-\dfrac{12}{\pi^2 t}} - \dfrac{12}{\pi^2 t}\log\dfrac{1+\sqrt{1-\frac{12}{\pi^2 t}}}{2}, & \text{for } \dfrac{12}{\pi^2} \leqslant t. \end{cases} \end{aligned} \tag{1·8}$$

The previous formula immediately yields (1·7). It also shows that μ_1 is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$ and has density

$$g_1(t) = \begin{cases} 0, & \text{for } 0 \leqslant t \leqslant \dfrac{3}{\pi^2}, \\ \dfrac{6}{\pi^2 t^2}\log\left(\dfrac{\pi^2 t}{3}\right), & \text{for } \dfrac{3}{\pi^2} \leqslant t \leqslant \dfrac{12}{\pi^2}, \\ \dfrac{12}{\pi^2 t^2}\log\left(\dfrac{\pi^2 t}{6}\left(1-\sqrt{1-\dfrac{12}{\pi^2 t}}\right)\right), & \text{for } \dfrac{12}{\pi^2} \leqslant t. \end{cases} \tag{1·9}$$

Extensive numerical computations revealed other special features of the measure μ_h. For example, its support $\mathscr{D}_h$, which is strictly smaller than $[3/\pi^2,\infty)^h$ for $h \geqslant 2$, looks very interesting. The set $\mathscr{D}_2$ is completely determined in Section 6. For $h \geqslant 3$, it follows from Theorem 1·1 and the definition of $L_i(x,y)$ that $\mathscr{D}_h$ is a countable union of surfaces. In particular, $\mathscr{D}_h$ is a measure zero subset of $[3/\pi^2,\infty)^h$.

To prove Theorem 1·1 we shall estimate, for each box $\mathscr{B} = \prod_{i=1}^h(\alpha_i,\beta_i) \subset (0,\infty)^h$,

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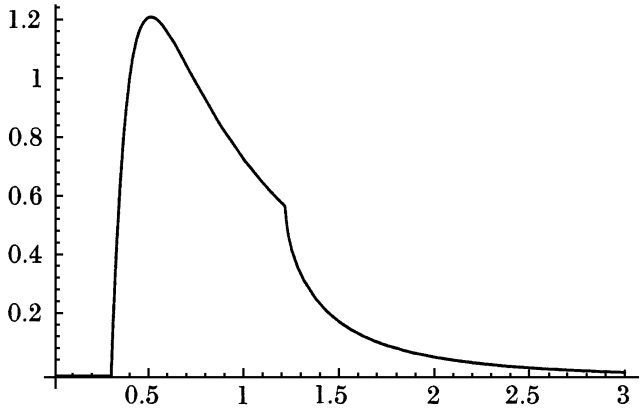


Fig. 1. The density of μ_1.

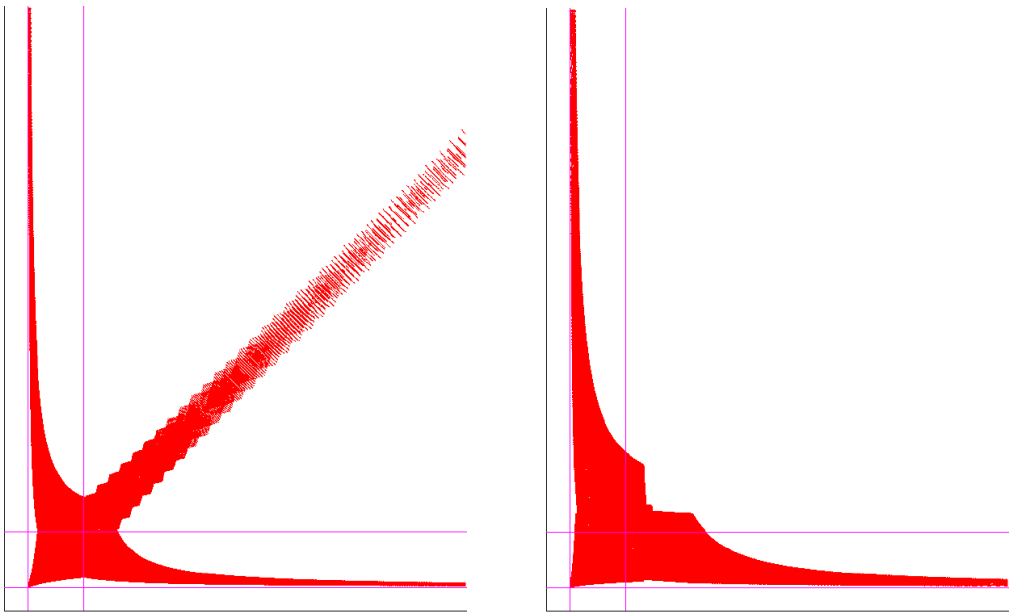


Fig. 2. The support of $\mu_{3000}^{2,[0,1]}$ and the projection on the $x-z$ coordinates of the support of $\mu_{3000}^{3,[0,1]}$.

the ratio

$$
\begin{aligned}
G_{\mathscr{B},I}(Q) &= \frac{1}{N_I(Q)}\,\#\bigcap_{i=1}^{h}\left\{\gamma_j\in\mathscr{F}_I(Q)\,;\quad \frac{\alpha_i|I|}{N_I(Q)}<\gamma_{j+i}-\gamma_{j+i-1}<\frac{\beta_i|I|}{N_I(Q)}\right\}\\
&= \frac{1}{N_I(Q)}\,\#\bigcap_{i=1}^{h}\left\{\gamma_j\in\mathscr{F}_I(Q)\,;\quad \frac{\alpha_i|I|}{N_I(Q)}<\frac{1}{q_{j+i}q_{j+i-1}}<\frac{\beta_i|I|}{N_I(Q)}\right\} \qquad (1\cdot10)\\
&= \frac{1}{N_I(Q)}\,\#\bigcap_{i=1}^{h}\left\{\gamma_j\in\mathscr{F}_I(Q)\,;\quad \frac{N_I(Q)}{\beta_i|I|}<q_{j+i}q_{j+i-1}<\frac{N_I(Q)}{\alpha_i|I|}\right\},
\end{aligned}
$$

as $Q\to\infty$.

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For each $1 \leqslant i \leqslant h$, we define

$$\left.\begin{aligned} &A_i(\mathscr{B}) = 3/(\pi^2\beta_i), \quad B_i(\mathscr{B}) = 3/(\pi^2\alpha_i), \\ &A_i(\mathscr{B}, I, Q) = \frac{N_I(Q)}{\beta_i|I|Q^2} = A_i(\mathscr{B}) + O\left(\frac{\log Q}{Q|I|}\right), \\ &B_i(\mathscr{B}, I, Q) = \frac{N_I(Q)}{\alpha_i|I|Q^2} = B_i(\mathscr{B}) + O\left(\frac{\log Q}{Q|I|}\right). \end{aligned}\right\} \tag{1·11}$$

2. *A certain transformation of the Farey triangle*

Let us first state an elementary lemma from [**2**].

Lemma 2·1. *The map*

$$T(x, y) = \left(y, \left[\frac{1+x}{y}\right]y - x\right), \quad (x, y) \in \mathscr{T}, \tag{2·1}$$

is a well-defined, bijective, area-preserving transformation of $\mathscr{T}$.

As a consequence of the fact that $\mathscr{T}$ is T-invariant and of relations (2·1) and (1·4) we obtain

Corollary 2·2. *For all* $i \geqslant 0$ *and* $(x, y) \in \mathscr{T}$ *one has*

$$(L_i(x, y), L_{i+1}(x, y)) = T^i(x, y).$$

We will employ a certain characterization of h-tuples of Farey fractions. It is known from [**7**] that the integers $c_0, c_1, \ldots, c_h$ are consecutive denominators of fractions from $\mathscr{F}(Q)$ if and only if the following three relations hold

$$c_0 + c_1, c_1 + c_2, \ldots, c_{h-1} + c_h > Q \geqslant c_0, c_1, \ldots, c_h \geqslant 1, \tag{2·2}$$

$$\gcd(c_0, c_1) = \gcd(c_1, c_2) = \cdots = \gcd(c_{h-1}, c_h) = 1, \tag{2·3}$$

$$c_{j+1} = k_j c_j - c_{j-1} \quad \text{for some } k_1, \ldots, k_h \in \mathbf{Z}. \tag{2·4}$$

The integers k_j are uniquely determined as $k_j = [(Q + c_{j-1})/c_j]$. Therefore, if $K \geqslant 1$ and $q_0, q_1, \ldots, q_K$ are denominators of consecutive Farey fractions of order Q then we have, for all $i, j \geqslant 0$ with $i + j \leqslant K$,

$$q_{j+i} = QL_i(q_j/Q, q_{j+1}/Q). \tag{2·5}$$

For each $k_1, \ldots, k_i \in \mathbf{Z}$, we consider the 2×2 matrix

$$B(k_1, \ldots, k_i) = \begin{pmatrix} k_i & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2·6}$$

Lemma 2·3. *Suppose that* $h \geqslant 2$ *and* $\mathbf{k} = (k_1, \ldots, k_{h-1}) \in (\mathbf{N}^*)^{h-1}$. *Then, for any integer* $1 \leqslant q \leqslant Q$, *the set*

$$S_{\mathbf{k}} = S_{\mathbf{k}}(Q, q) = \bigcap_{i=1}^{h-1} \left\{ t \in \left(1 - \frac{q}{Q}, 1\right]; \quad \left[\frac{1 + L_{i-1}(t, q/Q)}{L_i(t, q/Q)}\right] = k_i \right\}$$

has at most 2^{h-2} *connected components.*

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Proof. The proof is given by induction. For $h = 2$, this follows from the fact that

$$S_k = (1 - q/Q, 1] \bigcap [kq/Q - 1, (k+1)q/Q - 1)$$

is an interval (possibly empty) for all $k \in \mathbf{N}^*$. Assume then that $S_{(k_1,\dots,k_{h-1})}$ has at most 2^{h-2} components. If we put $B = B(k_1, \dots, k_{h-1}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, then the equality

$$\begin{pmatrix} L_h(t, q/Q) \\ L_{h-1}(t, q/Q) \end{pmatrix} = B \begin{pmatrix} L_1(t, q/Q) \\ L_0(t, q/Q) \end{pmatrix} = B \begin{pmatrix} q/Q \\ t \end{pmatrix} \tag{2·7}$$

provides, whenever $t \in S_k$,

$$L_h(t, q/Q) = B_{11} q/Q + B_{12} t \quad \text{and} \quad L_{h-1}(t, q/Q) = B_{21} q/Q + B_{22} t. \tag{2·8}$$

Hence,

$$\left[\frac{1 + L_{h-1}(t, q/Q)}{L_h(t, q/Q)} \right] = k_h$$

is equivalent to

$$\left[\frac{1 + B_{21} q/Q + B_{22} t}{B_{11} q/Q + B_{21} t} \right] = k_h. \tag{2·9}$$

The induction step is now completed by using

$$S_{(k_1,\dots,k_h)} = S_{(k_1,\dots,k_{h-1})} \bigcap \{ t \in \mathbf{R} \,;\, t \text{ solution of (2·9)} \}$$

and the fact that the set of solutions of an equation in t of type $[(at+b)/(ct+d)] = k$, where a, b, c, d are integers, is either an interval or the union of two intervals.

Lemma 2·4. *Assume that* $\lambda = (\lambda_1(Q), \dots, \lambda_h(Q))_Q$ *and* $\mu = (\mu_1(Q), \dots, \mu_h(Q))_Q$ *are two sequences from* $(0, \infty)^h$ *such that, as* $Q \to \infty$,

$$\lambda_i(Q) = \lambda_i + o(1), \quad \mu_i(Q) = \mu_i + o(1)$$

for some constants $0 < \lambda_i < \mu_i$, $1 \leqslant i \leqslant h$. *Then the number of connected components of the set*

$$J(\lambda, \mu, Q, q) = (Q - q, Q] \bigcap \bigcap_{i=1}^{h} \left\{ x \,;\, \lambda_i(Q) < L_{i-1}\Big(\frac{x}{Q}, \frac{q}{Q}\Big) L_i\Big(\frac{x}{Q}, \frac{q}{Q}\Big) < \mu_i(Q) \right\} \tag{2·10}$$

is $\leqslant 2^{3h-1}/(\lambda_1 \cdots \lambda_{h-1})$ *for any large* Q (*depending on* $\lambda_1, \dots, \lambda_h$ *only*) *and any* $1 \leqslant q \leqslant Q$.

Proof. We simply write $J(Q, q) = J(\lambda, \mu, Q, q)$ and denote, for $\mathbf{k} = (k_1, \dots, k_{h-1}) \in (\mathbf{N}^*)^{h-1}$, $B^{(\mathbf{k},i)} = B(k_1, \dots, k_{i-1})$ for $2 \leqslant i \leqslant h$ and $B^{(\mathbf{k},1)} = I$. It is useful to consider

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the sets

$$J'_{\mathbf{k}}(Q,q) = \bigcap_{i=1}^{h-1} \left\{ t \in \left(1-\frac{q}{Q},1\right];\ \left[\frac{1+L_{i-1}(t,q/Q)}{L_i(t,q/Q)}\right] = k_i \right\},$$

$$J''_{\mathbf{k}}(Q,q) = J''_{\mathbf{k}}(\lambda,\mu,Q,q)$$

$$= \bigcap_{i=1}^{h} \left\{ t;\ \lambda_i(Q) < \left(B_{11}^{(\mathbf{k},i)}\frac{q}{Q} + B_{12}^{(\mathbf{k},i)}t\right)\left(B_{21}^{(\mathbf{k},i)}\frac{q}{Q} + B_{22}^{(\mathbf{k},i)}t\right) < \mu_i(Q) \right\},$$

$$J_{\mathbf{k}}(Q,q) = J_{\mathbf{k}}(\lambda,\mu,Q,q) = J'_{\mathbf{k}}(Q,q) \cap J''_{\mathbf{k}}(Q,q).$$

We notice that

$$\bigcup_{\mathbf{k}\in(\mathbf{N}^*)^{h-1}} J_{\mathbf{k}}(Q,q) = \frac{1}{Q}\,J(Q,q). \tag{2·11}$$

On the other hand we have, for all $Q \geqslant Q_0 = Q_0(\lambda_1,\dots,\lambda_h)$, all $1 \leqslant q \leqslant Q$ and all $t \in J_{\mathbf{k}}(Q,q)$,

$$1 \leqslant k_i \leqslant k_i(t) = \left[\frac{1+L_{i-1}(t,q/Q)}{L_i(t,q/Q)}\right]$$

$$\leqslant \frac{2}{L_i(t,q/Q)} \leqslant \frac{2}{L_{i-1}(t,q/Q)L_i(t,q/Q)} \leqslant \frac{2}{\lambda_i(Q)} < \frac{4}{\lambda_i}. \tag{2·12}$$

Therefore, (2·11) transforms into

$$\frac{1}{Q}\,J(Q,q) = \bigcup_{\substack{\mathbf{k}\in(\mathbf{N}^*)^{h-1}\\ k_i<4/\lambda_i}} J_{\mathbf{k}}(Q,q). \tag{2·13}$$

But $J''_{\mathbf{k}}(Q,q)$ has, by its very definition, at most four connected components, whereas $J'_{\mathbf{k}}(Q,q)$ has, according to the previous lemma, at most 2^{h-2} components. Combining this with (2·13) we conclude that $J(Q,q)$ has at most $4^{h-1}\cdot 4\cdot 2^{h-1}/(\lambda_1\cdots\lambda_{h-1})$ connected components for all $Q \geqslant Q_0$ and all $1 \leqslant q \leqslant Q$.

3. *Some area estimates*

For each $0 < a < b$ we denote

$$\boldsymbol{\Omega}_{a,b} = \{(x,y) \in \mathscr{T};\ a < xy < b\} \quad \text{and} \quad \boldsymbol{\Omega}_a = \{(x,y) \in \mathscr{T};\ xy > a\}, \tag{3·1}$$

and consider, for each box $\mathscr{B} = \prod_{i=1}^{r}(\alpha_i,\beta_i) \subset (0,\infty)^r$, the set

$$\boldsymbol{\Omega}_{\mathscr{B}} = \bigcap_{i=1}^{r} \{(x,y) \in \mathscr{T};\ \alpha_i < L_{i-1}(x,y)L_i(x,y) < \beta_i\}.$$

Lemma 3·1. *Suppose that $0 < a < b$, $0 < a' < b'$ and $|a-a'|, |b-b'| \leqslant \varepsilon$. Then*

$$\text{Area}\,(\boldsymbol{\Omega}_{a,b} \triangle \boldsymbol{\Omega}_{a',b'}) \ll_{a,b} \varepsilon.$$

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Proof. The equality

$$\text{Area}\,(\Omega_a) = \begin{cases} 0, & \text{if } a \geqslant 1, \\ \displaystyle\int_a^1 \left(1 - \frac{a}{x}\right) dx = 1 - a + a\log a, & \text{if } \tfrac{1}{4} < a < 1, \\ \displaystyle\int_a^1 \left(1 - \frac{a}{x}\right) dx - \int_{(1-\sqrt{1-4a})/2}^{(1+\sqrt{1-4a})/2} \left(1 - x - \frac{a}{x}\right) dx & \\ \quad = 1 - a - \sqrt{1-4a}/2 + 2a\log((1+\sqrt{1-4a})/2), & \text{if } 0 < a \leqslant \tfrac{1}{4}, \end{cases} \tag{3·2}$$

and the mean value theorem yield, whenever $0 < a < b \leqslant 1$,

$$\begin{aligned} \text{Area}\,(\Omega_{a,b}) &\leqslant \int_b^1 \left(1 - \frac{b}{x}\right) dx - \int_a^1 \left(1 - \frac{a}{x}\right) dx \\ &\leqslant (b-a)\log b \leqslant (b-a)\left(\log a + \frac{b-a}{a}\right). \end{aligned}$$

With $c = \min(a, a')$, $c' = \max(a, a')$, $d = \min(b, b')$, $d' = \max(b, b')$, this provides further

$$\text{Area}\,(\Omega_{a,b} \triangle \Omega_{a',b'}) \leqslant \text{Area}\,(\Omega_{c,c'}) + \text{Area}\,(\Omega_{d,d'}) \ll_{a,b} (|a'-a| + |b'-b|) \leqslant 2\varepsilon.$$

Corollary 3·2. *For any boxes $\mathscr{B} = \prod_{i=1}^r (\alpha_i, \beta_i)$ and $\mathscr{B}' = \prod_{i=1}^r (\alpha_i', \beta_i')$ in $(0,\infty)^r$, we have*

$$\text{Area}\,(\Omega_{\mathscr{B}} \triangle \Omega_{\mathscr{B}'}) \ll_{\mathscr{B}} \max_{1 \leqslant i \leqslant r} \{|\alpha_i - \alpha_i'|, |\beta_i - \beta_i'|\}.$$

Proof. Denoting

$$\Omega_i(\mathscr{B}) = T^{-i+1}(\Omega_{\alpha_i,\beta_i}) = \{(u_1,u_2) \in \mathscr{T};\, \alpha_i < L_{i-1}(u_1,u_2)L_i(u_1,u_2) < \beta_i\},$$

it shall be clear that

$$\Omega_{\mathscr{B}} = \bigcap_{i=1}^r \Omega_i(\mathscr{B}). \tag{3·3}$$

Since T is area-preserving, the previous lemma provides, for all $1 \leqslant i \leqslant r$,

$$\text{Area}(\Omega_i(\mathscr{B}) \triangle \Omega_i(\mathscr{B}')) \ll_{\alpha_i,\beta_i} \max\{|\alpha_i - \alpha_i'|, |\beta_i - \beta_i'|\}. \tag{3·4}$$

The statement follows now from (3·3), the inclusion

$$\Omega_{\mathscr{B}} \triangle \Omega_{\mathscr{B}'} = \left(\bigcap_{i=1}^r \Omega_i(\mathscr{B})\right) \triangle \left(\bigcap_{i=1}^r \Omega_i(\mathscr{B}')\right) \subset \bigcup_{i=1}^r (\Omega_i(\mathscr{B}) \triangle \Omega_i(\mathscr{B}'))$$

and (3·4).

4. *The case of the full interval* $[0,1]$

In this section we estimate, for each box $\mathscr{B} = \prod_{i=1}^h (\alpha_i, \beta_i) \subset (0,\infty)^h$, the ratio

$$G_{\mathscr{B}}(Q) = \frac{1}{N(Q)} \#\bigcap_{i=1}^h \{1 \leqslant j \leqslant N(Q)\,;\, \alpha_i < N(Q)(\gamma_{j+i} - \gamma_{j+i-1}) < \beta_i\},$$

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as $Q \to \infty$. Using (2·5), we may express $N(Q)G_{\mathscr{B}}(Q)$ as

$$\#\bigcap_{i=1}^{h}\left\{(x,y) \in Q\mathscr{T} \cap \mathbf{Z}^2;\ \begin{array}{l}\gcd(x,y)=1,\\ \dfrac{\alpha_i Q^2}{N(Q)} < \dfrac{1}{L_{i-1}(x/Q,y/Q)L_i(x/Q,y/Q)} < \dfrac{\beta_i Q^2}{N(Q)}\end{array}\right\}.$$

Letting

$$\begin{aligned}\mathbf{\Omega}_{\mathscr{B}}(Q) &= \bigcap_{i=1}^{h}\left\{(x,y) \in Q\mathscr{T};\ \frac{N(Q)}{\beta_i Q^2} < L_{i-1}\Big(\frac{x}{Q},\frac{y}{Q}\Big)L_i\Big(\frac{x}{Q},\frac{y}{Q}\Big) < \frac{N(Q)}{\alpha_i Q^2}\right\}\\ &= \bigcap_{i=1}^{h}\left\{(Qx,Qy);\ (x,y) \in \mathscr{T},\ \frac{N(Q)}{\beta_i Q^2} < L_{i-1}(x,y)L_i(x,y) < \frac{N(Q)}{\alpha_i Q^2}\right\},\end{aligned}$$

we arrive at

$$G_{\mathscr{B}}(Q) = \frac{1}{N(Q)}\,\#\{(x,y) \in \mathbf{\Omega}_{\mathscr{B}}(Q) \cap \mathbf{Z}^2;\ \gcd(x,y)=1\}. \tag{4·1}$$

From [**2**, lemma 2] the number of integer points of coprime coordinates in $\mathbf{\Omega}_{\mathscr{B}}(Q)$ can be expressed as

$$\frac{6}{\pi^2}\ \mathrm{Area}(\mathbf{\Omega}_{\mathscr{B}}(Q)) + O(C_{\mathscr{B}}(Q)Q) + O(\mathrm{length}(\partial\mathbf{\Omega}_{\mathscr{B}}(Q))\log Q), \tag{4·2}$$

where $C_{\mathscr{B}}(Q)$ denotes the number of connected components of $\mathbf{\Omega}_{\mathscr{B}}(Q)$.

To estimate the error in (4·2), we first let $A_i = A_i(\mathscr{B})$, $B_i = B_i(\mathscr{B})$, $A_i(Q) = A_i\big(\mathscr{B},[0,1],Q\big)$, $B_i(Q) = B_i\big(\mathscr{B},[0,1],Q\big)$, where $A_i(\mathscr{B})$, $B_i(\mathscr{B})$, $A_i(\mathscr{B},I,Q)$ $B_i(\mathscr{B},I,Q)$ are given by (1·11). With this notation we may write

$$\frac{1}{Q}\,\mathbf{\Omega}_{\mathscr{B}}(Q) = \bigcap_{i=1}^{h}\{(x,y) \in \mathscr{T};\ A_i(Q) < L_{i-1}(x,y)L_i(x,y) < B_i(Q)\}$$

and give an upper bound for the length of $Q^{-1}\mathbf{\Omega}_{\mathscr{B}}(Q)$. To achieve this for $h \geqslant 2$ we consider, as in [**2**], the map

$$\mathbf{k}\colon \mathscr{T} \to (\mathbf{N}^*)^{h-1}, \quad \mathbf{k}(x,y) = (k_1(x,y),\dots,k_{h-1}(x,y)),$$

where

$$k_i(x,y) = \left[\frac{1+L_{i-1}(x,y)}{L_i(x,y)}\right],$$

and set, for each $\mathbf{k} \in (\mathbf{N}^*)^{h-1}$,

$$\mathscr{T}_{\mathbf{k}} = \{(x,y) \in \mathscr{T};\ \mathbf{k}(x,y) = \mathbf{k}\}.$$

It is clear that $\mathscr{T}_{\mathbf{k}} \cap \mathscr{T}_{\mathbf{k}'} = \varnothing$ for $\mathbf{k} \neq \mathbf{k}'$ and $\mathscr{T}$ is the union of $\mathscr{T}_{\mathbf{k}}$, $\mathbf{k} \in (\mathbf{N}^*)^{h-1}$.

The trivial estimate

$$k_i = \left[\frac{1+L_{i-1}(x,y)}{L_i(x,y)}\right] \leqslant \frac{1+L_{i-1}(x,y)}{L_i(x,y)} \leqslant \frac{2}{L_i(x,y)},$$

for $(x,y) \in \mathscr{T}_{\mathbf{k}}$, $1 \leqslant i \leqslant h-1$, shows that, for all $(x,y) \in \mathscr{T}_{\mathbf{k}} \cap \overline{Q^{-1}\mathbf{\Omega}_{\mathscr{B}}(Q)}$ and all

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$1 \leqslant i \leqslant h-1$, we have (setting $k_0 = 1$)

$$A_i(Q) \leqslant L_{i-1}(x,y)L_i(x,y) \leqslant 4/(k_{i-1}k_i) \leqslant 4/k_i,$$

so that, for all $1 \leqslant i \leqslant h-1$,

$$k_i \leqslant \frac{4}{A_i(Q)} = \frac{4}{3/(\pi^2\beta_i) + O_{\mathscr{B}}(\log Q/Q)}\,.$$

In particular, there exist $C_1(\mathscr{B}), \dots, C_{h-1}(\mathscr{B}) > 0$ such that

$$\Big\{\mathbf{k} \in (\mathbf{N}^*)^{h-1}\,;\, \mathscr{T}_{\mathbf{k}} \cap \overline{Q^{-1}\Omega_{\mathscr{B}}(Q)} \neq \varnothing\Big\} \subset \bigcap_{i=1}^{h-1} \{\mathbf{k} \in (\mathbf{N}^*)^{h-1}\,;\, 1 \leqslant k_i \leqslant C_i(\mathscr{B})\}, \quad (4{\cdot}3)$$

and $C_{\mathscr{B}}(Q) \leqslant 2^h C_1(B)\cdots C_{h-1}(\mathscr{B}) \ll_{\mathscr{B}} 1$.

On the other hand, the equality

$$\begin{pmatrix} L_{i-1}(x,y) \\ L_i(x,y)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & k_{i-1}\end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ -1 & k_1\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}, \quad (x,y) \in \mathscr{T}_{\mathbf{k}},\ 2 \leqslant i \leqslant h,$$

shows that, for all $\alpha > 0$, $\mathbf{k} \in (\mathbf{N}^*)^h$ and $1 \leqslant i \leqslant h$, we have

$$\operatorname{length}\big(\mathscr{T}_{\mathbf{k}} \cap \{(x,y) \in \mathscr{T}\,;\, L_{i-1}(x,y)L_i(x,y) = \alpha\}\big) \ll k_1\cdots k_{i-1}. \quad (4{\cdot}4)$$

From (4·3) and (4·4) we gather

$$\operatorname{length}\left(\frac{1}{Q}\,\Omega_{\mathscr{B}}(Q)\right) \ll C(\mathscr{B}) = \sum_{i=1}^{h}\sum_{k_i=1}^{C_i(\mathscr{B})}(1 + k_1\cdots k_{i-1}),$$

or simply

$$\operatorname{length}(\Omega_{\mathscr{B}}(Q)) \ll_{\mathscr{B}} Q. \quad (4{\cdot}5)$$

We now combine (4·5) with (4·2) and (4·1) to get

$$G_{\mathscr{B}}(Q) = \frac{6Q^2}{\pi^2 N(Q)}\operatorname{Area}\left(\frac{1}{Q}\,\Omega_{\mathscr{B}}(Q)\right) + O_{\mathscr{B}}\left(\frac{\log Q}{Q}\right). \quad (4{\cdot}6)$$

If we denote

$$\widetilde{\Omega}_{\mathscr{B}} = \Omega_{\prod_{i=1}^h (A_i,B_i)} = \bigcap_{i=1}^{h}\{(x,y) \in \mathscr{T}\,;\, A_i < L_{i-1}(x,y)L_i(x,y) < B_i\} \quad (4{\cdot}7)$$

and notice that

$$\frac{1}{Q}\,\Omega_{\mathscr{B}}(Q) = \Omega_{\prod_{i=1}^h (A_i(Q),B_i(Q))},$$

then Corollary 3.2, (4·7) and (1·11) provide

$$\begin{aligned}\operatorname{Area}\left(\frac{1}{Q}\,\Omega_{\mathscr{B}}(Q) \triangle \widetilde{\Omega}_{\mathscr{B}}\right) &= \operatorname{Area}\left(\Omega_{\prod_{i=1}^h (A_i(Q),B_i(Q))} \triangle \Omega_{\prod_{i=1}^h (A_i,B_i)}\right) \\ &\ll_{\mathscr{B}} \max_{1\leqslant i\leqslant h}\{|A_i(Q) - A_i|, |B_i(Q) - B_i|\} \ll_{\mathscr{B}} \frac{\log Q}{Q}\,.\end{aligned} \quad (4{\cdot}8)$$

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Finally, we insert (4·8) into (4·6) and find

$$G_{\mathscr{B}}(Q) = \frac{6Q^2}{\pi^2 N(Q)} \operatorname{Area}(\widetilde{\Omega}_{\mathscr{B}}) + O_{\mathscr{B}}\left(\frac{\log Q}{Q}\right). \tag{4·9}$$

In particular, (4·9) gives

$$\lim_{Q} G_{\mathscr{B}}(Q) = 2 \operatorname{Area}(\widetilde{\Omega}_{\mathscr{B}}). \tag{4·10}$$

5. *Repartition in short intervals*

In this section $I = [\alpha, \beta]$ is a subinterval of $[0,1]$ and $A_i = A_i(\mathscr{B})$, $B_i = B_i(\mathscr{B})$, $A_i(Q) = A_i(\mathscr{B}, I, Q)$, $B_i(Q) = B_i(\mathscr{B}, I, Q)$ (depending on I and on a box $\mathscr{B} \subset (0,\infty)^h$) and $\mathscr{F}_I(Q) = (\gamma_j)_{j=1}^{N_I(Q)}$ are as in the introduction. For coprime integers $1 \leqslant b < a$ we denote by $\bar{b} \pmod a$ the unique integer $\bar{b} \in \{1, \dots, a-1\}$ such that $b\bar{b} = 1 \pmod a$. The equality $\gamma_j - \gamma_{j-1} = a_j/q_j - a_{j-1}/q_{j-1} = 1/(q_{j-1}q_j)$ provides $a_j = \bar{q}_{j-1} \pmod{q_j}$. So $\gamma_j \in I$ is equivalent to $\bar{q}_{j-1} \in q_j I$. To estimate (1·10), we shall first evaluate

$$\#\bigcap_{i=1}^{h}\left\{(q_1,q_2) \in Q\mathscr{T} \cap \mathbf{Z}^2\,;\, \bar{q}_1 \in q_2 I,\ A_i(Q) < L_{i-1}\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right) L_i\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right) < B_i(Q)\right\}, \tag{5·1}$$

which is denoted by $\widetilde{G}_{\mathscr{B},I}(Q)$. We consider

$$J(Q,q) = J_{\mathscr{B}}(Q,q) = \bigcap_{i=1}^{h}\left\{x \in (Q-q, Q]\,;\, A_i(Q) < L_{i-1}\left(\frac{x}{Q}, \frac{q}{Q}\right) L_i\left(\frac{x}{Q}, \frac{q}{Q}\right) < B_i(Q)\right\}$$

as in Lemma 2·4 and set $I_q = (Q-q, Q]$ for $1 \leqslant q \leqslant Q$. Then the number $\widetilde{G}_{\mathscr{B},I}(Q)$ defined by (5·1) can be expressed as

$$\sum_{q_2=1}^{Q} \#\bigcap_{i=1}^{h}\left\{(x,y) \in I_{q_2} \times q_2 I;\ xy = 1 \pmod{q_2},\right.$$
$$\left. A_i(Q) < L_{i-1}\left(\frac{x}{Q}, \frac{q_2}{Q}\right) L_i\left(\frac{x}{Q}, \frac{q_2}{Q}\right) < B_i(Q)\right\},$$

or, with the notation above,

$$\widetilde{G}_{\mathscr{B},I}(Q) = \sum_{q=1}^{Q} \#\{(x,y) \in J(Q,q) \times qI\,;\ xy = 1 \pmod q\}. \tag{5·2}$$

For $\mathscr{I}_1$ and $\mathscr{I}_2$ subsets of $[0, q-1]$, we denote

$$N_q(\mathscr{I}_1, \mathscr{I}_2) = \#\{(x,y) \in \mathscr{I}_1 \times \mathscr{I}_2\,;\ xy = 1 \pmod q\}. \tag{5·3}$$

Taking into account (5·2) and (5·3) we may write

$$\widetilde{G}_{\mathscr{B},I}(Q) = \sum_{q=1}^{Q} N_q(J(Q,q), qI). \tag{5·4}$$

We shall make use of the following auxiliary result:

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Lemma 5·1 ([**1**, lemma 1·7]). *Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be subintervals of $[0, q-1]$. Then*

$$N_q(\mathscr{I}_1, \mathscr{I}_2) = \frac{\varphi(q)}{q^2}\,|\mathscr{I}_1|\cdot|\mathscr{I}_2| + O(\sigma_0(q)\sigma_{-1/2}(q)q^{\frac{1}{2}}\log^2 q),$$

where $\sigma_r(q) = \sum_{d|q} d^r$.

The set $J(Q,q)$ is, by Lemma 2·4, the union of at most $2^{3h-1}/(A_1\cdots A_{h-1})$ intervals. Hence, applying Lemma 5·1 to each connected component of $J(Q,q)$ and summing up afterwards, we obtain for all $\varepsilon > 0$,

$$\begin{aligned} N_q\big(J(Q,q), qI\big) &= \frac{\varphi(q)}{q^2}\,|J(Q,q)|\cdot q\,|I| + O_\varepsilon(\sigma_0(q)\sigma_{-1/2}(q)q^{\frac{1}{2}}\log^2 q) \\ &= \frac{\varphi(q)}{q}\,|I|\cdot|J(Q,q)| + O_\varepsilon(q^{\frac{1}{2}+\varepsilon}). \end{aligned} \tag{5·5}$$

From (5·4) and (5·5) we get

$$\widetilde{G}_{\mathscr{B},I}(Q) = |I|\sum_{q=1}^{Q}\frac{\varphi(q)}{q}\,|J(Q,q)| + O_\varepsilon(Q^{\frac{3}{2}+\varepsilon}). \tag{5·6}$$

The map

$$f(q) = |J(Q,q)| \tag{5·7}$$

is piecewise C^1 on $[1,Q]$, thus [**1**, lemma 2·3] provides

$$\sum_{q=1}^{Q}\frac{\varphi(q)}{q}\,f(q) = \frac{6}{\pi^2}\int_1^Q f(q)\,dq + O_\varepsilon\big(\|f\|_\infty\log Q + \log Q\int_1^Q |f'(q)|\,dq\big). \tag{5·8}$$

To estimate the error in the previous equality, we use the trivial bound $\|f\|_\infty \leqslant Q$. As in the proof of Lemma 2·4 we have, for large Q and all $1 \leqslant q \leqslant Q$,

$$J(Q,q) = \bigcup_{\substack{\mathbf{k}=(k_1,\dots,k_{h-1})\in(\mathbf{N}^*)^{h-1}\\ 1\leqslant k_i<4/A_i}} \widetilde{J}_{\mathbf{k},\mathscr{B}}(Q,q). \tag{5·9}$$

Here, we have defined $\widetilde{J}_{\mathbf{k},\mathscr{B}}(Q,q)$ as

$$\bigcap_{i=1}^{h}\left\{x\,;\ \begin{array}{l}(x,q)\in Q\mathscr{T}_{\mathbf{k}},\\ A_i(Q)Q^2 < \big(B_{12}^{(\mathbf{k},i)}x + B_{11}^{(\mathbf{k},i)}q\big)\big(B_{22}^{(\mathbf{k},i)}x + B_{21}^{(\mathbf{k},i)}q\big) < B_i(Q)Q^2\end{array}\right\}, \tag{5·10}$$

where $\mathscr{T}_{\mathbf{k}}$ are as in Section 4 and $B^{(\mathbf{k},i)} = B(k_1,\dots,k_{i-1})$ is given by (2·6). If we also let

$$K(\mathscr{B}) = \bigcap_{i=1}^{h}\left\{(k_1,\dots,k_{h-1})\,;\ 1\leqslant k_i < \frac{4}{A_i}\right\}$$

and

$$f_{\mathbf{k},\mathscr{B}}(q) = |\widetilde{J}_{\mathbf{k},\mathscr{B}}(Q,q)|,$$

then we may write

$$f(q) = \sum_{\mathbf{k}\in K(\mathscr{B})} f_{\mathbf{k},\mathscr{B}}(q).$$

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Since $\mathscr{T}_{\mathbf{k}}$ is a polygon which is independent of q and has at most 4^{h-1} sides, the definition of $\widetilde{J}_{\mathbf{k},\mathscr{B}}(Q,q)$ from (5·10) shows that for each $\mathbf{k} \in K(\mathscr{B})$, there exists a partition of $[1,Q]$ into $N(\mathbf{k})$ intervals such that the function $f_{\mathbf{k},\mathscr{B}}$ is monotonic on each of these intervals. Therefore, the variation of f on $[1,Q]$ is

$$\begin{aligned}\int_1^Q |f'(q)|\,dq &\leqslant \sum_{\mathbf{k}\in K(\mathscr{B})} \int_1^Q |f'_{\mathbf{k},\mathscr{B}}(q)|\,dq \leqslant \sum_{\mathbf{k}\in K(\mathscr{B})} 2N(\mathbf{k}) \max_{q\in[1,Q]} f_{\mathbf{k},\mathscr{B}}(q)\\ &\leqslant C(\mathscr{B})\|f\|_\infty \leqslant C(\mathscr{B})Q,\end{aligned} \tag{5·11}$$

for some constant $C(\mathscr{B}) > 0$. We now combine (5·8) and (5·11) to get

$$\sum_{q=1}^{Q} \frac{\varphi(q)}{q} f(q) = \frac{6}{\pi^2}\int_1^Q f(q)\,dq + O_{\mathscr{B}}(Q\log Q). \tag{5·12}$$

From (5·12), (5·6), (5·7) and Corollary 3·2 we gather for all $\varepsilon > 0$,

$$\begin{aligned}\widetilde{G}_{\mathscr{B},I}(Q) &= \frac{6|I|}{\pi^2}\int_1^Q f(q)\,dq + O_{\mathscr{B},\varepsilon}\big(Q^{\frac{3}{2}+\varepsilon}\big)\\ &= \frac{6|I|}{\pi^2}\,\mathrm{Area}\left(\bigcap_{i=1}^{h}\left\{(x,y)\in Q\mathscr{T};\ A_i(Q) < L_{i-1}\left(\frac{x}{Q},\frac{y}{Q}\right)L_i\left(\frac{x}{Q},\frac{y}{Q}\right) < B_i(Q)\right\}\right)\\ &\quad + O_{\mathscr{B},\varepsilon}\big(Q^{\frac{3}{2}+\varepsilon}\big)\\ &= \frac{6|I|Q^2}{\pi^2}\,\mathrm{Area}\left(\bigcap_{i=1}^{h}\{(u_1,u_2)\in\mathscr{T};\ A_i(Q) < L_{i-1}(u_1,u_2)L_i(u_1,u_2) < B_i(Q)\}\right)\\ &\quad + O_{\mathscr{B},\varepsilon}\big(Q^{\frac{3}{2}+\varepsilon}\big)\\ &= \frac{6|I|Q^2}{\pi^2}\,\mathrm{Area}\,(\widetilde{\Omega}_{\mathscr{B}}) + O_{\mathscr{B},\varepsilon}(Q^{\frac{3}{2}+\varepsilon}),\end{aligned}$$

which we combine with (1·2) to get

$$G_{\mathscr{B},I}(Q) = \frac{\widetilde{G}_{\mathscr{B},I}(Q)}{N_I(Q)} \sim 2\,\mathrm{Area}(\widetilde{\Omega}_{\mathscr{B}}). \tag{5·13}$$

6. *The support of the measure μ_2*

From Theorem 1·1 the support $\mathscr{D}_2$ of the measure μ_2 coincides with the closure of the range of the function $\Phi_2\colon \mathscr{T} \to (0,\infty)^2$ defined by (1·5), that is

$$\Phi_2(x,y) = \frac{3}{\pi^2}\left(\frac{1}{xy}, \frac{1}{y([(1+x)/y]y - x)}\right). \tag{6·1}$$

If we take as in Section 4, for any $k \in \mathbf{N}^*$,

$$\mathscr{T}_k = \left\{(x,y)\in\mathscr{T};\ \left[\frac{1+x}{y}\right] = k\right\} = \{(x,y)\in\mathscr{T};\ ky - x \leqslant 1 < (k+1)y - x\},$$

then we end up with a decomposition of $\mathscr{T}$ into mutually disjoint subsets $\mathscr{T}_k$. It is easily seen that $\mathscr{T}_1$ is the triangle bounded by $0 < x \leqslant 1$, $x + y = 1$ and $2y - x = 1$, with $y \in [\frac{2}{3}, 1)$. For $k \geqslant 2$, $\mathscr{T}_k$ is the quadrangle bounded by the segments $ky - x = 1$,

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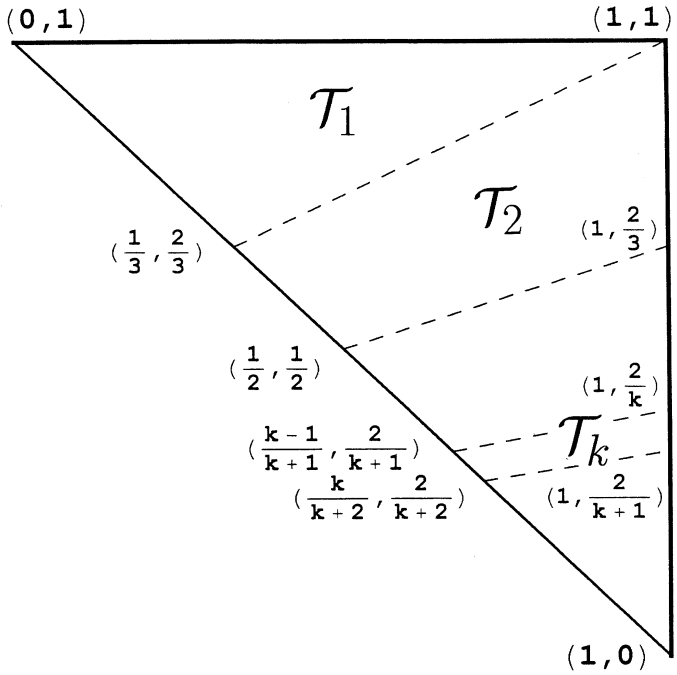


Fig. 3. The decomposition $\mathscr{T} = \bigcup_{k\geqslant 1}\mathscr{T}_k$.

$x = 1$, both with $y \in \big[2/(k+1), 2/k\big]$, and respectively $x + y = 1$, $(k+1)y - x = 1$, both with $y \in \big[2/(k+2), 2/(k+1)\big]$; see the picture in Figure 3.

A plain computation shows that the set $\widetilde{\mathscr{D}}_2 = \{(3x/\pi^2, 3y/\pi^2),\ (x,y) \in \mathscr{D}_2\}$ is the union of the region

$$S_1 = \left\{ \frac{3}{\pi^2}\left(\frac{1}{xy}, \frac{1}{y(y-x)}\right);\ 0 < x \leqslant 1,\ \max\left(1-x, \frac{x+1}{2}\right) \leqslant y \leqslant 1 \right\},$$

bounded by the hyperbola

$$g_0(t) = \frac{t}{t-1}, \quad t \in (1,\infty),$$

and the curves

$$g_1(t) = \frac{8}{1 - 4/t + (1+8/t)^{\frac{1}{2}}}, \quad t \in \left(1, \tfrac{9}{2}\right]$$

and

$$\widetilde{g}_1(t) = \frac{2}{1 - 4/t + (1-4/t)^{\frac{1}{2}}}, \quad t \in \left[\tfrac{9}{2}, \infty\right),$$

and, for $k \geqslant 2$, the 'diamonds'

$$S_k = \left\{ \frac{3}{\pi^2}\left(\frac{1}{xy}, \frac{1}{y(ky-x)}\right);\ \frac{k-1}{k+1} \leqslant x \leqslant 1,\ \max\left(1-x, \frac{x+1}{k+1}\right) \leqslant y \leqslant \frac{x+1}{k} \right\},$$

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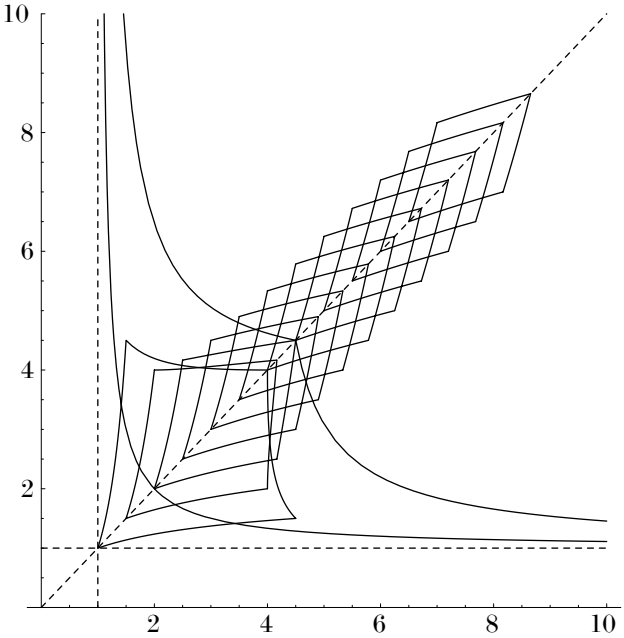


Fig. 4. The regions $S_1, S_2, \ldots, S_{10}$.

bounded by the curves defined by the algebraic functions

$$f_k(t) = \frac{t^2}{k-t}, \quad t \in \left[\frac{k}{2}, \frac{k+1}{2}\right],$$

$$\widetilde{f}_k(t) = \frac{2k}{1+(1+4k/t)^{\frac{1}{2}}}, \quad t \in \left[\frac{k}{2}, \frac{(k+1)^2}{2(k-1)}\right],$$

$$g_k(t) = \frac{2(k+1)^2}{k-2(k+1)/t+k\big(1+4(k+1)/t\big)^{\frac{1}{2}}}, \qquad t \in \left[\frac{k+1}{2}, \frac{(k+2)^2}{2k}\right],$$

$$\widetilde{g}_k(t) = \begin{cases} \dfrac{1}{1-3/t+(1-4/t)^{\frac{1}{2}}}, \quad t \in \left[4, \frac{9}{2}\right], & \text{for } k = 2 \\ \dfrac{2}{k-2(k+1)/t-k(1-4/t)^{\frac{1}{2}}}, \quad t \in \left[\dfrac{(k+1)^2}{2(k-1)}, \dfrac{(k+2)^2}{2k}\right], & \text{for } k \geqslant 3. \end{cases}$$

Notice that each region S_k is symmetric with respect to $y = x$, that is $\widetilde{f}_k = f_k^{-1}$ and $\widetilde{g}_k = g_k^{-1}$ for $k \geqslant 1$.

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